$$(\nabla^2+k^2)\psi(\vec r,\omega) = -4\pi f(\vec r,\omega) \qquad \text{(Helmholtz)}$$

定义 Green 函数

$$(\nabla^2+k^2)G(\vec r,\vec r') = -4\pi\delta(\vec r-\vec r'), \quad \vec R=\vec r-\vec r'$$

对上式两边 $\int f(\vec r',\omega)$ "上式" dV'

$$\int f(\vec r',\omega)(\nabla^2+k^2)G(\vec r,\vec r')\,d^3\vec r' = -4\pi\int f(\vec r',\omega)\,\delta(\vec r-\vec r')\,d^3\vec r'$$

$$(\nabla^2+k^2)\int f(\vec r',\omega)G(\vec r,\vec r')\,d^3\vec r' = -4\pi f(\vec r,\omega)$$

对比 Helmholtz 方程，则

$$\psi(\vec r,\omega) = \int f(\vec r',\omega)G(\vec r,\vec r')\,d^3\vec r'$$

先考虑齐次 Green 函数方程

$(\nabla^2+k^2)G_0(\vec R)=0$ 在球系下，再注意空间

空间同性，且 $\nabla=\nabla_R$ 因 $G(\vec R)=G(\vec r-\vec r')$

$$(\nabla_R^2+k^2)G_0(R)=0$$

$$\frac{1}{R}\frac{\partial^2}{\partial R^2}(RG_0)+k^2G_0=0$$

$$\Rightarrow \frac{\partial^2(RG_0)}{\partial R^2}+k^2(RG_0)=0$$

$\therefore RG_0=Ce^{\pm ikR}$，G的通解为

$$G_0=A\frac{e^{ikR}}{R}+B\frac{e^{-ikR}}{R}$$

试取 $G=\frac{e^{\pm ikR}}{R}$，代回 Green函数方程（非齐）

准备：$\nabla^2(uv)=(\nabla^2u)v+2\nabla u\cdot\nabla v+u\nabla^2 v$ "以下只计算上号，下号类似"

$$\nabla e^{ikr}=ik(\nabla r)e^{ikr}=ik\hat e_r e^{ikr}$$

$$\begin{aligned}\nabla^2 e^{ikr} &= \nabla\cdot(ik\hat e_r e^{ikr})\\ &= ik\hat e_r\cdot\nabla e^{ikr}+ik(\nabla\cdot\hat e_r)e^{ikr}\\ &= ik\hat e_r\cdot ik\hat e_r e^{ikr}+\frac{2ik}{r}e^{ikr}\\ &= -k^2e^{ikr}+\frac{2ik}{r}e^{ikr}\end{aligned}$$

$$\begin{aligned}(\nabla^2+k^2)G &= (\nabla_R^2+k^2)\frac{e^{ikR}}{R} = \nabla_R^2\frac{e^{ikR}}{R}+k^2\frac{e^{ikR}}{R}\\ &= (\nabla_R^2e^{ikR})\frac{1}{R}+2\nabla_R e^{ikR}\cdot\nabla_R\frac{1}{R}+e^{ikR}\nabla_R^2\frac{1}{R}+k^2\frac{e^{ikR}}{R}\\ &= -k^2\frac{e^{ikR}}{R}+\frac{2ik\,e^{ikR}}{R^2}+2ik\hat e_R e^{ikR}\cdot\left(-\frac{\hat e_R}{R^2}\right)+e^{ikR}\underline{\nabla_R^2\frac{1}{R}}+k^2\frac{e^{ikR}}{R}\\ &= -k^2\frac{e^{ikR}}{R}+2ik\frac{e^{ikR}}{R^2}-2ik\frac{e^{ikR}}{R^2}+e^{ikR}\left(\underline{-4\pi\delta(\vec R)}\right)+k^2\frac{e^{ikR}}{R}\\ &= -4\pi e^{ikR}\delta(\vec R) \;\hat{=}\; -4\pi\delta(\vec R) \qquad \square\end{aligned}$$

故 $G(\vec R)=\frac{e^{ikR}}{R}=\frac{e^{ik|\vec r-\vec r'|}}{|\vec r-\vec r'|}$

进一步，取 d'Alembert 方程的 Green 函数

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t}\right)G(x,x')=-4\pi\delta(x-x'), \quad x=(\vec r,t)$$

