差商格式

设空间步长和时间步长分别为h和τ，空间和时间步数序号记为i,k

一阶向前差商

$$\left.\frac{\partial u_{i,k}}{\partial x}\right|_{+}=\frac{u_{i+1,k}-u_{i,k}}{h} \qquad and \qquad \left.\frac{\partial u_{i,k}}{\partial t}\right|_{+}=\frac{u_{i,k+1}-u_{i,k}}{\tau} \tag{1}$$

一阶向后差商

$$\left.\frac{\partial u_{i,k}}{\partial x}\right|_{-}=\frac{u_{i,k}-u_{i-1,k}}{h} \tag{2}$$

二阶中心差商

$$\frac{\partial^2 u_{i,k}}{\partial x^2}=\frac{\left.\frac{\partial u_{i,k}}{\partial x}\right|_{+}-\left.\frac{\partial u_{i,k}}{\partial x}\right|_{-}}{h}=\frac{u_{i+1,k}-2u_{i,k}+u_{i-1,k}}{h^2} \tag{3}$$

边界条件

第一类边界条件

$$\begin{cases} u(0,t)=g_1(t) \\ u(l,t)=g_2(t) \end{cases} \quad 0\le t\le T \tag{4}$$

第二类边界条件

$$\begin{cases} \frac{\partial u(0,t)}{\partial x}=g_1(t) \\ \frac{\partial u(l,t)}{\partial x}=g_2(t) \quad 0\le t\le T \end{cases} \tag{5}$$

热扩散方程

1 一维

$$\frac{\partial u}{\partial t}=\lambda\frac{\partial^2 u}{\partial x^2} \quad \lambda=\frac{K}{c\rho}>0, 0<t\le T \tag{6}$$

边界条件

$$u(x,0)=\varphi(x) \quad 0\le x\le l \tag{7}$$

1.1 差商格式

差分有

$$\frac{u_{i,k+1}-u_{i,k}}{\tau}=\lambda\frac{u_{i+1,k}-2u_{i,k}+u_{i-1,k}}{h^2} \tag{8}$$

整理得

$$\begin{aligned} u_{i,k+1}&=\alpha u_{i+1,k}+(1-2\alpha)u_{i,k}+\alpha u_{i-1,k} \\ \alpha&=\frac{\tau\lambda}{h^2} \qquad i=1,2,\ldots,N-1, \quad k=0,1,2,\ldots,M \end{aligned} \tag{9}$$

边界条件差分有

$$\begin{aligned} &u_{i,0}=\varphi(ih) \quad i=1,2,\ldots,N-1 \\ &\begin{cases} u_{0,k}=g_1(k\tau) \\ u_{N,k}=g_2(k\tau) \end{cases} \end{aligned} \tag{10}$$

1.2 稳定性条件

一维热扩散差分格式稳定条件

$$\alpha = \frac{\tau\lambda}{h^2} \le \frac{1}{2} \tag{11}$$

一般给定α, h，再去计算τ

> 为啥稳定条件是这样的❓
>
> 根据最大模原理说明，如果f 是一个全纯函数且不是常数，那么它的模$|f|$在定义域内取不到局部最大值。
>
> ❓局域不会有最大值？平面波那个不就是反例么
>
> 对于递推公式
>
> $$u_{i,k+1} = \alpha u_{i+1,k} + (1-2\alpha)u_{i,k} + \alpha u_{i-1,k} \tag{12}$$
>
> 其系数和为1，若，$\alpha \le \frac{1}{2}$，则
>
> $$\left|u_j^{n+1}\right| \le \max\left(\left|u_{j-1}^n\right|, \left|u_j^n\right|, \left|u_{j+1}^n\right|\right), \forall j \tag{13}$$
>
> PDE有限差分方法(4)——稳定性 - 知乎 (zhihu.com)

2 二维

$$\begin{gathered}\frac{\partial u}{\partial t} = \lambda\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad , \quad \lambda = \frac{K}{c\rho} > 0, 0 < t \le T \\ 0 < x < l, 0 < y < s\end{gathered} \tag{14}$$

边界条件

$$u(x,y,0) = \varphi(x,y) \tag{15}$$

2.1 差商格式

$$\begin{cases}\dfrac{\partial u_{i,j,k}}{\partial t} = \dfrac{u_{i,j,k+1} - u_{i,j,k}}{\tau} \\ \dfrac{\partial^2 u_{i,j,k}}{\partial x^2} = \dfrac{u_{i+1,j,k} - 2u_{i,j,k} + u_{i-1,j,k}}{h^2} \\ \dfrac{\partial^2 u_{i,j,k}}{\partial y^2} = \dfrac{u_{i,j+1,k} - 2u_{i,j,k} + u_{i,j-1,k}}{h^2}\end{cases} \tag{16}$$

整理得到

$$\begin{aligned}u_{i,j,k+1} &= (1-4\alpha)u_{i,j,k} + \alpha\left(u_{i+1,j,k} + u_{i-1,j,k} + u_{i,j+1,k} + u_{i,j-1,k}\right) \\ \alpha = \frac{\tau\lambda}{h^2} \quad & i = 1,2,\ldots,N-1, \quad j = 1,2,\ldots,M-1, \quad k = 0,1,2,\ldots\end{aligned} \tag{17}$$

