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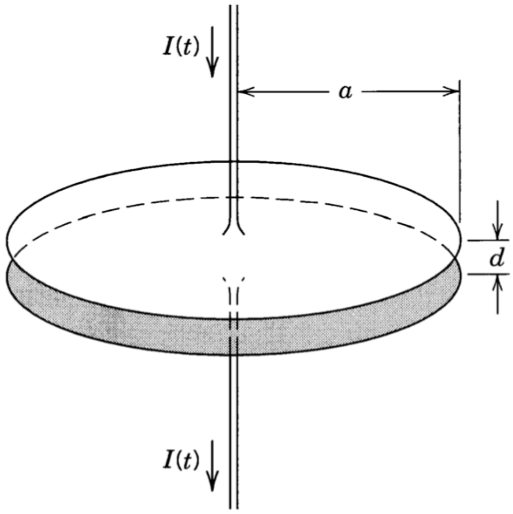
6.14

半径为a、间距为d的两理想圆形平行极板，各自通过轴向导线连接到电流源，如下图所示。导线中电流$I(t)=I_0\cos\omega t$。



（a）计算板间电场和磁场到频率（或波数）的二阶项，忽略场的边缘效应

（b）计算电抗X, (6.140)定义中的w_e 和w_m的体积分到频率的二阶，证明对于输入电流为$I_i=-i\omega Q$，其中Q为极板上的总电荷量，电磁场能量为

$$\int w_e d^3x=\frac{1}{4\pi\epsilon_0}\frac{|I_i|^2 d}{\omega^2 a^2},\int w_m d^3x=\frac{\mu_0}{4\pi}\frac{|I_i|^2 d}{8}\left(1+\frac{\omega^2 a^2}{12c^2}\right) \tag{1}$$

> $$w_e=\frac{1}{4}(\mathbf{E}\cdot\mathbf{D}^*),\quad w_m=\frac{1}{4}(\mathbf{B}\cdot\mathbf{H}^*) \tag{2}$$
> $$X\simeq\frac{4\omega}{|I_i|^2}\int_V(w_m-w_e)d^3x \tag{3}$$

> ❓频率的二阶什么意思

(a)

平板上带电荷

$$Q(t)=\int_0^t I(t')dt'=\frac{I_0}{\omega}\sin\omega t \tag{4}$$

记电场、磁场的前两阶为E_0,E_1,B_0,B_1，即

$$\vec{E}(\vec{r})=(E_0+E_1)\hat{z};\quad \vec{B}(\vec{r})=(B_0+B_1)\hat{\phi} \tag{5}$$

根据高斯定理有

$$\begin{gathered}\oiint \vec{E}_0\cdot d\vec{S}=\frac{Q}{\varepsilon_0}\\ \Downarrow\\ E_0=\frac{Q(t)}{\pi a^2\epsilon_0}=\frac{1}{\pi\epsilon_0}\frac{I_0}{\omega a^2}\sin(\omega t)\end{gathered} \tag{6}$$

根据安培定律

$$\nabla\times\vec{B}_0=\mu_0\epsilon_0\frac{\partial\vec{E}_0}{\partial t}$$

$$\Downarrow$$

$$
\begin{gathered}
\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_0) = \mu_0\epsilon_0 \frac{\partial E_0}{\partial t} = \frac{\mu_0 I_0}{\pi a^2}\cos(\omega t) \\
\Downarrow \\
B_0 = \frac{\mu_0 I_0}{2\pi a}\frac{\rho}{a}\cos(\omega t)
\end{gathered}
\tag{7}
$$

有E_1

$$
\begin{gathered}
\nabla \times \overrightarrow{E_1}^{2} = -\frac{\partial \vec{B}_0}{\partial t} \\
\Downarrow \\
\frac{\partial E_1}{\partial \rho} = -\frac{\mu_0 I}{2\pi a}\frac{\rho}{a}\omega \sin(\omega t) \\
\Downarrow \\
E_1 = -\frac{\mu_0 I_0}{4\pi}\frac{\rho^2}{a^2}\omega \sin(\omega t)
\end{gathered}
\tag{8}
$$

