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2.13

（a）内半径为b的空心导电圆柱体被小间隙分割成两半，电势分别固定为V_1, V_2。证明圆柱内部电势为

$$\Phi(\rho,\phi)=\frac{V_1+V_2}{2}+\frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right) \tag{1}$$

（b）计算每一个半圆柱上的表面电荷密度

(a)

利用2.12结论有

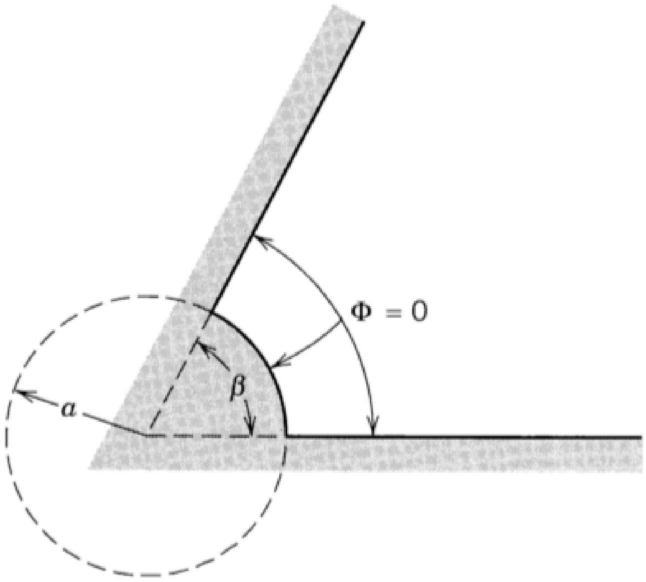
$$\begin{aligned}
\Phi(\rho,\varphi)&=\frac{1}{2\pi}\int_0^{2\pi}\Phi(\rho=b,\varphi)\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\varphi'-\varphi)}d\varphi'\\
&=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}V_1\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\varphi'-\varphi)}d\varphi'+\frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}V_2\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\varphi'-\varphi)}d\varphi'\\
&=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}V_1\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\varphi'-\varphi)}d\varphi'+\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}V_2\frac{b^2-\rho^2}{b^2+\rho^2+2b\rho\cos(\varphi'-\varphi)}d\varphi'\\
&=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\left[V_1\frac{\left(b^2-\rho^2\right)\left(b^2+\rho^2+2b\rho\cos(\varphi'-\varphi)\right)}{\left(b^2+\rho^2\right)^2-\left(2b\rho\cos(\varphi'-\varphi)\right)^2}+V_2\frac{\left(b^2-\rho^2\right)\left(b^2+\rho^2-2b\rho\cos(\varphi'-\varphi)\right)}{\left(b^2+\rho^2\right)^2-\left(2b\rho\cos(\varphi'-\varphi)\right)^2}\right]d\varphi'\\
&=\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{(V_1+V_2)\left(b^2-\rho^2\right)\left(b^2+\rho^2\right)}{\left(b^2+\rho^2\right)^2-4b^2\rho^2\cos^2(\varphi'-\varphi)}+\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{(V_1-V_2)2b\rho\cos(\varphi'-\varphi)}{\left(b^2+\rho^2\right)^2-4b^2\rho^2\cos^2(\varphi'-\varphi)}d\varphi'\\
&=\frac{1}{2\pi}\pi(V_1+V_2)+\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{(V_1-V_2)\left(b^2-\rho^2\right)2b\rho\cos(\varphi'-\varphi)}{b^4+2b^4\rho^2+(2b^2\rho^2-2b^2\rho^2)-4b^2\rho^2\cos^2(\varphi'-\varphi)}d\varphi'\\
&=\frac{1}{2\pi}\pi(V_1+V_2)+\frac{1}{2\pi}\frac{2(V_1-V_2)\left(b^2-\rho^2\right))}{b^2-\rho^2}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\varphi\right)\\
&=\frac{(V_1+V_2)}{2}+\frac{(V_1-V_2)}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\varphi\right)
\end{aligned} \tag{2}$$

(b)

$$\begin{aligned}
\sigma=-\varepsilon_0\frac{\partial\Phi}{\partial\rho}\bigg|_{\rho=b}&=-\varepsilon_0\frac{V_1-V_2}{\pi}\frac{1}{1+\left(\frac{2\rho b}{b^2-\rho^2}\cos\varphi\right)^2}\frac{(b^2-\rho^2)2b-2\rho b(-2\rho)}{\left(b^2-\rho^2\right)^2}\cos\varphi\Bigg|_{\rho=b}\\
&=-\varepsilon_0\frac{V_1-V_2}{\pi}\frac{4b\rho^2}{\left(b^2-\rho^2\right)^2+(2\rho b\cos\varphi)^2}\cos\varphi\Bigg|_{\rho=b}\\
&=-\varepsilon_0\frac{V_1-V_2}{\pi b\cos\varphi}
\end{aligned} \tag{3}$$