则 $G(x,x')=\frac{1}{2\pi}\int G(\vec r,\vec r')e^{-i\omega T}\,d\omega$

类似形式解为

$$\Psi(\vec r, t) = \int F(\vec r', t')\, G(x, x')\, d^4x'$$

其中 $G(x,x') = \frac{1}{2\pi}\int G(\vec r, \vec r')\, e^{-i\omega T}\, d\omega$

$$= \frac{1}{2\pi}\int \frac{e^{ikR}}{R}\, e^{-i\omega T}\, d\omega$$

$$= \frac{1}{2\pi}\int \frac{e^{-i\omega(T - \frac{R}{c})}}{R}\, d\omega$$

$$= \frac{1}{R}\,\frac{1}{2\pi}\int e^{-i\omega(T-\frac{R}{c})}\, d\omega$$

$$= \frac{1}{R}\,\delta\left(T - \frac{R}{c}\right), \quad T = t - t'$$

$$\therefore \Psi(\vec r, t) = \int \frac{1}{R}\,\delta\left(t - t' - \frac{R}{c}\right) F(\vec r', t')\, d^3\vec r'\, dt'$$

$$= \int \frac{F(\vec r', t - \frac{R}{c})}{R}\, d^3\vec r'$$

推迟解

2、Lorentz 规范下势方程的形式解、场之推解

$$\begin{cases} \nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho(\vec r, t)}{\varepsilon_0} \\ \nabla^2\vec A - \frac{1}{c^2}\frac{\partial^2 \vec A}{\partial t^2} = -\mu_0 \vec J(\vec r, t) \end{cases}$$

$$\Rightarrow \begin{cases} \varphi = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec r', t - \frac{R}{c})}{R}\, d^3\vec r' \\ \vec A = \frac{\mu_0}{4\pi}\int \frac{\vec J(\vec r', t - \frac{R}{c})}{R}\, d^3\vec r' \end{cases}$$

$$\begin{cases} \nabla\cdot\vec E = \frac{\rho}{\varepsilon_0} \\ \nabla\times\vec E = -\frac{\partial\vec B}{\partial t} \\ \nabla\cdot\vec B = 0 \\ \nabla\times\vec B = \mu_0\vec J + \mu_0\varepsilon_0\frac{\partial\vec E}{\partial t} \end{cases}$$

$$\nabla\times(\nabla\times\vec E) = -\frac{\partial}{\partial t}\nabla\times\vec B = -\mu_0\frac{\partial\vec J}{\partial t} - \mu_0\varepsilon_0\frac{\partial^2\vec E}{\partial t^2}, \quad \mu_0\varepsilon_0 \to \frac{1}{c^2}$$

$$\nabla\times(\nabla\times\vec E) = \nabla(\nabla\cdot\vec E) - \nabla^2\vec E, \quad \nabla\cdot\vec E = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla(\nabla\cdot\vec E) = \frac{\nabla\rho}{\varepsilon_0}$$

$$\nabla^2\vec E - \frac{1}{c^2}\frac{\partial^2\vec E}{\partial t^2} = \frac{1}{\varepsilon_0}\left(\nabla\rho + \frac{1}{c^2}\frac{\partial\vec J}{\partial t}\right)$$

$$\therefore \vec E = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{R}\left[-\nabla'\rho_{(\vec r', t')} - \frac{1}{c^2}\frac{\partial\vec J(\vec r', t')}{\partial t'}\right]_{ret} dV'$$

同理 $\vec B = \frac{\mu_0}{4\pi}\int\frac{1}{R}\left[\nabla'\times\vec J(\vec r', t')\right]_{ret} dV'$

但 $[\nabla'\rho]_{ret} = \nabla'[\rho]_{ret} + \left[\frac{\partial\rho}{\partial t'}\right]\nabla t' \qquad \rho = \rho(\vec r', t'),\ t' = t - \frac{R}{c}$

$$= \nabla'[\rho]_{ret} + \left[\frac{\partial\rho}{\partial t'}\right]_{ret}\left(-\frac{1}{c}\right)\nabla R$$

$$= \nabla'[\rho]_{ret} - \frac{\hat e_R}{c}\left[\frac{\partial\rho}{\partial t'}\right]_{ret}$$

结果 P247 (6.55)(6.56)

$$\begin{cases} \vec E(\vec r, t) = \frac{1}{4\pi\varepsilon_0}\int\left\{\frac{[\rho]_{ret}}{R^2}\hat e_R + \frac{\hat e_R}{cR}\left[\frac{\partial\rho}{\partial t'}\right]_{ret} - \frac{1}{c^2R}\left[\frac{\partial\vec J}{\partial t'}\right]_{ret}\right\} dV' \\ \vec B(\vec r, t) = \frac{\mu_0}{4\pi}\int\left\{\frac{[\vec J]_{ret}\times\hat e_R}{R^2} + \left[\frac{\partial\vec J}{\partial t'}\right]_{ret}\times\frac{\hat e_R}{cR}\right\} dV' \end{cases}$$

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