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2.2

用电像法讨论在内半径为a的中空接地导电球内的点电荷q的问题，试求：

(a) 球内的势

(b) 感生面电荷密度

(c) q所受作用力的大小与方向

(d) 如果球体保持在固定电位V，则解是否有变化?如果球体内外表面带有总电荷Q，解是否有变换?

(a)

记球心为坐标原点，

球内情形相当于球外情形调换像电荷和源电荷，即

$$\begin{gathered} r' = \frac{a^2}{\widetilde{r'}}, \quad q = -\frac{a}{\widetilde{r'}}\tilde{q} \\ \downarrow \\ \widetilde{r'} = \frac{a^2}{r'}, \quad \tilde{q} = -\frac{a}{r'}q \end{gathered} \tag{1}$$

有

$$\begin{aligned} R &= \left|\vec{r} - \vec{r}'\right| = \sqrt{r^2 + r'^2 - 2rr'\cos\alpha} \\ \widetilde{R} &= \left|\vec{r} - \widetilde{\vec{r}'}\right| \\ &= \sqrt{r^2 + \widetilde{r'}^2 - 2r\widetilde{r'}\cos\alpha} \\ &= \frac{1}{r'}\sqrt{r^2r'^2 + R_0^4 - 2R_0^2rr'\cos\alpha} \end{aligned} \tag{2}$$

所以

$$\begin{aligned} \varphi(\vec{r}) &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R} - \frac{R_0}{r'}\frac{1}{\widetilde{R}^2}\right) \\ &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2 + r' - 2rr'\cos\alpha}} - \frac{R_0}{\sqrt{r^2r^2 + R_0^4 - 2R_0^2rr'\cos\alpha}}\right) \end{aligned} \tag{3}$$

(b)

$$\sigma = -\varepsilon_0\frac{\partial\varphi}{\partial r}\bigg|_{r=a} = -\frac{q}{4\pi a^2}\left(\frac{a}{r'}\right)\frac{1 - \frac{a^2}{r'^2}}{\left(1 + \frac{a^2}{r'^2} - 2\frac{a}{r'}\cos\alpha\right)^{3/2}} \tag{4}$$

(c)

$$|\mathrm{F}| = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\left(\frac{a}{r'}\right)^3\left(1 - \frac{a^2}{r'^2}\right)^{-2} \tag{5}$$

方向指向像电荷

(d)

保持固定电位，则解加一常数

表面带有电荷，则在球心加一源电荷

2.7

考虑$z \geq 0$的半空间中的电势问题，在边界$z = 0$有Dirichlet边界条件（在∞处同样）

(a) 写出适当的格林函数$G(\vec{x},\vec{x}')$

(b) 如果在边界上$z=0$处，以原点为圆心、半径为a的圆内电势$\Phi=V$，圆外电势为0。求以柱坐标(ρ,ϕ,z)指定的P点处的电势的积分表达式

(c) 证明，沿圆轴$(\rho=0)$的电势为

$$\Phi=V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right) \tag{6}$$

(d) 证明，在远距离处$(\rho^2+z^2\gg a^2)$，电势可以对$(\rho^2+z^2)^{-1}$幂级数展开，首项为

$$\Phi=\frac{Va^2}{2}\frac{z}{\left(\rho^2+z^2\right)^{3/2}}\left[1-\frac{3a^2}{4\left(\rho^2+z^2\right)}+\frac{5\left(3\rho^2a^2+a^4\right)}{8(\rho^2+z^2)^2}+\cdots\right] \tag{7}$$

验证(c)、(d)的结果在共同的定义域内是一致的

(a)

$$G\left(\vec{x},\vec{x}'\right)=\left(\frac{1}{\sqrt{\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2}}-\frac{1}{\sqrt{\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z+z'\right)^2}}\right) \tag{8}$$

(b)

边界条件即为

$$\varphi|_{\rho\le a,z=0}=V \quad \varphi|_{\rho>a,z=0}=0 \tag{9}$$

有Dirichlet形式解

$$\begin{aligned}\varphi(\vec{r})&=\frac{1}{4\pi\varepsilon_0}\int_{V'}\rho\left(\vec{r'}\right)GdV'-\frac{1}{4\pi}\oint_{S'}\left[\varphi\frac{\partial G}{\partial n'}\right]dS'\\&=-\frac{1}{4\pi}\oint_{S'}\left[\varphi\frac{\partial G}{\partial n'}\right]dS'\end{aligned} \tag{10}$$

