Set: Collection of well defined objects.
cardinality: number of elements in a set.
Subset: a set which has few elements of a set.
Power set: Set of all subsets.

$|P(S)| = 2^n$

well defined: something which is already known.

Relation: b/w two sets A & B.

$$R = \{(a,b) \mid a = SD \in A, b \in B\}$$

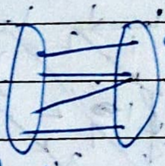
Cartesian Product

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

$R \subseteq A \times B$

function: mapping from $A \to B$

$f: A \to B$



↳ it's a relation where $\exists$ some association b/w a member of domain & codomain.

(1) $f(a) \in B \ \forall a \in A$

(2) if $b_1 = f(a)$ & $b_2 = f(a)$, then $b_1 = b_2 \ \forall a \in A$.

Equivalent Set

↳ two sets are equivalent if their no of elements are equal.

or $A \to B$, one to one correspondence (bijective)

One-One → every element is mapped to unique element.

Onto → Codomain = Range

Bijective = One-One + Onto

Q card$|\mathbb{N}|$ = card$|Even|$ = card$|odd|$

Proof: $\mathbb{N} = \{1, 2, 3, \ldots\}$

$E = \{2, 4, \ldots\}$

$f: \mathbb{N} \to E$ s.t $y = 2x$, $y \in E$, $x \in \mathbb{N}$

_/_/_

same for odd,

$f: \mathbb{N} \to O,\ y = 2k-1,\ y \in O,\ k \in \mathbb{N}$

Countable Set

↳ The set whose elements are countable.

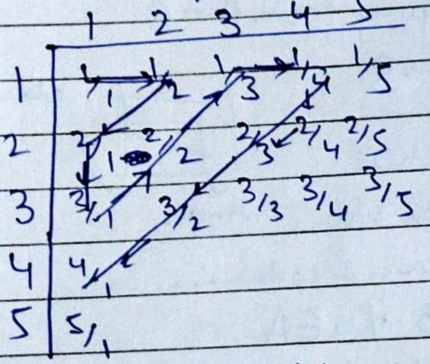
Association of set of $\mathbb{N}$ to the set of numbers, then the set is countable. i.e. if there exists a bijective fx^n from $\mathbb{N}$ to A, then A is countable

if $\mathbb{N} \to A$.

$|\mathbb{N}| = |E| = |O| = |\mathbb{Q}^+|$

Q $\mathbb{Q}^+$ is countable.

Ans $\mathbb{Q}^+ = \{p/q \mid p \in \mathbb{N}, q \in \mathbb{N}$



We can count like this and we find a $f: \mathbb{N} \to \mathbb{Q}^+$ hence it is countable

$|\mathbb{N}| = \aleph_0 \to$ aleph-Null

Infinite Algebra

① $\aleph_0 + n = \aleph_0$

② $\aleph_0 \times \aleph_0 = \aleph_0$

③ $\aleph_0 + \aleph_0 = \aleph_0$

④ $\aleph_0^n = \aleph_0$

_/_/_

Cantor's Thm

↳ Let f be a fx^n from a set A to its power set $P(A)$, then f is not surjective, consequently:

$card(A) < card(P(A))$

Proof: Let A be a set & $P(A)$ be its power set.

Let there be a fx^n $f: A \to B$.

Consider a set $B = \{x \in A \mid x \notin f(x)\}$

Assume f is surjective

$B \subseteq A$

$\exists\, a \in A$ s.t $f(a) = B$

$a \in B \Leftrightarrow a \notin f(a) = B$

contradiction (reduction ad absurdum)

So cardinality not same.

$card(A) < card(P(A))$

$\mathbb{N} = \aleph_0$

$P(\mathbb{N}) > \aleph_0 = \aleph_1 \quad P(\aleph_1) = \aleph_2$

$a^{\aleph_0} = \aleph_1$ With each power set, it increases.

$\aleph_0, \aleph_1, \aleph_2, \ldots$

Universal Set: Set of everything

Set: collection of well defined distinct elements having some common property.

Russell's Paradox (1901)

Let R be a set s.t $R = \{x \mid x \notin x\}$

Options: i) $R \in R$

if $R \in R$, then $R \notin R \Rightarrow$ contradiction

ii) $R \notin R$

if $R \notin R$, then $R \in R \Rightarrow$ contradiction.

