3.7

1 (a)

三电荷的电势为

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[-\frac{2}{r} + \frac{1}{|\vec{r} - a\hat{z}|} + \frac{1}{|\vec{r} + a\hat{z}|}\right] \tag{1}$$

利用勒让德多项式，

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \tag{1}$$

有

$$\begin{aligned}
\Phi &= \frac{q}{4\pi\epsilon_0}\left[-\frac{2}{r} + \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}}[P_l(\cos\theta) + P_l(-\cos\theta)]\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[-\frac{2}{r} + \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}}\left[1 + (-1)^l\right] P_l(\cos\theta)\right] \\
&= \frac{q}{2\pi\epsilon_0}\left[-\frac{1}{r} + \sum_{l \text{ even}} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)\right]
\end{aligned} \tag{2}$$

其中，

$$r_< = \min(r, a), \quad r_> = \max(r, a) \tag{3}$$

为取极限$a \to 0$，设，$r_< = a \quad r_> = r$，有

$$\Phi(r > a) = \frac{q}{2\pi\epsilon_0}\left[-\frac{1}{r} + \sum_{l \text{ even}} \frac{a^l}{r^{l+1}} P_l(\cos\theta)\right] = \frac{q}{2\pi\epsilon_0} \sum_{l=2,4,\ldots} \frac{a^l}{r^{l+1}} P_l(\cos\theta) \tag{4}$$

随$a \to 0$，$l = 2$项占主导地位，定义$qa^2 = Q$，有

$$\Phi \to \frac{Q}{2\pi\epsilon_0 r^3} P_2(\cos\theta) = \frac{Q}{4\pi\epsilon_0 r^3}\left(3\cos^2\theta - 1\right) \tag{5}$$

2 (b)

将(1)改写有

$$\Phi = \frac{q}{4\pi\epsilon_0}\left[-\frac{2}{r} + \frac{1}{|\vec{r} - a\hat{z}|} + \frac{1}{|\vec{r} + a\hat{z}|} + \frac{2}{b} - \frac{b/a}{|\vec{r} - (b^2/a)\hat{z}|} - \frac{b/a}{|\vec{r} + (b^2/a)\hat{z}|}\right] \tag{6}$$

勒让德多项式展开有

$$\begin{aligned}
\Phi &= \frac{q}{4\pi\epsilon_0}\left[\frac{2}{b} - \frac{2}{r} + \sum_{l=0}^{\infty}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{b}{a}\frac{r^l}{(b^2/a)^{l+1}}\right)[P_l(\cos\theta) + P_l(-\cos\theta)]\right] \\
&= \frac{q}{2\pi\epsilon_0}\left[\frac{1}{b} - \frac{1}{r} + \sum_{l \text{ even}}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{1}{b}\left(\frac{ar}{b^2}\right)^l\right) P_l(\cos\theta)\right]
\end{aligned} \tag{7}$$

对于$r > 0$，有

$$\begin{aligned}
\Phi(r > a) &= \frac{q}{2\pi\epsilon_0} \sum_{l=2,4,\ldots}\left(\frac{a^l}{r^{l+1}} - \frac{1}{b}\left(\frac{ar}{b^2}\right)^l\right) P_l(\cos\theta) \\
&= \frac{q}{2\pi\epsilon_0} \sum_{l=2,4,\ldots} \frac{a^l}{r^{l+1}}\left(1 - \left(\frac{r}{b}\right)^{2l+1}\right) P_l(\cos\theta)
\end{aligned} \tag{8}$$

取极限$a \to 0$时，仅$l = 2$项存在，有

$$\Phi \to \frac{Q}{2\pi\epsilon_0 r^3}\left(1-\left(\frac{r}{b}\right)^5\right)P_2(\cos\theta) = \frac{Q}{4\pi\epsilon_0 r^3}\left(1-\left(\frac{r}{b}\right)^5\right)\left(3\cos^2\theta - 1\right) \tag{9}$$

5.30

1 (a)

有失势

$$\mathbf{A}(\mathbf{x}) = \mathbf{e}_z \frac{\mu_0}{4\pi}\int_0^{2\pi} R\,\mathrm{d}\phi' \int_{-\infty}^{\infty} \mathrm{d}z' \frac{K(\phi')}{|\mathbf{x}-\mathbf{x}'|} \tag{10}$$

