Linear regression

Jacky Wang

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Theory

Math symbols

Using a single number for estimation: $\hat{y} = \theta_0$

Given $\mathbf{y} = [y^{(1)}, y^{(2)}, ..., y^{(n)}]^T$, Find

$$\theta_0 = \arg\min_{\hat{\theta}_0} RSS = \arg\min_{\hat{\theta}_0} \sum_{i=1}^{n} (y^{(i)} - \hat{\theta}_0)^2 \tag{1}$$

Solution:

- Rewrite RSS as $RSS = A\hat{\theta}_0^2 + B\hat{\theta}_0 + C$, note that $A = n > 0$, thus using the basic knowledge from high school, we get $\theta_0 = -\frac{B}{2A}$.
- Or we can:
$$\frac{\partial RSS}{\partial \theta_0} = \sum_{i=1}^{n} 2(y_i - \theta_0)(-1) = 0 \tag{2}$$
- Thus, $\theta_0 = mean(\mathbf{y})$, which is used in:
  - calculating TSS
  - *kNN regression* (For a observation's neighbors)
  - and *regression tree* (For each leaf/terminal node)

Using one predictor for estimation: $\hat{y} = \theta_0 + \theta_1 x_1$

Given:

- $\mathbf{X} = \begin{bmatrix} 1, x_1^{(1)} \\ 1, x_1^{(2)} \\ \vdots \\ 1, x_1^{(n)} \end{bmatrix} = \begin{bmatrix} x_0^{(1)}, x_1^{(1)} \\ x_0^{(2)}, x_1^{(2)} \\ \vdots \\ x_0^{(n)}, x_1^{(n)} \end{bmatrix}$

Table 1: Notation of Math symbols

Notation	Meaning
$\mathbf{x} = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \in \mathbb{R}^{(p+1)\times 1}$	input variables, features, predictors
$y = f(\mathbf{x}) + \epsilon$	output/target variable
$\hat{y} = \hat{f}(\mathbf{x}) = h(\mathbf{x}) = h_{\boldsymbol{\theta}}(\mathbf{x})$	hypothesis function of $\mathbf{x}$ with parameter $\boldsymbol{\theta}$
$\mathbf{x}^{(i)} = \begin{bmatrix} x_0^{(i)} \\ x_1^{(i)} \\ \vdots \\ x_p^{(i)} \end{bmatrix}$	i^{th} observation of $\mathbf{x}$ where $1 \leq i \leq n$
$y^{(i)}$	i^{th} observation of y where $1 \leq i \leq n$
$\mathbf{y} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(n)} \end{bmatrix} \in \mathbb{R}^{n\times 1}$	n observations of y
$\mathbf{X} = \begin{bmatrix} \left(\mathbf{x}^{(1)}\right)^T \\ \left(\mathbf{x}^{(2)}\right)^T \\ \vdots \\ \left(\mathbf{x}^{(n)}\right)^T \end{bmatrix} \in \mathbb{R}^{n\times(p+1)}$	n observations of $\mathbf{x}$
n	# of traing examples
p	# of traing p-redictors, features (excluding addtional 1 vector)
$\boldsymbol{\theta} = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}$	parameter of hypothesis funtion(model)

- $\mathbf{y} = [y^{(1)}, y^{(2)}, ..., y^{(n)}]^T$

Find

$$(\theta_0, \theta_1) = \arg\min_{\hat{\theta}_0, \hat{\theta}_1} RSS = \arg\min_{\hat{\theta}_0, \hat{\theta}_1} \sum_{i=1}^{n} (y^{(i)} - (\hat{\theta}_0 + \hat{\theta}_1 x_1^{(i)}))^2 \tag{3}$$