_/_/__

Axiomatic Set Theory

① **Axiom of Extension**

↳ two sets are considered equal if they have the same elements

Logic: $\forall A \forall B (A = B \Leftrightarrow \forall x (x \in A \Leftrightarrow x \in B))$

② **Axiom of Specification**

↳ for any set you can create a subset containing exactly those elements that satisfy the same property (φ).

Logic: $\forall B \exists A \forall x (x \in A \Leftrightarrow x \in B \wedge \varphi(x))$

③ **Axiom of Pairing**

↳ for any two sets, $\exists$ a set that contains exactly those two sets as elements.

Logic: $\forall A \forall B \exists C \forall x (x \in C \Leftrightarrow (x = A \vee x = B))$

④ **Axiom of Choice**

↳ Given a collection of non-empty sets, it is possible to select exactly one element from each set.

Logic: $\forall A (\forall x \in A \exists y (y \in x) \Rightarrow \exists f \forall x \in A (f(x) \in x))$

⑤ **Axiom of Union**

↳ for any set of sets, $\exists$ a set that contains all elements of those sets.

Logic: $\forall A \exists B \forall x (x \in B \Leftrightarrow \exists C \in A (x \in C))$

⑥ **Axiom of Power**

↳ for any set, $\exists$ a set of all its subsets

Logic: $\forall A \exists B \forall C (C \subseteq A \Leftrightarrow C \in B)$

—/—/—

(7) Axiom of infinity

↳ $\exists$ a set that contains the empty set and is closed under the operation of adding single element

Logic : $\exists A(\phi \in A \wedge \forall x(x \in A \Rightarrow (x \vee \{x\} \in A)))$

$\boxed{\text{Logic}}$

~~Answers What? why? How?~~

$X \vdash \alpha$

from a set of statements (x) we derive (A)

logic → is how we reach A from X.

Theory of Truth

↳ 'P' is true iff P

Defn: $X \vdash \alpha$ holds iff for all situations whenever all the elements of X are true then α is also True.

Propositional Logic

α, β → propositional variables

Connectors → $\wedge$ (and)

$\vee$ (or)

$\rightarrow$ (if ... then ...)

$\sim$ (NOT)

$V : X \rightarrow \{T, F\}$ [valuation function]

$V(\alpha \wedge \beta) = \begin{cases} 1 & \text{if } V(\alpha) = V(\beta) = 1 \\ 0 & \text{otherwise} \end{cases}$

$V(\alpha \vee \beta) = \begin{cases} 0 & \text{if } V(\alpha) = V(\beta) = 0 \\ 1 & \text{otherwise} \end{cases}$

$V(\sim \alpha) = \begin{cases} 0 & \text{if } V(\alpha) = 1 \\ 1 & \text{otherwise} \end{cases}$

$V(\alpha \rightarrow \beta) = \begin{cases} 0 & \text{if } V(\alpha) = 1,\ V(\beta) = 0 \\ 1 & \text{otherwise} \end{cases}$

Formation Rule

$\Sigma = \{P, |, \sim, (,), \rightarrow\}$

Valuation func → $V : P| \rightarrow \{T, F\}$

$X \vdash \alpha$

proposition conclusion

Truth Table

for eg $X = \{P|,\ P| \rightarrow P||\}$

$\alpha = P||$

Ans

$P|$	$P||$	$P| \rightarrow P||$	$P||$
0	0	1	0
0	1	1	1
1	0	0	0
1	1	1	1

Axiomatic Defn of Logic

Ax1 : $\alpha \rightarrow (\beta \rightarrow \alpha)$

Ax2 : $(\alpha \rightarrow (\beta \rightarrow \alpha)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma))$

Ax3 : $((\sim\alpha) \rightarrow \sim\beta) \rightarrow (\beta \rightarrow \alpha)$

Modus Ponese : $\alpha, \alpha \rightarrow \beta / \beta$ [if $\alpha, \alpha \rightarrow \beta$, then β]

Ex1 $\{\alpha, \alpha \rightarrow \beta\} \vdash \beta$

Ans

1	$\alpha \rightarrow \beta$	in X
2	α	in X
3	β	MP 1 2

Ex2 $\{\alpha, \alpha \rightarrow \beta, \beta \rightarrow \gamma\} \vdash \gamma$

Ans

1	α	in X
2	$\alpha \rightarrow \beta$	in X
3	β	M.P 1 2
4	$\beta \rightarrow \gamma$	in X
5	γ	MP 3 4

Theorem: if $X = \phi$, α is a theorem!

