$$A_y = \frac{\mu_0 I a}{4\pi}\int_0^{2\pi}\frac{\cos\phi'}{\sqrt{a^2+r^2-2ar\cos\theta\sin\phi'}}d\phi'$$

可对于 $a \ll r$ 和 $a \gg r$ 展开

$$\frac{1}{\sqrt{a^2+r^2-2ar\cos\theta\sin\phi}} = \frac{1}{\sqrt{a^2+r^2}}\frac{1}{\sqrt{1-\frac{2ar}{a^2+r^2}\sin\theta\cos\phi'}}$$

$$\doteq \frac{1}{\sqrt{a^2+r^2}}\left(1+\frac{ar}{a^2+r^2}\sin\theta\cos\phi'+\cdots\right)$$

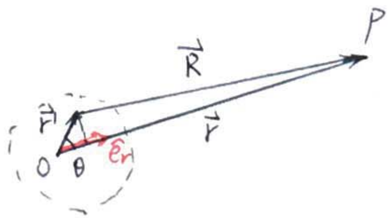
$$\therefore A_y = \frac{\mu_0 I a}{4\pi}\frac{1}{\sqrt{a^2+r^2}}\left[\underbrace{\int_0^{2\pi}\cos\phi' d\phi'}_{=0} + \frac{ar}{a^2+r^2}\sin\theta\underbrace{\int_0^{2\pi}\cos^2\phi' d\phi'}_{=\pi}\right]$$

$$= \frac{\mu_0 I a^2 r}{4\pi(a^2+r^2)^{\frac{3}{2}}}\sin\theta = e_\phi A_\phi$$

$$\therefore \vec{A} = \frac{\mu_0 I a^2 r}{\quad}\sin\theta\,\hat{e}_\phi$$

再由 $\vec{B} = \nabla\times\vec{A}$ 可[illegible]

4. 矢磁多极展开，求磁矩



$$R \approx r - r'\cos\theta$$

$$= r - \vec{r'}\cdot\hat{e}_r$$

$$= r\left(1-\frac{\vec{r'}\cdot\hat{e}_r}{r}\right)$$

[illegible]展开 $\frac{1}{R} = \frac{1}{r}\frac{1}{1-\frac{\vec{r'}\cdot\hat{e}_r}{r}}$ "$\frac{1}{1-x} = 1+x+x^2\cdots$"

$$= \frac{1}{r}\left[1+\frac{\vec{r'}\cdot\hat{e}_r}{r}+\left(\frac{\vec{r'}\cdot\hat{e}_r}{r}\right)^2+\cdots\right]$$

$$\approx \frac{1}{r}+\frac{\vec{r'}\cdot\hat{e}_r}{r^2}$$

矢势 $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r'})}{R}dV'$

$$= \frac{\mu_0}{4\pi}\left[\int\frac{\vec{J}(\vec{r'})}{r}dV' + \int \vec{J}(\vec{r'})\vec{r'}\cdot\frac{\hat{e}_r}{r^2}dV'\right]$$

$$= \frac{\mu_0}{4\pi}\left[\int\vec{J}(\vec{r'})dV'\,\frac{1}{r} + \int\vec{J}(\vec{r'})\vec{r'}dV'\cdot\frac{\hat{e}_r}{r}\right]$$

① 由 $\nabla\cdot(\vec{A}\vec{B}) = (\nabla\cdot\vec{A})\vec{B} + (\vec{A}\cdot\nabla)\vec{B}$

取 $\vec{A}=\vec{J}$，$\vec{B}=\vec{r}$，注意 $\nabla\vec{r} = \vec{I}$，则

$$\nabla\cdot(\vec{J}\vec{r}) = (\nabla\cdot\vec{J})\vec{r} + \underbrace{(\vec{J}\cdot\nabla)\vec{r}}_{\vec{I}} = (\nabla\cdot\vec{J})\vec{r}+\vec{J}$$

$$\therefore \int\vec{J}(\vec{r'})dV' = \int\nabla'\cdot(\vec{J'}\vec{r'})dV' - \int(\nabla'\cdot\vec{J'})\vec{r'}dV'$$

稳恒电流 $\frac{\partial\rho}{\partial t}=0$

$$\boxed{\frac{\partial\rho}{\partial t}+\nabla\cdot\vec{J}=0}$$

$$= \oint_S d\vec{S'}\cdot\vec{J'}\vec{r'} - 0$$

$\xrightarrow{S足够大} = 0$

② 由 $\nabla\cdot(\vec{A}\vec{B}\vec{C}) = (\nabla\cdot\vec{A})\vec{B}\vec{C} + \underbrace{(\vec{A}\cdot\nabla)}_{标量算符}\vec{B}\vec{C}$

$$= (\nabla\cdot\vec{A})\vec{B}\vec{C} + (\vec{A}\cdot\nabla\vec{B})\vec{C} + \vec{B}(\vec{A}\cdot\nabla\vec{C})$$

