§5. 球外空间(内)的 Dirichlet 问题

1. 球外(无源区)的 Green 函数、势的形式解

$$G(\vec r,\vec r') = \frac{1}{|\vec r-\vec r'|} - \frac{R_0}{r'|\vec r-\bar{\vec r}'|}$$

其中 $\bar r' \cdot r' = R_0^2$

$$= \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{R_0}{r'\sqrt{r^2+\bar r'^2-2r\bar r'\cos\gamma}}$$

$$= \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{R_0}{\sqrt{r^2r'^2+R_0^4-2R_0^2rr'\cos\gamma}}$$

$$\frac{\partial G}{\partial n'} = -\left.\frac{\partial G}{\partial r'}\right|_{r'=R_0}$$

在 Gauss 公式 (Green 定理) 中，V 与 S 的正方向约定：研究对象 V 在 S 内，$\vec n'$ 由内指向外。若 …… S 外，$\vec n'$ 由外指向内。

$$= -\frac{\partial}{\partial r'}\left(\frac{1}{\sqrt{\cdots}} - \frac{R_0}{\sqrt{\cdots}}\right)\Bigg|_{r'=R_0}$$

$$= -\left[\left(-\frac{1}{2}\right)\frac{2r'-2r\cos\gamma}{(r^2+r'^2-2rr'\cos\gamma)^{\frac32}} - \left(-\frac{1}{2}\right)\frac{R_0(2r^2r'-2R_0^2r\cos\gamma)}{(r^2r'^2+R_0^4-2R_0^2rr'\cos\gamma)^{\frac32}}\right]\Bigg|_{r'=R_0}$$

$$= \left[\frac{r'-r\cos\gamma}{(r^2+r'^2-2rr'\cos\gamma)^{\frac32}} - \frac{R_0(r^2r'-R_0^2r\cos\gamma)}{(r^2r'^2+R_0^4-2R_0^2rr'\cos\gamma)^{\frac32}}\right]\Bigg|_{r'=R_0}$$

$$= \frac{R_0-r\cos\gamma}{(r^2+R_0^2-2R_0r\cos\gamma)^{\frac32}} - \frac{R_0^2(r^2-R_0r\cos\gamma)}{(r^2R_0^2+R_0^4-2R_0^3r\cos\gamma)^{\frac32}}$$

$$= \cdots\cdots - \frac{r^2-R_0r\cos\gamma}{R_0(r^2+R_0^2-2R_0r\cos\gamma)^{\frac32}}$$

$$= \frac{1}{R_0(r^2+R_0^2-2R_0r\cos\gamma)^{\frac32}}\left(R_0^2-\cancel{R_0r\cos\gamma}-r^2+\cancel{R_0r\cos\gamma}\right)$$

$$= \frac{R_0^2-r^2}{R_0(R_0^2+r^2-2R_0r\cos\gamma)^{\frac32}}$$

故 Dirichlet 问题 (特别对于无源区) 球外

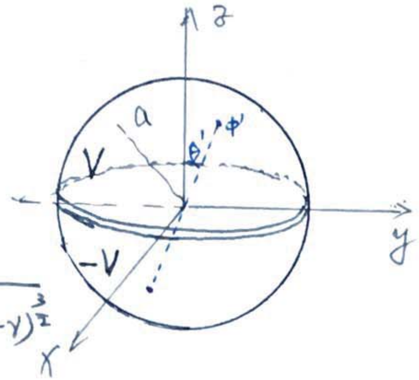
$$\varphi(\vec r) = -\frac{1}{4\pi}\oint\!\!\!\oint \varphi(\vec r')\frac{\partial G(\vec r,\vec r')}{\partial n'}dS'$$

$$= \frac{R_0^2-r^2}{4\pi R_0}\oint\!\!\!\oint \varphi(\vec r')\frac{dS'}{(R_0^2+r^2-2R_0r\cos\gamma)^{\frac32}}$$

球面 $dS' = R_0^2\,d\Omega'$
$= R_0^2\sin\theta'\,d\theta'\,d\varphi'$

$$= \frac{R_0(R_0^2-r^2)}{4\pi}\oint\!\!\!\oint \frac{\varphi(\vec r')\,d\Omega'}{(R_0^2+r^2-2R_0r\cos\gamma)^{\frac32}}$$