Ex $\alpha \to \alpha$ is a thm.

Ans ① $\alpha \to ((\alpha \to \alpha) \to \alpha)$ Ax1, $\beta = (\alpha \to \alpha)$

② $\alpha \to ((\alpha \to \alpha) \to \alpha) \to ((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha))$ Ax2

③ $(\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha)$ MP ① ②

④ $\alpha \to (\alpha \to \alpha)$ Ax1

⑤ $\alpha \to \alpha$ MP ③ ④

Predicate Logic / First order logic

~~M(x) — predicate val~~

predicate

Quantifiers

$\forall \to$ forall

$\exists \to$ there exists

Logic

a) $\sim \forall x\, \phi(x) \equiv \exists x \sim \phi(x)$

b) $\sim \exists x\, \phi(x) \equiv \forall x \sim \phi(x)$

c) $\exists x\, \phi(x) \equiv \sim \forall x \sim \phi(x)$

$\boxed{\text{Relations}}$

Ordered Pair

$(a,b) \neq (b,a)$

$(a,b) \Rightarrow \{\{a\},\{a,b\}\}$

↳ defined ordering of element.

Cartesian Product

$D \times C = \{ x \mid x = (d,c)$ s.t $d \in D$ and $c \in C\}$

eg $D = \phi$ $C = \{a,b\}$

↳ $D \times C = \phi$

$R \subseteq D \times C$ ← codomain

codomain

Relation from D to C.

if $D = C$, (unary relation)

$R \subseteq D \times D$

$R \subseteq ((A_1 \times A_2) \times A_3 \ldots)$

n-ary relation

Properties

i) Reflexive :- $\forall x \in A, (x,x) \in R$

or $\forall x \in A, xRx$

ii) Symmetric :- $xRy \rightarrow yRx, \forall x, y \in A$

or if $(x,y) \in R$, then $(y,x) \in R$

iii) Asymmetric :- $\forall x, y \in A : xRy \rightarrow \sim(yRx)$

or if $(x,y) \in R$, $(y,x) \notin R$

iv) Antisymmetric :- $\forall x, y \in A$, $xRy \wedge yRx \rightarrow x = y$

↳ basically not sym or asym

v) Transitive : $\forall x, y, z \in A$, $xRy \wedge yRy \rightarrow xRz$

Equivalent : if a relation is Reflexive, Symmetric and transitive.

eg $R = \{(x,y) \mid x, y \in \mathbb{N} \wedge x \equiv y \pmod 4\}$

Ans ① Reflexive

$xRx \in R$?

$x \equiv x \pmod 4 \Rightarrow$ true.

② Symmetric

$xRy \rightarrow yRx$?

$x \equiv y \pmod 4$

$\Rightarrow 4 \mid x - y \Rightarrow 4 \mid (x-y) \times (-1)$

$\Rightarrow 4 \mid y - x$

$\Rightarrow y \equiv x \pmod 4$

↳ true

③ Transitive

$xRy \land yRz \to xRz$?

$x \equiv y \pmod 4 \land y \equiv z \pmod 4$

$\Rightarrow 4 | x-y \land \Rightarrow 4 | y-z$

since, both are true,

$4 | (x-y) + (y-z)$

$\Rightarrow 4 | x-z \Rightarrow x \equiv z \pmod 4 \to$ true.

Equivalent Class

↳ if R is a equivalent relation of A. The set of all elements related to a of A are called equivalent class.

→ denoted as $[a]_R$

$[a]_R = \{s \mid (a,s) \in R\}$

Eg find eq calass of 0, 1, 2, 3 of last sum.