Again,

- $$\frac{\partial RSS}{\partial \theta_0} = 0 \tag{4}$$

- $$\frac{\partial RSS}{\partial \theta_1} = 0 \tag{5}$$

We'll get

- $\theta_1 = cor(\mathbf{x}_1, \mathbf{y})\frac{sd(\mathbf{y})}{sd(\mathbf{x}_1)} = \frac{\sum_{i=1}^{n}(x_1^{(i)}-\bar{x}_1)(y^{(i)}-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_1^{(i)}-\bar{x}_1)^2}\sqrt{\sum_{i=1}^{n}(y^{(i)}-\bar{y}_1)^2}} \frac{\frac{1}{n}\sqrt{\sum_{i=1}^{n}(y^{(i)}-\bar{y})^2}}{\frac{1}{n}\sqrt{\sum_{i=1}^{n}(x_1^{(i)}-\bar{x}_1)^2}}$
- $\theta_0 = mean(\mathbf{y}) - \theta_1 mean(\mathbf{x}_1) = \bar{y} - \theta_1\bar{x}_1$

Property of residual

Degree of freedom $n-p-1 = n-2$ when $p=1$

- $\sum_{i=1}^{n} x_0^{(i)}(y^{(i)} - \hat{y}^{(i)}) = \sum_{i=1}^{n}(y^{(i)} - \hat{y}^{(i)}) = 0$
- $\sum_{i=1}^{n} x_1^{(i)}(y^{(i)} - \hat{y}^{(i)}) = 0$

Uncertainty of model

Note:

- $\mathbf{X}$ is known
- $y = f(\mathbf{x}) + \epsilon$ where ϵ is a random variable, thus y is also a *random variable*
- θ_0, θ_1 is a funciton of $\mathbf{y} = (y^{(1)}, y^{(2)}, ..., y^{(n)})$, so also a *random variable*

We have:

- $\hat{\sigma}^2 = \frac{RSS}{n-p-1} = \frac{RSS}{n-2}$
- $\sigma_{\theta_0}^2 = \sigma^2\big(\frac{1}{n} + \frac{\bar{x}_1^2}{\sum_{i=1}^{n}(x_1^{(i)}-\bar{x}_1)^2}\big) \approx \hat{\sigma}^2\big(\frac{1}{n} + \frac{\bar{x}_1^2}{\sum_{i=1}^{n}(x_1^{(i)}-\bar{x}_1)^2}\big)$
- $\sigma_{\theta_1}^2 = \sigma^2\frac{1}{\sum_{i=1}^{n}(x_1^{(i)}-\bar{x}_1)^2} \approx \hat{\sigma}^2\frac{1}{\sum_{i=1}^{n}(x_1^{(i)}-\bar{x}_1)^2}$
- $SE_{prediction} = \hat{\sigma}\sqrt{1 + \frac{1}{n} + \frac{(x_1^{'}-\bar{x}_1)^2}{\sum_{i=1}^{n}(x_1^{(i)}-\bar{x}_1)^2}}$ at $x_1 = x_1^{'}$
- $SE_{line} = \hat{\sigma}\sqrt{\frac{1}{n} + \frac{(x_1^{'}-\bar{x}_1)^2}{\sum_{i=1}^{n}(x_1^{(i)}-\bar{x}_1)^2}} = \hat{\sigma}\sqrt{\text{leverage statistic}(x_1^{'})}$ at $x_1 = x_1^{'}$

To understand:

- If $\bar{x}_1 = 0$, $\sigma_{\theta_0}^2 = \frac{\sigma^2}{n}$
- As $n \to \infty$:
  - $\sigma_{\theta_0}^2 = \sigma_{\theta_1}^2 = SE_{line} = 0$, uncertainty surrounding a particular y at $x_1^{'}$, related to uncertainty of θ_0, θ_1
  - $SE_{prediction} = \sigma$, uncertainty surrounding the average y at $x_1^{'}$, which includes irreducible error

General regression: $\hat{y} = \theta_0 + \theta_1 x_1 + ... + \theta_p x_p$

- Given Experience($\mathbf{E}$):
  - n observations $\mathbf{X} \in \mathcal{R}^{n\times p}$
  - $\mathbf{y} \in \mathcal{R}^{n\times 1}$
- The task($\mathbf{T}$) is: predicting y using model $h(\mathbf{x}) = h_{\boldsymbol{\theta}}(\mathbf{x})$,
- Performance($\mathbf{P}$): Which minimizes loss(error) fuction $J(\boldsymbol{\theta})$