根据(3.149)有

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{4}{\pi}\int_0^{\infty} \mathrm{d}k \cos\left[k\left(z-z'\right)\right]\left\{\frac{1}{2}I_0\left(k\rho_<\right)K_0\left(k\rho_>\right) + \sum_{m=1}^{\infty}\cos\left[m\left(\phi-\phi'\right)\right]I_m\left(k\rho_<\right)K_m\left(k\rho_>\right)\right\} \tag{11}$$

代回有

$$\mathbf{A}(\mathbf{x}) = \mathbf{e}_z \frac{\mu_0 I}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}z' \int_0^{\infty}\mathrm{d}k' \cos\left[k\left(z-z'\right)\right]\cos\phi I_1\left(k\rho_<\right)K_1\left(k\rho_>\right) \tag{12}$$

利用

$$\int_{-\infty}^{\infty}\cos\left[k\left(z-z'\right)\right]\mathrm{d}z' = \mathrm{Re}\left[e^{ikz}\int_{-\infty}^{\infty}e^{-ikz'}\mathrm{d}z'\right] = \mathrm{Re}\left[2pi\delta(k)e^{ikz}\right] = 2\pi\delta(k)\cos(kz) \tag{13}$$

有

$$\mathbf{A}(\mathbf{x}) = e_z\mu_0 I\cos\phi\frac{1}{2}\lim_{k\to 0}\cos(kz) I_1\left(k\rho_<\right)K_1\left(k\rho_>\right) \tag{14}$$

又因为

$$I_1\left(k\rho_<\right)K_1\left(k\rho_>\right) \xrightarrow{k\to 0} \frac{1}{2}\frac{\rho_<}{\rho_>}\frac{\Gamma(2)}{\Gamma(1)} = \frac{1}{2}\frac{\rho_<}{\rho_>} \tag{15}$$

所以

$$\mathbf{A}(\mathbf{x}) = e_z\frac{\mu_0 I}{4}\frac{\rho_<}{\rho_>}\cos\phi \tag{16}$$

因此有

$$\begin{aligned}\mathbf{B}(\mathbf{x}) = \boldsymbol{\nabla}\times\mathbf{A} &= \frac{\partial A_z}{\partial y}e_x - \frac{\partial A_z}{\partial x}e_y \\ &= -\frac{\mu_0 I}{4}\begin{cases}\frac{1}{R}e_y \\ R\left(\frac{2xy}{\left(x^2+y^2\right)^2}e_x - \frac{x^2-y^2}{\left(x^2+y^2\right)^2}e_y\right)\end{cases} \\ &= -\frac{\mu_0 I}{4}\begin{cases}\frac{1}{R}e_y \\ \frac{R}{\rho^2}(\sin(2\phi)e_x - \cos(2\phi)e_y)\end{cases}\end{aligned} \tag{17}$$

符合偶极子形式

2 (b)

根据半径划分内外场，各自有能量密度

$$\begin{aligned} W_{\text{inside}} &= \frac{1}{2\mu_0}\int_0^R \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi \frac{\mu_0^2 I^2}{16R^2} = \frac{\mu_0 \pi I^2}{32} \\ W_{\text{outside}} &= \frac{1}{2\mu_0}\int_R^\infty \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi \frac{\mu_0^2 I^2 R^2}{16}\frac{1}{\rho^2}\left[\sin^2 2\phi + \cos^2 2\phi\right] = \frac{\mu_0 \pi I^2}{32} \end{aligned} \tag{18}$$

所以

$$W = W_{\text{inside}} + W_{\text{outside}} = \frac{\mu_0 \pi I^2}{16} \tag{19}$$

3 (c)

$$\begin{aligned} J &= \int_{-\pi/2}^{\pi/2} K(\phi) R \, \mathrm{d}\phi = I \\ L &= \frac{W}{I^2} = \frac{\mu_0 \pi}{8} \end{aligned} \tag{20}$$

可以看成一个回路

6.4

1 (a)

设球心位于坐标原点，转动轴跟z轴重合，$\vec{m} = m\hat{z}$，有

$$\mathbf{m} = \int \mathbf{M}(\mathbf{x}) d\mathbf{x} = \mathbf{M}V = \mathbf{M}\frac{4}{3}\pi R^3 \tag{21}$$