边界条件

$$\begin{aligned}&u_{i,j,0} = \varphi(ih, jh) \quad i = 1,2,\ldots,N-1, j = 1,2,\ldots,M-1 \\ &\begin{cases}u_{0,j,k} = g_1(k\tau, jh) \\ u_{N,j,k} = g_2(k\tau, jh)\end{cases} \quad k = 0,1,2,\ldots, j = 1,2,\ldots,M-1 \\ &\begin{cases}u_{i,0,k} = g_3(k\tau, ih) \\ u_{i,N,k} = g_4(k\tau, ih)\end{cases}\end{aligned} \tag{18}$$

2.2 稳定性条件

$$\alpha = \frac{\tau\lambda}{h^2} \leq \frac{1}{4} \tag{19}$$

一般给定α, h，再去计算τ

波动方程

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}, \quad 0 < x < l, 0 < t \leq T \tag{20}$$

1 差分格式

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= \frac{y_{i+1,k} - 2y_{i,k} + y_{i-1,k}}{h^2} \\ \frac{\partial^2 y}{\partial t^2} &= \frac{y_{i,k+1} - 2y_{i,k} + y_{i,k-1}}{\tau^2} \end{aligned} \tag{21}$$

整理得到

$$y_{i,k+1} = 2\left(1-\alpha^2\right)y_{i,k} + \alpha^2\left(y_{i+1,k} + y_{i-1,k}\right) - y_{i,k-1} \tag{22}$$

其中，$\alpha = \frac{\tau v}{h} \leq 1$

初始条件

$$\begin{cases} y(x,0) = \varphi(x) \\ \frac{\partial y(x,0)}{\partial t} = \psi(x) \end{cases} \quad 0 \leq x \leq l \tag{23}$$

边界条件

$$\begin{cases} y(0,t) = g_1(t) \\ y(N,t) = g_2(t) \end{cases} \quad 0 \leq t \leq T \tag{24}$$

1.1 一阶向前差分

若对于初始时刻考虑**一阶向前差分**，有

$$\begin{gathered} \frac{\partial y_{i,0}}{\partial t} = \frac{y_{i,1} - y_{i,0}}{\tau} \\ y_{i,1} = y_{i,0} + \tau\psi(ih) \end{gathered} \tag{25}$$

整理有，

$$\begin{cases} y_{i,k+1} = 2\left(1-\alpha^2\right)y_{i,k} + \alpha^2\left(y_{i+1,k} + y_{i-1,k}\right) - y_{i,k-1} & \\ i = 1,2,\ldots,N-1 & k = 1,2,\ldots,M-1 \\ y_{i,0} = \varphi(ih) & i = 0,1,\ldots,N \\ y_{i,1} = \varphi(ih) + \tau\psi(ih) & i = 0,1,\ldots,N \\ y_{0,k} = g_1(k\tau) & k = 1,2,\ldots,M \\ y_{N,k} = g_2(k\tau) & k = 1,2,\ldots,M \end{cases} \tag{26}$$

1.2 一阶中心差分

若对于初始时刻考虑**一阶中心差分**，有

$$\begin{gathered} \frac{\partial y_{i,0}}{\partial t} = \frac{y_{i,1} - y_{i,-1}}{2\tau} \\ y_{i,1} = y_{i,-1} + 2\tau\psi(ih) \end{gathered} \tag{27}$$

为消去通项公式中的$y_{i,-1}$项，联立

由于差分网格一般是从$t=0$开始的，所以网格里一般没有$y_{i,-1}$这个个点，需要用下述办法表示出来

$$\begin{gathered}y_{i,k+1}=2\left(1-\alpha^2\right)y_{i,k}+\alpha^2\left(y_{i+1,k}+y_{i-1,k}\right)-y_{i,k-1}\\ y_{i,1}=2\left(1-\alpha^2\right)y_{i,0}+\alpha^2\left(y_{i+1,0}+y_{i-1,0}\right)-y_{i,-1}\end{gathered}\tag{28}$$

有

$$y_{i,1}=\left(1-\alpha^2\right)y_{i,0}+\frac{\alpha^2}{2}(y_{i+1,0}+y_{i-1,0})+\tau\psi(ih)\tag{29}$$

整理有，

$$\begin{cases}y_{i,k+1}=2\left(1-\alpha^2\right)y_{i,k}+\alpha^2\left(y_{i+1,k}+y_{i-1,k}\right)-y_{i,k-1}\\ i=1,2,\ldots,N-1\quad k=1,2,\ldots,M-1\\ y_{i,0}=\varphi(ih)\quad i=0,1,\ldots,N\\ y_{i,1}=\left(1-\alpha^2\right)\varphi(ih)+\frac{\alpha^2}{2}[\varphi((i+1)h)+\varphi((i-1)h)]+\tau\psi(ih)\\ i=0,1,\ldots,N-1\\ y_{0,k}=g_1(k\tau)\quad k=1,2,\ldots,M\\ y_{N,k}=g_2(k\tau)\quad k=1,2,\ldots,M\end{cases}\tag{30}$$

2 稳定性条件

$$\alpha=\frac{\tau v}{h}\leq 1\tag{31}$$