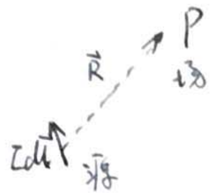
# 第二章 静磁学、电磁感应

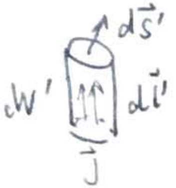
## §1. 静磁学基本规律

### 1. Biot-Savart 定律




$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{l}' \times \hat{e}_R}{R^2}$$


$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{\vec{J}(\vec{r}') \times \hat{e}_R}{R^2} dV'$$



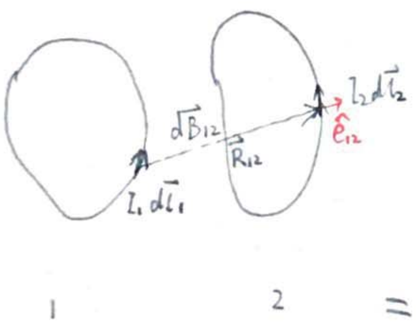

$$I d\vec{l}' = (\vec{J}\cdot d\vec{S}')\, d\vec{l}' \quad \langle \vec{J} \parallel d\vec{l}' \rangle$$

$$= (d\vec{l}'\cdot d\vec{S}')\,\vec{J} = \vec{J}\, dV'$$

两电流环之间的相互作用

Ampere 力 $d\vec{F} = I d\vec{l} \times \vec{B}$

$$\left\langle \vec{F}_L = q(\vec{E} + \vec{v}\times\vec{B}),\ \vec{f}_L = \rho\vec{E} + \vec{J}\times\vec{B} \right\rangle$$




$$d\vec{B}_{12} = \frac{\mu_0}{4\pi}\frac{I_1 d\vec{l}_1 \times \hat{e}_{12}}{R_{12}^2}$$

$$d\vec{F}_{12} = I_2 d\vec{l}_2 \times d\vec{B}_{12}$$

$$= \frac{\mu_0 I_1 I_2}{4\pi}\frac{d\vec{l}_2 \times (d\vec{l}_1 \times \hat{e}_{12})}{R_{12}^2}$$

$$= \frac{\mu_0 I_1 I_2}{4\pi} d\vec{l}_1 \frac{d\vec{l}_2\cdot\hat{e}_{12}}{R_{12}^2} - \frac{\mu_0 I_1 I_2}{4\pi}\frac{(d\vec{l}_1\cdot d\vec{l}_2)\,\hat{e}_{12}}{R_{12}^2}$$

$$\vec{F}_{12} = \frac{\mu_0 I_1 I_2}{4\pi}\oint_{l_1}\oint_{l_2} d\vec{l}_1 \frac{d\vec{l}_2\cdot\hat{e}_{12}}{R_{12}^2} - \frac{\mu_0 I_1 I_2}{4\pi}\oint_{l_1}\oint_{l_2}\frac{d\vec{l}_1\cdot d\vec{l}_2}{R_{12}^2}\hat{e}_{12}$$

其中第一项先积分，注意 $d\vec{l}_2\cdot\hat{e}_{12} = dR_{12}$

$$\oint_{l_2}\frac{d\vec{l}_2\cdot\hat{e}_{12}}{R_{12}^2} = \oint_{l_2}\frac{dR_{12}}{R_{12}^2} = 0$$ ，则

$$\vec{F}_{12} = -\frac{\mu_0 I_1 I_2}{4\pi}\oint_{l_1}\oint_{l_2}\frac{d\vec{l}_1\cdot d\vec{l}_2}{R_{12}^2}\hat{e}_{12}$$

易见，$\vec{F}_{21} = -\vec{F}_{12}$ ，满足牛三律。

### 2. 磁场的无源性、矢势

考虑 $$\nabla\times\frac{\vec{J}(\vec{r}')}{R} = \left(\nabla_R\frac{1}{R}\right)\times\vec{J}(\vec{r}') + \frac{1}{R}\underbrace{\nabla\times\vec{J}(\vec{r}')}_{=0}$$

