§2. 电(磁)介质的性质、边值关系运用

1. 电/磁介质

$\vec{P} = \dfrac{\sum \vec{p}_i}{\Delta V}$　　均匀各向同性、线性介质 $\vec{P} = \chi_e \varepsilon_0 \vec{E}$

极化强度　　　　　　　　　　　　　　　　　极化率

$$\nabla\cdot\vec{P} = -\rho_P \Longrightarrow \nabla\cdot\vec{D} = \rho_f$$

电位移

其中 $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$

$$= \underbrace{(1+\chi_e)}_{\varepsilon_r}\varepsilon_0\vec{E} = \varepsilon\vec{E}$$

$\vec{M} = \dfrac{\sum \vec{m}_i}{\Delta V}$　- - - - - -　$\vec{M} = \chi_m \vec{H}$

磁化率

$$\nabla\times\vec{M} = \vec{J}_M \Longrightarrow \nabla\times\vec{H} = \vec{J}_f$$

其中 $\vec{B} = \mu_0(\vec{H}+\vec{M})$

$$= \underbrace{(1+\chi_m)}_{\mu_r}\mu_0\vec{H} = \mu\vec{H}$$

2. 边值关系应用

典型例题：Laplace 方程

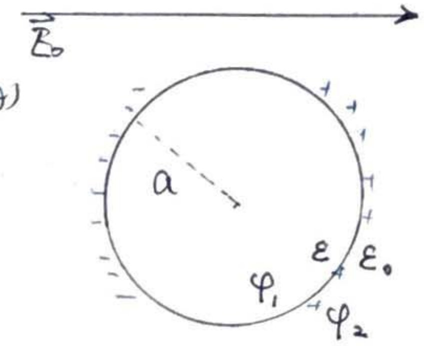
势方程 $\nabla^2\varphi = -\dfrac{\rho}{\varepsilon_0}$ 在无源区，化为

$\nabla^2\varphi = 0$　　Laplace

在球系下 (r, θ, ϕ)，进一步考虑是轴对称

通解　　Legendre

$$\varphi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$



例：半径为a介质球。

求内外电势分布

小定理：$\rho_P = -\left(1-\dfrac{\varepsilon_0}{\varepsilon}\right)\rho_f$

则介质球内任何一点，$\because \rho_f = 0 \therefore \rho_P = 0$

$$\nabla^2\varphi_1 = 0, \quad \nabla^2\varphi_2 = 0$$

$$\begin{cases} \varphi_1 = \displaystyle\sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \\ \varphi_2 = \displaystyle\sum_{l=0}^{\infty}\left(C_l r^l + \frac{D_l}{r^{l+1}}\right)P_l(\cos\theta) \end{cases}$$

边界：$r\to 0$ 时 φ_1 有限，则 $B_l = 0$。$r\to\infty$ 时

$\varphi_2 = -E_0 r\cos\theta$，即 $C_1 \neq 0$，$C_l = 0\ (l\neq 1)$

化为 $$\begin{cases} \varphi_1 = \displaystyle\sum_l A_l r^l P_l(\cos\theta) \\ \varphi_2 = -E_0 r\cos\theta + \displaystyle\sum_{l=0}^{\infty}\frac{D_l}{r^{l+1}}P_l(\cos\theta) \end{cases}$$

边值关系：

$$\begin{cases} \hat{n}\cdot(\vec{D}_2 - \vec{D}_1) = 0 \\ \hat{n}\times(\vec{E}_2 - \vec{E}_1) = 0 \end{cases} \Rightarrow \begin{cases} -\varepsilon_0\dfrac{\partial\varphi_2}{\partial r}\Big|_a + \varepsilon\dfrac{\partial\varphi_1}{\partial r}\Big|_a = 0 \\ -\dfrac{1}{r}\dfrac{\partial\varphi_2}{\partial\theta}\Big|_{r=a} + \dfrac{1}{r}\dfrac{\partial\varphi_1}{\partial\theta}\Big|_{r=a} = 0 \end{cases}$$

即 $\begin{cases} -E_0\cos\theta - \sum_{l=0}^{\infty}(l+1)\dfrac{D_l}{a^{l+2}}P_l(\cos\theta) = \dfrac{\varepsilon}{\varepsilon_0}\sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos\theta) \\ -E_0 a(\cos\theta)' + \sum_{l=0}^{\infty}\dfrac{D_l}{a^{l+1}}P_l'(\cos\theta) = \sum_{l=0}^{\infty} A_l a^l P_l'(\cos\theta) \end{cases}$

注意 $P_1(\cos\theta) = \cos\theta$，各 P_l 相互独立函数

$$\Rightarrow \begin{cases} -E_0P_1 - \sum_{l=0}^{\infty}\dfrac{(l+1)D_l}{a^{l+2}}P_l = \dfrac{\varepsilon}{\varepsilon_0}\sum_{l=0}^{\infty} l A_l a^{l-1} P_l \\ -E_0 P_1' + \sum_{l=0}^{\infty}\dfrac{D_l}{a^{l+2}}P_l' = \sum_{l=0}^{\infty} A_l a^{l-1} P_l' \end{cases}$$

