A toy implementation of the BP-bar complex in Sage

Christian Nassau

We describe a small Sage library that implements enough of the Bar complex for BP to compute some low-lying Novikov-Ext groups. More specifically, we implement a class `BPBar` that represents the infinite tensor power

$$\mathcal{B} = \pi_* \left(BP \wedge BP \wedge \cdots \right) = \Gamma \otimes_A \Gamma \otimes_A \Gamma \otimes_A \cdots$$

and also a class `HBPBar` which represents

$$\mathcal{H} = H_* \left(BP \wedge BP \wedge \cdots \right) \subset \mathbb{Q} \otimes \mathcal{B}$$

Here $(A, \Gamma) = (BP_*, BP_* BP)$ is the BP-Hopf algebroid and we think of $BP^{\wedge\infty}$ as the colimit of the $BP^{\wedge k}$ where $BP^{\wedge k} \to BP^{\wedge(k+1)}$ is given by $x \mapsto x \wedge 1$.

This code was an attempt to recreate some computations by Glen Wilson in Sage, and we thank him heartily for sharing his code and for many inspiring communications during the Homotopy Theory Summer in Berlin in 2018.

1 Overview of some classes

The $\mathcal{H}$ and $\mathcal{B}$ come equipped with classes `BasisEnumerator`, `BarBasis` that allow to loop through a basis of a $BP^{\wedge s}$ in a fixed dimension. Specifically, an instance of `BasisEnumerator` knows how to

1. loop through the weighted partitions $P_w(n)$ of an integer n where $w = (w_1, \ldots, w_k)$ is an arbitrary list of weights
2. efficiently compute the reverse map $P_w(n) \to \mathbb{N}_0$ (function `seqno`)

The `BarBasis` translates this information from partitions to elements of $\mathcal{B}$ and $\mathcal{H}$.

The structure maps of $\mathcal{B}$ and $\mathcal{H}$ are all derived from the map η_R which on $BP^{\wedge\infty}$ is given by $x \mapsto 1 \wedge x$. For the generators m_k, t_j of $\mathcal{H}$ this is given by

$$\eta_R(m_n) = \sum_{i+j=n} m_i t_j^{p^i}, \quad \eta_R(\underbrace{1|\cdots|1|t_k}_{p \text{ ones}}) = \underbrace{1|\cdots|1|t_k}_{p+1 \text{ ones}}.$$

This is later used to compute the $\Delta(t_n)$ via $\Delta\eta_R(m_n) = (\eta_R \otimes \mathrm{id})\eta_R(m_n)$.

In Sage one can compute these as follows (here for $p = 3$):

```
sage: M = HBPBar(3) ; m,t = M.gens()
sage: e,D = M.etaR,M.Delta
sage: m[1], e(m[1])
(m1, t1 + m1)
sage: m[2], e(m[2])
(m2, t2 + m2 + m1*t1**3)
sage: m[3], e(m[3])
(m3, t3 + m3 + m2*t1**9 + m1*t2**3)
sage: # an iterated eta_R of t2
sage: t[2], e(t[2]), e(e(t[2]))
(t2, 1|t2, 1|1|t2)
sage: # two coproducts
sage: D(t[1])
1|t1 + t1
sage: D(t[2])
1|t2 + t2 + t1|t1**3 - 3*m1*t1|t1**2 - 3*m1*t1**2|t1
```

There is a class `ArakiGens` that implements the isomorphisms $\mathcal{B} \otimes \mathbb{Q} \leftrightarrow \mathcal{H} \otimes \mathbb{Q}$. This is used to transport the structure formulas from `HBPBar` to `BPBar`:

```
sage: A = ArakiGens(3)
sage: A.mapm2v()
Generic morphism:
  From: Homology bar complex for BP with coefficients in
      Rational Field at prime 3
  To:   Bar complex for BP with coefficients in Rational Field
       at prime 3
sage: A.mapm2v()(m[1])
-1/24*v1
sage: A.mapm2v()(m[2])
-1/19680*v2 + 1/472320*v1**4
sage: A.mapv2m()
Generic morphism:
  From: Bar complex for BP with coefficients in Rational Field
       at prime 3
  To:   Homology bar complex for BP with coefficients in
      Rational Field at prime 3
sage: A.mapv2m()(A.mapm2v()(m[2]))
m2
```

