$$u(x,t) = \frac{1}{2\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty} A(k)\, e^{i(kx-\omega t)}\, dk + \int_{-\infty}^{+\infty} A^*(k)\, e^{-i(kx-\omega t)}\, dk\right]$$

得出， $A^*(-k) = A(k)$ 且 $\omega(-k) = \omega(k)$

计算 $$\frac{\partial u(x,0)}{\partial t} = \frac{1}{2\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty} -i\omega(k) A(k)\, e^{ikx}\, dk + \int_{-\infty}^{+\infty} i\omega(k) A^*(k)\, e^{-ikx}\, dk\right]$$

令 $$u(x,0) + \frac{i}{\omega}\frac{\partial u(x,0)}{\partial t} = \frac{1}{2\sqrt{2\pi}}\left\{\int_{-\infty}^{+\infty}\left[A(k')\, e^{ik'x} + A^*(k')\, e^{-ik'x}\right] dk' + \frac{1}{\omega(k)}\int_{-\infty}^{+\infty}\left[\omega(k') A(k')\, e^{ik'x} - \omega(k') A^*(k')\, e^{-ik'x}\right] dk'\right\}$$

作 $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx}\left[u(x,0) + \frac{i}{\omega}\frac{\partial u(x,0)}{\partial t}\right] dx$$ 注意利用 $\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\, dx = \delta(k)$

$$= \frac{1}{4\pi}\left\{\int_{-\infty}^{+\infty}\left[A(k')\, e^{i(k'-k)x} + A^*(k')\, e^{-i(k'+k)x}\right] dk'\, dx + \frac{1}{\omega(k)}\int_{-\infty}^{+\infty}\left[\omega(k') A(k')\, e^{i(k'-k)x} - \omega(k') A^*(k')\, e^{-i(k'+k)x}\right] dk'\, dx\right\}$$

$$= \frac{1}{2}\left\{\int_{-\infty}^{+\infty}\left[A(k')\,\delta(k'-k) + A^*(k')\,\delta(k'+k)\right] dk' + \frac{1}{\omega(k)}\int_{-\infty}^{+\infty}\left[\omega(k') A(k')\,\delta(k'-k) - \omega(k') A^*(k')\,\delta(k'+k)\right] dk'\right\}$$

$$= \frac{1}{2}\left\{A(k) + A^*(-k) + \frac{1}{\omega(k)}\left[\omega(k) A(k) - \omega(-k) A^*(-k)\right]\right\}$$

$$= \frac{1}{2}\left[A(k) + \cancel{A^*(-k)} + A(k) - \cancel{A^*(-k)}\right] = A(k)$$

化为初值问题 $$A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx}\left[u(x,0) + \frac{i}{\omega}\frac{\partial u(x,0)}{\partial t}\right] dx$$

例： Gauss 型波包的演化， $u(x,0) = e^{-\frac{x^2}{2L^2}}\cos k_0 x$ 并取 $\frac{\partial u(x,0)}{\partial t} = 0$ 思考?

色散关系为 $\omega(k) = \nu\left(1 + \frac{a^2 k^2}{2}\right)$

解: $A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx} e^{-\frac{x^2}{2L^2}} \cos k_0 x \, dx = \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx} e^{-\frac{x^2}{2L^2}} \left(e^{ik_0 x} + e^{-ik_0 x}\right) dx$

$$= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty} \left[e^{-\frac{x^2}{2L^2}} e^{-i(k-k_0)x} + e^{-\frac{x^2}{2L^2}} e^{-i(k+k_0)x}\right] dx$$

$$= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty} \left\{e^{-\frac{1}{2L^2}\left[x^2+i2L^2(k-k_0)x\right]} + e^{-\frac{1}{2L^2}\left[x^2+i2L^2(k+k_0)x\right]}\right\} dx$$

$$= \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{+\infty} \left\{e^{-\frac{1}{2L^2}\left[x+iL^2(k-k_0)\right]^2} e^{-\frac{1}{2L^2}L^4(k-k_0)^2} + e^{-\frac{1}{2L^2}\left[x+iL^2(k+k_0)\right]^2} e^{-\frac{1}{2L^2}L^4(k+k_0)^2}\right\} dx$$

$$= \frac{1}{2\sqrt{2\pi}}\left\{e^{-\frac{L^2}{2}(k-k_0)^2}\int_{-\infty}^{+\infty} e^{-\frac{1}{2L^2}\left[x+iL^2(k-k_0)^2\right]^2} dx + e^{-\frac{L^2}{2}(k+k_0)^2}\int_{-\infty}^{+\infty} e^{-\frac{1}{2L^2}\left[x+iL^2(k+k_0)\right]^2} dx\right\}$$

$$= \frac{1}{\cancel{4}\sqrt{2\pi}}\left\{e^{-\frac{L^2}{2}(k-k_0)^2}\int_0^{\infty} e^{-\frac{y_1^2}{2L^2}} dy_1 + e^{-\frac{L^2}{2}(k+k_0)^2}\int_0^{\infty} e^{-\frac{y_2^2}{2L^2}} dy_2\right\}$$