根据5.105有

$$\begin{aligned} \vec{B} &= \frac{2\mu_0}{3}\vec{M} \\ &= \frac{2\mu_0}{3}\left(\frac{3m\hat{z}}{4\pi R^3}\right) \\ &= \frac{\mu_0 m}{2\pi R^3}\hat{z} \end{aligned} \tag{22}$$

考虑旋转坐标系内无电流，且，根据5.142有

$$\begin{gathered} \vec{E}' = \vec{E} + \vec{v} \times \vec{B} = 0 \\ \Downarrow \\ \mathbf{E} = -\mathbf{v} \times \mathbf{B} \\ \mathbf{E} = -(\boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{B} \\ \mathbf{E} = -\omega(\hat{\mathbf{z}} \times \mathbf{r}) \times \mathbf{B} \\ \mathbf{E} = \tfrac{-\mu_0 m\omega}{2\pi R^3}(\hat{\mathbf{z}} \times \mathbf{r}) \times \hat{\mathbf{z}} \\ \mathbf{E} = \tfrac{-\mu_0 m\omega}{2\pi R^3}\rho\hat{\boldsymbol{\rho}} \end{gathered} \tag{23}$$

利用高斯定律，有

$$\begin{aligned} \rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\ \rho &= \epsilon_0 \tfrac{\partial E_\rho}{\partial \rho} \\ \rho &= -\tfrac{\epsilon_0 \mu_0 m\omega}{2\pi R^3} \\ \rho &= -\tfrac{m\omega}{2\pi c^2 R^3} \end{aligned} \tag{24}$$

2 (b)

由于球体电中性，单极矩为0，且外场为基函数，l的奇数次项为0，

所以$l=2$为最小非0项，

有电势

$$\begin{aligned}\Phi(\vec{x}) &= -\int \vec{E}\cdot d\vec{l} \\ &= -\left(-\frac{\mu_0 m\omega r^2}{2\pi R^3}\right) \\ &= \frac{\mu_0 m\omega r^2 \sin^2\theta}{2\pi R^3}\end{aligned} \tag{25}$$

利用，$\sin^2\theta = \frac{1}{3}[P_0(\cos\theta) - P_2(\cos\theta)]$有

$$\Phi(\vec{x}) = \frac{\mu_0 m\omega r^2}{2\pi R^3}\frac{1}{3}[P_0(\cos\theta) - P_2(\cos\theta)] \tag{26}$$

对于$l=2$项，

$$\Phi_{\ell=2}(r=R) = -\frac{\mu_0 m\omega}{6\pi R}P_2(\cos\theta) \tag{27}$$

由4.1，4.6有

$$\begin{aligned}q_{2,0} &= \frac{\varepsilon_0 5R^3}{Y_{1,0}(\theta,\varphi)}\left(-\frac{\mu_0 m\omega}{6\pi R}P_2(\cos\theta)\right) \\ &= -\frac{5m\omega R^2}{6\pi c^2}\frac{P_2(\cos\theta)}{Y_{1,0}(\theta,\varphi)} \\ &= -\frac{5m\omega r^3}{6\pi c^2 R^3}\frac{\frac{1}{2}\left(3\cos^2\theta - 1\right)}{\frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\cos^2\theta - 1\right)} \\ &= -\frac{5m\omega R^2}{3c^2\pi}\sqrt{\frac{\pi}{5}}\end{aligned} \tag{28}$$

$$\begin{aligned}Q_{3,3} &= 2\sqrt{\frac{4\pi}{5}}q_{2,0} \\ Q_{3,3} &= 2\sqrt{\frac{4\pi}{5}}\left(-\frac{5m\omega R^2}{3c^2\pi}\sqrt{\frac{\pi}{5}}\right) \\ Q_{3,3} &= -\frac{4m\omega R^2}{3c^2}\end{aligned} \tag{29}$$

由于四极矩无迹，$Q_{1,1}+Q_{2,2}+Q_{3,3}$，且$x-y$对称，$Q_{1,1}=Q_{2,2}$

有，$Q_{1,1}=Q_{2,2}=-\frac{1}{2}Q_{3,3}$

3 (c)