For linear regression:

- $J(\boldsymbol{\theta}) = \sum_{i=1}^{n} \left(h_{\boldsymbol{\theta}}(\mathbf{x}^{(i)}) - y^{(i)}\right)^2 = RSS(\boldsymbol{\theta})$
- $h_{\boldsymbol{\theta}}(\mathbf{x}) = \langle \boldsymbol{\theta}, \mathbf{x} \rangle = \mathbf{x}^T \boldsymbol{\theta} = \theta_0 + \theta_1 x_1 + ... + \theta_p x_p$

We can get $\boldsymbol{\theta}$ through: $\frac{\partial J(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = 0$

Maximum likelihood estimation

Assuming:

- $y = \theta_0 + \theta_1 x_1 + ... + \theta_p x_p + \epsilon$
- $\epsilon \sim N(0, \sigma^2)$, i.i.d.
- $\mathbf{X}$ is known

Then likelihood funciton: $P(\mathbf{y}|\boldsymbol{\theta}, \mathbf{X}) = \prod_{i=1}^{n} P(\mathbf{y_i}|\boldsymbol{\theta}, \mathbf{X})$
$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp^{-\frac{(y_i - (\theta_0 + \theta_1 x_1 + ... + \theta_p x_p))^2}{2\sigma^2}}$
Log likelihood function: $\log(P(\mathbf{y_i}|\boldsymbol{\theta}, \mathbf{X})) = \sum_{i=1}^{n} \left(\log(\frac{1}{\sqrt{2\pi}\sigma}) - \frac{(y_i - (\theta_0 + \theta_1 x_1 + ... + \theta_p x_p))^2}{2\sigma^2}\right)$
$= n \log(\frac{1}{\sqrt{2\pi}\sigma}) + \sum_{i=1}^{n} \left(-\frac{(y_i - (\theta_0 + \theta_1 x_1 + ... + \theta_p x_p))^2}{2\sigma^2}\right)$
$= n \log(\frac{1}{\sqrt{2\pi}\sigma}) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - (\theta_0 + \theta_1 x_1 + ... + \theta_p x_p))^2$
$= n \log(\frac{1}{\sqrt{2\pi}\sigma}) - \frac{1}{2\sigma^2} \mathrm{RSS}(\boldsymbol{\theta})$
Thus maximum likelihood estimatoin of $\boldsymbol{\theta}$ is equivalent to LS.

Assumations

Additive the effect of change in x_j on y is independent of the other predictors

Linear change in Y is constant due to a one-unite change in x_j

Model related concept

Model assessment parameters, which are shown in *summary(lm())*

- $R^2 = 1 - \frac{RSS}{TSS}$ where $RSS = \sum_i (y^{(i)} - \hat{y}^{(i)})^2$ and $TSS = \sum_i (y^{(i)} - \bar{y})^2$
  - $R^2 = 1$ if $RSS = 0$
  - $R^2 = 0$ if $RSS = TSS$, i.e, using $\bar{y}$ as the estimate for each observation.

  - $R^2 = \rho^2$ where $\rho = cor(\mathbf{x}_1, \mathbf{y})$ if there are only one predictor, i.e, $\mathbf{X} \in \mathcal{R}^{n\times 1}$.

- $RSE = \sqrt{\frac{\sum_{i=1}^{n}(y^{(i)}-\hat{y}^{(i)})^2}{n-p-1}} = \sigma$, sd of ϵ, RSE^2 is the power of noise
- $\mathrm{E}RSE = 0$
- If F-statistic » 1, reject H_0
  - all regression coefficents are 0, $H_0 : \theta_0 = \theta_1 = ...\theta_p = 0$
  - H_a: at least one θ_j is non zero
- summary to a single number R^2: will throw out lots of information.