For $\mathcal{B}$ at $p = 3$ we then get, for example

```
sage: B = BPBar(3) ; v,t = B.gens()
sage: B.etaR(v[1])
-24*t1 + v1
sage: B.etaR(v[2])
-19680*t2 + 13824*t1**4 + v2 - 1484*v1*t1**3 + 144*v1**2*t1**2
     - 4*v1**3*t1
sage: B.Delta(t[1])
1|t1 + t1
sage: B.Delta(t[2])
1|t2 + t2 + t1|t1**3 + 1/8*v1*t1|t1**2 + 1/8*v1*t1**2|t1
```

One can use a different base ring for `BPBar` to compute with a reduction of $\mathcal{B}$ modulo some p^N:

```
sage: B = BPBar(3,IntegerModRing(3**2)) ; v,t = B.gens()
sage: B.Delta(t[3])
1|t3 + t3 + t2|t1**9 + t1|t2**3 + 6*v2*t1|t1**8 + 6*v2*t1**2|
    t1**7 + 8*v2*t1**3|t1**6 + 3*v2*t1**4|t1**5 + 3*v2*t1**5|t1
    **4 + 8*v2*t1**6|t1**3 + 6*v2*t1**7|t1**2 + 6*v2*t1**8|t1 +
     8*v1*t2|t2**2 + 8*v1*t2**2|t2 + 8*v1*t1|t1**3*t2**2 + 7*v1
    *t1*t2|t1**3*t2 + 8*v1*t1*t2**2|t1**3 + 8*v1*t1**2|t1**6*t2
     + 8*v1*t1**2*t2|t1**6 + v1**2*t1|t1**2*t2**2 + 2*v1**2*t1*
    t2|t1**2*t2 + v1**2*t1*t2**2|t1**2 + v1**2*t1**2|t1*t2**2 +
     2*v1**2*t1**2|t1**5*t2 + 2*v1**2*t1**2*t2|t1*t2 + 2*v1**2*
    t1**2*t2|t1**5 + v1**2*t1**2*t2**2|t1 + 2*v1**2*t1**3|t1
    **4*t2 + v1**2*t1**3|t1**8 + 2*v1**2*t1**3*t2|t1**4 + v1
    **2*t1**4|t1**7 + 8*v1**3*t1**2|t1**4*t2 + 8*v1**3*t1**2*t2
    |t1**4 + 7*v1**3*t1**3|t1**3*t2 + 8*v1**3*t1**3|t1**7 + 7*
    v1**3*t1**3*t2|t1**3 + 8*v1**3*t1**4|t1**2*t2 + 7*v1**3*t1
    **4|t1**6 + 8*v1**3*t1**4*t2|t1**2 + 8*v1**3*t1**5|t1**5 +
    8*v1**4*t1|t1**8 + 5*v1**4*t1**2|t1**7 + 5*v1**4*t1**4|t1
    **5 + 5*v1**4*t1**5|t1**4 + 5*v1**4*t1**7|t1**2 + 8*v1**4*
    t1**8|t1
```

For the bar complex we also need the "higher coproducts" Δ_k that are induced by the map

$$a_1 \wedge a_2 \wedge \cdots \mapsto a_1 \wedge \cdots a_k \wedge 1 \wedge a_{k+1} \wedge \cdots$$

on spectra. These are available as `Delta_map(k)`:

```
sage: X = BPBar(2) ; v,t = X.gens()
sage: e,D = X.etaR, X.Delta_map
sage: D(3)
Generic endomorphism of Bar complex for BP with coefficients
    in Rational Field at prime 2
sage: D(2)(t[1])
t1
sage: D(2)(e(t[1]))
1|1|t1 + 1|t1
sage: D(2)(e(e(t[1])))
1|1|1|t1
sage: D(2)(e(e(e(t[1]))))
1|1|1|1|t1
```

