(5) Scaling property

if $g(t) \Leftrightarrow G(\omega)$ then

for any const a, $g(at) \Leftrightarrow \frac{1}{|a|} G\left(\frac{\omega}{a}\right)$

Proof:

for +ve const a,

$$F(g(at)) = \int_{-\infty}^{\infty} g(at) e^{-j\omega t} dt$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} g(x) e^{-j\omega/a \, t} dx \qquad x = at$$

$$= \frac{1}{a} G\left(\frac{\omega}{a}\right)$$

if $a < 0$,

$$g(at) \Leftrightarrow -\frac{1}{a} G\left(\frac{\omega}{a}\right)$$

Convolution

The convolution of two funcs $g(t)$ & $w(t)$, denoted by $g(t) * w(t)$ is,

$$g(t) * w(t) = \int_{-\infty}^{\infty} g(t) \, w(t-\tau) dt$$

if $g_1(t) \Leftrightarrow G_1(t)$ & $g_2(t) \Leftrightarrow G_2(t)$

then, $g_1(t) * g_2(t) \Leftrightarrow G_1(\omega) G_2(\omega)$ [Time]

" $\Leftrightarrow \frac{1}{2\pi}$ " [Freq]

by defn,

$$F\left|g_1(t) * g_2(t)\right| = \int_{-\infty}^{\infty} e^{-j\omega t} \left[\int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) dt\right] dt$$

$$= \int_{-\infty}^{\infty} g_1(\tau) \left[\int_{-\infty}^{\infty} e^{-j\omega t} g_2(t-\tau) dt\right] d\tau$$

$$= \int_{-\infty}^{\infty} g_1(\tau) e^{-j\omega t} G_2(\omega) d\tau$$

$$= G_2(\omega) \int_{-\infty}^{\infty} g_1(\tau) e^{-j\omega t} d\tau = G_1(\omega) G_2(\omega)$$

Thm: $g(t) \Leftrightarrow G(\omega)$

$$\int_{-\infty}^{t} g(\tau)\,d\tau \Leftrightarrow \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega)$$

Proof

(1) Time conv

$$g(t) * u(t) = \int_{-\infty}^{\infty} g(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} g(\tau)\,d\tau$$

$$g(t) * (u(t)) \Leftrightarrow G(\omega)\,U(\omega)$$

$$= G(\omega)\left[\frac{1}{j\omega} + \pi\,\delta(\omega)\right]$$

$$= \frac{G(\omega)}{j\omega} + \pi G(0)\,\delta(\omega)$$

(2) Time diff & integration

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}\,d\omega$$

$$\frac{dg}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega G(\omega)\,e^{j\omega t}\,d\omega$$

$$\frac{dg}{dt} \Leftrightarrow j\omega G(\omega)$$

$$\frac{d^n g}{dt^n} \Leftrightarrow (j\omega)^n G(\omega)$$

Properties

1. Commutative
2. Associative
3. Distributive

if $g(t)$ → real

$$G(\omega) = G^*(\omega)$$

$$\int_{-\infty}^{\infty} g(t)\, e^{-j\omega t} = G(\omega)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t} = g(t)$$

Properties

① Linearity

$$x_1(t) \rightarrow X_1(j\omega)$$

$$x_2(t) \rightarrow X_2(j\omega)$$

$$\alpha x_1(t) + \beta x_2(t) \rightleftarrows \alpha X_1(\omega) + \beta X_2(\omega)$$

Proof $\quad x(t) \rightleftarrows X(\omega)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\alpha x_1(t) + \beta x_2(t)\right] e^{-j\omega t}\, dt$$

$$= \alpha X_1 + \beta X_2$$

② Conjugation

$$x(t) \rightleftarrows X(\omega)$$

$$x^*(t) \rightleftarrows X^*(\omega)$$

③ Area under x(t)

$\int x(t)\,dt$

$\boxed{X(0)} = \int_{-\infty}^{\infty} x(t)\,dt$ at $\omega = 0$

④ Area under $X(\omega)$:

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} - - \,dt$.

$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\,d\omega$. $t = 0$

$\boxed{2\pi\, x(0)}$

5) Time reversal

$x(t) \Leftrightarrow X(\omega)$

$x(-t) \Leftrightarrow X(-\omega)$

Proof $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt$

$= \int_{+\infty}^{-\infty} x(-t) e^{j\omega t}\,dt$

$= \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\,dt$

$= X(-\omega)$

6) Time scaling

$$x(at) = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega \tau/a} \frac{d\tau}{a}$$