Ans ⓪ $a \equiv 0 \pmod 4$

$[0] = \{ \ldots, -4, 0, 4, 8, \ldots \}$

① $a \equiv 1 \pmod 4$

$[1] = \{ \ldots, -3, 1, 5, \ldots \}$

② $a \equiv 2 \pmod 4$

$[2] = \{ \ldots, -6, -2, 2, 6, \ldots \}$

③ $a \equiv 3 \pmod 4$

$[3] = \{ \ldots, -1, 3, 7, \ldots \}$

Theorem Let $R \subseteq A \times A$ and equivalent

ii) $[a] = [b]$ i) aRb iii) $[a] \cap [b] \neq \phi$

Ans if aRb, (assume true) i implies ii

Let $c \in [a]$

since aRb and R is sym, bRa

by transitive, bRa and $aRc \Rightarrow bRc$

hence $c \in [b]$

Hence $[a] \subseteq [b]$

Similarly $[b] \subseteq [a]$

So $[a] = [b]$.

if $[a] = [b]$, $[a] \cap [b] \neq \phi$ (ii implies iii)

as $a \in [a]$ (reflexive)

Let $[a] \cap [b] \neq \phi$

but $c \in [a]$ & $c \in [b]$

$\Rightarrow aRc$ & bRc (iii implies i)

$\Rightarrow cRb$ (symmetric)

$\Rightarrow aRc$ & $cRb \Rightarrow aRb$ (Transitive)

Theorem Let S be a set and E be an equivalent relation on S. Then E partitions S into k parts (distinct classes) & $k \geq 1$

Proof We know,

$$\bigcup_{a \in A} [a]_R = A.$$

from earlier,

$[a] \cap [b] = \phi$ if disjoint. & $[a] \neq [b]$

This shows, that there are many such disjoint subsets which are all equivalent classes.

Collection of subsets A_i, $i \in I$ forms a partition of S iff ① $A_i \neq \phi \ \forall i \in I$

② $A_i \cap A_j = \phi$ when $i = j$

③ $\bigcup_{i \in I} A_i = S$

Partial order

① reflexive
② antisymmetric
③ transitive

} relation.

$(S, R) \Rightarrow$ (POSET, Relation).

Hasse Diagram

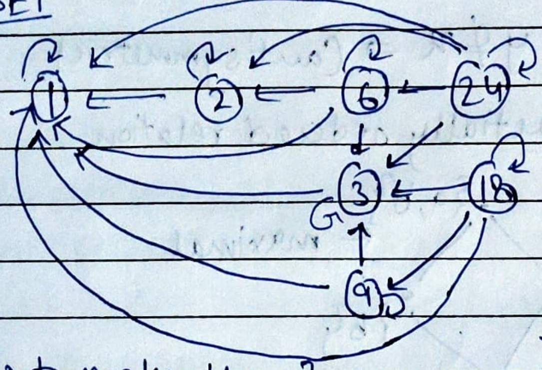
↳ pictorial representation to depict relations within a poset.

- ↳ element is a point.
- ↳ only direct connection, no transitive stuff
- ↳ order low to high.

Eg $R = \{\text{divides}, /\}$

$A = \{1, 2, 3, 6, 9, 18, 24\}$

Any POSET



all relations

→ antisym ✓

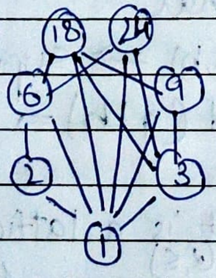
→ transitive ✓

→ reflexive ✓

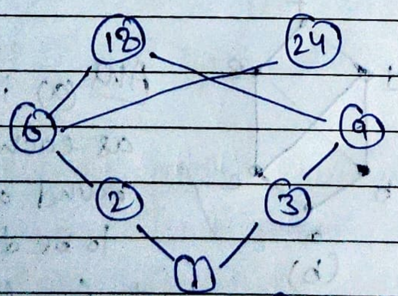
It's a POSET

How to make Hasse?

① write from low to high (dir remove as its anti sym) and remove self (as its reflexive)



② Remove transitive stuff



This is my Hasse digram.

~~Minimal element: a if ∃ no x s.t x ≤ a~~

~~Maximal element: b if ∃ no x s.t b ≤ x~~

Maximal element: $\exists$ no x s.t $a \leq x$, then a is max

Minimal element: $\exists$ no x s.t $x < b$, Then b is min

Eg Let $R \Rightarrow$ subset

$S = \{a, b\}$ $P(S) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

$(P(S), R) \rightarrow$ Hasse Diagram?