"$\vec{R} = \vec{r} - \vec{r}'$"

$$= -\frac{\hat{e}_R}{R^2}\times\vec{J}(\vec{r}')$$

$$= \vec{J}\times\frac{\hat{e}_R}{R^2}$$

$$\therefore\ d\vec{B} = \frac{\mu_0}{4\pi}\nabla\times\frac{\vec{J}(\vec{r}')}{R}\,dV'$$

$$\vec{B} = \frac{\mu_0}{4\pi}\nabla\times\int_V\frac{\vec{J}(\vec{r}')}{R}\,dV'$$

定义"矢势" $\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')}{R}\,dV'$ 则 $\vec{B} = \nabla\times\vec{A}$

显然 $\nabla\cdot\vec{B}=0 \qquad \nabla\cdot(\nabla\times\vec{A})=0$

$$\nabla\times\vec{B}=\frac{\mu_0}{4\pi}\nabla\times\left[\nabla\times\int_V\frac{\vec{J}(\vec{r}')}{R}dV'\right]$$

利用 $\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}$

$$=\frac{\mu_0}{4\pi}\nabla\int_V\nabla\cdot\frac{\vec{J}(\vec{r}')}{R}dV'-\frac{\mu_0}{4\pi}\int\nabla^2\frac{\vec{J}(\vec{r}')}{R}dV'$$

$$=\frac{\mu_0}{4\pi}\nabla\int_V\vec{J}(\vec{r}')\cdot\nabla\frac{1}{R}dV'-\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}')\nabla^2\frac{1}{R}dV'$$

注意 $\vec{R}=\vec{r}-\vec{r}'$，则 $\nabla\odot f(\vec{R})=\nabla_R\odot f(\vec{R})=-\nabla'\odot f(\vec{R})$

$$=-\frac{\mu_0}{4\pi}\nabla\int\vec{J}(\vec{r}')\cdot\nabla'\frac{1}{R}dV'-\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}')\nabla_R^2\frac{1}{R}dV'$$

$$=\frac{\mu_0}{4\pi}\nabla\int\left\{[\nabla'\cdot\vec{J}(\vec{r}')]\frac{1}{R}-\nabla'\cdot\left[\frac{\vec{J}(\vec{r}')}{R}\right]\right\}dV'-\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}')\nabla_R^2\frac{1}{R}dV'$$

利用 $\frac{\partial\rho}{\partial t}+\nabla\cdot\vec{J}=0$ 方程. 对于稳恒 $0$，则 $0$

化为 $\oint dS'$，表面 $\vec{J}(\vec{r}')=0$

利用 $\nabla^2\frac{1}{r}=-4\pi\delta(\vec{r})$

$$=0-\frac{\mu_0}{4\pi}\cdot(-4\pi)\int\vec{J}(\vec{r}')\,\delta(\vec{R})\,dV' \qquad \delta(\vec{r}-\vec{r}')$$

$$=\mu_0\vec{J}(\vec{r})$$

由此，可得 Ampere 环路

$$\iint\nabla\times\vec{B}\cdot d\vec{S}=\mu_0\iint\vec{J}\cdot d\vec{S}$$

利用 Stokes 公式

$$\oint_L\vec{B}\cdot d\vec{l}=\mu_0 I$$

3. 矢势方程及其边值关系

$$\begin{cases}\nabla\cdot\vec{B}=0 \longrightarrow \vec{B}=\nabla\times\vec{A}\\ \nabla\times\vec{B}=\mu_0\vec{J}\end{cases}$$

再利用

$$\nabla\times(\nabla\times\vec{A})=\mu_0\vec{J}$$

$$\Rightarrow\ \nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}=\mu_0\vec{J}$$