其中，

$$\begin{aligned}\frac{\partial G}{\partial n'}&=-\frac{\partial G}{\partial z'}\bigg|_{z'=0}\\&=-\frac{2z}{\left(\left(x-x'\right)^2+\left(y-y'\right)^2+z^2\right)^{3/2}}\end{aligned} \tag{11}$$

有

$$\begin{aligned}\varphi(\vec{r})&=\frac{1}{4\pi}\oint_{S'}\left[\frac{2\varphi z}{\left(\left(x-x'\right)^2+\left(y-y'\right)^2+z^2\right)^{3/2}}\right]dS'\\&=\frac{1}{4\pi}\oint_{S'}\left[\frac{2\varphi z}{\left(-2\rho\rho'\cos\left(\phi-\phi'\right)+\rho^2+\rho'^2+z^2\right)^{3/2}}\right]dS'\\&=\frac{V}{2\pi}\oint_{S'}\left[\frac{z}{\left(-2\rho\rho'\cos\left(\phi-\phi'\right)+\rho^2+\rho'^2+z^2\right)^{3/2}}\right]dS'\end{aligned} \tag{12}$$

> 下边界上有面元?
>
> $$dS'=\rho'd\phi'd\rho' \tag{13}$$

$$\varphi(\vec{r})=\frac{V}{2\pi}\int_0^a\int_0^{2\pi}\left[\frac{z}{\left(-2\rho\rho'\cos\left(\phi-\phi'\right)+\rho^2+\rho'^2+z^2\right)^{3/2}}\right]\rho'd\phi'd\rho' \tag{14}$$

(c)

$$\begin{aligned}\varphi|_{\rho=0}&=\frac{V}{2\pi}\int_0^a\int_0^{2\pi}\frac{z}{\left(\rho'^2+z^2\right)^{3/2}}\rho'd\phi'd\rho'\\&=\frac{V}{2\pi}\cdot2\pi\cdot\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)\end{aligned} \tag{15}$$

$$= V \cdot \left(1 - \frac{\sim}{\sqrt{a^2+z^2}} \right)$$

(d)

当$\rho^2+z^2 \gg a^2$时，利用，$(1+x)^{-3/2} = O\left(x^5\right) + \frac{315x^4}{128} - \frac{35x^3}{16} + \frac{15x^2}{8} - \frac{3x}{2} + 1$

$$\begin{aligned} \varphi(\vec{r}) &= \frac{V}{2\pi}\int_0^a\int_0^{2\pi}\left[\frac{z}{\left(-2\rho\rho'\cos\left(\phi-\phi'\right)+\rho^2+\rho'^2+z^2\right)^{3/2}}\right]\rho' d\phi' d\rho' \\ &= \frac{V}{2\pi}\cdot\frac{z}{\left(\rho^2+z^2\right)^{3/2}}\cdot\iint\left(1-\frac{\rho'^2-2\rho\rho'\cos\left(\phi-\phi'\right)}{\rho^2+z^2}\right)^{-3/2}\rho' d\phi' d\rho' \\ &= \frac{V}{2\pi}\cdot\frac{z}{\left(\rho^2+z^2\right)^{3/2}}\cdot\iint\left[1-\frac{3}{2}\cdot\frac{\rho'^2-2\rho\rho'\cos\left(\phi-\phi'\right)}{\rho^2+z^2}+\cdots\right]\rho' d\phi' d\rho' \\ &= \frac{V}{2\pi}\cdot\frac{z}{\left(\rho^2+z^2\right)^{3/2}}\cdot\left(\frac{a^2}{2}\cdot 2\pi-\frac{3}{2}\cdot\frac{1}{\rho^2+z^2}\cdot\frac{\pi a^4}{2}+\cdots\right) \\ &= \frac{Va^2}{2}\cdot\frac{z}{\left(\rho^2+z^2\right)^{3/2}}\cdot\left(1-\frac{3a^2}{4\left(\rho^2+z^2\right)}+\cdots\right) \end{aligned} \tag{16}$$

QED

考虑z轴上（$\rho=0$），上式化为

$$\varphi(\vec{r})|_{z\to\infty} = \frac{Va^2}{2}\cdot\frac{z}{\left(\rho^2+z^2\right)^{3/2}}\cdot\left(1-\frac{3a^2}{4\left(\rho^2+z^2\right)}+\cdots\right) \tag{17}$$

对于（c），有

$$\begin{aligned} \varphi &= V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right) \\ &= V\left(1-\left(1+\frac{a^2}{z^2}\right)^{-1/2}\right) \\ &\approx \frac{Va^2}{2z^2}\left(1-\frac{3a^2}{4z^2}+\frac{5a^4}{8z^4}\right) \end{aligned} \tag{18}$$

