定步长积分

对于定积分

$$I=\int_a^b f(x)dx \tag{1}$$

1 矩形法

$$I \approx \sum_{i=0}^{N-1} f\left(x_i\right) \Delta x \tag{2}$$

2 梯形法

$$I \approx \sum_{i=0}^{N} C_i f\left(x_i\right) \Delta x \tag{3}$$

其中，$C_0=C_N=\frac{1}{2}, C_1=C_2=\cdots=C_{N-1}=1$

具有一阶精度

3 辛普森法

$$\begin{gathered}I \approx \sum_{i=0}^{N} C_i f\left(x_i\right) \Delta x \\ C_i=\begin{cases}\frac{1}{3} & i=0, N \\ \frac{4}{3} & i=1,3, \cdots \\ \frac{2}{3} & i=2,4, \cdots\end{cases}\end{gathered} \tag{4}$$

对于每三项有

$$S_{N/2}=\int_{x_{N-2}}^{x_N} f(x) dx \approx \frac{\Delta x}{3}\left[f\left(x_{N-2}\right)+4 f\left(x_{N-1}\right)+f\left(x_N\right)\right] \tag{5}$$

具有三阶精度

变步长积分

1 变步长梯形法

$$\begin{gathered}T_1=\frac{h}{2}\left[f\left(x_k\right)+f\left(x_{k+1}\right)\right] \\ T_2=\frac{1}{2} T_1+\frac{h}{2} f\left(x_{k+\frac{1}{2}}\right) \\ T_{2n}=\frac{1}{2} T_n+\frac{h}{2} \sum_{k=0}^{n-1} f\left(a+\left(k+\frac{1}{2}\right) h\right)\end{gathered} \tag{6}$$

算法简单，但精度差，收敛慢

2 变步长辛普森法

$$S_n=\frac{4}{3} T_{2n}-\frac{1}{3} T_n=T_{2n}+\frac{1}{3}\left(T_{2n}-T_n\right) \tag{7}$$

重复上述积分过程，将积分区间逐步折半，$\frac{h}{2} \Rightarrow h, 2n \Rightarrow n$

知道相邻两次积分值S_{2n}, S_n满足

$$\begin{array}{ll}\left|S_{2n}-S_n\right|<\varepsilon & \left|S_{2n}\right| \leq 1 \\ \left|\frac{S_{2n}-S_n}{S_{2n}}\right|<\varepsilon & \left|S_{2n}\right|>1\end{array} \tag{8}$$

3 龙贝格法

$$
\begin{aligned}
C_n &= S_{2n} + \frac{1}{15}(S_{2n} - S_n) \\
R_n &= C_{2n} + \frac{1}{63}(C_{2n} - C_n)
\end{aligned}
\tag{9}
$$

二分步长，重复积分过程，使

$$
\begin{array}{ll}
\left|R_{2n} - R_n\right| < \varepsilon & \left|R_{2n}\right| \leq 1 \\
\left|\frac{R_{2n} - R_n}{R_{2n}}\right| < \varepsilon & \left|R_{2n}\right| > 1
\end{array}
\tag{10}
$$

高斯型代数求积

1 定理

考虑$[-1,1]$上的积分

$$\int_{-1}^{1} f(x) dx \approx \sum_{k=0}^{n} A_k f\left(x_k\right) \tag{11}$$

如果节点$x_0, x_1, \cdots, x_n$为$n+1$次多项式$\omega(x)$的根，

$$\omega(x) = \left(x - x_0\right)\left(x - x_1\right)\left(x - x_2\right) \cdots \left(x - x_n\right) \tag{12}$$

且，$\omega(x)$与任一次数不超过n的多项式$q(x)$正交，

$$\int_{-1}^{1} \omega(x) q(x) dx = 0 \tag{13}$$

则，求积公式对一切次数不超过$2n+1$的多项式都准确成立

其求积系数满足

$$A_k = \int_{-1}^{1} \frac{\omega(x)}{\left(x - x_k\right) \omega'\left(x_k\right)} dx \tag{14}$$

2 节点x_k的选取

由特殊函数可知，勒让德多项式满足在$[-1,1]$上正交

$$
\begin{gathered}
p_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}\left[\left(x^2 - 1\right)^n\right] \\
\int_{-1}^{1} p_n(x) p_{n+1}(x) dx = 0
\end{gathered}
\tag{15}
$$

$p_{n+1}(x)$的首项系数为$\frac{[2(n+1)]!}{2^{n+1}[(n+1)!]^2}$

故，取

$$\omega(x) = \frac{2^{n+1}[(n+1)!]^2}{[2(n+1)]!} p_{n+1}(x) = \frac{(n+1)!}{[2(n+1)]!} \cdot \frac{d^{n+1}}{dx^{n+1}}\left[\left(x^2 - 1\right)^{n+1}\right] \tag{16}$$