Other

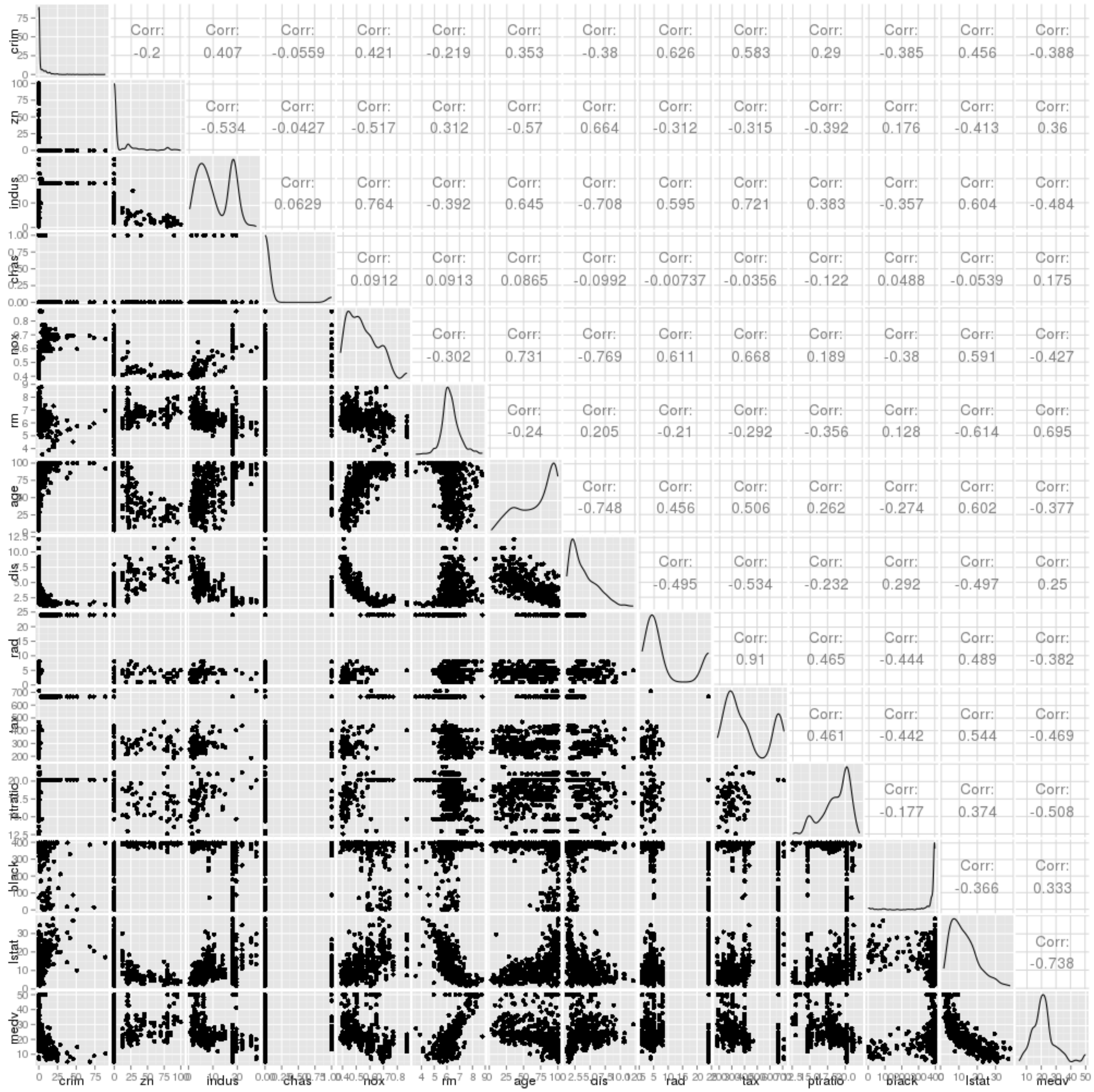
- Confidence interval VS prediction interval?
- VIF: $VIF(\theta_j) = \frac{1}{1-R^2_{x_j|x_{-j}}} = f(R^2_{x_j|x_{-j}})$ VS 5. The larger VIF for x_j, the more collinearity x_j with other predictors x_{-j}
- Leverage: $h^{(i)} = \frac{1}{n} + \frac{||\mathbf{x}^{(i)}-\hat{\mathbf{x}}^{(i)}||^2}{\sum_{i'=1}^{n}||\mathbf{x}^{(i)}-\hat{\mathbf{x}}^{(i)}||^2}$
- Outlier: high residual