2.26

如图所示，二维区域内，$\rho\geq a, 0\leq\phi\leq\beta$的区域被在$\phi=0,\rho=a$和$\phi=\beta$处的接地导体表面分隔出来，在大$\rho$情况下，电势由电荷和固定电势导体之间的结构决定



电势有通解

$$\Phi=\left(A_0+B_0\ln\rho\right)\left(C_0+D_0\phi\right)+\sum_\nu\left(A_\nu\rho^\nu+B_\nu\rho^{-\nu}\right)\left(C_\nu\cos\nu\phi+D_\nu\sin\nu\phi\right)\tag{4}$$

有边界条件

① $\Phi(\phi=0)=0$

② $\Phi(\phi=\beta)=0$

③ $\Phi(\rho=a)=0$

考虑边界条件①有

$$\begin{gathered}\forall\rho\qquad 0=\left(A_0+B_0\ln\rho\right)C_0+\sum_\nu\left(A_\nu\rho^\nu+B_\nu\rho^{-\nu}\right)\left(C_\nu\cos\nu0+D_\nu\sin\nu0\right)\\ \downarrow\\ C_0=0,\qquad C_\nu=0\end{gathered}\tag{5}$$

有解

$$\Phi=\left(A_0+B_0\ln\rho\right)\left(D_0\phi\right)+\sum_\nu\left(A_\nu\rho^\nu+B_\nu\rho^{-\nu}\right)D_\nu\sin\nu\phi\tag{6}$$

考虑边界条件②有

$$\begin{gathered}\forall\rho\qquad 0=\left(A_0+B_0\ln\rho\right)\left(D_0\beta\right)+\sum_\nu\left(A_\nu\rho^\nu+B_\nu\rho^{-\nu}\right)D_\nu\sin\nu\beta\\ \downarrow\\ D_0=0\qquad \nu=\frac{n\pi}{\beta}\end{gathered}\tag{7}$$

有通解

$$\Phi=\left(D_0\phi\right)+\sum_\nu\left(A_n\rho^{\frac{n\pi}{\beta}}+B_n\rho^{-\frac{n\pi}{\beta}}\right)D_n\sin\frac{n\pi}{\beta}\phi\tag{8}$$

考虑边界条件③有

$$\begin{gathered}\forall\theta\qquad 0=\left(D_0\phi\right)+\sum_\nu\left(A_na^{\frac{n\pi}{\beta}}+B_na^{-\frac{n\pi}{\beta}}\right)D_n\sin\frac{n\pi}{\beta}\phi\\ \downarrow\\ D_0=0\qquad B_n=-A_na^{2n\pi/\beta}\end{gathered}\tag{9}$$

有解

$$\Phi(\rho,\phi)=\sum_{n=1}^{\infty}A_n\left(\left(\frac{\rho}{a}\right)^{n\pi/\beta}-\left(\frac{\rho}{a}\right)^{-n\pi/\beta}\right)\sin\left(\frac{n\pi\phi}{\beta}\right)\tag{10}$$

3.5

内半径为a的空心球表面电势固定为$\Phi = V(\theta, \phi)$，证明球内电势的两种形式解等价

(a)

$$\Phi(\mathbf{x}) = \frac{a\left(a^2 - r^2\right)}{4\pi} \int \frac{V\left(\theta', \phi'\right)}{\left(r^2 + a^2 - 2ar\cos\gamma\right)^{3/2}} d\Omega' \tag{11}$$

其中，$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\left(\phi - \phi'\right)$

(b)

$$\Phi(\mathbf{x}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}\left(\frac{r}{a}\right)^l Y_{lm}(\theta, \phi) \tag{12}$$

其中，$A_{lm} = \int d\Omega' Y_{lm}^*\left(\theta', \phi'\right) V\left(\theta', \phi'\right)$

对于$r < r_0$，有球格林函数

$$\frac{1}{|\mathbf{r} - \mathbf{r}_0|} = \frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\gamma}} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r^l}{r_0^{l+1}} Y_{lm}^*\left(\theta', \phi'\right) Y_{lm}(\theta, \phi) \tag{13}$$