$$= \frac{1}{a}\int x(\tau)\, e^{-j\omega/a\, \tau}\, d\tau$$

$$X(\omega) = \frac{1}{a} X\left(\frac{j\omega}{a}\right) \quad +ve$$

$$X(\omega) = \frac{-1}{a} X(j\omega/a) \quad -ve$$

7) Time shifting

$$x(t+t_0) \Leftrightarrow X(j\omega)\, e^{\pm j\omega_0 t}$$

proof: $\int x(t+t_0)\, e^{-j\omega t}$

$$\int x(t)\, e^{-j\omega(t-t_0)}$$

$$e^{+j\omega t_0} X(\omega)$$

8) Freq shifting

$$x(t) \Leftrightarrow X(j\omega)$$

$$e^{\pm j\omega t} x(t) \Leftrightarrow X(\omega \pm \omega_0)$$

9) Conv in time

$$x_1(t) * x_2(t) \leftrightarrow X_1(\omega) \cdot X_2(\omega)$$

Proof given earlier

10) Duality

$$x(t) \Longleftrightarrow 2\pi X(-\omega)$$

$$X(-\omega) = \frac{\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}{2\pi}$$

$$2\pi X(-\omega) = X(t)$$

FTs

① $\delta(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t-0)\, \cancel{e^{-j\omega(0)}}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

② $e^{-at}u(t)$ / $e^{at}u(-t)$ / $e^{a|t|}$

$$X_1(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t)\, e^{-j\omega t}\, dt$$

$$= \int_0^{\infty} e^{-at} e^{-j\omega t}\, dt = \int_0^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \frac{1}{a+j\omega}$$

$X_2(\omega)$ - same but $\frac{1}{a-j\omega}$

$$X_3(\omega) = X_1(\omega) + X_2(\omega)$$

$$= \frac{2a}{a^2+\omega^2}$$

④ Sgn(x)

$\int_{-\infty}^{\infty} |sgn(x)| \, dt \nless \infty$

so, $sgn(x) = u(t) - u(-t)$

$$X(\omega) = \lim_{a \to 0} \left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega} \right)$$

$$= \lim_{a \to 0} \frac{-2j\omega}{a^2+\omega^2} = \frac{-2j}{\omega} = \frac{2}{j\omega}$$

⑤ Step (u(t))

$$u(t) = \frac{1 + sgn(x)}{2}$$

$$= \frac{1}{2} + \frac{sgn(u)}{2}$$

$$F(u(t)) = F\left(\frac{1}{2}\right) + \frac{1}{2} F(sgn(t))$$

$$= 2\pi\delta(\omega) + \frac{2}{j\omega} \cdot \frac{1}{2}$$

$$= \pi\delta(\omega) + \frac{1}{j\omega}$$

classmate

Date
Page

⑥ $e^{j\omega t} / \delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0) e^{j\omega_0 t}\, d\omega,$$

$$= \frac{1}{2\pi} \times e^{j\omega_0 t} \underbrace{\int_{-\infty}^{\infty} \delta(\omega-\omega_0)}_{1}$$

$$= \frac{1}{2\pi} e^{j\omega_0 t}$$

⑦ $\cos \omega_0 t$

$\hookrightarrow \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$

$$X(\omega) = \frac{\cancel{2}\pi}{\cancel{2}}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)$$

⑧ $rect(t/\tau)$

$$X(\omega) = \int_{T_0} e^{-j\omega t} dt \Rightarrow \frac{-1}{j\omega}(e^{-j\omega T/2} - e^{j\omega T/2}$$

$$= \frac{2(\sin\frac{\omega T}{2})}{\omega}$$

$$= T \operatorname{sinc} \frac{\omega T}{2}.$$

___/___/___

Systems

↳ entity that processes a set of signals to yield another set of signals

↳ mapping of signals

eg. hydraulic / algorithm / electrical / mechanical

$$\begin{matrix} f_1(x,y) \\ f_2(x,y) \\ \vdots \end{matrix} \longrightarrow \boxed{\text{System}} \longrightarrow \begin{matrix} g_1(x,y) \\ g_2(x,y) \\ \vdots \end{matrix}$$

Linear systems

↳ ~~is a system which inputs~~ linearly as in

~~if $x_1(t)$ i/p, gives $y_1(t)$ o/p and x_2 i/p, gives y_2 o/p~~

$\Rightarrow$ $ax_1(t) + bx_2(t)$ i/p, gives $by_2(t) + ay_1(t)$ o/p

where a, b are constants.

$$x_1(t) \longrightarrow \boxed{\text{Sys}} \longrightarrow y_1(t)$$

$$x_2(t) \longrightarrow \boxed{\text{Sys}} \longrightarrow y_2(t)$$

$$\begin{matrix} ax_1(t) + \\ bx_2(t) \end{matrix} \longrightarrow \boxed{\text{Sys}} \longrightarrow \begin{matrix} ay_1(t) + \\ by_2(t) \end{matrix}$$

linear systems satisfy (i) homogenity & (ii) additivity

Eg $y(t) = 10\,x(t)$ determine linearity.

Ans $x_1(t) = u(t)$ $\quad x_2(t) = \delta(t)$ $\quad y_1(t) = 10u(t)$ $\quad y_2 = 10\delta(t)$

$x(t) = ax_1(t) + bx_2(t)$

$y(t) = 10x(t) = 10ax_1(t) + 10bx_2(t)$

$\quad = ay_1(t) + by_2(t)$

linear!

