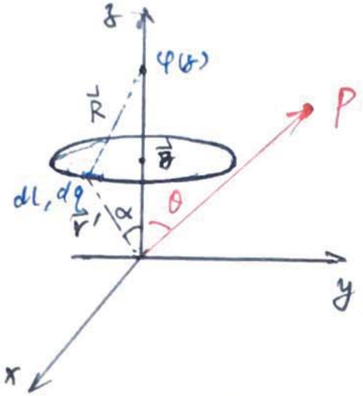
例：环电量 q（均匀）

其他 α，r' 见图。



解：体系为轴对称

dq 对 z 点的电势

$$d\varphi(z)=\frac{1}{4\pi\varepsilon_0}\frac{dq}{R}=\frac{1}{4\pi\varepsilon_0}\frac{dq}{|\vec z-\vec r'|}$$

$$\varphi(z)=\int d\varphi(z)=\frac{1}{4\pi\varepsilon_0}\frac{\int dq}{R}=\frac{q}{4\pi\varepsilon_0}\frac{1}{|\vec z-\vec r'|}$$

利用 $\frac{1}{|\vec r-\vec r'|}=\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)$，有

$$\varphi(r,0)=\frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\alpha)$$

利用 $\varphi(r,\theta)=\sum_{l=0}^{\infty}\varphi_l(r,0)P_l(\cos\theta)$

$$\varphi(r,\theta)=\frac{q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\alpha)P_l(\cos\theta)$$

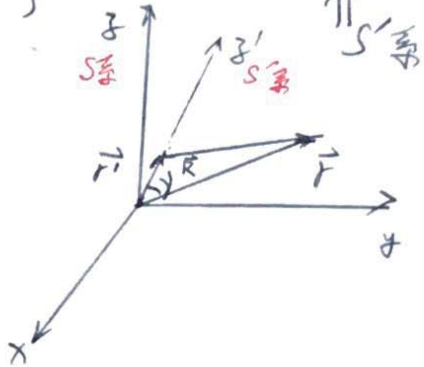
$$=\begin{cases}\dfrac{1}{4\pi\varepsilon_0}\sum\limits_{l=0}^{\infty}\dfrac{r'^l}{r^{l+1}}P_l(\cos\alpha)P_l(\cos\theta) & r>r'\\[2ex] \dfrac{1}{4\pi\varepsilon_0}\sum\limits_{l=0}^{\infty}\dfrac{r^l}{r'^{l+1}}\cdots\cdots & r<r'\end{cases}$$

2. 关于 Green 函数按球函数展开（球函数加法定理）

$$P_l(\cos\gamma)=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

其中 (θ,ϕ) 方位角描 $\vec r$
(θ',ϕ') …… $\vec r'$
$\Uparrow$ S系

(γ,β) 描 $(x'y'z'$系$)$
$\Uparrow$ S′系



球函数展开

$$g(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}Y_{lm}(\theta,\phi)$$

$$A_{lm}=\int Y^*_{lm}(\theta,\phi)g(\theta,\phi)\,d\Omega$$

对于 z 轴上的点 $\theta=0$，ϕ 任意，即 $m=0$，注意

$$Y_{l0}(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$ 那么

$$g(\theta,\phi)=\sum_{l=0}^{\infty}A_{l0}Y_{l0}(\theta,\phi)=\sum_{l=0}^{\infty}\sqrt{\frac{2l+1}{4\pi}}A_{l0}$$

其中 $$A_{l0}=\int Y^*_{l0}(\theta,\phi)g(\theta,\phi)\,d\Omega=\sqrt{\frac{2l+1}{4\pi}}\int P_l(\cos\theta)g(\theta,\phi)\,d\Omega$$

一方面，展 $P_l(\cos\gamma)$，显然 $\gamma=\gamma(\theta,\phi,\theta',\phi')$

$$P_l(\cos\gamma)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}A_{km}(\theta',\phi')Y_{km}(\theta,\phi)$$

一方满球函数方程

$$\frac{1}{r^2}\frac{1}{\sin\gamma}\frac{d}{d\gamma}\left[\sin\gamma\frac{dP_l(\gamma)}{d\gamma}\right]+\frac{l(l+1)}{r^2}P_l(\gamma)=0$$

即 $$\nabla'^2P_l(\cos\gamma)+\frac{l(l+1)}{r^2}P_l(\cos\gamma)=0$$

由于 $\nabla'^2 = \nabla^2$ 为不变量(算符)，则

$$\nabla^2 P_l(\cos\gamma) + \frac{l(l+1)}{r^2} P_l(\cos\gamma) = 0$$ 具体写出

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial P_l(\cos\gamma)}{\partial\theta}\right] + \frac{1}{\sin^2\theta}\frac{\partial^2 P_l(\cos\gamma)}{\partial\phi^2} + l(l+1)P_l(\cos\gamma) = 0$$

正是 l 阶球函数方程，只能有

$$P_l(\cos\gamma) = \sum_{m=-l}^{l} A_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

$$A_{lm}(\theta',\phi') = \int \underline{Y^*_{lm}(\theta,\phi)\, P_l(\cos\gamma)\, d\Omega}$$

我们不确定 l,m

对比

另一方面

$$\begin{cases} g(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m'=-l}^{l} A_{lm'}(\theta',\phi')\, Y_{lm'}(\gamma,\beta) \\ A_{lm'}(\theta',\phi') = \int Y^*_{lm'}(\gamma,\beta)\, g(\theta,\phi)\, d\Omega \end{cases}$$

