第3章 电磁场普遍规律

§1. 位移电流、Maxwell方程组

1. 位移电流

$$
\begin{cases}
\nabla\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0} \\
\nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\
\nabla\cdot\vec{B} = 0 \\
\nabla\times\vec{B} = \mu_0\vec{J}
\end{cases}
$$

$\longrightarrow$ 问题：表四

$$\underbrace{\nabla\cdot(\nabla\times\vec{B})}_{0} = \mu_0\nabla\cdot\vec{J}$$

$$\therefore \nabla\cdot\vec{J} = 0$$

电荷守恒 $\dfrac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$ } 矛盾?

由电荷守恒出发，结合表一，得

$$\varepsilon_0\nabla\cdot\frac{\partial\vec{E}}{\partial t} + \nabla\cdot\vec{J} = 0$$

$$\Rightarrow \nabla\cdot\Big(\vec{J} + \underbrace{\varepsilon_0\frac{\partial\vec{E}}{\partial t}}_{\vec{J}_D}\Big) = 0$$

$\vec{J}_{tot} = \vec{J} + \vec{J}_D$ $\dfrac{\text{稳恒}}{\text{恒定}}$

表四改为 $\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$

2. 势及其方程

表三 定义 $\vec{B} = \nabla\times\vec{A}$，代回表二

$$\nabla\times\vec{E} + \frac{\partial}{\partial t}\nabla\times\vec{A} = 0$$

$$\Rightarrow \nabla\times\Big(\vec{E} + \frac{\partial\vec{A}}{\partial t}\Big) = 0$$ 从而定义

$$\vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t}$$

$$
\begin{cases}
\nabla^2\varphi + \nabla\cdot\dfrac{\partial\vec{A}}{\partial t} = -\dfrac{\rho}{\varepsilon_0} \\
\nabla^2\vec{A} - \mu_0\varepsilon_0\dfrac{\partial^2\vec{A}}{\partial t^2} - \nabla\Big(\nabla\cdot\vec{A} + \mu_0\varepsilon_0\dfrac{\partial\varphi}{\partial t}\Big) = -\mu_0\vec{J}
\end{cases}
$$

关于规范：势不确定 — $\vec{A}\to\vec{A}+\nabla\chi$，相应 $\varphi\to\varphi-\dfrac{\partial\chi}{\partial t}$

① Coulomb 规范 "$\nabla\cdot\vec{A} = 0$"

$$
\begin{cases}
\nabla^2\varphi = -\dfrac{\rho}{\varepsilon_0} \\
\nabla^2\vec{A} - \mu_0\varepsilon_0\dfrac{\partial^2\vec{A}}{\partial t^2} - \cancel{\nabla\nabla\cdot\vec{A}}\,\mu_0\varepsilon_0\nabla\dfrac{\partial\varphi}{\partial t} = -\mu_0\vec{J}
\end{cases}
$$

② Lorentz 规范 $\nabla\cdot\vec{A} + \mu_0\varepsilon_0\dfrac{\partial\varphi}{\partial t} = 0$

$$
\begin{cases}
\nabla^2\varphi - \mu_0\varepsilon_0\dfrac{\partial^2\varphi}{\partial t^2} = -\dfrac{\rho}{\varepsilon_0} \\
\nabla^2\vec{A} - \mu_0\varepsilon_0\dfrac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}
\end{cases}
$$

特别、对于Coulomb 规范

$$\varphi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}d^3r'$$

将$\vec{J}$分解为 $\vec{J} = \underset{(\text{横})}{\vec{J}_t} + \underset{(\text{纵})}{\vec{J}_l}$

$\nabla\cdot\vec{J}_t = 0$ $\qquad$ $\nabla\times\vec{J}_l = 0$

取
$$
\begin{cases}
\vec{J}_l = -\dfrac{1}{4\pi}\nabla\Big(\nabla\cdot\displaystyle\int\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}d\vec{r}'\Big) = -\dfrac{1}{4\pi}\nabla(\nabla\cdot\vec{V}) \\
\vec{J}_t = \dfrac{1}{4\pi}\nabla\times\Big(\nabla\times\displaystyle\int\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3r'\Big) = \dfrac{1}{4\pi}\nabla\times(\nabla\times\vec{V})
\end{cases}
$$