Eg $y(t) = x^2(t)$ determine linearity.

Ans $x_1 = u(t)$ $\quad x_2 = \delta(t)$ $\quad y_1(t) = u^2(t)$ $\quad y_2(t) = \delta^2(t)$

$x(t) = ax_1(t) + bx_2(t)$

$y(t) = x^2(t) = a^2x_1^2(t) + b^2x_2^2(t) + 2ab\,x_1(t)\,x_2(t)$

$\neq ay_1(t) + b(y_2(t))$

Non-linear!

Time invariant Systems

↳ Systems whose parameters don't change with time.

$$x(t) \to \boxed{Sys} \to y(t)$$

$$x(t-t_d) \to \boxed{Sys} \to y(t-t_d)$$

Eg ① $y(t) = 2x(t-5)$ ② $y(t) = 2x(3t)$

Ans ① $y(t) = 2x(t-5)$

$y_1(t) = 2x_1(t-5)$

$x_2(t) = x_1(t-t_d)$

$y_2(t) = 2x_2(t-5) = 2x(t-5-t_d)$

$y_1(t-t_d) = 2x_1(t-5-t_d)$

$y_2(t) = y_1(t-t_d)$

invariant

② $y_1(t) = 2x_1(3t)$; $x_2(t) = x_1(t-t_d)$

$y_2(t) = 2x_2(3t) = 2x_1(3t-t_d)$

$y_1(t-t_d) = 2x(3(t-t_d))$

$\neq y_2(t)$

variant

Causal Systems

↳ System where the output $y(t)$ at any time t depends only on the current output $x(t)$ at t and $x(t-1)$, ...
If it depends on $x(t+1)$, ..., it is non-causal.

i) $y(t) = x(t) + 0.2x(t-2) \Rightarrow$ causal

ii) $y(t) = \frac{1}{3}x(t+2) + \frac{1}{3}x(t) \Rightarrow$ non-causal

Impulse Response

↳ An LTI system can be repped by its unit impulse response, which is the system's response due to the impulse input $\delta(t)$

$$\delta(t) \to \boxed{LTI} \to h(t)$$

$$x(t) \to \boxed{h(t)} \to y(t)$$

$$y(t) = h(t) * x(t)$$

Q Let LTI by

$$y(t) = 0.5\,x(t) + 0.25\,x(t-1) \quad / \quad x(-1) = 0$$

(a) find $h(t)$ (b) draw block (c) write output.

Ans Let $x(t) = \delta(t)$

$$h(t) = 0.5\,x(t) + 0.25\,x(t-1)$$

$$h(t) = \begin{cases} 0.5 & t = 0 \\ 0.25 & t = 1 \\ 0 & \text{else} \end{cases}$$

$$y(t) = h(0)\,x(t) + h(1)\,x(t-1)$$

$$x(t) \longrightarrow \boxed{h(t) = 0.5\,\delta(t) + 0.25\,\delta(t-1)} \longrightarrow y(t)$$

Signal Transmission in LTI

for an LTI system

$$g(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) \qquad y(t) = h(t) * g(t)$$

$$y(t) \Leftrightarrow Y(\omega),\ h(t) \Leftrightarrow H(\omega)\ \&\ g(t) \Leftrightarrow G(\omega)$$

$$Y(\omega) = H(\omega)\,G(\omega)$$

Signal Distortion

$$|Y(\omega)|\,e^{j\theta_y(\omega)} = |G(\omega)|\,|H(\omega)|\,e^{j(\theta_g(\omega) + \theta_h(\omega))}$$

$$|Y(\omega)| = |g(\omega)|\,|H(\omega)| \quad \& \quad \theta_{y(\omega)} = \theta_{g(\omega)} + \theta_{h(\omega)}$$

Distortionless transmission

$$y(t) = G_0\,x(t - t_d)$$

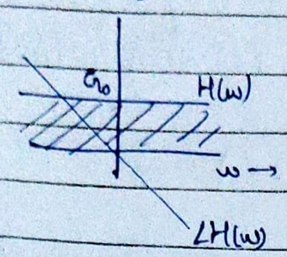
$$Y(\omega) = G_0\,X(\omega)\,e^{-j\omega t_d}$$

$$Y(\omega) = X(\omega)\,H(\omega)$$

$$H(\omega) = G_0\,e^{-j\omega t_d}$$

$$|H(\omega)| = G_0 \qquad \angle H(\omega) = -\omega t_d$$

Measure of time delay with freq



$$\theta_h(w) = -wt_d$$

$$t_d(w) = \frac{-d\theta_h}{dw}$$

Every freq component is altered by G_0 & time delay of t_d.

Even flat frequency response is not enough for distortion less travel.