球体内的静电势如上一问所示：

$$\begin{aligned}\Phi_{\text{in}}(\vec{x}) &= \frac{\mu_0 m\omega r^2}{2\pi R^3}\frac{1}{3}[P_0(\cos\theta) - P_2(\cos\theta)] \\ \therefore \vec{E}^r_{\text{in}} &= -\frac{\mu_0 m\omega r}{\pi R^3}\frac{1}{3}[P_0(\cos\theta) - P_2(\cos\theta)]\end{aligned} \tag{30}$$

因为低于$l=2$的项在球外不存在，所以静电势球外为

$$\begin{aligned}\Phi_{\text{out}}(\vec{x}) &= -\frac{\mu_0 m\omega R^2}{2\pi r^3}\frac{1}{3}P_2(\cos\theta) \\ \therefore \vec{E}^r_{\text{out}} &= -\frac{\mu_0 m\omega R^2}{2\pi r^4}P_2(\cos\theta)\end{aligned} \tag{31}$$

$$
\begin{aligned}
\sigma(\theta) &= \varepsilon_0[E^r_{\text{out}} - E^r_{\text{in}}]_{r=R} \\
&= \varepsilon_0\left[-\frac{\mu_0 m\omega R^2}{2\pi r^4}P_2(\cos\theta) - \left(-\frac{\mu_0 m\omega r}{\pi R^3}\frac{1}{3}[1-P_2(\cos\theta)]\right)\right]_{r=R} \\
&= \frac{m\omega}{\pi c^2 R^2}\left(-\frac{1}{2}P_2(\cos\theta) + \frac{1}{3}[1-P_2(\cos\theta)]\right) \\
\sigma(\theta) &= \frac{m\omega}{3\pi c^2 R^2}\left(1 - \frac{5}{2}P_2(\cos\theta)\right)
\end{aligned}
\tag{32}
$$

4 (d)

$$
\begin{aligned}
\mathcal{E} &= \int_{\theta=\pi/2}^{0} \vec{E}\cdot d\vec{\ell} = [-\Phi_{\text{out}}]_{\theta=\pi/2}^{0}\Big|_{r=R} \\
&= \frac{\mu_0 m\omega}{6\pi R} + \frac{\mu_0 m\omega}{12\pi R} \\
\mathcal{E} &= \frac{\mu_0 m\omega}{4\pi R}
\end{aligned}
\tag{33}
$$

7.16

1 (a)

在无源区，有

$$
i\vec{k}\times\vec{H} = -i\omega\vec{D}, \quad i\vec{k}\times\vec{E} - i\omega\vec{B} = 0 \tag{34}
$$

利用法拉第定律和$\vec{B} = \mu_0\vec{H}$，有

$$
i\vec{k}\times(i\vec{k}\times\vec{E}) - i\mu_0\omega(i\vec{k}\times\vec{H}) = 0 \tag{35}
$$

对第二项利用安培定律有

$$
\vec{k}\times(\vec{k}\times\vec{E}) + \mu_0\omega^2\vec{D} = 0 \tag{36}
$$

2 (b)

设$\vec{k} = k\hat{n}$，有

$$
\hat{n}(\hat{n}\cdot\vec{E}) - \vec{E} + \mu_0 v^2\vec{D} = 0 \tag{37}
$$

可以写作矩阵形式

$$
\begin{gathered}
A_{ij} = n_i n_j - \delta_{ij}, \quad W_{ij} = \delta_{ij}\mu_0\epsilon_j = \delta_{ij}/v_j^2 \\
\mathbf{A}\vec{E} = -v^2\mathbf{W}\vec{E}
\end{gathered}
\tag{38}
$$

其本征值即为传播速度，解其久期方程

$$
0 = \det\left(\mathbf{A} + v^2\mathbf{W}\right) \tag{38}
$$

有

$$
\begin{gathered}
v = 0 \\
or \\
\sum_i \frac{n_i^2}{v^2 - v_i^2} = 0
\end{gathered}
\tag{39}
$$

3 (c)

对(38)考虑不同的特征值，有方程

$$(\mathbf{A}+v_a^2\mathbf{W})\vec{E}_a=0,\quad (\mathbf{A}+v_b^2\mathbf{W})\vec{E}_b=0 \tag{40}$$