可见(c)、(d)在共同定义域内结果一致

2.9

半径为a的绝缘、球形、导体壳处于均匀电场E_0中。如果球体被垂直于电场的平面切割成两个半球，求阻止两球分离的力。

(a) 如果壳层不带电

(b) 如果壳层总电荷量为Q

(a)

考虑电场沿z负方向，球体被xOy平面切开

根据课本(2.15)，有感应电荷密度

$$\sigma = -\epsilon_0\frac{\partial\Phi}{\partial r}\bigg|_{r=a} = 3\epsilon_0 E_0\cos\theta \tag{19}$$

因此球面上面元$d\vec{S}$受力，利用$d\vec{F}=\vec{E}dq=\frac{\sigma^2 d\vec{S}}{2\epsilon_0}$

> $\mathbf{E}_{ind} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{r}}$
>
> $\mathbf{F} = \frac{1}{2\epsilon_0}\int\sigma^2\hat{\mathbf{r}}da$
>
> W.J.Duffin.Electricity and Magnetism.McGraw-Hill Book Company,fourth edition,1990 P51

$$d\vec{F}_z = d\vec{F}\cos\theta = \frac{9}{2}\epsilon_0 E_0^2\cos^3\theta d\vec{S} \tag{20}$$

左右两半球积分有

$$\begin{aligned} \vec{F}_{\mathbf{z}} &= 2\int_0^{2\pi}\int_0^{\pi/2}\frac{9}{2}\epsilon_0 E_0^2\cos^3\theta a^2\sin\theta d\theta d\phi\hat{e_z} \\ &= 9\pi a^2\epsilon_0 E_0^2\int_0^{\pi/2}\cos^3\theta\sin\theta d\theta\hat{e_z} \\ &= 9\pi a^2\epsilon_0 E_0^2\left[-1/4\cos^4\theta\right]^{\pi/2}\hat{e} \end{aligned} \tag{21}$$

$$
\begin{aligned}
&= 9\pi a \epsilon_0 E_0 [-1/4 \cos \theta]_0 \, e_z \\
&= \frac{9}{4}\pi a^2 \epsilon_0 E_0^2 \hat{e_z}
\end{aligned}
$$

(b)

考虑壳层带电Q，均匀分布于球面，有面密度

$$
\sigma_a = \frac{Q}{4\pi a^2} \tag{22}
$$

再考虑感应电荷，有总电荷密度

$$
\sigma = 3\epsilon_0 E_0 \cos\theta + \frac{Q}{4\pi a^2} \tag{23}
$$

面源$d\vec{S}$受力

$$
d\vec{F}_z = d\vec{F}\cos\theta = \left(\frac{Q^2}{32\pi^2 a^4 \epsilon} + \frac{3eQ\cos(\theta)}{4\pi a^2} + \frac{9}{2}e^2\epsilon\cos^2(\theta)\right) d\vec{S} \tag{24}
$$

同(a)积分有

$$
\vec{F_z} = \left(\frac{9}{4}\pi a^2 \epsilon_0 E_0^2 + \frac{Q^2}{32\pi\epsilon_0 a^2} + \frac{E_0 Q}{2}\right)\hat{e_z} \tag{25}
$$

2.12

从二维电势问题的的级数解（2.71）出发，在半径为b的圆柱体表面指定电势，正式评估系数，将其代入级数，然后求和，以泊松积分的形式得到圆柱体内的电势。

$$
\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi\left(b,\phi'\right) \frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos\left(\phi'-\phi\right)} d\phi' \tag{26}
$$

如果想得到在圆柱体和无穷大的空间所围区域内的电势，需要做什么修改？

> (2.71)
>
> $$
> \begin{aligned}
> \Phi(\rho,\phi) = & a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin\left(n\phi + \alpha_n\right) \\
> & + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin\left(n\phi + \beta_n\right)
> \end{aligned} \tag{27}
> $$
>
> **柯西积分公式**
>
> 设 Ω 是复平面 $\mathbb{C}$ 的一个单连通的开子集。 $f:\Omega\to\mathbb{C}$ 是一个 Ω 上的全纯函数。设 γ 是 Ω 内的一个简单闭合的可求长曲线 (即连续而不自交并且能定义长度的闭合曲线)，那么函数 f 在 γ 内部的点 a 上的值是:
>
> $$
> f(a) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a} dz \tag{28}
> $$

