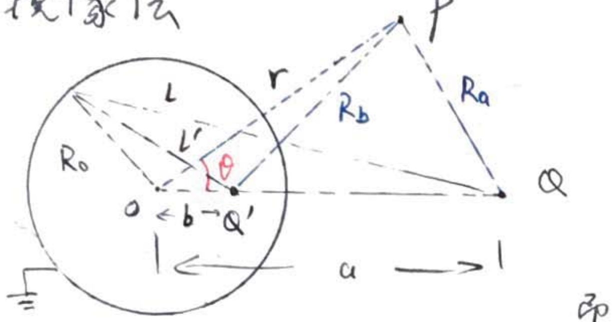
§4 镜像法球外Green函数

1. 镜像法



球面上势

$$\varphi=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{l}+\frac{Q'}{l'}\right)\equiv 0$$

即 $\dfrac{l'}{l}=-\dfrac{Q'}{Q}=\text{const.}$

相似△： $\dfrac{l'}{l}=\dfrac{b}{R_0}=\dfrac{R_0}{a}\Rightarrow b=\dfrac{R_0^2}{a}$

$\dfrac{Q'}{Q}=-\dfrac{R_0}{a}$，则 $Q'=-\dfrac{R_0}{a}Q$

∴对于球外任一点P，势

$$\varphi=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R_a}+\frac{Q'}{R_b}\right)$$

$$=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\sqrt{r^2+a^2-2ra\cos\theta}}-\frac{\frac{R_0}{a}Q}{\sqrt{r^2+b^2-2rb\cos\theta}}\right)$$

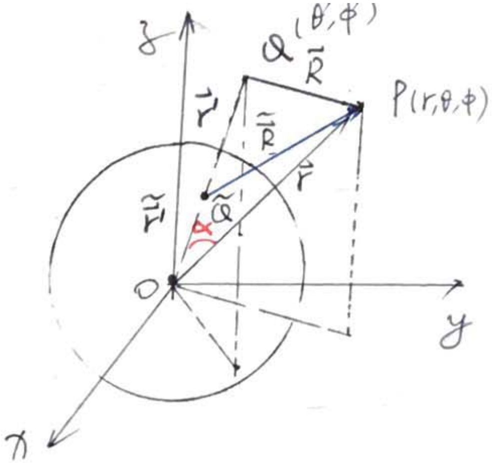
其中，$b=\dfrac{R_0^2}{a}$

更一般情况下，三维，图中P点电势

$$\varphi=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{R}+\frac{\tilde{Q}}{\tilde{R}}\right)=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{|\vec{r}-\vec{r}'|}+\frac{\tilde{Q}}{|\vec{r}-\tilde{\vec{r}}'|}\right)$$

其中：$\tilde{r}'=\dfrac{R_0^2}{r'}$

且 $\tilde{Q}=-\dfrac{R_0}{r'}Q$



$$R=|\vec{r}-\vec{r}'|=\sqrt{r^2+r'^2-2rr'\cos\alpha}$$

$$\tilde{R}=|\vec{r}-\tilde{\vec{r}}'|$$

$$=\sqrt{r^2+\tilde{r}'^2-2r\tilde{r}'\cos\alpha}$$

$$=\frac{1}{r'}\sqrt{r^2r'^2+R_0^4-2R_0^2rr'\cos\alpha}$$

源Q坐标 $\vec{r}'(x',y',z')$ 像$\tilde{Q}$坐标 $\tilde{\vec{r}}'(\tilde{x}',\tilde{y}',\tilde{z}')$

$$\varphi(\vec{r})=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R}-\frac{R_0}{r'}\frac{1}{\tilde{R}}\right)$$

$$=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\alpha}}-\frac{R_0}{\sqrt{r^2r'^2+R_0^4-2R_0^2rr'\cos\alpha}}\right)$$