取 $\vec A = \vec J$, $\vec B = \vec C = \vec r$, 则

$$\nabla\cdot(\vec J\vec r\vec r) = \underbrace{(\nabla\cdot\vec J)}_{=0}\vec r\vec r + (\vec J\cdot\underbrace{\nabla\vec r}_{I})\vec r + \vec r(\vec J\cdot\underbrace{\nabla\vec r}_{I})$$

$$= \vec J\vec r + \vec r\vec J \neq 2\vec J\vec r$$

但 $\int(\vec J\vec r + \vec r\vec J)\,dV = \int\nabla\cdot(\vec J\vec r\vec r)\,dV = \underbrace{\oint d\vec S\cdot(\vec J\vec r\vec r)}_{=0}$

$$\int\vec J(\vec r')\vec r'\,dV'\cdot\vec r = \frac12\left[\underbrace{\int(\vec J'\vec r' + \vec r'\vec J')\,dV'}_{=0} + \int(\vec J'\vec r' - \vec r'\vec J')\,dV'\right]\cdot\vec r$$

$$= \frac12\int(\vec J'\vec r' - \vec r'\vec J')\,dV'\cdot\vec r$$

$$= \frac12\int(\vec J'\vec r'\cdot\vec r - \vec r'\vec J'\cdot\vec r)\,dV'$$

$$= \frac12\int(\vec r'\times\vec J')\times\vec r\,dV'$$

$$\cancel{= -\frac12\,\vec r\times\int(\vec r'\times\vec J')\,dV'}$$

$$\therefore\ \vec A = \frac{\mu_0}{4\pi}\left(\frac12\int\vec r'\times\vec J'\,dV'\right)\times\frac{\hat e_r}{r^2}$$

定义磁矩 $\vec m = \frac12\int\vec r'\times\vec J'\,dV'$

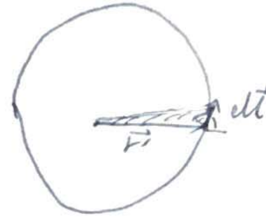
$$\therefore\ = \frac{\mu_0}{4\pi}\,\frac{\vec m\times\hat e_r}{r^2}$$

特别，对于面线圈 $\vec m = \frac12\iiint\vec r'\times\vec J'\,dV'$

$$= \frac I2\oint\vec r'\times d\vec l'$$

$\frac12\,\vec r'\times d\vec l' = d\vec S'$

$\therefore\ \vec m = I\oint d\vec S' = I\vec S$



§7. 电磁感应. 磁能

1. Faraday

$$\mathcal E = -k\frac{d\Phi}{dt}\ ,\quad k = 1$$

$$\Phi = \iint\vec B\cdot d\vec S$$

$$\Longrightarrow\quad \nabla\times\vec E = -\frac{\partial\vec B}{\partial t}\qquad(\text{麦二})$$

左 $\mathcal E = \oint\vec E_c\cdot d\vec l$ ，右 $= -\iint\frac{\partial\vec B}{\partial t}\cdot d\vec S$

↓ stokes

$= \iint(\nabla\times\vec E_c)\cdot d\vec S$ 　等

2. 磁能

能量密度

$(w_e = \frac12\vec E\cdot\vec D)$ 　 $w_m = \frac12(\vec H\cdot\vec B)$

“待第6章讲”

对于静磁场
$$\begin{cases}\vec B = \nabla\times\vec A\\ \nabla\times\vec H = \vec J_f\end{cases}$$

~~##~~ $\vec H\cdot\vec B = \vec H\cdot(\nabla\times\vec A)$

再由 ~~$\nabla\times(\vec H\times\vec A) = \vec H\,\nabla$~~

$$\nabla\cdot(\vec H\times\vec A) = \vec A\cdot(\nabla\times\vec H) - \vec H\cdot(\nabla\times\vec A)$$

$$\therefore\ w_m = \frac12\left[\vec A\cdot(\nabla\times\vec H) - \nabla\cdot(\vec H\times\vec A)\right]$$

$$W_m = \int dw_m = \frac12\int\vec A\cdot(\nabla\times\vec H)\,dV - \frac12\int\nabla\cdot(\vec H\times\vec A)\,dV$$

$$= \frac12\int\vec A\cdot\vec J_f\,dV - \frac12\oint d\vec S\cdot(\vec H\times\vec A)$$

当电流分布于有限区域，由于 $S\to\infty$ 处 $\vec H,\vec A\to0$

$$\therefore\ w_m = \frac12\vec A\cdot\vec J_f$$

（类似 $w_e = \frac12\varphi\rho_f$）

3. 自感与互感

$$W_m = \frac12\int\vec J(\vec r)\cdot\vec A(\vec r)\,dV$$

$$= \frac{\mu_0}{4\pi}\cdot\frac12\int\frac{\vec J(\vec r)\cdot\vec J(\vec r')}{|\vec r-\vec r'|}\,dV'\,dV$$

$$= \frac12\underbrace{\frac{\mu_0}{4\pi I_iI_j}\int\frac{\vec J(\vec r)\cdot\vec J(\vec r')}{|\vec r-\vec r'|}\,dV'\,dV}_{M_{12}}\ I_iI_j$$

互感/自感

$$= \frac12 M_{ij}I_iI_j$$

① 当 “1≠2” 互感 M_{ij}，$W_m = \frac12 M_{ij}I_iI_j$

② 当 “1≠2” 自感，~~##~~ $L = M_{ii}$，$W_m = \frac12 LI^2$

特别，对于线形回路

$$M_{ij} = \frac{\mu_0}{4\pi I_iI_j}\oint_{l_1}\oint_{l_2}\frac{I_iI_j\,d\vec l_i\cdot d\vec l_j}{|\vec r_i-\vec r_j|}$$

$$= \frac{\mu_0}{4\pi}\oint_{l_1}\oint_{l_2}\frac{d\vec l_i\cdot d\vec l_j}{|\vec r_i-\vec r_j|}$$

作业：5.3　5.6　5.13　5.26