Nature of audio and Video distortion

→ Humans can notice amplitude distortion, but phase dist is harder to hear

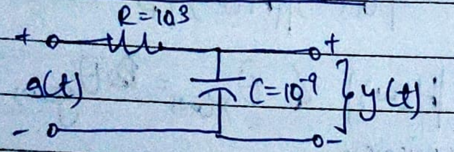
→ For phase detection to be visible, t_d has to be huge

→ Human eye is sensitive to phase distortion

→ Amplitude distortion causes very little degradation on image quality.

Low Pass Filter Problem

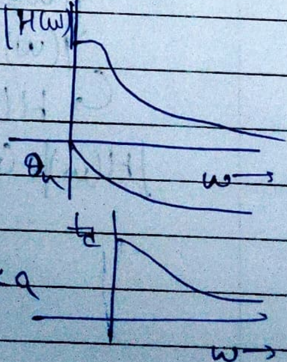
Q



Find $H(w)$, $|H(w)|$, $\theta_h(w)$, $t_d(w)$. amp(5%) & θ(2%) allowed

Ans $H(w) = \dfrac{\frac{1}{jwC}}{R + \frac{1}{jwC}} = \dfrac{Y(w)}{X(w)}$ $\quad a = (RC)^{-1}$

$= \dfrac{a}{a + jw}$



$|H(w)| = \dfrac{a}{\sqrt{a^2 + w^2}} \approx 1 \quad w << a$

$\theta_h(w) = -\tan^{-1}\dfrac{w}{a} \approx -\dfrac{w}{a} \quad w << a$

$t_d(w) = \dfrac{d\theta_h}{dw} = \dfrac{a}{a^2 + w^2} \approx \dfrac{1}{a} = 10^{-6}$

Nature of audio & video dist

$|H(\omega_0)| \geq .98 \quad t_d \geq 0.95/a$

$\frac{a}{\sqrt{a^2+\omega_0^2}} \geq .98 \Rightarrow \omega_0 \leq 0.2039a$

$\frac{a}{\omega_0^2+a^2} \geq \frac{0.95}{a} \Rightarrow \omega_0 \leq 0.2294a$

$t_d \approx \frac{1}{10^6} \quad y(t) = g(t-10^{-6})$

Signal energy & Energy spectral Density

Parseval's Theorem

$$E_g = \int_{-\infty}^{\infty} g(t)\, g^*(t)\,dt = \int_{-\infty}^{\infty} g(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(\omega)\, e^{-j\omega t}\, d\omega\right] dt$$

Q Verify for $g(t) = e^{-at}u(t)$ $(a>0)$

Ans $E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(\omega)\left[\int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt\right] d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, G^*(\omega)\, d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2\, d\omega$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{a^2+\omega^2}\, d\omega = \frac{1}{2\pi a}\tan^{-1}\frac{\omega}{a}\Big|_{-\infty}^{\infty} = \frac{1}{2a}$ Verified

$E_g = \int_{-\infty}^{\infty} g^2(t)\, dt$

$= \int_{0}^{\infty} e^{-2at}\, dt = \frac{1}{2a}$

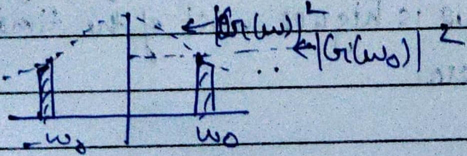
Energy Spectral Density (ESD)

$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega) H(\omega)|^2\, d\omega$

$= 2 \cdot |G(\omega_0)|^2 \frac{d\omega}{2\pi} = 2|G(\omega_0)|^2\, df$

$\boxed{\Psi_g(\omega) = |G(\omega)|^2}$ ← ESD

$E_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_g(\omega)\, d\omega = \int_{-\infty}^{\infty} \Psi_g(\omega)\, df$



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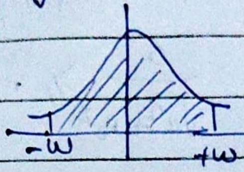
Q Estimate essential Bandwidth of $g(t) = e^{-at}u(t)$ if band should contain 95% of energy.

Ans

$G(\omega) = \frac{1}{a+j\omega}$



$ESD = |G(\omega)|^2 = \frac{1}{a^2+\omega^2}$

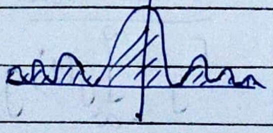
$\frac{0.95}{2a} = \frac{1}{2\pi}\int_{-W}^{W} \frac{d\omega}{\omega^2+a^2} = \frac{1}{2\pi a}\tan^{-1}\frac{\omega}{a}\Big|_{-W}^{W} = \frac{1}{\pi a}\tan^{-1}\frac{W}{a}$