由此，$p_{n+1}(x)$的$n+1$个零点，即为积分式$\int_{-1}^{1} f(x) dx \approx \sum_{k=0}^{n} A_k f\left(x_k\right)$的节点$x_0, x_1, \cdots, x_n$

p_{n+1}有$n+1$个多项式相乘，共有$n+1$个零点

3 系数A_k的选取

根据上述情况，由求积系数

$$\begin{aligned}A_k&=\int_{-1}^{1}\frac{\omega(x)}{(x-x_k)\omega'(x_k)}dx\\&=\frac{2}{\left(1-x_k^2\right)\left[p_{n+1}'(x_k)\right]^2}\end{aligned}\tag{17}$$

且截断误差为

$$R(f)=\frac{2^{2n+3}}{2n+3}\cdot\frac{[(n+1)!]^4}{[(2n+2)!]^3}f^{(2n+2)}(\eta),\quad \eta\in[-1,1]\tag{18}$$

4 各阶公式

4.1 $n=0$时（1点公式），

$$p_1(x)=\frac{1}{2}\cdot\frac{d^1}{dx^1}\left[\left(x^2-1\right)^1\right]=x\tag{19}$$

$$p_1'(x)=1\tag{20}$$

根据$p_1(x)=0$，解得$x_0=0$

有系数

$$A_0=\frac{2}{\left(1-x_0^2\right)[p_1'(x_0)]^2}=2\tag{21}$$

得到积分公式

$$\int_{-1}^{1}f(x)dx\approx\sum_{k=0}^{n}A_kf(x_k)\quad\longrightarrow\quad\int_{-1}^{1}f(x)dx\approx 2f(0)\tag{22}$$

有截断误差

$$R(f)=\frac{1}{3}f''(\eta)\tag{23}$$

4.2 $n=1$时，（2点公式）

$$p_2(x)=\frac{1}{2^2 2!}\cdot\frac{d^2}{dx^2}\left[\left(x^2-1\right)^2\right]=\frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)^2=\frac{1}{2}\cdot(3x^2-1)\tag{24}$$

$$p_2'(x)=3x\tag{25}$$

取$p_2(x)=0$，解得节点

$$x_0=-\frac{1}{\sqrt{3}}\qquad x_1=\frac{1}{\sqrt{3}}\tag{26}$$

有

$$\begin{aligned}A_0&=\frac{2}{\left(1-x_0^2\right)[p_2'(x_0)]^2}=1\\A_1&=\frac{2}{\left(1-x_1^2\right)[p_2'(x_1)]^2}=1\end{aligned}\tag{27}$$

得到积分公式

$$\int_{-1}^{1}f(x)dx\approx\sum_{k=0}^{1}A_kf(x_k)=f\left(-\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)\tag{28}$$

截断误差

$$R(f)=\frac{1}{135}f^{(4)}(\eta)\tag{29}$$

4.3 n=2时，（3点公式）

$$
\begin{aligned}
p_3(x) &= \frac{1}{2^3 3!} \cdot \frac{d^3}{dx^3}\left[\left(x^2-1\right)^3\right] = \frac{1}{2}x\left(5x^2-3\right) \\
p_3'(x) &= \frac{3}{2}\left(5x^2-1\right)
\end{aligned} \tag{30}
$$

取$p_3(x)=0$，解得节点

$$
x_0 = 0 \quad x_1 = -\sqrt{\frac{3}{5}} \quad x_2 = \sqrt{\frac{3}{5}} \tag{31}
$$

有

$$
\begin{aligned}
A_0 &= \frac{2}{\left(1-x_0^2\right)\left[p_3'\left(x_0\right)\right]^2} = \frac{8}{9} \\
A_1 &= \frac{2}{\left(1-x_1^2\right)\left[p_3'\left(x_1\right)\right]^2} = \frac{5}{9} \\
A_2 &= \frac{2}{\left(1-x_2^2\right)\left[p_3'\left(x_2\right)\right]^2} = \frac{5}{9}
\end{aligned} \tag{32}
$$

得到积分公式

$$
\int_{-1}^{1} f(x)dx \approx \frac{5}{9} f\left(-\frac{\sqrt{15}}{5}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\frac{\sqrt{15}}{5}\right) \tag{33}
$$

截断误差

$$
R(f) = \frac{1}{15750} f^{(6)}(\eta) \tag{34}
$$

4.4 $n=3$时，（4点公式）

$$
\begin{aligned}
p_4(x) &= \frac{1}{2^4 4!} \cdot \frac{d^4}{dx^4}\left[\left(x^2-1\right)^4\right] = \frac{1}{8}\left(35x^4-30x^2+3\right) \\
p_4'(x) &= \frac{1}{8}\left(140x^3-60x\right)
\end{aligned} \tag{35}
$$

