3. 倏逝波. (复折射率)

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1} = \frac{1}{n_{12}} \qquad n_{12} = \frac{n_1}{n_2} \text{ 相对折射率}$$

临界 θ_c ， $\sin\theta_c = \frac{1}{n_{12}}$ ，则因 $\sin\theta_2 = n_{12}\sin\theta_1 > 1$

$$\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = \pm i\sqrt{n_{12}^2\sin^2\theta_1 - 1}$$

$$k''_x = k_x = \omega\sqrt{\mu_1\varepsilon_1}\sin\theta_1 \ , \quad k''_y = k''\cos\theta_2 = \pm i\omega\sqrt{\mu_2\varepsilon_2}\sqrt{n_{12}^2\sin^2\theta_1 - 1}$$

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则折射波场 $\vec{E}'' = \vec{E}_0'' e^{i(\vec{k}''\cdot\vec{r} - \omega t)}$

$$= \vec{E}_0'' e^{-\omega\sqrt{\mu_2\varepsilon_2}\sqrt{n_{12}^2\sin^2\theta_1 - 1}\cdot z} e^{i(\omega\sqrt{\mu_1\varepsilon_1}\sin\theta_1 x - \omega t)}$$

§3. 波的传播叠加、群速度

1. 群速度

考虑两等幅波叠加： $u_1 = Ae^{i\varphi_1}$ $\quad u_2 = Ae^{i\varphi_2}$

$$u = u_1 + u_2 = A(e^{i\varphi_1} + e^{i\varphi_2}) = A[(\cos\varphi_1 + \cos\varphi_2) + i(\sin\varphi_1 + \sin\varphi_2)]$$

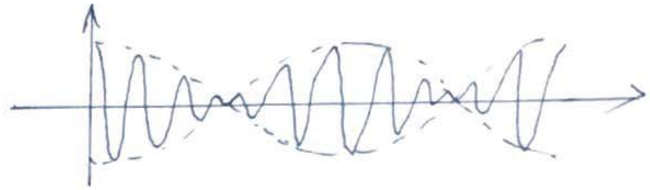
$$= 2A\left(\cos\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2} + i\sin\frac{\varphi_1+\varphi_2}{2}\cos\frac{\varphi_1-\varphi_2}{2}\right)$$

$$= 2A\cos\frac{\varphi_1-\varphi_2}{2}\, e^{i\frac{\varphi_1+\varphi_2}{2}}$$

设两波 $\varphi_1 = \vec{k}_1\cdot\vec{r} - \omega_1 t$ $\qquad \varphi_2 = \vec{k}_2\cdot\vec{r} - \omega_2 t$

$$= (\vec{k} - \tfrac{d\vec{k}}{2})\cdot\vec{r} - (\omega - \tfrac{d\omega}{2})t \qquad = (\vec{k} + \tfrac{d\vec{k}}{2})\cdot\vec{r} - (\omega + \tfrac{d\omega}{2})t$$

则 $u = 2A\cos\left(\frac{d\vec{k}}{2}\cdot\vec{r} - \frac{d\omega}{2}t\right) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$



波包

相速度 $\vec{v}_p = \frac{\omega}{k}\hat{n}$ $\qquad$ 群速度 $\vec{v}_g = \frac{d\omega}{dk}\hat{n}$

也可 $\vec{v}_g = \frac{d\omega}{d\vec{k}}$ 形式写法

$$= \frac{\partial\omega}{\partial k_x}\hat{i} + \frac{\partial\omega}{\partial k_y}\hat{j} + \frac{\partial\omega}{\partial k_z}\hat{k} = \nabla_k\omega$$

~~在k~~ 小范围展开 $\qquad \omega = \omega(\vec{k})$

$$\omega \doteq \omega_0 + (\vec{k} - \vec{k}_0)\cdot\nabla_k\omega|_{k_0}$$

一般而言，任意波包 $u(\vec{r}, t)$ 可按单色波展开

$$\boxed{u(\vec{r}, t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int a(\vec{k})e^{i(\vec{k}\cdot\vec{r} - \omega t)}\, d^3\vec{k}}$$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}}\int a(\vec{k})\, e^{i(\vec{k}\cdot\vec{r} - \omega_0 t)} e^{-i(\vec{k} - \vec{k}_0)\cdot\nabla_k\omega|_0 t}\, d^3\vec{k}$$

$$= e^{-i(\omega_0 - \vec{k}_0\cdot\nabla_k\omega_0)t}\frac{1}{(2\pi)^{\frac{3}{2}}}\int a(\vec{k})\, e^{i\vec{k}\cdot(\vec{r} - \nabla_k\omega|_0 t)}\, d^3\vec{k}$$

$$= e^{-i(\omega_0 - \vec{k}_0\cdot\vec{v}_{g0})t}\frac{1}{(2\pi)^{\frac{3}{2}}}\int a(\vec{k})\, e^{i\vec{k}\cdot(\vec{r} - \vec{v}_{g0}t)}\, d^3\vec{k}$$

$$= e^{-i(\omega_0 - \vec{k}_0\cdot\vec{v}_{g0})t}\, u(\vec{r} - \vec{v}_{g0}t, 0)$$

特别，若无色散 $v_g = v_p$ （$\omega = v_p k$），则

$$u(\vec{r},t) = e^{-i(\omega_0 - \vec{k}_0\cdot\vec{v}_p)t}\, u(\vec{r} - \vec{v}_p t, 0)$$

$$= u(\vec{r} - \vec{v}_p t, 0)$$

例：对Gauss型波包，如 $u(x,0) = e^{-\frac{x^2}{2L^2}} \cos k_0 x$

改为复数形式 $u(x,0) = e^{-\frac{x^2}{2L^2}} e^{ik_0 x}$

满足色散关系：$\omega(k) = v\left(1 + \frac{a^2k^2}{2}\right)$

则 $v_{g0} = \frac{d\omega}{dk}\Big|_{k_0} = va^2 k_0$

$$u(x,t) = e^{-i(\omega_0 - k_0 v_{g0})t}\, e^{-\frac{(x - v_g t)^2}{2L^2}}\, e^{ik_0(x - v_g t)}$$

$$= e^{-\frac{(x - va^2k_0 t)^2}{2L^2}}\, e^{ik_0 x}\, e^{-iv\left(1 + \frac{a^2k_0^2}{2}\right)t}$$

即7.98式（严格解）中取 $1 + \frac{ia^2vt}{L^2} \approx 1$

2、波包的演化（一维情况）

对于一个实波包

$$u(x,t) = \frac{1}{2\sqrt{2\pi}}\left\{\int_{-\infty}^{+\infty} A(k)\, e^{i(kx-\omega t)}\, dk + c.c.\right\}$$

$$= \frac{1}{2\sqrt{2\pi}}\left\{\int_{-\infty}^{+\infty} A(k)\, e^{i(kx-\omega t)}\, dk + \int_{-\infty}^{+\infty} A^*(k)\, e^{-i(kx-\omega t)}\, dk\right\}$$

作代换 $k \to -k$，$\omega \to -\omega$

$$u(x,t) = \frac{1}{2\sqrt{2\pi}}\left[\int_{-\infty}^{+\infty} A(-k)\, e^{-i(kx-\omega t)}\, dk + \int_{-\infty}^{+\infty} A^*(-k)\, e^{i(kx-\omega t)}\, dk\right]$$

函数不变，则 $A^*(-k) = A(k)$ 及 $\omega(k)$ 为奇函数。