① $l=1$，则

$$\begin{cases} -E_0 - \dfrac{2D_1}{a^3} = \dfrac{\varepsilon}{\varepsilon_0}A_1 \\ -E_0 + \dfrac{D_1}{a^3} = A_1 \end{cases} \Rightarrow \begin{cases} A_1 = -\dfrac{3\varepsilon_0}{\varepsilon+2\varepsilon_0}E_0 \\ D_1 = \dfrac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}E_0 a^3 \end{cases}$$

② $l\neq 1$，则 $A_l = D_l = 0$

$$\therefore \begin{cases} \varphi_1 = -\dfrac{3\varepsilon_0}{\varepsilon+2\varepsilon_0}E_0 r\cos\theta \\ \varphi_2 = -E_0 r\cos\theta + \dfrac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\dfrac{E_0 a^3}{r^2}\cos\theta \end{cases}$$

§3. Green 函数与形式解

1. Green 定理

$$\int (\psi\nabla^2\varphi - \varphi\nabla^2\psi)\,dV = \oint\left(\psi\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\psi}{\partial n}\right)dS$$

$$\begin{aligned} &\psi\nabla^2\varphi - \varphi\nabla^2\psi \\ &= \psi\,\nabla\cdot\nabla\varphi - \varphi\,\nabla\cdot\nabla\psi \\ &= \nabla\cdot(\psi\nabla\varphi) - \nabla\psi\cdot\nabla\varphi - \nabla\cdot(\varphi\nabla\psi) + \nabla\varphi\cdot\nabla\psi \\ &= \nabla\cdot(\psi\nabla\varphi - \varphi\nabla\psi) \end{aligned}$$

$$\begin{aligned} \therefore \int(\psi\nabla^2\varphi - \varphi\nabla^2\psi)\,dV &= \int\nabla\cdot(\psi\nabla\varphi - \varphi\nabla\psi)\,dV \\ &= \oint d\vec{S}\cdot(\psi\nabla\varphi - \varphi\nabla\psi) \\ &= \oint\left(\psi\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\psi}{\partial n}\right)dS \end{aligned}$$

，注意 $d\vec{S} = \hat{n}\,dS$

$\hat{n}\cdot\nabla = \dfrac{\partial}{\partial n}$

2. 静电势的形式解

$$\nabla^2\varphi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}$$

定义 Green 函数：$\nabla^2 G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$

注意：G 对于 $\vec{r},\vec{r}'$ 对称，则 ∇'^2 也成立

取 Green 定理中，$\psi = G(\vec{r},\vec{r}')$，及 $\varphi = \varphi(\vec{r}')$

$$\begin{aligned} &\int\left[G(\vec{r},\vec{r}')\nabla'^2\varphi(\vec{r}') - \varphi(\vec{r}')\nabla'^2 G(\vec{r},\vec{r}')\right]dV' \\ &= \oint\left[G(\vec{r},\vec{r}')\frac{\partial\varphi(\vec{r}')}{\partial n'} - \varphi(\vec{r}')\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\right]dS' \end{aligned}$$

$$\therefore \varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int G(\vec{r},\vec{r}')\rho(\vec{r}')\,dV' + \frac{1}{4\pi}\oint\left[G(\vec{r},\vec{r}')\frac{\partial\varphi(\vec{r}')}{\partial n'} - \varphi(\vec{r}')\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\right]dS'$$

对于无界空间 $\nabla^2 \frac{1}{r} = -4\pi\delta(\vec{r})$，则

$$G(\vec{r},\vec{r}') = \frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{R}$$，代回

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{R}\,dV' + \frac{1}{4\pi}\oint\left[\frac{1}{R}\frac{\partial\varphi(\vec{r}')}{\partial n'} - \varphi(\vec{r}')\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right]dS'$$

两类边值问题

(1) Dirichlet 问题 已知 $\varphi|_{S'}$ 值。取 $G_D(\vec{r},\vec{r}')|_{S'} = 0$

则 $$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int G(\vec{r},\vec{r}')\rho(\vec{r}')\,dV' - \frac{1}{4\pi}\oint \varphi(\vec{r}')\frac{\partial G_D(\vec{r},\vec{r}')}{\partial n'}\,dS'$$

(2) Neumann 问题：已知 $\frac{\partial\varphi}{\partial n'}\Big|_{S'}$ 值，取 $\frac{\partial G_N(\vec{r},\vec{r}')}{\partial n'}\Big|_{S'} = -\frac{4\pi}{S}$

则 $$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int G(\vec{r},\vec{r}')\rho(\vec{r}')\,dV' + \frac{1}{4\pi}\oint G_N(\vec{r},\vec{r}')\frac{\partial\varphi(\vec{r}')}{\partial n'}\,dS' + \langle\varphi\rangle_S$$

作业： 1.3， 1.4， 1.5 ， 1.10 ， 1.11 ， 1.12 ， *1.14