有边界条件

①：$\Phi(\rho=0)=\infty$

②：$\Phi(\rho=b)=V$

利用边界条件①有

$$
b_0 = b_n = 0 \tag{29}
$$

有解

$$
\begin{aligned}
\Phi(\rho,\phi) &= a_0 + \sum_{n=1}^{\infty} a_n \rho^n \sin\left(n\phi - \alpha_n\right) \\
&= a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{\rho}{b}\right)^n \sin\left(n\phi - \alpha_n\right)
\end{aligned} \tag{30}
$$

利用，$\sin\theta = \left(e^{i\theta} - e^{-i\theta}\right)/(2i)$，有

$$
\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2i}\left(\frac{\rho}{b}\right)^n \left[e^{-i\alpha_n} e^{in\phi} - e^{i\alpha_n} e^{-in\phi}\right]
$$

$$
\begin{aligned}
&= a_0 + \sum_{n=1} \left(\frac{\rho}{b}\right)^n \left[c_n e^{in\phi} + d_n e^{-in\phi}\right] \\
&= \sum_{n=-\infty}^{\infty} c_n \left(\frac{\rho}{b}\right)^{|n|} e^{in\phi}
\end{aligned} \tag{31}
$$

利用边界条件②有

$$
\begin{aligned}
V &= \sum_{n=-\infty}^{\infty} c_n e^{in\phi} \\
\int_0^{2\pi} V(\phi) e^{-in'\phi} d\phi &= \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{i(n-n')\phi} d\phi \\
\int_0^{2\pi} V(\phi) e^{-in'\phi} d\phi &= \sum_{n=-\infty}^{\infty} c_n 2\pi \delta_{nn'}
\end{aligned} \tag{32}
$$

有

$$
c_n = \frac{1}{2\pi} \int_0^{2\pi} V(\phi) e^{-in\phi} d\phi \tag{33}
$$

有圆柱内区域电势

$$
\begin{aligned}
\Phi(\rho, \phi) &= \frac{1}{2\pi} \int_0^{2\pi} d\phi' V\left(\phi'\right) \sum_{n=-\infty}^{\infty} \left(\frac{\rho}{b}\right)^{|n|} e^{in(\phi-\phi')} \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\phi' V\left(\phi'\right) \left[-1 + \sum_{n=0}^{\infty} \left(\frac{\rho}{b}\right)^n e^{in(\phi-\phi')} + \sum_{n=0}^{\infty} \left(\frac{\rho}{b}\right)^n e^{-in(\phi-\phi')}\right] \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\phi' V\left(\phi'\right) \left[-1 + \sum_{n=0}^{\infty} \left[\left(\frac{\rho}{b}\right) e^{i(\phi-\phi')}\right]^n + \sum_{n=0}^{\infty} \left[\left(\frac{\rho}{b}\right) e^{-i(\phi-\phi')}\right]^n\right]
\end{aligned} \tag{34}
$$

利用，$\sum_0^\infty r^n = \frac{1}{1-r}$，有

$$
\begin{aligned}
\Phi(\rho, \phi) &= \frac{1}{2\pi} \int_0^{2\pi} d\phi' V\left(\phi'\right) \left[-1 + \frac{1}{1-\left(\frac{\rho}{b}\right) e^{i(\phi-\phi')}} + \frac{1}{1-\left(\frac{\rho}{b}\right) e^{-i(\phi-\phi')}}\right] \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\phi' V\left(\phi'\right) \left[\frac{\left(\left(1-\left(\frac{\rho}{b}\right) e^{-i(\phi-\Phi')}\right) + \left(1-\left(\frac{\rho}{b}\right) e^{i(\phi-\phi')}\right) - \left(1-\left(\frac{\rho}{b}\right) e^{i(\phi-\Phi')}\right)\left(1-\left(\frac{\rho}{b}\right) e^{-i(\phi'\Phi')}\right)\right)}{\left(1-\left(\frac{\rho}{b}\right) e^{i(\phi-\phi')}\right)\left(1-\left(\frac{\rho}{b}\right) e^{-i(\phi-\phi')}\right)}\right] \\
&= \frac{1}{2\pi} \int_0^{2\pi} \Phi\left(b, \phi'\right) \frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos\left(\phi'-\phi\right)} d\phi'
\end{aligned} \tag{35}
$$

对于圆柱外区域，交换b, ρ即可，有

$$
\Phi(\rho, \phi) = \frac{1}{2\pi} \int_0^{2\pi} \Phi\left(b, \phi'\right) \frac{\rho^2-b^2}{b^2+\rho^2-2b\rho\cos\left(\phi'-\phi\right)} d\phi' \tag{36}
$$