取$p_3(x)=0$，解得节点

$$
\begin{aligned}
x_0 = -\sqrt{\frac{1}{35}\left(15-2\sqrt{30}\right)} = -0.339981 \quad x_1 = \sqrt{\frac{1}{35}\left(15-2\sqrt{30}\right)} = 0.339981 \\
x_2 = -\sqrt{\frac{1}{35}\left(2\sqrt{30}+15\right)} = -0.861136 \quad x_3 = \sqrt{\frac{1}{35}\left(2\sqrt{30}+15\right)} = 0.861136
\end{aligned} \tag{36}
$$

有

$$
\begin{aligned}
A_0 &= \frac{2}{\left(1-x_0^2\right)\left[p_4'\left(x_0\right)\right]^2} = 0.652145 \\
A_1 &= \frac{2}{\left(1-x_1^2\right)\left[p_4'\left(x_1\right)\right]^2} = 0.652145 \\
A_2 &= \frac{2}{\left(1-x_2^2\right)\left[p_4'\left(x_2\right)\right]^2} = 0.347855 \\
A_3 &= \frac{3}{\left(1-x_3^2\right)\left[p_4'\left(x_3\right)\right]^2} = 0.347855
\end{aligned} \tag{37}
$$

得到积分公式

$$
\int_{-1}^{1} f(x)dx \approx 0.652145 \cdot f\left(-0.339981\right) + 0.652145 \cdot f(0.339981) + 0.347855 \cdot f(-0.861136) + 0.347855 \cdot f(0.861136) \tag{38}
$$

截断误差

$$
R(f) = \frac{1}{34872875} f^{(8)}(\eta) \tag{39}
$$

5 区间变换

利用两点高斯公式求积分的近似值

$$I = \int_0^1 \sqrt{1+x^2}dx \tag{40}$$

区间变换

拟合办法求得区间变换公式

```
In[62]:= Fit[{{0, -1}, {1, 1}}, {1, x}, x];
Solve[% == t, x]

Out[63]= {{x -> 0.5 (1. + 1. t)}}
```

或

$$\begin{cases} x = \frac{1}{2}(b_2+a_2) + \frac{1}{2}(b_2-a_2)u \\ y = \frac{1}{2}(b_1+a_1) + \frac{1}{2}(b_1-a_1)v \end{cases} \tag{41}$$

或

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^1 f\left(\frac{b-a}{2}x + \frac{a+b}{2}\right)dx \tag{42}$$

得到变换，$x = \frac{1}{2} + \frac{1}{2}t = \frac{1+t}{2}$

得到积分公式

$$\begin{aligned} I = \int_0^1 \sqrt{1+x^2}dx &= \frac{1}{2}\int_{-1}^1 \sqrt{1+\frac{1}{4}(1+t)^2}dt \\ &\approx \frac{1}{2}\cdot\left[\sqrt{1+\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right)^2} + \sqrt{1+\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right)^2}\right] \\ &= 1.147833092 \end{aligned} \tag{43}$$

6 二维高斯求积法

利用高斯求积法计算

$$I = \int_{1.4}^{2.0}(b_2)\int_{1.0}^{1.5}(a_1)\ln(x+2y)dxdy \tag{44}$$

积分区间变换

$$\begin{gathered} \begin{cases} x = \frac{1}{2}(b_2+a_2) + \frac{1}{2}(b_2-a_2)u \\ y = \frac{1}{2}(b_1+a_1) + \frac{1}{2}(b_1-a_1)v \end{cases} \\ \mathrm{d}x = \frac{1}{2}(b_2-a_2)\,\mathrm{d}u \\ \mathrm{d}y = \frac{1}{2}(b_1-a_1)\,\mathrm{d}u \end{gathered} \tag{45}$$

$$\begin{aligned} R &= \{(x,y) \mid 1.4 \le x \le 2.0, 1.0 \le y \le 1.5\} \\ R' &= \{(u,v) \mid -1 \le u \le 1, -1 \le v \le 1\} \end{aligned}$$

有积分

$$\begin{aligned} I &= \int_{1.4}^{2.0}\int_{1.0}^{1.5}\ln(x+2y)dxdy \\ &= 0.075\int^1\int^1 \ln(0.3u+0.5v+4.2)dudv \end{aligned} \tag{46}$$

$$\int_{-1} \int_{-1}$$

分别对x,y进行高斯求积

$$
\begin{aligned}
&u_0 = v_0 = -0.7745967 \quad u_1 = v_1 = 0 \quad u_2 = v_2 = 0.7745967 \\
&A_0 = A_2 = 0.5555556 \quad\ A_1 = 0.8888889
\end{aligned} \tag{47}
$$

有

$$
\begin{aligned}
I &= \int_{1.4}^{2.0} {}_{(a_2)} \int_{1.0}^{1.5} (a_1) \ln{(x+2y)} dxdy \\
&= 0.075 \sum_{i=0}^{2} \sum_{j=0}^{2} A_i A_j \ln{(0.3u_i + 0.5v_j + 4.2)} \\
&= 0.4295545
\end{aligned} \tag{48}
$$