Under the hood a lot of (hopefully well-positioned) caching is going on, to make sure that the basic structure formulas (including the $\eta_R(v_j)$ and $\Delta_k(t_j)$) are only computed once. This is particularly relevant when such a structure map is evaluated on a monomial:

```
sage: D(2)(e(t[1]**3*t[2]))
1|1|t1**3*t2 + 1|t2|t1**3 + 3*1|t1|t1**2*t2 + 1|t1|t1**5 +
    3*1|t1*t2|t1**2 + 3*1|t1**2|t1*t2 + 3*1|t1**2|t1**4 + 3*1|
    t1**2*t2|t1 + 1|t1**3|t2 + 3*1|t1**3|t1**3 + 1|t1**3*t2 +
    1|t1**4|t1**2 - 2*t1|t1|t1**4 - 6*t1|t1**2|t1**3 - 6*t1|t1
    **3|t1**2 - 2*t1|t1**4|t1 + v1|t1|t1**4 + 3*v1|t1**2|t1**3
    + 3*v1|t1**3|t1**2 + v1|t1**4|t1
```

2 Cohomology

The `BPBar` supports a method `matrix_differential` that computes the matrix of the bar differential $\delta : \Gamma^{\otimes s} \to \Gamma^{\otimes(s+1)}$, i. e. the alternating sum of the Δ_k.

```
sage: BPBar(3).matrix_differential(2,8)
```

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -24 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -24 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 576 & -48 & 0 \end{pmatrix}$$

Here the matrix is computed with respect to the following bases:

```
sage: list(BPBar(3).bar_basis(2,8))
```

$$\left[1|t_1^2, t_1|t_1, v_1|t_1, t_1^2, v_1t_1, v_1^2\right]$$

```
sage: list(BPBar(3).bar_basis(3,8))
```

$$\left[1|1|t_1^2, 1|t_1|t_1, t_1|1|t_1, v_1|1|t_1, 1|t_1^2, t_1|t_1, v_1|t_1, t_1^2, v_1t_1, v_1^2\right]$$

The method `Ext_smith` goes one step further and computes the Smith normal form of the differential; for example, in bidegree (2,16) we find one diagonal coefficient that is divisible by p:

```
sage: d,u,v = BPBar(3).matrix_differential(2,16).smith_form(
    integral=True)
sage: d.diagonal()
[1/8, 1/8, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 6, 0, 0, 0, 0, 0]
```

This coefficient corresponds to an $x \in \Gamma^{\otimes 2}$ such that $\delta(x) \equiv 0 \mod 6$; we therefore find a non-zero cocycle $y = \frac{1}{6}\delta(x)$. The `Ext_smith` routine uses the transformation matrix `u` to compute this cocycle:

```
sage: L = list(BPBar(3).Ext_smith(3,16,withcocycle=True))
sage: # L is a list of tuples (cf,x,y) with delta(x) = cf*y
sage: (c,x,y), = L
```

Then

$$c = 6, \quad x = 24 \cdot t_1|t_1^3 + 2 \cdot t_1^3|t_1 + 3 \cdot v_1t_1|t_1^2$$

and

$$y = 11 \cdot t_1|t_1^2|t_1 - t_1^2|t_1|t_1 + v_1t_1|t_1|t_1.$$

We conclude with some tables that illustrate the timing of such Ext computations and the size/growth of the bar resolution.