分别左乘$\vec{E}_b,\vec{E}_a$有

$$\vec{E}_b\mathbf{A}\vec{E}_a+v_a^2\vec{E}_b\mathbf{W}\vec{E}_a=0,\quad \vec{E}_a\mathbf{A}\vec{E}_b+v_b^2\vec{E}_a\mathbf{W}\vec{E}_b=0 \tag{41}$$

由于A,W厄米，用第二个方程减去第一个方程的共轭有

$$(v_b^2-v_a^2)\vec{E}_a\mathbf{W}\vec{E}_b=0 \tag{42}$$

由于$v_a\neq v_b$，则有$\vec{E}_a\mathbf{W}\vec{E}_b=0$

由于$W_{ij}=\delta_{ij}\mu_0\epsilon_j$，则有

$$\vec{E}_a\cdot\vec{D}_b=\vec{E}_b\cdot\vec{D}_a=0 \tag{43}$$

且$\mathbf{A}^2=-\mathbf{A}$，有

$$\vec{D}_a\cdot\vec{D}_b=\vec{E}_a\mathbf{\Sigma}^2\vec{E}_b=\frac{1}{\mu_0^2}\vec{E}_a\mathbf{W}^2\vec{E}_b=\frac{1}{\mu_0^2v_a^2v_b^2}\vec{E}_a\mathbf{A}^2\vec{E}_b=-\frac{1}{\mu_0^2v_a^2v_b^2}\vec{E}_a\mathbf{A}\vec{E}_b \tag{44}$$

但因为$\mathbf{A}\vec{E}_b=-v_b^2\mathbf{W}\vec{E}_b$，得到

$$\vec{D}_a\cdot\vec{D}_b=\frac{1}{\mu_0^2v_a^2}\vec{E}_a\mathbf{W}\vec{E}_b=\frac{1}{\mu_0v_a^2}\vec{E}_a\cdot\vec{D}_b=0 \tag{45}$$

9.11

有电荷和电流密度

$$\begin{aligned}\rho&=q[2\delta(z)-\delta(z-a\cos\omega t)-\delta(z+a\cos\omega t)]\delta(x)\delta(y)\\ \vec{J}&=\hat{z}qa\omega\sin\omega t[\delta(z-a\cos\omega t)-\delta(z+a\cos\omega t)]\delta(x)\delta(y)\end{aligned} \tag{46}$$

因为$ka\ll 1$，可以计算

$$\begin{aligned}\vec{p}(t)&=\textstyle\int\vec{x}\rho d^3x=-q(a\cos\omega t-a\cos\omega t)=0\\ \vec{m}(t)&=\tfrac{1}{2}\textstyle\int\vec{x}\times\vec{J}d^3x=0\end{aligned} \tag{47}$$

$$\begin{aligned}Q_{ij}(t)&=\int\left(3x_ix_j-r^2\delta_{ij}\right)\rho(t)d^3x=-qa^2\cos^2\omega t\,(3\delta_{i3}\delta_{j3}-\delta_{ij})\\ Q_{33}(t)&=-2Q_{11}(t)=-2Q_{22}(t)=-4qa^2\cos^2\omega t\end{aligned} \tag{48}$$

因为电荷直线运动，所以所有磁多极矩都不存在，电极矩中，电四极矩存在

电四极矩可变形为

$$Q_{33}(t)=-2qa^2[1+\cos(2\omega t)]=\mathrm{Re}\left[-2qa^2\left(1+e^{-2i\omega t}\right)\right] \tag{49}$$

由于零频项不辐射，因此可以假设一个四极矩谐波

$$Q_{33}=-2Q_{11}=-2Q_{22}=-2qa^2 \tag{50}$$

以角频率2ω振荡，

其辐射角分布为

$$\frac{dP}{d\Omega}=\frac{c^2Z_0k^6}{512\pi^2}|Q_{33}|^2\sin^2\theta\cos^2\theta=\frac{Z_0q^2}{128\pi^2}(ck)^2(ka)^4\sin^2\theta\cos^2\theta \tag{51}$$

利用，$ck=2\omega$，有

$$\frac{dP}{d\Omega}=\frac{Z_0q^2\omega^2}{32\pi^2}(ka)^4\sin^2\theta\cos^2\theta \tag{52}$$