Lab: linear regression

Explore data

Use *ggpairs()* to plot the scatter matrix. It's show that *lstat* (-0.738) and *rm* (0.695) has the largest cor() with *medv*.

```
library(ISLR)
library(ggplot2)
library(GGally)
library(MASS)
ggpairs(Boston)
```



Simple linear regression

Regressing *medv* (房价中位数) onto *lstat* (低社会经济阶层家庭百分比) using simple linear regression.

```r
### * Load library
library(MASS)          # large collections of datasets and functions
library(ISLR)          # functions and datasets assoicated with ISL book

### * Explore data
str(Boston)

```

```r
### * Fit
simple_linear_regression_model <- lm(medv ~ lstat, data = Boston)
```

```
Attaching package: 'MASS'

The following object is masked from 'package:dplyr':

    select
'data.frame':	506 obs. of  14 variables:
 $ crim   : num  0.00632 0.02731 0.02729 0.03237 0.06905 ...
 $ zn     : num  18 0 0 0 0 0 12.5 12.5 12.5 12.5 ...
 $ indus  : num  2.31 7.07 7.07 2.18 2.18 2.18 7.87 7.87 7.87 7.87 ...
 $ chas   : int  0 0 0 0 0 0 0 0 0 0 ...
 $ nox    : num  0.538 0.469 0.469 0.458 0.458 0.458 0.524 0.524 0.524 0.524 ...
 $ rm     : num  6.58 6.42 7.18 7 7.15 ...
 $ age    : num  65.2 78.9 61.1 45.8 54.2 58.7 66.6 96.1 100 85.9 ...
 $ dis    : num  4.09 4.97 4.97 6.06 6.06 ...
 $ rad    : int  1 2 2 3 3 3 5 5 5 5 ...
 $ tax    : num  296 242 242 222 222 222 311 311 311 311 ...
 $ ptratio: num  15.3 17.8 17.8 18.7 18.7 18.7 15.2 15.2 15.2 15.2 ...
 $ black  : num  397 397 393 395 397 ...
 $ lstat  : num  4.98 9.14 4.03 2.94 5.33 ...
 $ medv   : num  24 21.6 34.7 33.4 36.2 28.7 22.9 27.1 16.5 18.9 ...
```

Model assessment result using \$RSE, R^2, F\$-statistic:

- $RSE = 6.216$
- $R^2 = 54.32F - statistic = 601.1 >> 1, soweareabletorejectH_0 : \theta_0 = \theta_1 = 0$
- p value for both *intercept* θ_0 and *lstat* θ_1 are very small, so we are able to reject $H_0 : \theta_0 = 0$ and $H_0 : \theta_1 = 0$. Later we will use CI(Confidence interval) to get the same result.

```r
### ** RSE, R^2, F-statistic
## RSE: residual standard error, estimate of sigma_epsilon
## R^2: [0, 1],  1 - RSS/TSS,  0 ~ denotes the same performance of mean(y), 1 denotes perfect
## F-statistic: H_0: beta_1 = beta_2 = beta_p = 0, if > 1, reject H_0
summary(simple_linear_regression_model)
```

```
Call:
lm(formula = medv ~ lstat, data = Boston)

Residuals:
    Min      1Q  Median      3Q     Max
-15.168  -3.990  -1.318   2.034  24.500

Coefficients:
```

```
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 34.55384    0.56263   61.41   <2e-16 ***
lstat       -0.95005    0.03873  -24.53   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 6.216 on 504 degrees of freedom
Multiple R-squared:  0.5441,Adjusted R-squared:  0.5432
F-statistic: 601.6 on 1 and 504 DF,  p-value: < 2.2e-16
```

Get the coefficients using *coef()* and their confidence interval using *confint()*:

- $[33.44845735.6592247]$ doesn't include 0, thus reject $H_0 : \theta_0 = 0$
- $[-1.026148 - 0.8739505]$ doesn't include 0, thus reject $H_0 : \theta_1 = lstat = 0$

```r
coef(simple_linear_regression_model)

## Confidence interval of beta, more understandabel than p-value
confint(simple_linear_regression_model) # beta_0 + c( -1, 1) * qt(.975, dof) * se(beta_0)
```

```
Error in coef(simple_linear_regression_model) :
  object 'simple_linear_regression_model' not found
Error in confint(simple_linear_regression_model) :
  object 'simple_linear_regression_model' not found
```

Predict and get the confidence interval and predict interval.?

```r
### * Predict
predict(simple_linear_regression_model, data.frame(lstat = c(5, 10, 15)))
predict(simple_linear_regression_model, data.frame(lstat = c(5, 10, 15)), interval="prediction")
predict(simple_linear_regression_model, data.frame(lstat = c(5, 10, 15)), interval="confidence")
```

```
       1        2        3
29.80359 25.05335 20.30310
       fit       lwr      upr
1 29.80359 17.565675 42.04151
2 25.05335 12.827626 37.27907
3 20.30310  8.077742 32.52846
       fit      lwr      upr
1 29.80359 29.00741 30.59978
2 25.05335 24.47413 25.63256
3 20.30310 19.73159 20.87461
```

Plot the training dataset and fitted line:

```r
plot(Boston$lstat, Boston$medv, main = "simple linear regression(Training data)")
abline(simple_linear_regression_model, col = "red")
```

[width=.9]../images/olm$_t raing_d ata$

Using plot(lm_model) to get the four diagnostic plots. (normal QQ plot?)

- residuals: $y^{(i)} - \hat{y}^{(i)}$
- studentized residuals: $\frac{y^{(i)} - \hat{y}^{(i)}}{SE(y^{(i)} - \hat{y}^{(i)})}$
- leverage statistics: $h^{(i)} = \frac{1}{n} + \frac{||\mathbf{x}^{(i)} - \hat{\mathbf{x}}^{(i)}||^2}{\sum_{i'=1}^{n} ||\mathbf{x}^{(i)} - \hat{\mathbf{x}}^{(i)}||^2}$ (to check for p>1)

```r
par(mfrow = c(2, 2))
plot(simple_linear_regression_model)
```

[width=.9]../images/olm$_p lot$

We can also plot the diagnostic plot mannually.

```r
par(mfrow = c(2, 2))
plot(predict(simple_linear_regression_model), residuals(simple_linear_regression_model), main = "res
plot(predict(simple_linear_regression_model), rstudent(simple_linear_regression_model), main = "stud
plot(hatvalues(simple_linear_regression_model), main = "leverage statistics")
plot(hatvalues(simple_linear_regression_model), rstudent(simple_linear_regression_model),
     xlab = "leverage statistics",
     ylab = "studentlized residuals",
     main = "levearge statistics VS studentlized residuals")
```

[width=.9]../images/olm$_r esidual_l everage$

Multiple linear regression

Since $lstat, rm$ has largest ρ related to $medv$, we regress $medv$ to them:

```r
lm_two_feature <- lm(medv ~ lstat +  rm,  data = Boston)
summary(lm_two_feature)
```

```
Call:
lm(formula = medv ~ lstat + rm, data = Boston)

Residuals:
    Min      1Q  Median      3Q     Max
-18.076  -3.516  -1.010   1.909  28.131

Coefficients:
```

```
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -1.35827    3.17283  -0.428    0.669
lstat       -0.64236    0.04373 -14.689   <2e-16 ***
rm           5.09479    0.44447  11.463   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.54 on 503 degrees of freedom
Multiple R-squared:  0.6386,Adjusted R-squared:  0.6371
F-statistic: 444.3 on 2 and 503 DF,  p-value: < 2.2e-16
```