6	0.0s	0.0s	0.1s	0.5s	3.9s	2 32.5s	436.5s	? skipped	? skipped	? skipped	? skipped	? skipped
5	0.0s	0.0s	0.0s	0.2s	2 1.1s	5.1s	2 54.4s	2 419.5s	? skipped	? skipped	? skipped	? skipped
4	0.0s	0.0s	0.0s	2 0.1s	0.3s	2 0.9s	2 6.0s	2 32.0s	2 198.6s	2 1369.8s	? skipped	? skipped
3	0.0s	0.0s	2 0.0s	0.0s	2 0.1s	2+2 0.2s	2 0.6s	2 2.4s	2+2 9.3s	2+2+2 48.2s	2+2 237.8s	2+2+2 1258.9s
2	0.0s	2 0.0s	0.0s	2+2 0.0s	2+2 0.0s	2 0.0s	2 0.1s	2+2+2 0.1s	2+2 0.4s	2+4 1.2s	2+2 3.9s	2 13.5s
1	2 0.0s	4 0.0s	2 0.0s	16 0.0s	2 0.0s	8 0.0s	2 0.0s	32 0.0s	2 0.0s	8 0.0s	2 0.1s	16 0.2s
	2	4	6	8	10	12	14	16	18	20	22	24

Fig. 1: Novikov Ext for the prime 2

6	0.0s	0.0s	0.1s	0.5s	3.0s	20.5s	158.6s	2136.6s	? skipped	? skipped	? skipped
5	0.0s	0.0s	0.0s	0.2s	0.7s	3.9s	3 21.1s	154.8s	1275.6s	? skipped	? skipped
4	0.0s	0.0s	0.0s	0.0s	0.2s	3 0.7s	2.5s	14.8s	75.6s	3 480.9s	? skipped
3	0.0s	0.0s	0.0s	3 0.0s	0.0s	0.1s	0.3s	3 0.9s	3.3s	3 13.1s	64.1s
2	0.0s	0.0s	3 0.0s	0.0s	0.0s	0.0s	3 0.0s	0.1s	3 0.1s	3 0.4s	3 0.9s
1	3 0.0s	3 0.0s	9 0.0s	3 0.0s	3 0.0s	9 0.0s	3 0.0s	3 0.0s	27 0.0s	3 0.0s	3 0.0s
	4	8	12	16	20	24	28	32	36	40	44

Fig. 2: Novikov Ext for the prime 3

	8	16	24	32	40	48	56	64	72	80	88	96	104
6	0.0s	0.0s	0.1s	0.4s	1.9s	18.8s	113.4s	657.8s	? skipped	? skipped	? skipped	? skipped	? skipped
5	0.0s	0.0s	0.0s	0.1s	0.4s	2.5s	12.4s	61.8s	301.6s	? skipped	? skipped	? skipped	? skipped
4	0.0s	0.0s	0.0s	0.0s	0.1s	0.5s	1.5s	5.3s	21.0s	5 93.7s	395.8s	? skipped	? skipped
3	0.0s	0.0s	0.0s	0.0s	0.0s	5 0.1s	0.1s	0.4s	1.0s	2.9s	8.5s	5 36.1s	111.9s
2	0.0s	0.0s	0.0s	0.0s	5 0.0s	0.0s	0.0s	0.0s	0.1s	0.1s	5 0.2s	0.6s	1.5s
1	5 0.0s	5 0.0s	5 0.0s	5 0.0s	25 0.0s	5 0.0s	5 0.0s	5 0.0s	5 0.0s	25 0.0s	5 0.0s	5 0.0s	5 0.0s