显然 $\nabla\times\vec{J}_l = 0$，及 $\nabla\cdot\vec{J}_t = 0$，再

$$\vec{J}_l+\vec{J}_t = \frac{1}{4\pi}\left[\nabla\times(\nabla\times\vec{V}) - \nabla(\nabla\cdot\vec{V})\right] = -\frac{1}{4\pi}\nabla^2\vec{V}$$

$$= -\frac{1}{4\pi}\int \nabla^2\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3\vec{r}'$$

$$= -\frac{1}{4\pi}\int \vec{J}(\vec{r}')\underbrace{\nabla^2\frac{1}{|\vec{r}-\vec{r}'|}}_{=-4\pi\delta(\vec{r}-\vec{r}')}d^3\vec{r}'$$

$$= -\frac{1}{4\pi}(-4\pi)\int \vec{J}(\vec{r}')\,\delta(\vec{r}-\vec{r}')\,d^3\vec{r}'$$

$$= \vec{J}(\vec{r})$$

单独讨论 $\vec{J}_l$

$$\vec{J}_l = -\frac{1}{4\pi}\nabla\int \vec{J}(\vec{r}')\cdot\nabla\frac{1}{|\vec{r}-\vec{r}'|}d^3\vec{r}'$$

$$= +\frac{1}{4\pi}\nabla\int \vec{J}(\vec{r}')\cdot\nabla'\frac{1}{|\vec{r}-\vec{r}'|}d^3\vec{r}'$$

$$= \frac{1}{4\pi}\nabla\left[\int\nabla'\cdot\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3\vec{r}' - \int\frac{1}{|\vec{r}-\vec{r}'|}\nabla'\cdot\vec{J}(\vec{r}')\,d^3\vec{r}'\right]$$

$$= \frac{1}{4\pi}\nabla\left[\underbrace{\oint d\vec{S}'\cdot\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}}_{=0} - \cdots\right]$$

$$= -\frac{1}{4\pi}\nabla\int\frac{\nabla'\cdot\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3\vec{r}'$$

再由 $\frac{\partial\rho}{\partial t}+\nabla'\cdot\vec{J}(\vec{r}')=0$

作业 6.14(c)× $w_e=\frac{1}{2}\varepsilon_0E^2$ $w_m=\frac{1}{2}\mu_0H^2$

$$\therefore \mu_0\vec{J}_l = \frac{\mu_0}{4\pi}\nabla\frac{\partial}{\partial t}\int\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3\vec{r}'$$

$$= \mu_0\varepsilon_0\nabla\frac{\partial}{\partial t}\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}d^3\vec{r}'$$

$$= \mu_0\varepsilon_0\nabla\frac{\partial\varphi}{\partial t}$$

则 $\nabla^2\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\varphi}{\partial t}$

$$= -\mu_0\vec{J}+\mu_0\vec{J}_l = -\mu_0\vec{J}_t$$

§2. 波动方程的Green函数.

1. d'Alembert方程的形式解

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi F(\vec{r},t) \qquad c=\frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

$$\begin{cases}\Psi(\vec{r},t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi(\vec{r},\omega)e^{-i\omega t}d\omega\\ F(\vec{r},t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(\vec{r},\omega)e^{-i\omega t}d\omega\end{cases}$$ 代回

$$\Rightarrow\begin{cases}\psi(\vec{r},\omega)=\int_{-\infty}^{+\infty}\Psi(\vec{r},t)e^{i\omega t}dt\\ f(\vec{r},\omega)=\int_{-\infty}^{+\infty}F(\vec{r},t)e^{i\omega t}dt\end{cases}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\nabla^2\psi(\vec{r},\omega)\cancel{e^{-i\omega t}}d\omega+\frac{\omega^2}{c^2}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi(\vec{r},\omega)\cancel{e^{-i\omega t}}d\omega=-4\pi\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(\vec{r},\omega)\cancel{e^{-i\omega t}}d\omega$$

Helmholtz方程 $(\nabla^2+k^2)\psi(\vec{r},\omega)=-4\pi f(\vec{r},\omega)$，$k=\frac{\omega}{c}$