How about regress using all predictors?

```r
lm_all_feature <- lm(medv ~ .,  data = Boston) # . denotes other variable except medv?
summary(lm_all_feature)        # Find the feature with largest p value
```

```
    crim       zn    indus     chas      nox       rm      age      dis
1.792192 2.298758 3.991596 1.073995 4.393720 1.933744 3.100826 3.955945
     rad      tax  ptratio    black    lstat
7.484496 9.008554 1.799084 1.348521 2.941491
```

```r
## Remove features that has large p value,  i.e, 0.95 confidence intervel has 0
confint(lm_all_feature)                  # See age and indus
lm_all_but_age <- lm(medv ~ . - age, data = Boston)
summary(lm_all_but_age)

lm_all_but_age_indus <- lm(medv ~ . - age - indus, data = Boston)
summary(lm_all_but_age_indus)
```

```
                    2.5 %       97.5 %
(Intercept)  26.432226009  46.486750761
crim         -0.172584412  -0.043438304
zn            0.019448778   0.073392139
indus        -0.100267941   0.141385193
chas          0.993904193   4.379563446
nox         -25.271633564 -10.261588893
rm            2.988726773   4.631003640
age          -0.025262320   0.026646769
dis          -1.867454981  -1.083678710
rad           0.175692169   0.436406789
tax          -0.019723286  -0.004945902
ptratio      -1.209795296  -0.695699168
black         0.004034306   0.014589060
lstat        -0.624403622  -0.425113133

Call:
```

```
lm(formula = medv ~ . - age, data = Boston)

Residuals:
     Min       1Q   Median       3Q      Max 
-15.6054  -2.7313  -0.5188   1.7601  26.2243 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept)  36.436927   5.080119   7.172 2.72e-12 ***
crim         -0.108006   0.032832  -3.290 0.001075 ** 
zn            0.046334   0.013613   3.404 0.000719 ***
indus         0.020562   0.061433   0.335 0.737989    
chas          2.689026   0.859598   3.128 0.001863 ** 
nox         -17.713540   3.679308  -4.814 1.97e-06 ***
rm            3.814394   0.408480   9.338  < 2e-16 ***
dis          -1.478612   0.190611  -7.757 5.03e-14 ***
rad           0.305786   0.066089   4.627 4.75e-06 ***
tax          -0.012329   0.003755  -3.283 0.001099 ** 
ptratio      -0.952211   0.130294  -7.308 1.10e-12 ***
black         0.009321   0.002678   3.481 0.000544 ***
lstat        -0.523852   0.047625 -10.999  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.74 on 493 degrees of freedom
Multiple R-squared:  0.7406,Adjusted R-squared:  0.7343 
F-statistic: 117.3 on 12 and 493 DF,  p-value: < 2.2e-16

Call:
lm(formula = medv ~ . - age - indus, data = Boston)

Residuals:
     Min       1Q   Median       3Q      Max 
-15.5984  -2.7386  -0.5046   1.7273  26.2373 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept)  36.341145   5.067492   7.171 2.73e-12 ***
crim         -0.108413   0.032779  -3.307 0.001010 ** 
zn            0.045845   0.013523   3.390 0.000754 ***
chas          2.718716   0.854240   3.183 0.001551 ** 
nox         -17.376023   3.535243  -4.915 1.21e-06 ***
rm            3.801579   0.406316   9.356  < 2e-16 ***
dis          -1.492711   0.185731  -8.037 6.84e-15 ***
rad           0.299608   0.063402   4.726 3.00e-06 ***
tax          -0.011778   0.003372  -3.493 0.000521 ***
ptratio      -0.946525   0.129066  -7.334 9.24e-13 ***
black         0.009291   0.002674   3.475 0.000557 ***
```

```
lstat        -0.522553   0.047424 -11.019  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.736 on 494 degrees of freedom
Multiple R-squared:  0.7406,Adjusted R-squared:  0.7348
F-statistic: 128.2 on 11 and 494 DF,  p-value: < 2.2e-16
```

Performance on training data:

- R^2 increases as the number of predictors used in regression, even the predictor is the noise. (p 元、n 个方程, p 增达到 n, 方程有解, $RSS = 0, R^2 = 1$), lead to overfit.
- $RSE, R^2_{adjusted}$ penalize if p is too large

method	R^2	$R^2_{adjusted}$	RSE	F-statistic
1 predictor	0.5441	0.5432	6.216	601.6
2 perdictors	0.6386	0.6371	5.540	444.3
all predictors	0.74064266410940938	0.7343	4.745	117.3
all but age	0.74064121655051440	0.7343	4.740	117.3
all but age, indus	0.74058228025695738	0.7348	4.736	128.2