有B_1

$$
\begin{gathered}
\nabla \times \overrightarrow{B_1} = \mu_0\epsilon_0 \frac{\partial E_1}{\partial t} \\
\Downarrow \\
\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_1) = -\frac{\mu_0 I_0}{4\pi c^2}\frac{\omega \rho^2}{a^2}\cos(\omega t) \\
\Downarrow \\
B_1 = -\frac{\mu_0 I_0}{16\pi c^2}\frac{\rho^3}{a^2}\omega^2 \cos(\omega t)
\end{gathered}
\tag{9}
$$

综上有

$$
\begin{aligned}
\vec{E}(\vec{r}) = (E_0 + E_1)\hat{z} &= \left\{\frac{1}{\pi\epsilon_0}\frac{I_0}{\omega a^2}\sin(\omega t) - \frac{\mu_0 I_0}{4\pi}\frac{\rho^2}{a^2}\omega \sin(\omega t)\right\}\hat{z} = \frac{1}{\pi\epsilon_0}\frac{I_0}{\omega a^2}\sin(\omega t)\left\{1 - \frac{\rho^2}{4c^2}\omega^2\right\}\hat{z} \\
\vec{B}(\vec{r}) = (B_0 + B_1)\hat{\phi} &= \left\{\frac{\mu_0 I_0}{2\pi a}\frac{\rho}{a}\cos(\omega t) - \frac{\mu_0 I_0}{16\pi c^2}\frac{\rho^3}{a^2}\omega^2 \cos(\omega t)\right\}\hat{\phi} = \frac{\mu_0 I_0}{2\pi a}\frac{\rho}{a}\cos(\omega t)\left\{1 - \frac{\rho^2}{8c^2}\omega^2\right\}\hat{\phi}
\end{aligned}
\tag{10}
$$

(b)

根据(2)有

$$
\begin{aligned}
w_e &= \frac{1}{4}\vec{E}\cdot\vec{D}^* = \frac{\epsilon_0}{4}|\vec{E}|^2 = \frac{|I_0|^2}{4\pi^2\epsilon_0 a^4\omega^2}\left(1 - \frac{\rho^2\omega^2}{2c^2}\right) \\
w_m &= \frac{1}{4}\vec{B}\cdot\vec{H}^* = \frac{1}{4\mu_0}|\vec{B}|^2 = \frac{|I_0|^2\rho^2}{16\pi^2\epsilon_0 a^4 c^2}\left(1 - \frac{\rho^2\omega^2}{4c^2}\right)
\end{aligned}
\tag{11}
$$

在电容器上体积分有

$$
\begin{aligned}
\int w_e d^3x &= \frac{|I_0|^2 d}{4\pi\epsilon_0 a^2\omega^2}\left(1 - \frac{a^2\omega^2}{4c^2} + \cdots\right) \\
\int w_m d^3x &= \frac{\mu_0 |I_0|^2 d}{32\pi}\left(1 - \frac{a^2\omega^2}{6c^2} + \cdots\right)
\end{aligned}
\tag{12}
$$

根据高斯定理，有平板带电量

$$
\begin{aligned}
Q &= \epsilon_0 \oiint \vec{E}_0 \cdot d\vec{S} \\
&= 2\pi\epsilon_0 \frac{i}{\pi\epsilon_0}\frac{I_0}{\omega a^2}e^{-i\omega t}\int_0^a \left(1 - \frac{\rho^2}{4c^2}\omega^2\right)\rho d\rho \\
&= i\frac{I_0}{\omega}\left\{1 - \frac{a^2}{8c^2}\omega^2\right\}e^{-i\omega t}
\end{aligned}
\tag{13}
$$

有电流关系

$$\begin{aligned} I_i &= -i\omega Q = I_0\left(1 - \tfrac{a^2\omega^2}{8c^2} + \cdots\right) \\ |I_i|^2 &= |I_0|^2\left(1 - \tfrac{a^2\omega^2}{4c^2} + \cdots\right) \end{aligned} \tag{14}$$

代回得到

$$\int w_e d^3x = \frac{|I_i|^2 d}{4\pi\epsilon_0 a^2\omega^2}, \quad \int w_m d^3x = \frac{\mu_0 |I_i|^2 d}{32\pi}\left(1 + \frac{a^2\omega^2}{12c^2}\right) \tag{15}$$