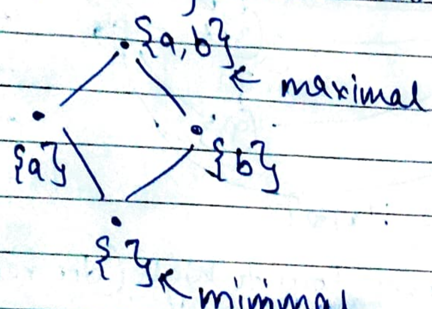
Ans ① $x \subseteq x \Rightarrow$ (reflexive)

② $x \subseteq y$ & $y \subseteq z$

$\Rightarrow x \leq z$ $\Rightarrow$ (transitive)

③ $x \subseteq y \Rightarrow y \not\subseteq x \Rightarrow$ (antisymmetric)

it's a partially ordered relation.

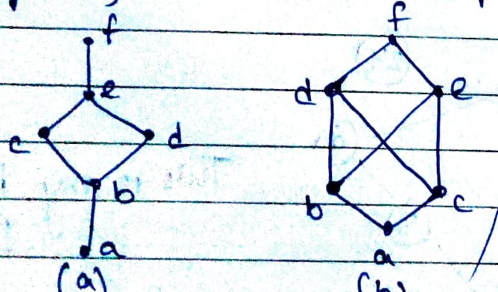


least upper bound (l.u.b) $\Rightarrow$ supremum

greatest lower bound (g.l.b) $\Rightarrow$ infremum

A partially ordered set is a lattice if each pair of $x, y \in \alpha$ $(\alpha, \leq)$ has a supremum & infremum.

Eg



(a) (b)

Ans (a) is a lattice as each pt has an upper bound and lower bound ~~to be differentiated~~

but for (b), check

⑤, ⑥, above it d, e, f have same precedence, hence upper bound has no lowest.

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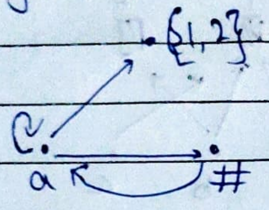
Graphs

~~Let there be a sets.~~

$R \subseteq S \times S$ (binary relation)

graph is a pictorial representation of a binary relation over a single set.



(graph)

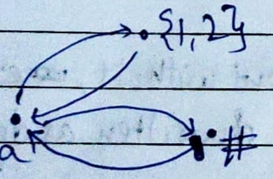
$R = \{(a,a), (a,\#), (\#,a), (a,\{1,2\})\}$

$S = \{a, \#, \{1,2\}\}$

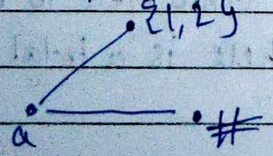
Graph $G = (V,E)$ contains V, a non-empty set of vertices (nodes) and E, a set of edges. Each ~~each~~ ~~to~~ edge has either 1 or 2 vertices attached to it called endpts. edge connects edgepoints.

we can remove ⇄ to add — if relation is symmetric.

Directed graph: has non-empty set of V and set of E. Each directed edge is associated with a ordered pair of vertices. (u,v) means starts at u and ends at v.



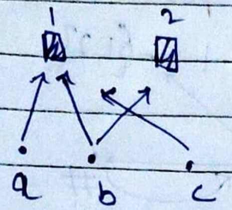
Undirected graph: for symmetric relations, (u,v) means starts at u and ends at v and also starts at v and ends at u.



Bipartite graphs

↳ a graph G is bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 s.t every edge in the graph connects a vertex in V_1 and vertex in V_2. (V_1, V_2) is a bipartition of V of G.

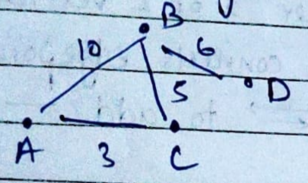
$V_1 = \{a, b, c\}$
$V_2 = \{1, 2\}$



$G = (V_1, V_2, E)$

Weighted Graphs

↳ if some weigh is given on the edges of a graph



path ⇒ set of vertices | for eg → (A, C, D)

if start vertex = end vertex, we have a cycle.

cycle ⇒ infinite no of paths

Degree of vertex

↳ no of edges incident with it, excluding the self ~~graph~~ edge cases. written as deg(v)

Thm 1 Sum of all degrees = even. if $G=(V,E)$ with m edges

$$\sum_{v \in V} \deg(v) = 2m$$

Proof each edge is connected to two vertices, hence for m edges, there is a total of 2m degree sum.