Fig. 3: Novikov Ext for the prime 5

Fig. 4: Rank of $BP_*BP^{\otimes s}$ for the prime 2

dim	0	1	2	3	4	5	6
2	1	2	3	4	5	6	7
4	1	3	6	10	15	21	28
6	2	6	13	24	40	62	91
8	2	9	24	51	95	162	259
10	2	12	39	96	201	378	658
12	3	18	64	174	400	819	1540
14	4	26	102	304	760	1680	3389
16	4	34	153	505	1375	3276	7070
18	5	46	227	816	2400	6132	14105
20	6	62	333	1292	4076	11109	27125
22	6	78	471	1988	6730	19530	50498
24	7	99	656	2994	10845	33432	91343
26	8	126	906	4440	17135	55944	161126
28	9	156	1227	6466	26560	91686	277866
30	11	194	1643	9272	40457	147426	469420
32	12	240	2184	13138	60715	233073	778435
34	13	292	2868	18384	89850	362796	1269163
36	15	355	3732	25430	131240	556641	2037112
38	16	428	4821	34836	189475	842904	3223003
40	17	510	6171	47262	270582	1260945	5031817
42	20	610	7846	63548	382465	1864976	7758928
44	22	726	9918	84774	535560	2729397	11826605
46	23	854	12447	112220	743360	3955308	17833235
48	26	1007	15538	147491	1023260	5678980	26619446
50	28	1182	19302	192592	1397732	8083314	39357941
52	29	1372	23829	249878	1895375	11411946	57672923
54	32	1594	29279	322264	2552505	15987286	83797938
56	35	1849	35829	413340	3415250	22234191	120785730
58	37	2128	43623	527296	4541515	30709056	172783240

Fig. 5: Rank of $BP_*BP^{\otimes s}$ for the prime 3

dim	0	1	2	3	4	5	6
4	1	2	3	4	5	6	7
8	1	3	6	10	15	21	28
12	1	4	10	20	35	56	84
16	2	7	18	39	75	132	217
20	2	10	30	72	151	288	511
24	2	13	46	124	285	588	1120
28	2	16	66	200	505	1128	2304
32	3	22	96	315	860	2064	4501
36	3	28	136	484	1420	3640	8435
40	3	34	186	722	2276	6216	15260
44	3	40	246	1044	3540	10296	26740
48	4	50	326	1485	5380	16612	45549
52	5	62	429	2084	8025	26214	75698
56	5	74	555	2880	11765	40560	123089
60	5	86	704	3912	16951	61606	196196
64	6	103	891	5254	24065	92022	307083
68	7	124	1125	6996	33745	135444	472808
72	7	145	1406	9228	46785	196726	717213
76	7	166	1734	12040	64135	282192	1073100
80	8	193	2130	15578	87027	400140	1585255
84	9	226	2612	20028	117050	561472	2314516
88	9	259	3180	25576	156150	780324	3342885
92	9	292	3834	32408	206630	1074696	4779684
96	10	332	4602	40794	271360	1467544	6769679
100	11	380	5514	51084	353952	1988208	9503760
104	12	431	6576	63638	458775	2673861	13232205
108	12	482	7788	78816	590955	3570958	18280528
112	13	541	9186	97098	756705	4737477	25069627
116	14	610	10815	119104	963675	6245916	34141702

Fig. 6: Rank of $BP_*BP^{\otimes s}$ for the prime 5

dim	0	1	2	3	4	5	6
8	1	2	3	4	5	6	7
16	1	3	6	10	15	21	28
24	1	4	10	20	35	56	84
32	1	5	15	35	70	126	210
40	1	6	21	56	126	252	462
48	2	9	31	88	215	468	931
56	2	12	45	136	355	828	1765
64	2	15	63	205	570	1413	3199
72	2	18	85	300	890	2338	5593
80	2	21	111	426	1351	3759	9478
88	2	24	141	588	1995	5880	15610
96	3	30	181	801	2885	8981	25060
104	3	36	231	1080	4105	13446	39340
112	3	42	291	1440	5760	19791	60565
120	3	48	361	1896	7976	28692	91651
128	3	54	441	2463	10900	41013	136549
136	3	60	531	3156	14700	57834	200515
144	4	70	641	4010	19600	80535	290500
152	4	80	771	5060	25880	110880	415660
160	4	90	921	6341	33876	151101	587986
168	4	100	1091	7888	43980	203982	823054
176	4	110	1281	9736	56640	272943	1140895
184	4	120	1491	11920	72360	362124	1566985
192	5	135	1736	14510	91770	476595	2133565
200	5	150	2016	17576	115626	622566	2881291
208	5	165	2331	21188	144810	807597	3861214
216	5	180	2681	25416	180330	1040808	5137090
224	5	195	3066	30330	223320	1333089	6788020
232	5	210	3486	36000	275040	1697310	8911420