2. 双半球导体外的电势



$$\varphi(\vec r)=\frac{a(a^2-r^2)V}{4\pi}\times\left[\int_{\Omega^+}\frac{d\Omega'}{(a^2+r^2-2ar\cos\gamma)^{\frac32}}-\int_{\Omega^-}\frac{d\Omega'}{(a^2+r^2-2ar\cos\gamma)^{\frac32}}\right]$$

$$=\frac{Va(a^2-r^2)}{4\pi}\int_0^{2\pi}d\varphi'\left[\int_0^{\frac{\pi}{2}}\frac{\sin\theta'd\theta'}{(a^2+r^2-2ar\cos\gamma)^{\frac32}}-\int_{\frac{\pi}{2}}^{\pi}\frac{\sin\theta'd\theta'}{(a^2+r^2-2ar\cos\gamma)^{\frac32}}\right]$$

其中 $\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')$

$$=\frac{Va(a^2-r^2)}{4\pi}\int_0^{2\pi}d\varphi'\left[\int_0^{1}\frac{d(\cos\theta')}{(a^2+r^2-2ar\cos\gamma)^{\frac32}}-\int_{-1}^{0}\frac{d(\cos\theta')}{(a^2+r^2-2ar\cos\gamma)^{\frac32}}\right]$$

第二项作代换(反演) $\theta'=\pi-\theta''$, $\varphi'=\varphi''+\pi$

$$\cos\gamma=-\cos\theta\cos\theta''-\sin\theta\sin(\theta''-\pi)\cos(\varphi-\varphi''-\pi)$$

$$=-\cos\theta\cos\theta''-\sin\theta\sin\theta''\cos(\varphi-\varphi'')=-\cos\gamma'$$

及 $d(\cos\theta')=-d(\cos\theta'')$

$$\int_{-1}^{0}\cdots=\int_{1}^{0}\qquad \int_0^{2\pi}\cdots=\int_{\pi}^{3\pi}$$

$$\therefore\ \varphi(\vec r)=\frac{Va(a^2-r^2)}{4\pi}\int_0^{2\pi}d\varphi'\int_0^1 d(\cos\theta')\left[\frac{1}{(a^2+r^2-2ar\cos\gamma)^{\frac32}}-\frac{1}{(a^2+r^2+2ar\cos\gamma)^{\frac32}}\right]$$

$$=\frac{V}{4\pi}\frac{a(a^2-r^2)}{(a^2+r^2)^{\frac32}}\int_0^{2\pi}d\varphi'\int_0^1 d(\cos\theta')\left[\frac{1}{(1-2\alpha\cos\gamma)^{\frac32}}-\frac{1}{(1+2\alpha\cos\gamma)^{\frac32}}\right]$$

其中 $\alpha=\dfrac{ar}{a^2+r^2}$

展开 $(1\pm x)^{-\frac32}=1\mp\frac32 x+\frac{1}{2!}\cdot\frac{15}{4}x^2\mp\frac{1}{3!}\cdot\frac{105}{8}x^3+\cdots$

$$(1-2\alpha\cos\gamma)^{-\frac32}-(1+2\alpha\cos\gamma)^{-\frac32}$$

$$=6\alpha\cos\gamma+35\alpha^3\cos^3\gamma$$

$$\therefore\ \varphi(\vec r)=\frac{V}{4\pi}\frac{a(a^2-r^2)}{(a^2+r^2)^{\frac32}}\int_0^{2\pi}d\varphi'\int_0^1 d(\cos\theta')(6\alpha\cos\gamma+35\alpha^3\cos^3\gamma)$$

其中 $\int_0^{2\pi}d\varphi'\int_0^1 d(\cos\theta')\cdot\cos\gamma=\int_0^{2\pi}d\varphi'\int_0^1 d(\cos\theta')\cos\theta'\cdot\cos\theta$

$$+\underbrace{\int_0^{2\pi}d\varphi'\cos(\varphi-\varphi')}_{=0}\int_0^1 d(\cos\theta')\sin\theta'\sin\theta$$

$$=2\pi\cdot\cos\theta\cdot\frac12\cos^2\theta'\Big|_0^1=\pi\cos\theta$$

§6. 分离变量法之柱坐标

1. 柱坐标下 Laplace 方程通解

$\nabla^2\varphi=0$ → 柱

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\phi^2}=0$$

取 $\varphi=R(\rho)\Phi(\phi)$

$$\frac{1}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)=-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}\equiv\nu^2$$

作业: 2.2, 2.7, 2.9, 2.12