$\cancel{W = 12.706\,a}$

Q $g(t) = rect(t/\tau)$; 90% band find essential bandwidth

Ans $\cancel{rect(t/\tau) \Leftrightarrow \tau\, sinc(\frac{\omega\tau}{2})}$

$\Psi_g(\omega) = |G(\omega)|^2 = T^2 sinc^2(\frac{\omega T}{2})$



$E_W = \frac{1}{2\pi}\int_{-\infty}^{\infty} T^2 sinc^2(\frac{\omega T}{2})\, d\omega$

$\cancel{= \frac{1}{\pi}\int_0^{WT} sinc^2(\frac{x}{2})dx}$

Energy of Modulated Signal

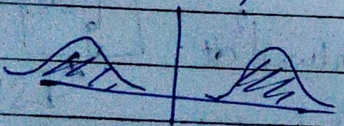
~~→ modulation = shifting by ω_0 L/R~~

~~eg Let $g(t)$ = signal & $\phi(t) = g(t)\cos\omega_0 t$~~

$\Rightarrow \phi(\omega) = \frac{1}{2}[G(\omega+\omega_0) + G(\omega-\omega_0)]$

$\Rightarrow ESD = \frac{1}{4}[G(\omega+\omega_0) + G(\omega-\omega_0)]^2 = \frac{1}{4}[\Psi$

if $\omega_0 \geq 2\pi B$, non overlapping



Energy is proportional to square of amplitude, for $g(t)$ the value is higher most of the times while $\phi(t)$ drops to zero.

Time auto correlation & ESD

~~Cross corelation function~~ $\to \varphi_{g_1 z}(T)$

Auto corelation function $\to \varphi_{gg}(T)$

$$ACF \Rightarrow \int_{-\infty}^{\infty} g(t)\, g(t+\tau)\, dt$$

~~Let $t+\tau = x$~~ $\Rightarrow \int_{-\infty}^{\infty} g(x)\, g(x-\tau)\, dx$

Hence ACF $\Rightarrow \int_{-\infty}^{\infty} g(t)\, g(t \pm \tau)\, dt$

$\Rightarrow$ ACF is an even function as $\varphi_g(T) = \varphi_g(-T)$

At $\tau = 0$, $ACF = \int_{-\infty}^{\infty} g(t)\, g(t)\, dt = E_g$

$\Rightarrow$ Let's find its fourier transform [ACF]

$$F[\varphi_g(T)] = \int_{-\infty}^{\infty} e^{-j\omega t}\left[\int_{-\infty}^{\infty} g(t)\, g(t+T)\, dt\right] d\tau$$

$$= \int_{-\infty}^{\infty} g(t)\left[\int_{-\infty}^{\infty} g(t+\tau)\, e^{-j\omega\tau}\, d\tau\right] dt$$

$$= G(\omega)\int_{-\infty}^{\infty} g(t)\, e^{j\omega t}\, dt = G(\omega)\, G(-\omega)$$

$$= |G(\omega)|^2 = \varphi_g(\omega)$$

$$\varphi_g(t) \Leftrightarrow \varphi_g(\omega)$$

$\Rightarrow$ ACF $\varphi_g(t)$ is the conv of $g(\tau)$ & $g(-\tau)$.

$$g(\tau) * g(-\tau) = \int_{-\infty}^{\infty} g(x)\, g(-(\tau - x))\, dx$$

$$= \int_{-\infty}^{\infty} g(x)\, g(x-\tau)\, dx = \varphi_g(\tau)$$

Eg Find the ACF of $g(t) = e^{-at} u(t)$ also find ESD

Ans $g(t) = e^{-at} u(t)$ $\quad g(t-\tau) = e^{-a(t-\tau)} u(t-\tau)$

$$\varphi_g(t) = \int_{-\infty}^{\infty} g(t)\, g(t-\tau)\, dt = e^{a\tau}\int_{\tau}^{\infty} e^{-2at}\, dt = \frac{1}{2a} e^{-at}$$

$$\varphi_g(\omega) = \frac{1}{a^2 + \omega^2}$$

__/__/__

ESD of input/output

$Y(\omega) = H(\omega)\,X(\omega)$

$|Y(\omega)|^2 = |H(\omega)|^2\,|X(\omega)|^2$

$\Psi_y(\omega) = |H(\omega)|^2\,\Psi_x(\omega)$

o/p is $|H(\omega)|^2$ times i/p.

Signal power

For power signal $g(t)$, measure is power

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t)\,dt \quad \left[\text{truncate signal from } -T/2 \text{ to } T/2\right]$$

Power Spectral Density (PSD)

$$E_g = \int_{-\infty}^{\infty} g_T^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G_T(\omega)|^2\,d\omega$$

$$P_g = \lim_{T\to\infty} \frac{1}{T} E_g = \lim_{T\to\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} |G_T(\omega)|^2\,d\omega\right]$$

$$S_g(\omega) = \lim_{T\to\infty} \frac{|G_T(\omega)|^2}{T}$$

$$P_g = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_g(\omega)\,d\omega = \frac{1}{\pi}\int_0^{\pi} S_g(\omega)\,d\omega$$

ACF of Power Signal

$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,g(t+\tau)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,g(t+\tau)\,dt \Rightarrow R_g(\tau) = R_g(-\tau)$$

Hence even func

Similar to last,

$$R_g(\tau) \Leftrightarrow \lim \frac{|G_T(\omega)|^2}{T} = S_g(\omega)$$

_/_/_

Random signal

↳ a signal whose physical description is known completely by a mathematical/graphical method is called deterministic signal.