利用 Gauss 积分公式 $I(0) = \int_0^{\infty} e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$

$$= \frac{1}{\cancel{\sqrt{2\pi}}}\left[e^{-\frac{L^2}{2}(k-k_0)^2}\frac{1}{2}\sqrt{\cancel{2\pi}L^2} + e^{-\frac{L^2}{2}(k+k_0)^2}\frac{1}{2}\sqrt{\cancel{2\pi}L^2}\right]$$

$$= \frac{L}{2}\left[e^{-\frac{L^2}{2}(k-k_0)^2} + e^{-\frac{L^2}{2}(k+k_0)^2}\right]$$

群速度 $v_g = \frac{d\omega}{dk} = \nu a^2 k$ ，进一步求出波函数 $u(x,t)$

$$u(x,t) = \frac{L}{4\sqrt{2\pi}}\left\{\int_{-\infty}^{+\infty}\left[e^{-\frac{L^2}{2}(k-k_0)^2} + e^{-\frac{L^2}{2}(k+k_0)^2}\right] e^{i(kx-\nu t)} e^{-i\nu\frac{a^2k^2}{2}t} dk + c.c.\right\}$$

$$= \frac{L}{2\sqrt{2\pi}}\mathrm{Re}\left\{\int_{-\infty}^{+\infty}\left[e^{-\frac{L^2}{2}(k-k_0)^2} + e^{-\frac{L^2}{2}(k+k_0)^2}\right] e^{i(kx-\nu t)} e^{-i\nu\frac{a^2k^2}{2}t} dk\right\}$$

$$= \frac{L}{2\sqrt{2\pi}}\mathrm{Re}\ \cancel{\int_{-\infty}^{+\infty}}\left\{e^{i(k_0x-\nu t)}\int_{-\infty}^{+\infty} e^{-\frac{L^2}{2}(k-k_0)^2} e^{i(k-k_0)x} e^{-i\nu\frac{a^2k^2}{2}t} dk + (k_0 \to -k_0)\right\}$$

其中 $-\frac{L^2}{2}(k-k_0)^2 + i(k-k_0)x - iv\frac{a^2k^2}{2}t = -\frac{1}{2}(L^2+iva^2t)k^2 + (L^2k_0+ix)k - \frac{L^2k_0^2}{2} - ik_0x$

$$= -\frac{1}{2}(L^2+iva^2t)\left[k^2 - \frac{2(L^2k_0+ix)}{L^2+iva^2t}k + \frac{L^2k_0^2+2ik_0x}{L^2+iva^2t}\right]$$

$$= -\frac{1}{2}(L^2+iva^2t)\underbrace{\left(k - \frac{L^2k_0+ix}{L^2+iva^2t}\right)}_{k'}^2 + \frac{1}{2}(L^2+iva^2t)\left[\frac{(L^2k_0+ix)^2}{(L^2+iva^2t)^2} - \frac{L^2k_0^2+2ik_0x}{L^2+iva^2t}\right]$$

$$= -\frac{1}{2}(L^2+iva^2t)k'^2 - \frac{(x-va^2k_0t)^2}{2(L^2+iva^2t)} - \frac{i}{2}va^2k_0^2t$$

故 $u(x,t) = \frac{1}{2}\mathrm{Re}\left\{\frac{L}{\cancel{\sqrt{2\pi}}}\, e^{i(k_0x-vt)}\, e^{-\frac{(x-va^2k_0t)^2}{2(L^2+iva^2t)}}\, e^{-\frac{i}{2}va^2k_0^2t}\underbrace{\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(L^2+iva^2t)k'^2}dk'}_{\int_0^\infty\ \cancel{2}\times\frac{1}{\cancel{2}}\sqrt{\frac{\cancel{2\pi}}{L^2+iva^2t}}} + (k_0\to -k_0)\right\}$

$$= \frac{1}{2}\mathrm{Re}\left\{\sqrt{\frac{L^2}{L^2+iva^2t}}\, e^{-\frac{(x-va^2k_0t)^2}{2(L^2+iva^2t)}}\, e^{-\frac{i}{2}va^2k_0^2t}\, e^{i(k_0x-vt)} + (k_0\to -k_0)\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{e^{-\frac{(x^2-va^2k_0t)^2}{2L^2\left(1+\frac{iva^2t}{L^2}\right)}}}{\sqrt{1+\frac{iva^2t}{L^2}}}\, e^{-\frac{i}{2}va^2k_0^2t}\, e^{i(k_0x-vt)} + (k\to -k_0)\right\}$$

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