如取 $g(\theta,\phi) = \sqrt{\frac{4\pi}{2l+1}}\, Y^*_{lm}(\theta,\phi)$，$g$ 也满足同 l 阶球方程

$$\begin{cases} \sqrt{\frac{4\pi}{2l+1}}\, Y^*_{lm}(\theta,\phi) = \sum_{m'=-l}^{l} A_{lm',m}(\theta',\phi')\, Y_{lm'}(\gamma,\beta) \\ A_{lm',m}(\theta',\phi') = \sqrt{\frac{4\pi}{2l+1}} \int Y^*_{lm'}(\gamma,\beta)\, Y^*_{lm}(\theta,\phi)\, d\Omega \end{cases}$$

对于 z' 轴上一点 $\gamma=0$，β 任意，即 $m'=0$。以及

$$Y_{l0}(\gamma,\beta) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\gamma)$$

以及 $\left[\theta(\gamma,\beta), \phi(\gamma,\beta)\right]\Big|_{\gamma=0} = [\theta',\phi']$

$$\begin{cases} \sqrt{\frac{4\pi}{2l+1}}\, Y^*_{lm}(\theta',\phi') = \sqrt{\frac{2l+1}{4\pi}}\, A_{l0,m}(\theta',\phi') \\ A_{l0,m}(\theta',\phi') = \int P_l(\cos\gamma)\, Y^*_{lm}(\gamma,\beta)\, d\Omega \end{cases}$$

$$\therefore A_{lm}(\theta',\phi') = \frac{4\pi}{2l+1}\, Y^*_{lm}(\theta',\phi') \qquad \square$$

最后 $$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

将 Green 函数写成

$$\frac{1}{|\vec r - \vec r'|} = 4\pi \sum_{l=0}^{\infty} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}} \sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

3. 球内/外空间的球 Green 函数

球内或外的 Green 函数

$$G(\vec r,\vec r') = \underbrace{\frac{1}{|\vec r - \vec r'|}}_{\text{球展开}} - \frac{R_0}{r'}\underbrace{\frac{1}{|\vec r - \tilde{\vec r}'|}}_{\text{球展开}} \qquad r'\tilde r' = R_0^2$$

$$G_{in} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left(\begin{cases}\frac{r^l}{r'^{l+1}} & r<r' \\ \frac{r'^l}{r^{l+1}} & r>r'\end{cases} - \frac{r'^l r^l}{R_0^{2l+1}}\right) Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

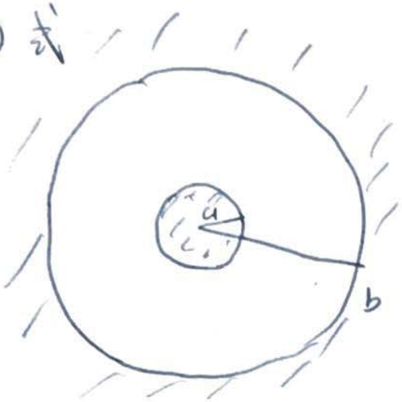
$$G_{out} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left(\begin{cases}\frac{r^l}{r'^{l+1}} & r<r' \\ \frac{r'^l}{r^{l+1}} & r>r'\end{cases} - \frac{R_0^{2l+1}}{r'^{l+1} r^{l+1}}\right) Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

如教材书 P_{122} (3.125) 式

其中 a, b 球

$a \to 0$ 时　球内问题

$b \to \infty$ 时　球外问题



$$P_{2k+1}(0) = 0 \qquad P_{2k}(0) = \frac{(-1)^k (2k-1)!!}{(2k)!!}$$

$$\varphi(\vec{r}) = \frac{Q}{4\pi\varepsilon_0} \sum_{k=0}^{\infty} \frac{(-1)^k (2k-1)!!}{(2k)!!} \begin{Bmatrix} r^{2k} \dfrac{1}{a^{2k+1}} - \dfrac{a^{2k}}{b^{4k+1}} \\ a^{2k} \dfrac{1}{r^{2k+1}} - \dfrac{r^{2k}}{b^{4k+1}} \end{Bmatrix} P_{2k}(\cos\theta)$$

作业：2.13，2.26

3.5，3.6

例《P_{123}》

$$\rho = \frac{Q}{2\pi a^2} \delta(r'-a)\, \delta(\cos\theta')$$

轴对称

$$\sum Y^* Y = \sum P P$$

$$G_{in} = \sum_{l=0}^{\infty} \left(\begin{Bmatrix} \dfrac{r^l}{r'^{l+1}} \\ \dfrac{r'^l}{r^{l+1}} \end{Bmatrix} - \frac{r'^l r^l}{b^{2l+1}} \right) P_l(\cos\theta') P_l(\cos\theta)$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}')\, G(\vec{r}, \vec{r}')\, dV'$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q}{2\pi a^2} \int \delta(r'-a)\, \delta(\cos\theta') \sum_{l=0}^{\infty} \left(\begin{Bmatrix} \\ \end{Bmatrix} - \quad \right) P_l(\cos\theta') P_l(\cos\theta)\, dV'$$

$$dV' = r'^2 dr' \underbrace{\sin\theta'\, d\theta'}_{-d(\cos\theta')} \underline{d\phi'} \qquad \int_0^{2\pi} d\phi' = 2\pi$$

$$= \frac{Q}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \begin{Bmatrix} r^l \left(\dfrac{1}{a^{l+1}} - \dfrac{a^l}{b^{2l+1}} \right) \\ a^l \left(\dfrac{1}{r^{l+1}} - \dfrac{r^l}{b^{2l+1}} \right) \end{Bmatrix} P_l(0)\, P_l(\cos\theta)$$