↳ A signal only known through its probabilistic description such as mean etc rather than it's complete mathematical/graphical description is called random signal.

Study Probality Theory

- ↳ Samplespace
- ↳ Basic Probality
- ↳ Conditional Probability
- ↳ Bernoulli/Gaussian pdf

PCM Repeater Error Probability

↳ regenerative repeaters are used to detect pulses and retransmit new clean pulses, this decreases pulse distortion & noise.

↳ it has n links in tandem. The pulses are detected at the end of each link and new pulse is sent over the next link. If P_e is the prob of error in detecting pulse at any link, total probability should be $\leq nP_e$ i.e $P_E \leq nP_e$

in o→o→o→o — — →o out
1^{st} 2^{nd} 3^{rd} k^{th}

Proof) One link ⇒ $1-P_e$ [correct detection]

n links ⇒ $1-P_E$ [" "]

A pulse will be correctly detected iff no error or even numbers of errors are present.

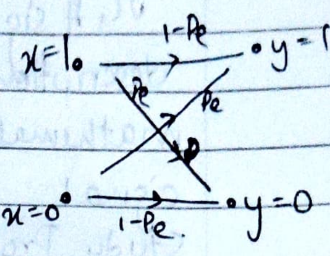
$$1-P_E = P(\text{no error}) + P(2\text{ error}) + P(4\text{ error}) + \ldots$$

P(correct on all links) = $(1-P_e)^n$

& P(error on k links) = ${}^nC_k \, P_e^k (1-P_e)^{n-k}$

$$1-P_E = (1-P_e)^n + \sum_{n=3,4..}^{n} {}^nC_k \, P_e^k (1-P_e)^{n-k}$$

as $P_e \lll 1$.



$$1-P_E \simeq (1-P_e)^n + {}^nC_2 P_e^2$$

$$= (1-P_e)^n + \frac{n(n-1)}{2} P_e^2$$

if $nP_e \lll 1$

$$1-P_E \simeq (1-P_e)^n$$

$$= 1-nP_e \quad [nP_e \lll 1]$$

$\Rightarrow P_E \simeq nP_e$ proved

Binary transmission

$\hookrightarrow$ init, $0 \to 00/000$ and $1 \to 11/111$ these are the norms of transmission.

if P_e is the error prob of 1 digit, P(E) is the prob of getting a wrong output

Then $P(E) = \sum_{k=2}^{3} {}^3C_k P_e^k (1-P_e)^{k} \ldots P_e^3$

$$= {}^3C_2 P_e^2 (1-P_e) + P_e^3$$

$$\simeq 3P_e^2$$

$$P(E) \simeq 3P_e^2$$

Random Variables

$\hookrightarrow$ Outcomes might be real numbers

$\hookrightarrow$ some numerical values might be assigned to each sample point

$\hookrightarrow$ these variables are random variables

Study Distribution Function

Joint probality

↳ prob when $x = x_i$ & $y = y_j$

↳ $P_{xy}(x_i, y_j)$

$$\sum_i \sum_j P_{xy}(x_i, y_j) = 1$$

$$\sum_i P_{xy}(x_i, y_j) = \sum_i P_{x/y}(x_i/y_j)\, P_y(y_j)$$

$$= P_y(y_j) \sum_i P_{x/y}(x_i/y_j)$$

$$= P_y(y_j) \quad \text{Marginal Probability of } y$$

$$P_x(x_i) = \sum_j P_{xy}(x_i, y_j) \quad \text{Marginal Probability of } x$$

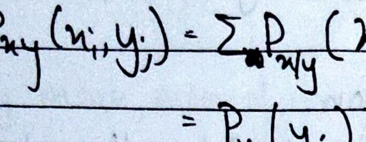
Binary Symmetric Channel

↳ error probability is P_e

↳ Prob of transmitting 1 is Q and for 0 is $1-Q$

Determine prob of getting 0 and 1 at receiver.