立体角积分得总功率

$$P=\frac{Z_0q^2\omega^2}{60\pi}(ka)^4 \tag{53}$$

11.5

根据11.31，有

$$\begin{aligned}
a_\parallel(t)&=\frac{du_\parallel}{dt}=\frac{d}{dt}\left(\frac{u'_\parallel+v}{1+\frac{u'_\parallel v}{c^2}}\right)\\
&=\frac{1}{\left(1+\frac{u'_\parallel v}{c^2}\right)^2}\left[\left(1+\frac{u'_\parallel v}{c^2}\right)\frac{du'_\parallel}{dt}-\left(u'_\parallel+v\right)\frac{v}{c^2}\frac{du'_\parallel}{dt}\right]\\
&=\frac{1}{\left(1+\frac{u'_\parallel v}{c^2}\right)^2}\left[1+\frac{u'_\parallel v}{c^2}-\frac{u'_\parallel v}{c^2}-\frac{v^2}{c^2}\right]\left[\frac{dt'}{dt}\right]\frac{du'_\parallel}{dt'}\\
&=\frac{1}{\left(1+\frac{u'_\parallel v}{c^2}\right)^2}\frac{1}{\gamma^2}\left[\frac{dt'}{dt}\right]a'_\parallel\\
&=\frac{a'_\parallel}{\gamma^2\left(1+\frac{u'_\parallel v}{c^2}\right)^2}\left[\frac{d}{dt}\left(\gamma t-\frac{\beta\gamma x_\parallel(t)}{c}\right)\right]\\
&=\frac{a'_\parallel}{\gamma\left(1+\frac{u'_\parallel v}{c^2}\right)^2}\left[1-\frac{v}{c^2}\frac{dx_\parallel}{dt}\right]\\
&=\frac{a'_\parallel}{\gamma\left(1+\frac{u'_\parallel v}{c^2}\right)^2}\left[1-\frac{v}{c^2}u_\parallel\right]\\
&=\frac{a'_\parallel}{\gamma\left(1+\frac{u'_\parallel v}{c^2}\right)^2}\left[1-\frac{v}{c^2}\left(\frac{u'_\parallel+v}{1+\frac{u'_\parallel v}{c^2}}\right)\right]\\
&=\frac{a'_\parallel}{\gamma\left(1+\frac{u'_\parallel v}{c^2}\right)^3}\left[1+\frac{u'_\parallel v}{c^2}-\frac{v\left(u'_\parallel+v\right)}{c^2}\right]\\
&=\frac{a'_\parallel}{\gamma^3\left(1+\frac{u'_\parallel v}{c^2}\right)^3}
\end{aligned} \tag{54}$$

化为矢量式为

$$\mathbf{a}_\parallel(t)=\frac{\mathbf{a}'_\parallel}{\gamma^3\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)^3} \tag{55}$$

对于法向

$$\mathbf{a}_\perp(t)=\frac{1}{\gamma}\frac{d}{dt}\left(\frac{\mathbf{u}'_\perp}{1+\frac{u'_\parallel v}{c^2}}\right)$$

