$$\begin{cases} \rho^2 \dfrac{\partial^2 R}{\partial\rho^2} + \rho\dfrac{\partial R}{\partial\rho} = \nu^2 R \\[2ex] \dfrac{\partial^2\Phi}{\partial\phi^2} + \nu^2\Phi = 0\end{cases}$$

① $\nu = 0$

$$\begin{cases} \rho^2 \dfrac{\partial^2 R}{\partial\rho^2} + \rho\dfrac{\partial R}{\partial\rho} = 0 \\[2ex] \dfrac{\partial^2\Phi}{\partial\phi^2} = 0\end{cases}$$

$$\Longrightarrow \begin{cases} R = A_0 + B_0 \ln\rho \\ \Phi = C_0 + D_0\phi \end{cases}$$

$$\varphi_0 = (A_0 + B_0\ln\rho)(C_0 + D_0\phi)$$

② $\nu \neq 0$

令 $\rho = e^t$，$t = \ln\rho$，$\dfrac{\partial t}{\partial\rho} = \dfrac{1}{\rho}$

$$\frac{\partial R}{\partial\rho} = \frac{1}{\rho}\frac{\partial R}{\partial t}, \quad \frac{\partial^2 R}{\partial\rho^2} = -\frac{1}{\rho^2}\frac{\partial R}{\partial t} + \frac{1}{\rho^2}\frac{\partial^2 R}{\partial t^2}$$

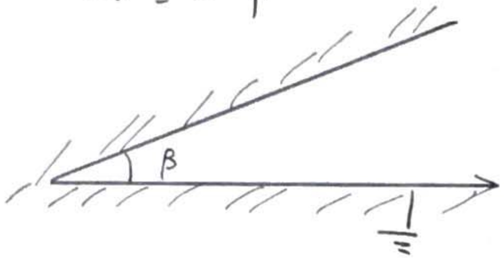
方程一化为 $\dfrac{\partial^2 R}{\partial t^2} - \nu^2 R = 0$

$$R_\nu = A_\nu e^{\nu t} + B_\nu e^{-\nu t} = A_\nu\rho^\nu + B_\nu\rho^{-\nu}$$

而 $\Phi_\nu = C_\nu\cos\nu\phi + D_\nu\sin\nu\phi$

$$\varphi = (A_0 + B_0\ln\rho)(C_0 + D_0\phi) + \sum_\nu (A_\nu\rho^\nu + B_\nu\rho^{-\nu})(C_\nu\cos\nu\phi + D_\nu\sin\nu\phi)$$

例：劈角（保持电势 V）



$0 < \phi < \beta$ 范围内，当 $\rho \to 0$ 时，φ 有限

$$B_0 = B_\nu = 0$$

$$\varphi = C_0 + D_0\phi + \sum_\nu \rho^\nu (C_\nu\cos\nu\phi + D_\nu\sin\nu\phi)$$

$\phi = 0$ 时 $\varphi \equiv V$

$$C_0 + \sum_\nu \rho^\nu C_\nu \equiv V \longrightarrow C_0 = V, \quad C_\nu = 0$$

$$\therefore \varphi = V + D_0\phi + \sum_\nu \rho^\nu D_\nu \sin\nu\phi$$

$\phi = \beta$ 时 $\varphi \equiv V$

$$V + D_0\beta + \sum_\nu \rho^\nu D_\nu\sin\nu\beta = V$$

则 $D_0 = 0$　$\sin\nu\beta = 0 \longrightarrow \nu = \dfrac{m\pi}{\beta}$，$m = 1, 2, 3\cdots$

$$\therefore \varphi(\rho,\phi) = V + \sum_{m=1}^{\infty} a_m \rho^{\frac{m\pi}{\beta}} \sin\frac{m\pi}{\beta}\phi$$

最低阶　$\varphi_1(\rho,\phi) = V + a_1\rho^{\frac{\pi}{\beta}}\sin\dfrac{\pi}{\beta}\phi$

电场 $\begin{cases} E_\rho(\rho,\phi) = -\dfrac{\partial\varphi}{\partial\rho} = \cdots \\[2ex] E_\phi(\rho,\phi) = -\dfrac{1}{\rho}\dfrac{\partial\varphi}{\partial\phi} = \cdots \end{cases}$

表面电荷密度（面）$\sigma(\rho) = -\dfrac{\varepsilon_0\pi a_1}{\beta}\rho^{\frac{\pi}{\beta}-1}$

§7 分离变量法之球坐标

1. 球谐函数

(1) 球系下 Laplace 方程通解 $\nabla^2\varphi=0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}=0$$

分离变量 $\varphi=R(r)Y(\theta,\phi)$，得

$$\begin{cases}\dfrac{d}{dr}\left(r^2\dfrac{dR}{dr}\right)-l(l+1)R=0\\[2mm] \dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial Y}{\partial\theta}\right)+\dfrac{1}{\sin^2\theta}\dfrac{\partial^2 Y}{\partial\phi^2}+l(l+1)Y=0\end{cases}$$

球方程

$Y_{lm}(\theta,\phi)$

第一式可化为 Euler 型常微

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-l(l+1)R=0$$

通解： $R(r)=Cr^l+\dfrac{D}{r^{l+1}}$

进一步，分变 $Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$，得

$$\begin{cases}\dfrac{d^2\Phi}{d\phi^2}+\lambda\Phi=0\\[2mm] \sin\theta\dfrac{d}{d\theta}\left(\sin\theta\dfrac{d\Theta}{d\theta}\right)+\left[l(l+1)\sin^2\theta-\lambda\right]\Theta=0\end{cases}$$