Ans



$P_{y|x}(0|1) = P_{y|x}(1|0) = P_e$

$P_{y|x}(1|1) = P_{y|x}(0|0) = 1 - P_e$

$$P_x(1) = Q \quad P_x(0) = 1 - Q$$

$$P_y(y_i) = P_x(0)\, P_{y|x}(1/0) + P_x(1)\, P_{y|x}(1/1)$$

$$= (1-Q)P_e + Q(1-P_e)$$

Similarly,

$$P_x(x_i) = \cancel{Q(1-P_e) + (1-Q)P_e}$$

$$= QP_e + (1-Q)(1-P_e)$$

Q Prob of 0 transmitted is 0.4
" " 1 " " 0.6

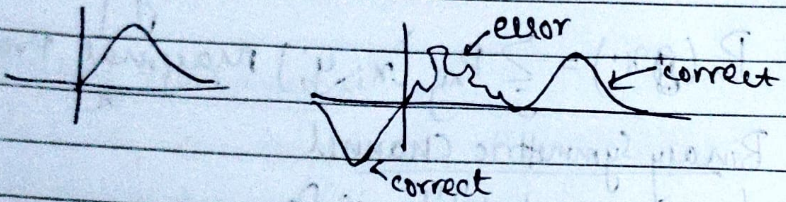
given
$P(\epsilon|0) = 10^{-6}$ $P(\epsilon|1) = 10^{-4}$
$P(\epsilon) = ?$

Ans $P(\epsilon) = P_x(0)\, P(\epsilon|0) + P_x(1)\,(\epsilon|1)$
$= 0.4\times10^{-6} + 0.6\times10^{-4}$
$= 0.604\times10^{-4}$

Study pdf and cdfs and gaussian pdf

Probability of Detetion

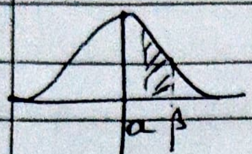
↳ The recived signal has the desired signal and has some random noise. This leads to pulse error detection.



In absence of noise, value of positive pulse is $+A_p$ and else $-A_p$.
for additive noise, it is $\pm A_p + n$
due to symmetry, threshold is always 0.
for +ve pulse, if $n > A_p$ and -ve, it is detected as a 0 and is wrong.

Error prob in Polar Signals

Let $P(\epsilon|0) \Rightarrow n > A_p$
$P(\epsilon|1) \Rightarrow n < -A_p$



$$p(\alpha < n < \beta) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\infty}^{\beta} e^{-\frac{n^2}{2\sigma^2}} dn.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{y}^{\infty} e^{-x^2/2} du.$$

$$P(\varepsilon|0) = \frac{1}{\sqrt{2\pi}} \int_{Ap/\sigma_n^2}^{\infty} e^{-u^2/2} du \qquad \uparrow Q(x)$$

$$P(\varepsilon) = P(1)P(\varepsilon|1) + P(0)P(\varepsilon|0)$$
$$= Q\left(\frac{Ap}{\sigma_n}\right)$$

Study mean, mean of sum and product

Rayleigh Density

↳ The rayleigh density is characterised by the pdf:

$$R(r) = \begin{cases} \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} & ; r \geq 0 \\ 0 & ; r < 0 \end{cases}$$

$$P_x(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \qquad P_y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-y^2/2\sigma^2}$$

$$P_{xy}(x,y) = P_x(x) P_y(y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

Central Limit Theorem

↳ Under some conditions the sum of RVs take a gaussian pdf. The rigorous tendencies of such order is called central limit theorem.

→ Consider two RVS $z = x + y \Rightarrow y = z - x$.

$$F_z(z) = P(z \leq Z) = P(x \leq \infty, y \leq z - x)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} P_{xy}(xy)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{z-x} P_{xy}(xy)\, dy$$

$$P_z(z) = \frac{dF_z(z)}{dz} = \int_{-\infty}^{\infty} P_{xy}(x, z-x)\, dx$$

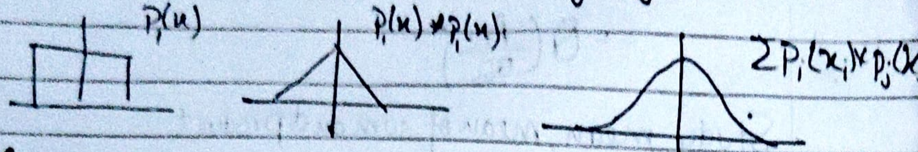
If x, y are independent,

$$P_{xy}(x, z-x) = P_x(x) P_y(z-x)$$

$$= \int_{-\infty}^{\infty} P_x(x) P_y(z-x) dx$$

$$= P_x(x) * P_y(y)$$

Keep on convoluting till inf to get gaussian curve.



Correlation

↳ nothing but the covariance is the dependence b/w RVs.

$$\sigma_{xy} = \overline{(x-\bar{x})(y-\bar{y})}$$

Uncorellated → $\sigma_{xy} = 0$ or $\overline{yx} = \bar{x}\bar{y}$

Independent RVs are uncorellated but the converse is not true.

