Homework 4: Bayesics of Inference

Solutions to this exercise sheet are to be handed in before the lecture on Wednesday, XY.XY.12. Code (.m-file) and output (.mat-file, figures as .pdf-files) for the matlab-questions should be either submitted by email to `XY.XY@student.uni-tuebingen.de` (subject: [ML1] Exercise 4) or uploaded to Ilias. Use comments to explain your code. Please adhere to the file naming convention: `Homework4_<YourName>.<ext>`.

This exercise sheet will concentrate on the Poisson distribution, which is a very important model in neuroscience. The Poisson distribution $P(X = x|\theta)$ denotes the probability of x events happening in some observation interval (e.g. the number of action potentials fired by a neuron in response to a stimulus), assuming that these events happen at random times and with a mean rate of θ. We have

$$P(X = x|\theta) = \frac{1}{x!}\theta^x \exp(-\theta) \tag{1}$$

1. (15 points) **Inference of the mean rate in the Poisson distribution**
   (a) Suppose you are given data $x_1, x_2, \ldots x_N$. Show that the maximum likelihood estimate of θ is given by $\hat{\theta} = \frac{1}{N}\sum_n x_n$ [Hint: maximize the log-likelihood by setting its derivative w.r.t to θ to zero. You are not required to show that this is a maximum (rather than a minimum or stationary point).]
   (b) Show that the Poisson distribution is in the exponential family, by identifying the functions $g(\theta)$, $f(x)$, $\phi(\theta)$ and $S(x)$ discussed in the lecture.
   (c) The conjugate prior of the Poisson distribution is the Gamma distribution with parameters α and β, Gamma(α, β),

   $$p(\theta|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}\exp(-\beta\theta) \tag{2}$$

   and mean $\mathrm{E}(\theta) = \alpha/\beta$ and variance $\mathrm{Var}(\theta) = \alpha/beta^2$. Show that the posterior distribution over θ after observing data $D = \{x_1, x_2, x_3, \ldots x_n\}$ is given by

   $$\theta|D \sim \mathrm{Gamma}(\alpha + \sum_{n=1}^{N} x_n, \beta + N) \tag{3}$$

2. (15 points) **Inference in the Poisson distribution, Application [MATLAB]**
   (a) Load the variables in the file `Homework2.mat`. Assuming that these data are generated by a Poisson distribution with parameter θ, and using a Gamma-distribution with parameters $\alpha = 2$ and $\beta = 1$, calculate and plot the posterior distribution over θ. [You do not need the statistics toolbox, but if you use the function `gampdf` in it , please note that it uses a different parameterization of the gamma distribution.]
   (b) Calculate both the MLE and the posterior mean for θ using only the first n datapoints, and plot both as a function of n. Report the MLE and the posterior mean of θ for $n = 10$.
   (c) Plot the posterior variance as a function of n, and report the value for $n = 10$.
   (d) [optional] Calculate the predictive distribution for the $n + 1$th observation, and (numerically or analytically) calculate its mean and variance.