其中 $\lambda=m^2$， $m=0,1,2\ldots$

第二方程 代换 $x=\cos\theta$，化为

$$(1-x^2)\frac{d^2\Theta}{dx^2}-2x\frac{d\Theta}{dx}+\left[l(l+1)-\frac{m^2}{1-x^2}\right]\Theta=0$$

连带 Legendre 方程，特别 $m=0$ 时

$$(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+l(l+1)P=0$$

关于 Legendre 多项式的各种性质。

(3.26) 式改写

$$a_l=(-1)^{\frac{l-1}{2}}\frac{(2l+1)(l-2)!!}{2^{\frac{l+1}{2}}\left(\frac{l+1}{2}\right)!}=(-1)^{\frac{l-1}{2}}\frac{(2l+1)(l-2)!!}{(l+1)!!}$$

令 $l=2k-1$， $k=1,2,3\ldots$

$$\text{分母}=2^k k!=\underbrace{2\cdot2\cdot2\cdots}_{k\text{个}}\times\underbrace{k(k-1)\cdots2\cdot1}_{k\text{个}}$$

$$=2k\cdot(2k-2)\cdot\cdots4\cdot2=(2k)!!$$

(2) 关于正交函数展开

$$\int_a^b V_n^*(\xi)V_m(\xi)\,d\xi=\delta_{nm}\qquad \hat e_i\cdot\hat e_j=\delta_{ij}$$

$$\begin{cases}f(\xi)=\sum\limits_n a_n V_n(\xi)\\[2mm] a_n=\displaystyle\int_a^b V_n^*(\xi)f(\xi)\,d\xi\end{cases}$$

(3) 轴对称条件下

$$\varphi(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

例：双半球面，$V(\theta)=\begin{cases}V & (z>0)\quad \theta\in(0,\frac{\pi}{2})\\ -V & (z<0)\quad \theta\in(\frac{\pi}{2},\pi)\end{cases}$

$$V(\theta)=\sum_{l=0}^{\infty}C_lP_l(\cos\theta),\quad C_l=\frac{2l+1}{2}\int_0^{\pi}V(\theta)P_l(\cos\theta)\sin\theta\, d\theta$$

解出：$V(\theta)=V\left(\frac{3}{2}P_1-\frac{7}{8}P_3+\frac{11}{16}P_5+\cdots\right)=\sum_l C_lP_l$

球内外电势 $\begin{cases}\varphi_{in}=\sum_l A_l r^l P_l(\cos\theta)\\ \varphi_{out}=\sum_l \frac{B_l}{r^{l+1}}P_l(\cos\theta)\end{cases}$

边值关系 $\varphi|_R=V(\theta)$

$$\sum_l A_l R^l P_l=\sum_l C_lP_l=\sum\frac{B_l}{R^{l+1}}P_l$$

$$\Rightarrow A_l=\frac{C_l}{R^l},\quad B_l=C_lR^{l+1}$$

$$\therefore \varphi_{in}=\cancel{V\frac{3}{2}\frac{r}{a}P_1}\sum_l C_l\left(\frac{r}{a}\right)^l P_l$$

$$=V\left(\frac{3}{2}\frac{r}{a}P_1-\frac{7}{8}\left(\frac{r}{a}\right)^3P_3+\cdots\right)$$

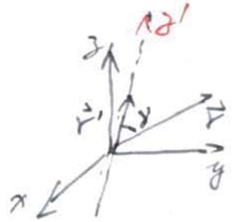
§8 球Green函数

1. 无界Green函数，按Legendre多项式展开

对于 $\gamma=0$ 的特殊位置（z正半轴）

$$\frac{1}{|\vec r-\vec r'|}=\frac{1}{\sqrt{r^2+r'^2-2rr'}}=\frac{1}{|r-r'|}$$

不妨先考虑 $r>r'$



$$=\frac{1}{r}\frac{1}{1-\frac{r'}{r}}\quad <1$$

利用展开式 $\frac{1}{1-x}=1+x+x^2+x^3+\cdots=\sum_n x^n$

$$\therefore \left.\frac{1}{|\vec r-\vec r'|}\right|_{\gamma=0}=\frac{1}{r}\sum_n\left(\frac{r'}{r}\right)^n=\sum_n\frac{r'^n}{r^{n+1}}$$

同理，对于 $r<r'$，$\left.\frac{1}{|\vec r-\vec r'|}\right|_{\gamma=0}=\sum_n\frac{r^n}{r'^{n+1}}$

统一形式为 $\left.\frac{1}{|\vec r-\vec r'|}\right|_{\gamma=0}=\sum_n\frac{r_<^n}{r_>^{n+1}}$

如果体系轴对称，有通解

$$\varphi(r,\theta)=\sum_{l=0}^{\infty}\left(A_lr^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

性质：$P_l(1)=1$

当和 $\theta=\gamma=0$ 时

$$\varphi(z) = \varphi(r,0) = \sum_{l=0}^{\infty} A_l r^l + \frac{B_l}{r^{l+1}} = \sum_{l=0}^{\infty} \varphi_l$$

反之，若已知 z 轴上（正半）$\varphi(z) = \sum_{l=0}^{\infty} \varphi_l$ ——多项式展开

则有

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} \varphi_l P_l(\cos\theta)$$

亦用于 Green 函数

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta)$$