eg $g_1(t) = \sin at$ $g_2(t) = \cos at$

Coefficient of corellation

↳ $\rho_{xy} = \dfrac{\sigma_{xy}}{\sigma_x \sigma_y}$

MSE

if x, y are uncorellated, $z = x+y$

$$\sigma_z^2 = \overline{(x-\bar{x}+y-\bar{y})^2}$$

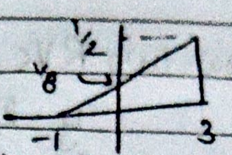
$$= \overline{(x-\bar{x})^2} + \overline{(y-\bar{y})^2} + 2\overline{(x-\bar{x})(y-\bar{y})}$$

$$= \sigma_x^2 + \sigma_y^2 + 2\sigma_x\sigma_y$$

$$= \sigma_x^2 + \sigma_y^2$$

Important Questions

Q Find mean, MSE, var of pdf



Ans ~~Area should be 1~~

$$\bar{x} = \int_{-1}^{3} x P(x) = \int_{-1}^{3} \frac{1}{8}(y)(y-1)dy = \frac{1}{8}\left(\frac{64}{3} - \frac{16}{2}\right) = \frac{5}{3}$$

$$\overline{x^2} = \frac{1}{8}\int_{-1}^{3} x^2(x^2+1)dx = \frac{11}{3}$$

$$\sigma_x^2 = \overline{x^2} - \bar{x}^2 = 8/9$$

Q Find mean, MSE and var of $x \leftarrow$ dice roll sum

Ans $\bar{x} = \sum_{2}^{12} x P(x) = \frac{1}{36}(2) + \frac{2}{36}(3) - \quad + \frac{1}{36}(12)$

$= 7$

$\overline{x^2} = \sum_{2}^{12} x P(u) = \frac{1}{36}(4) + \frac{2}{36}(9) + \quad + \frac{1}{36}(144)$

$= 54.83$

$\sigma_x^2 = x^2 - (\bar{x})^2 = 54.83 - 49 = 5.83$

Q Find mean, MSE, var of $x \leftarrow$ dice roll

Ans $\bar{x} = \sum_{1}^{6} x(P(x)) = \frac{1}{6}(1) + \frac{1}{6}(2) + \quad + \frac{1}{6}(3)$

$= 7/2$

$\overline{x^2} = \sum_{1}^{6} x^2 P(x) = 91/6$

$\sigma_x^2 = \overline{x^2} - (\bar{x})^2 = 35/12$

Q Show $\bar{z} = \bar{x} + \bar{y}$ for gaussian & $\sigma_z^2 = \sigma_x^2 + \sigma_y^2$

Ans $P_z(z) = P_x(x) * P_y(y)$

$= e^{-(\sigma_x^2+\sigma_y^2)w^2} e^{-jw(\bar{x}+\bar{y})}$

$= \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}} e^{-(z-(\bar{x}+\bar{y}))^2/2(\sigma_x^2+\sigma_y^2)}$

proved

Q Prove $\rho_{xy} \le 1$

Ans $[a(x-\bar{x}) - (y-\bar{y})]^2 \ge 0$

$a^2\sigma_x^2 + \sigma_y^2 - 2a\sigma_{xy} \ge 0$

$4\sigma_{xy}^2 - 4\sigma_x^2\sigma_y^2 \le 1$

$\left|\frac{\sigma_{xy}}{\sigma_x\sigma_y}\right| = \rho_{xy} \le 1$

Q Show two RVs x, y are

$y = k_1 x_0 + k_2$

$\rho_{xy} = 1$ iff $k_1 > 0$ and $\rho_{xy} = -1$ iff $k_1 < 0$

Ans $\sigma_y^2 = k_1^2 \sigma_x^2$ $\quad \sigma_{xy} = k_1 \sigma_x^2$

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{k_1 \sigma_x^2}{k_1 \sigma_x^2} = 1 \text{ if } k_1 > 0$$

↳ σ_x & σ_y are both positive, hence $\rho_{xy} = -1$

Q $x = \cos\theta$

$y = \sin\theta$ Show uncorellated but not independent

Ans $\bar{x} = \int_0^{2\pi} \sin\theta \, d\theta = 0$ $\quad \bar{y} = \int_0^{2\pi} \cos\theta \, d\theta = 0.$

$\sigma_{xy} = \overline{(x - \bar{x})(y - \bar{y})} = \bar{x} . \bar{y} = 0.0 = 0$

but $x^2 + y^2 = 1$

$y = \sqrt{1 - x^2}$ ← not independent.

Q A random signal has two vals 3 & 0 with random prob a gaussial noise $n(t)$ is added giving $y(t)$ find pdf of y. ↳ mean(8) rms(8)

Ans $P_x(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-3)$

$P_n(n) = \frac{1}{2\sqrt{2\pi}} e^{-n^2/8}$ $\quad y = x + n$

$P_y(y) = P_x(x) * P_n(n) = \left[\frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-3)\right] * \frac{1}{2\sqrt{2\pi}} e^{-x^2/8}$

$= \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\left[\frac{1}{2\sqrt{2\pi}} e^{-(y-n)^2/8}\right] dn + \frac{1}{2}\int$ same

$= \frac{1}{4\sqrt{2\pi}} e^{-y^2/8} + \frac{1}{4\sqrt{2\pi}} e^{-(y-3)^2/8}$