$$
\begin{aligned}
&= \frac{1}{\gamma\left(1+\frac{u'_\parallel v}{c^2}\right)^2}\left[\left(1+\frac{\parallel}{c^2}\right)\frac{a}{dt}\mathbf{u}'_\perp - \mathbf{u}'_\perp\frac{a}{dt}\left(1+\frac{\parallel}{c^2}\right)\right] \\
&= \frac{1}{\gamma\left(1+\frac{u'_\parallel v}{c^2}\right)^2}\left[\frac{dt'}{dt}\right]\left[\left(1+\frac{u'_\parallel v}{c^2}\right)\frac{d}{dt'}\mathbf{u}'_\perp - \mathbf{u}'_\perp\frac{v}{c^2}\frac{d}{dt'}u'_\parallel\right] \\
&= \frac{1}{\gamma^2\left(1+\frac{u'_\parallel v}{c^2}\right)^3}\left[\left(1+\frac{u'_\parallel v}{c^2}\right)\frac{d}{dt'}\mathbf{u}'_\perp - \mathbf{u}'_\perp\frac{v}{c^2}\frac{d}{dt'}u'_\parallel\right] \\
&= \frac{1}{\gamma^2\left(1+\frac{u'_\parallel v}{c^2}\right)^3}\left[\left(1+\frac{u'_\parallel v}{c^2}\right)\mathbf{a}'_\perp - \mathbf{u}'_\perp\frac{v}{c^2}a'_\parallel\right] \\
&= \frac{1}{\gamma^2\left(1+\frac{u'_\parallel v}{c^2}\right)^3}\left[\mathbf{a}'_\perp + \mathbf{a}'_\perp\frac{u'_\parallel v}{c^2} - \mathbf{u}'_\perp\frac{v}{c^2}a'_\parallel\right] \\
&= \frac{1}{\gamma^2\left(1+\frac{u'_\parallel v}{c^2}\right)^3}\left[\mathbf{a}'_\perp + \left(\mathbf{a}' - a'_\parallel\hat{\mathbf{v}}\right)\frac{u'_\parallel v}{c^2} - \mathbf{u}'_\perp\frac{v}{c^2}a'_\parallel\right] \\
&= \frac{1}{\gamma^2\left(1+\frac{u'_\parallel v}{c^2}\right)^3}\left[\mathbf{a}'_\perp + \mathbf{a}'\frac{u'_\parallel v}{c^2} + \frac{va'_\parallel}{c^2}\left(-\mathbf{u}'_\perp - \hat{\mathbf{v}}u'_\parallel\right)\right] \\
&= \frac{1}{\gamma^2\left(1+\frac{u'_\parallel v}{c^2}\right)^3}\left[\mathbf{a}'_\perp + \frac{1}{c^2}\left(\mathbf{a}'u'_\parallel v - va'_\parallel\mathbf{u}'\right)\right] \\
&= \frac{1}{\gamma^2\left(1+\frac{u'_\parallel v}{c^2}\right)^3}\left[\mathbf{a}'_\perp + \frac{1}{c^2}\left(\mathbf{a}'\left(\mathbf{u}'\cdot\mathbf{v}\right) - \left(\mathbf{a}'\cdot\mathbf{v}\right)\mathbf{u}'\right)\right] \\
&= \frac{1}{\gamma^2\left(1+\frac{\mathbf{u}'\cdot\mathbf{v}}{c^2}\right)^3}\left[\mathbf{a}'_\perp + \frac{1}{c^2}\mathbf{v}\times\left(\mathbf{a}'\times\mathbf{u}'\right)\right] \quad \text{q.e.d}
\end{aligned} \tag{56}
$$

11.23

1 (a)

设$\mathcal{P}\mathcal{P}'$分别为lab, cm座标系内四矢量，有

$$
\begin{gathered}
\mathcal{P}_1 = \left(E_1, \vec{p}_{\mathrm{LAB}}\right), \quad \mathcal{P}_2 = \left(m_2, \overrightarrow{0}\right) \\
\mathcal{P}'_1 = \left(E'_1, \overrightarrow{p'}\right), \quad \mathcal{P}'_2 = \left(E'_2, -\overrightarrow{p'}\right)
\end{gathered} \tag{57}
$$

lab系内，利用动量守恒、能量守恒有

$$
\begin{gathered}
\mathcal{P}_1 + \mathcal{P}_2 = \mathcal{P}_3 + \mathcal{P}_4 \\
W^2 = \left(E'_1 + E'_2\right)^2 = \left(E'_1 + E'_2\right)^2 - \left(\overrightarrow{p'}_1 + \overrightarrow{p'}_2\right)^2 = \left(\mathcal{P}'_1 + \mathcal{P}'_2\right)^2
\end{gathered} \tag{58}
$$

其中，$\left(\mathcal{P}'_1 + \mathcal{P}'_2\right)^2$为洛伦兹不变量

进一步有

$$
W^2 = \left(\mathcal{P}'_1 + \mathcal{P}'_2\right)^2 = \left(\mathcal{P}_1 + \mathcal{P}_2\right)^2 = \mathcal{P}_1^2 + \mathcal{P}_2^2 + 2\mathcal{P}_1\cdot\mathcal{P}_2 = m_1^2 + m_2^2 + 2m_2E_1 \tag{59}
$$