其中，$\vec{r}(a, \theta, \phi), \vec{r'}(a, \theta', \phi')$

(13)两边求导再乘r有，

$$\frac{ar\cos\gamma - r^2}{\left(r^2 + a^2 - 2ar\cos\gamma\right)^{3/2}} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l}{2l+1}\frac{r^l}{a^{l+1}} Y_{lm}^*\left(\theta', \phi'\right) Y_{lm}(\theta, \phi) \tag{14}$$

(13)两边对a求导再乘$-a$有

$$\frac{-ar\cos\gamma + a^2}{\left(r^2 + a^2 - 2ar\cos\gamma\right)^{3/2}} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l+1}{2l+1}\frac{r^l}{a^{l+1}} Y_{lm}^*\left(\theta', \phi'\right) Y_{lm}(\theta, \phi) \tag{15}$$

上述两式相加有

$$\frac{a^2 - r^2}{\left(r^2 + a^2 - 2ar\cos\gamma\right)^{3/2}} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r^l}{a^{l+1}} Y_{lm}^*\left(\theta', \phi'\right) Y_{lm}(\theta, \phi) \tag{16}$$

代入(a)有

$$\Phi(\mathbf{x}) = \int V\left(\theta', \phi'\right)\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r^l}{a^l} Y_{lm}^*\left(\theta', \phi'\right) Y_{lm}(\theta, \phi) d\Omega' \tag{17}$$

即(b)形式，QED

3.6

两个点电荷$q, -q$分别位于$z = a, z = -a$处

(a) 求包括在$r > a$和$r < a$情况下，球谐函数展开形式的电势和对r幂级数展开形式的电势

(b) 保持乘积$qa = p/2$为常数，取极限$a \to 0$，求$r \neq 0$处的电势。这就是沿z轴偶极子的电势

(c) 设，(b))中的偶极子被与原点同心的半径为b的接地球壳包围。通过线性叠加求壳内任意点的电势

(a)

取$\vec{a} = a \cdot \hat{k}$，对于两点电荷有电势

$$\Phi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\vec{x} - \vec{a}|} - \frac{1}{|\vec{x} + \vec{a}|}\right) \tag{18}$$

对于球格林函数有

$$\frac{1}{|\vec{x}-\vec{x}'|}=4\pi\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*\left(\hat{x}'\right)Y_{lm}(\hat{x}) \tag{19}$$

且考虑对称性$m=0$，得到

> $Y_{l0}(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$

$$\begin{aligned}\Phi&=\frac{q}{\epsilon_0}\sum_{l,0}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\left[Y_{lm}^*\left(0,\phi'\right)-Y_{l0}^*\left(\pi,\phi'\right)\right]Y_{l0}(\theta,\phi)\\&=\frac{q}{2\pi\epsilon_0}\sum_{l\ \mathrm{odd}}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta)\end{aligned} \tag{20}$$

(b)

取$r_<=a,r_>=r$，有

$$\Phi=\frac{qa}{2\pi\epsilon_0r^2}\sum_{k=0}^{\infty}\left(\frac{a}{r}\right)^{2k}P_{2k+1}(\cos\theta) \tag{21}$$

取$a\to 0,qa=p/2$，则

$$\Phi=\frac{p}{4\pi\epsilon_0}\frac{1}{r^2}P_1(\cos\theta)=\frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2} \tag{22}$$

(c)

考虑偶极子解(22)叠加一个球谐函数解，即为本问解，

取$r_<=r$，有

> 因为不知道电荷位置，所以采用待定系数的形式

$$\Phi=\frac{p}{4\pi\epsilon_0}\left[\frac{1}{r^2}P_1(\cos\theta)+\sum_{l=0}^{\infty}\left[A_lr^l+B_lr^{-(l+1)}\right]P_l(\cos\theta)\right] \tag{23}$$

有边界条件

① $\Phi(\rho=0)=0$

② $\Phi(\rho=b)=0$

考虑边界条件①有$B_l=0$

考虑边界条件②有

$$\begin{gathered}\sum_{l=0}^{\infty}A_lb^{l+2}P_l(\cos\theta)+P_1(\cos\theta)=0\\\downarrow\\A_l=-\frac{1}{b^{l+2}},\quad l=0\end{gathered} \tag{24}$$

有解

$$\Phi=\frac{p}{4\pi\epsilon_0}\left(\frac{1}{r^2}-\frac{r}{b^3}\right)\cos\theta \tag{25}$$