Thm2 number of vertices in a graph with odd degree is even.

Proof We know, $\sum_{v \in V} deg(v) = even$

$$\sum_{v \in V_{ev}} deg(v) + \sum_{v \in V_{odd}} deg(v) = even$$

$$\sum_{v_{odd} \in V} deg(v) = even - \sum_{v \in V_{ev}} deg(v)$$

Claim: $\sum$ even nums = even.

Proof: Let $a_1, \ldots, a_n$ be even

So, $a_1 = 2k_1; a_2 = 2k_2; \ldots; a_n = 2k_n$

$$\sum_{i \in N} a_i = 2(k_1 + k_2 + \ldots + k_n)$$

$$2 \mid \sum a_i$$

So, even - even $\Rightarrow$ even.

Hence, $\sum_{v_{odd} \in V} deg(v) = even$.

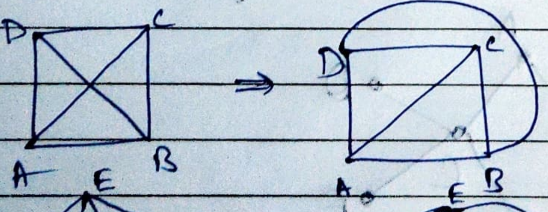
Indirected graph

Indegree ($deg^-(v)$) $\rightarrow$ number of edges with v as terminal vertex

Outdegree ($deg^+(v)$) $\rightarrow$ number of edges with v as starting vertex.

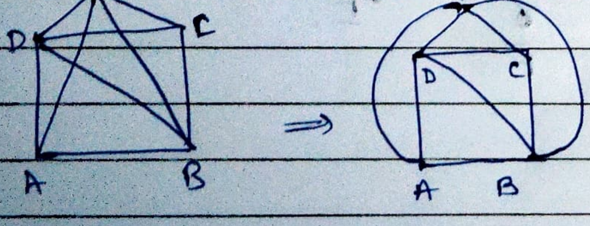
Graph is planar if it can be drawn on a plane

Eg ①

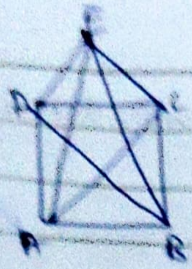


planar

②



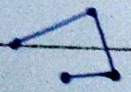
planar

3 

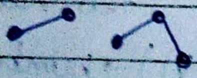
Not planar.

Complete graph
↳ undirected graph whose all vertices are connected to each other with edges
↳ represented as K_n.
1 was K_4 & 3 was K_5
⇒ Only till K_4, complete graphs are planar.

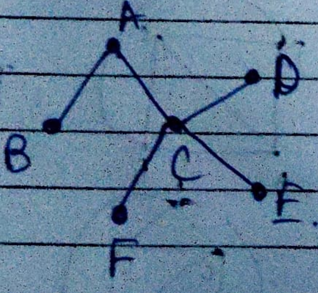
Connected graph
↳ $\exists$ a path b/w all vertices $\forall v \in V$.



Disconnected graph
↳ there is no path from some pts to some other.



Tree
↳ special type of graph
↳ it is a connected, acyclic, undirected graph



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Theorem A tree with n nodes as n-1 edges.

Proof $\boxed{\text{by induction}}$

for $n=1$

graph $\quad n=1$ $\quad \boxed{\text{correct}}$

$e = 1-1 = 0$

for $n=2$

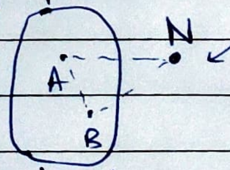
graph $\quad n=2$ $\quad \boxed{\text{correct}}$

$e = 2-1 = 1$

for $n=k \Rightarrow n=k+1$

So, for tree with k nodes, it has k-1 edges

Let the ~~graph~~ tree be in this blob



~~and this isthe extra~~ node.

k nodes

if we add one edge, it's okay

if we add two edge, it's no more tree as it will form cycle. $(A \to N \to B \to A)$

So, we can insert at most 1 edge.

so for k+1 nodes, we have k edges.

hence proved