考虑

$$
\left(\mathcal{P}_1\cdot\mathcal{P}_2\right)^2 = \left(m_2E_1\right)^2 = m_2^2\left(p_1^2 + m_1^2\right) = m_2^2p_1^2 + m_1^2m_2^2 \tag{60}
$$

$$
\begin{aligned}
\left(\mathcal{P}'_1\cdot\mathcal{P}'_2\right)^2 &= \left(E'_1E'_2 + p'^2\right)^2 = E'^2_1E'^2_2 + 2E'_1E'_2p'^2 + p'^4 \\
&= \left(p'^2 + m_1^2\right)\left(p'^2 + m_2^2\right) + 2E'_1E'_2p'^2 + p'^4
\end{aligned}
$$

$$
\begin{aligned}
&= 2p'^4 + (m_1^2 + m_2^2)p'^2 + 2E_1'E_2'p'^2 + m_1^2 m_2^2 \\
&= p'^2\left(2p'^2 + m_1^2 + m_2^2 + 2E_1'E_2'\right) + m_1^2 m_2^2 \\
&= p'^2\left(E_1'^2 + 2E_1'E_2' + E_2'^2\right) + m_1^2 m_2^2 \\
&= p'^2 W^2 + m_1^2 m_2^2
\end{aligned} \tag{61}
$$

由于洛伦兹不变性，有

$$
\left(\mathcal{P}_1 \cdot \mathcal{P}_2\right)^2 = \left(\mathcal{P}_1' \cdot \mathcal{P}_2'\right)^2 \Rightarrow m_2^2 p_1^2 = p'^2 W^2 \Rightarrow p' = \frac{m_2}{W} p_1 \tag{62}
$$

由于$\vec{p}_1$,$\vec{p}'$同向，有

$$
\overrightarrow{p'} = \frac{m_2}{W}\vec{p}_1 \tag{63}
$$

2 (b)

从$\vec{p}'$洛伦兹变换有

$$
p' = \gamma_{\mathrm{cm}}\left(p_1 - \beta_{\mathrm{cm}} E_1\right); \quad \left(-p'\right) = \gamma_{\mathrm{cm}}\left(-\beta_{\mathrm{cm}} m_2\right) \tag{64}
$$

因此有

$$
\begin{gathered}
\beta_{\mathrm{cm}} = \tfrac{p_1}{m_2+E_1}, \quad \Rightarrow \quad \vec{\beta}_{\mathrm{cm}} = \tfrac{\vec{p}_1}{m_2+E_1} \\
\gamma_{\mathrm{cm}} = \tfrac{1}{\sqrt{1-\beta_{\mathrm{cm}}^2}} = \tfrac{m_2+E_1}{\sqrt{(m_2+E_1)^2-p_1^2}} = \tfrac{m_2+E_1}{\sqrt{m_2^2+2m_2E_1+E_1^2-p_1^2}} = \tfrac{m_2+E_1}{W}
\end{gathered} \tag{65}
$$

3 (c)

非相对论极限下有

$$
E_1 \approx m_1 + \frac{p_1^2}{2m_1} \tag{66}
$$

因此有

$$
\begin{aligned}
W^2 &\approx m_1^2 + m_2^2 + 2m_2\left(m_1 + \frac{p_1^2}{2m_1}\right) \\
&= \left(m_1 + m_2\right)^2 + \frac{m_2}{m_1}p_1^2 \\
&= \left(m_1 + m_2\right)^2\left\{1 + \frac{m_2}{\left(m_1+m_2\right)^2}\frac{p_1^2}{m_1}\right\}
\end{aligned} \tag{67}
$$

$$
\begin{aligned}
W &= \left(m_1 + m_2\right)\sqrt{1 + \frac{m_2}{\left(m_1+m_2\right)^2}\frac{p_1^2}{m_1}} \\
&\approx \left(m_1 + m_2\right)\left\{1 + \frac{m_2}{\left(m_1+m_2\right)^2}\frac{p_1^2}{2m_1}\right\} \\
&= m_1 + m_2 + \frac{m_2}{m_1+m_2}\frac{p_1^2}{2m_1}
\end{aligned} \tag{68}
$$

同样有

$$
\begin{aligned}
\vec{p}' &= \frac{m_2}{W}\vec{p}_1 \approx \frac{m_2}{m_1+m_2}\vec{p}_1 \\
\vec{\beta}_{\mathrm{cm}} &= \frac{\vec{p}_1}{m_2+E_1} \approx \frac{\vec{p}_1}{m_1+m_2}
\end{aligned} \tag{69}
$$

即伽利略变换下的结果