圆心角公式

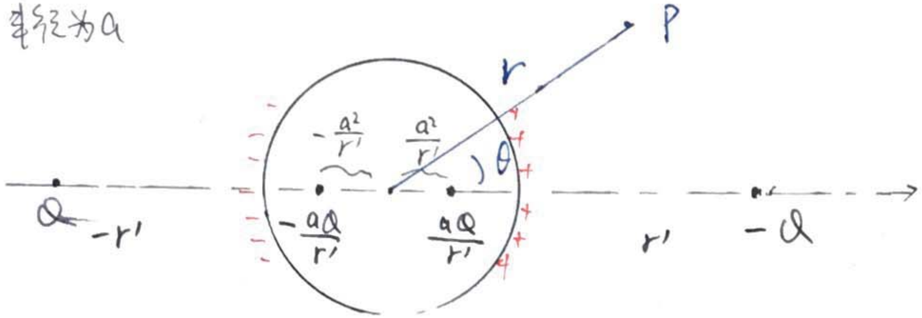
$$\cos\alpha=\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\varphi_2-\varphi_1)$$

关于电荷（感应）面密度

$$\sigma=-\varepsilon\frac{\partial\varphi}{\partial n}\Big|_{r=R_0}$$

2. 镜像法的应用（匀强场极限） 如何取近似

半径为a



球外空间电势 $\varphi = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\alpha}} - \frac{R_0}{r'\sqrt{r^2+\frac{R_0^4}{r'^2}-2\frac{R_0^2 r}{r'}\cos\alpha}}\right)$

两源两像共产生势

$$\varphi = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}} - \frac{a}{r'\sqrt{r^2+\frac{a^4}{r'^2}-2\frac{a^2r}{r'}\cos\theta}}\right)$$

$$+\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+r'^2+2rr'\cos\theta}} - \frac{a}{r'\sqrt{r^2+\frac{a^4}{r'^2}+2\frac{a^2r}{r'}\cos\theta}}\right)$$

远离±Q处， $a \ll r'$， $r \ll r'$

$$\approx \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r'\sqrt{1+2\frac{r}{r'}\cos\theta}} - \frac{a}{r'\sqrt{r^2+2a^2\frac{r}{r'}\cos\theta}}\right)$$

$$-\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r'\sqrt{1-2\frac{r}{r'}\cos\theta}} - \frac{a}{r'\sqrt{r^2-2a^2\frac{r}{r'}\cos\theta}}\right)$$

利用 $\frac{1}{\sqrt{1\pm x}} \approx 1\mp\frac{1}{2}x + O(x^2)$

$$\varphi \doteq \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r'}\left(1-\frac{r}{r'}\cos\theta\right) - \frac{a}{r'r}\left(1-\frac{a^2}{rr'}\cos\theta\right)\right]$$

$$-\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r'}\left(1+\frac{r}{r'}\cos\theta\right) - \frac{a}{r'r}\left(1+\frac{a^2}{rr'}\cos\theta\right)\right]$$

$$= \frac{Q}{2\pi\varepsilon_0}\left(-\frac{r}{r'^2}\cos\theta + \frac{a^3}{r^2r'^2}\cos\theta\right)$$

$$= \frac{Q}{2\pi\varepsilon_0 r'^2}\left(-r\cos\theta + \frac{a^3}{r^2}\cos\theta\right)$$

令 $E_0 = \frac{Q}{2\pi\varepsilon_0 r'^2}$，则 $(r'\to\infty,\ Q\to\infty)$

$$\varphi = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta$$

匀强电场的势　　偶极矩势

“单极矩” $\varphi^{(0)} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$

“偶极矩” $\varphi^{(1)} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{e}_r}{r^2} = \frac{p}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}$

3. 球外空间Green函数

$\nabla^2 G = -4\pi\delta(\vec{R})$ 对比 $\nabla^2\varphi = -\frac{\rho}{\varepsilon_0} = -\frac{Q\delta(\vec{R})}{\varepsilon_0}$ 点电荷

$\varphi \to G$，取 $\frac{Q}{\varepsilon_0}\to 4\pi$ 或 $\frac{Q}{4\pi\varepsilon_0}\to 1$

$$G = \frac{1}{|\vec{r}-\vec{r}'|} - \frac{R_0}{r'|\vec{r}-\vec{r}'|}$$