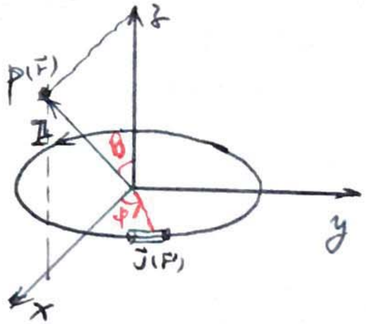
$\vec{A}$ 不唯一，$\vec{A}'=\vec{A}+\nabla\chi$

可取 Coulomb 规范 $\nabla\cdot\vec{A}=0$，则

$$\nabla^2\vec{A}=-\mu_0\vec{J}$$

边值 $\begin{cases}\hat{n}\cdot(\vec{B}_2-\vec{B}_1)=0\\ \hat{n}\times(\vec{H}_2-\vec{H}_1)=\vec{\alpha}_f\end{cases}\Rightarrow\Big\{$

例：求空间中的势与场




$$\vec{A}=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')}{R}dV'$$

$$\vec{J}(\vec{r}')=C\,\delta(\theta'-\tfrac{\pi}{2})\,\delta(r'-a)$$

$$\iint\vec{J}\cdot d\vec{S}=I,\qquad dS_\phi=r'd\theta'dr'$$

$$C\int\delta(\theta'-\tfrac{\pi}{2})\delta(r'-a)\,r'd\theta'dr'=I$$

$$\therefore C=\frac{I}{a}$$

则 $\vec{J}(\vec{r}) = \dfrac{I}{a}\,\delta(\theta'-\frac{\pi}{2})\,\delta(r'-a)\,\hat{e}_{\phi'}$

电流环在xy平面 $z'=0$

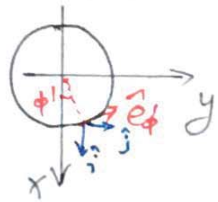
场点 $P(x,0,z)$ 处的矢势为

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi a}\int \frac{\delta(\theta'-\frac{\pi}{2})\,\delta(r'-a)}{\sqrt{(x-x')^2+y'^2+z^2}}\,\hat{e}_\phi\, r'^2 \sin\theta'\,dr'\,d\theta'\,d\phi'$$

$$\left\{\begin{array}{ll} x = r\sin\theta & x' = a\cos\phi' \\ z = r\cos\theta & y' = a\sin\phi' \end{array}\right.$$

$$= \frac{\mu_0 I}{4\pi a}\int \frac{\delta(\theta'-\frac{\pi}{2})\,\delta(r'-a)}{\sqrt{(r\sin\theta - a\cos\phi')^2+(a\sin\phi')^2+(r\cos\theta)^2}}\,\hat{e}_\phi\, r'^2\sin\theta'\,dr'\,d\theta'\,d\phi'$$

$$= \frac{\mu_0 I}{4\pi a}\int \frac{\delta(\theta'-\frac{\pi}{2})\,\delta(r'-a)}{\sqrt{a^2+r^2-2ar\sin\theta\cos\phi'}}\,\hat{e}_\phi\, r'^2\sin\theta'\,dr'\,d\theta'\,d\phi'$$




$$\left\{\ \hat{e}_{\phi'} = \hat{j}\cos\phi' - \hat{i}\sin\phi'\right.$$

$$= \frac{\mu_0 I \times a^2}{4\pi a}\int \frac{-\hat{i}\sin\phi' + \hat{j}\cos\phi'}{\sqrt{a^2+r^2-2ar\sin\theta\cos\phi'}}\,d\phi'$$

$$A_x \propto \hat{i}\int_{0,\,-\pi}^{2\pi,\,\pi} \frac{\sin\phi'}{\sqrt{\ }}\,d\phi' \quad \text{奇函数} \equiv 0$$

$$A_y = \frac{\mu_0 I a}{4\pi}\int \frac{\cos\phi'}{\sqrt{a^2+r^2-2ar\sin\theta\cos\phi'}}\,d\phi'$$