Interacton terms: [removing additive assumation]

```r
summary(lm(medv ~ lstat * age, data = Boston)) # i.e,  lstat + age + lstat:age
```

```
Call:
lm(formula = medv ~ lstat * age, data = Boston)

Residuals:
   Min      1Q Median      3Q     Max 
-15.81   -4.04   -1.33    2.08   27.55 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) 36.088536   1.469835   24.55  < 2e-16 ***
lstat       -1.392117   0.167456   -8.31  8.8e-16 ***
age         -0.000721   0.019879   -0.04    0.971    
lstat:age    0.004156   0.001852    2.24    0.025 *  
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.15 on 502 degrees of freedom
Multiple R-squared:  0.556,Adjusted R-squared:  0.553 
F-statistic:  209 on 3 and 502 DF,  p-value: <2e-16
```

Non-linear transformations of the predictors

It's shown that $medv, lstat$ in not linear relationship in scatter matrix plot. Thus, we try to regression onto $lstat, lstat^2$.

```r
lm_quad_lstat <- lm(medv ~ lstat + I(lstat^2), data = Boston)
summary(lm_quad_lstat)
```

```
Call:
lm(formula = medv ~ lstat + I(lstat^2), data = Boston)

Residuals:
   Min     1Q Median     3Q    Max 
-15.28  -3.83  -0.53   2.31  25.41 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 42.86201    0.87208    49.1   <2e-16 ***
lstat       -2.33282    0.12380   -18.8   <2e-16 ***
I(lstat^2)   0.04355    0.00375    11.6   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.52 on 503 degrees of freedom
Multiple R-squared:  0.641,Adjusted R-squared:  0.639 
F-statistic:  449 on 2 and 503 DF,  p-value: <2e-16
```

Small value of p value of F, thus

- Reject H_0 : model-1 and model-2 are equal
- Accept H_a : model-2 is better than model-1

```r
anova(simple_linear_regression_model, lm_quad_lstat)
```

```
Analysis of Variance Table

Model 1: medv ~ lstat
Model 2: medv ~ lstat + I(lstat^2)
  Res.Df   RSS Df Sum of Sq   F Pr(>F)    
1    504 19472                            
2    503 15347  1      4125 135 <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```r
par(mfrow = c(2, 2))
plot(lm_quad_lstat)
```

[width=.9]../images/olm$_{q}uad_{l}sat$

How about $lstat^3, lstat^4$, ...? It's shown that order greater than 5 have big p values, so we are NOT able to reject:

- $H_0 : \theta_{lstat^6} = 0$
- $H_0 : \theta_{lstat^7} = 0$
- $H_0 : \theta_{lstat^8} = 0$

```r
lm_poly_lstat_8 <- lm(medv ~ poly(lstat, 8), data = Boston)
summary(lm_poly_lstat_8)
```

```
Call:
lm(formula = medv ~ poly(lstat, 8), data = Boston)

Residuals:
    Min      1Q  Median      3Q     Max
-13.739  -3.147  -0.733   2.096  26.992

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)       22.533      0.232   97.23  < 2e-16 ***
poly(lstat, 8)1 -152.460      5.213  -29.25  < 2e-16 ***
poly(lstat, 8)2   64.227      5.213   12.32  < 2e-16 ***
poly(lstat, 8)3  -27.051      5.213   -5.19  3.1e-07 ***
poly(lstat, 8)4   25.452      5.213    4.88  1.4e-06 ***
poly(lstat, 8)5  -19.252      5.213   -3.69  0.00025 ***
poly(lstat, 8)6    6.509      5.213    1.25  0.21240
poly(lstat, 8)7    1.942      5.213    0.37  0.70970
poly(lstat, 8)8   -6.730      5.213   -1.29  0.19730
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.21 on 497 degrees of freedom
Multiple R-squared:  0.684,Adjusted R-squared:  0.679
F-statistic:  134 on 8 and 497 DF,  p-value: <2e-16
```

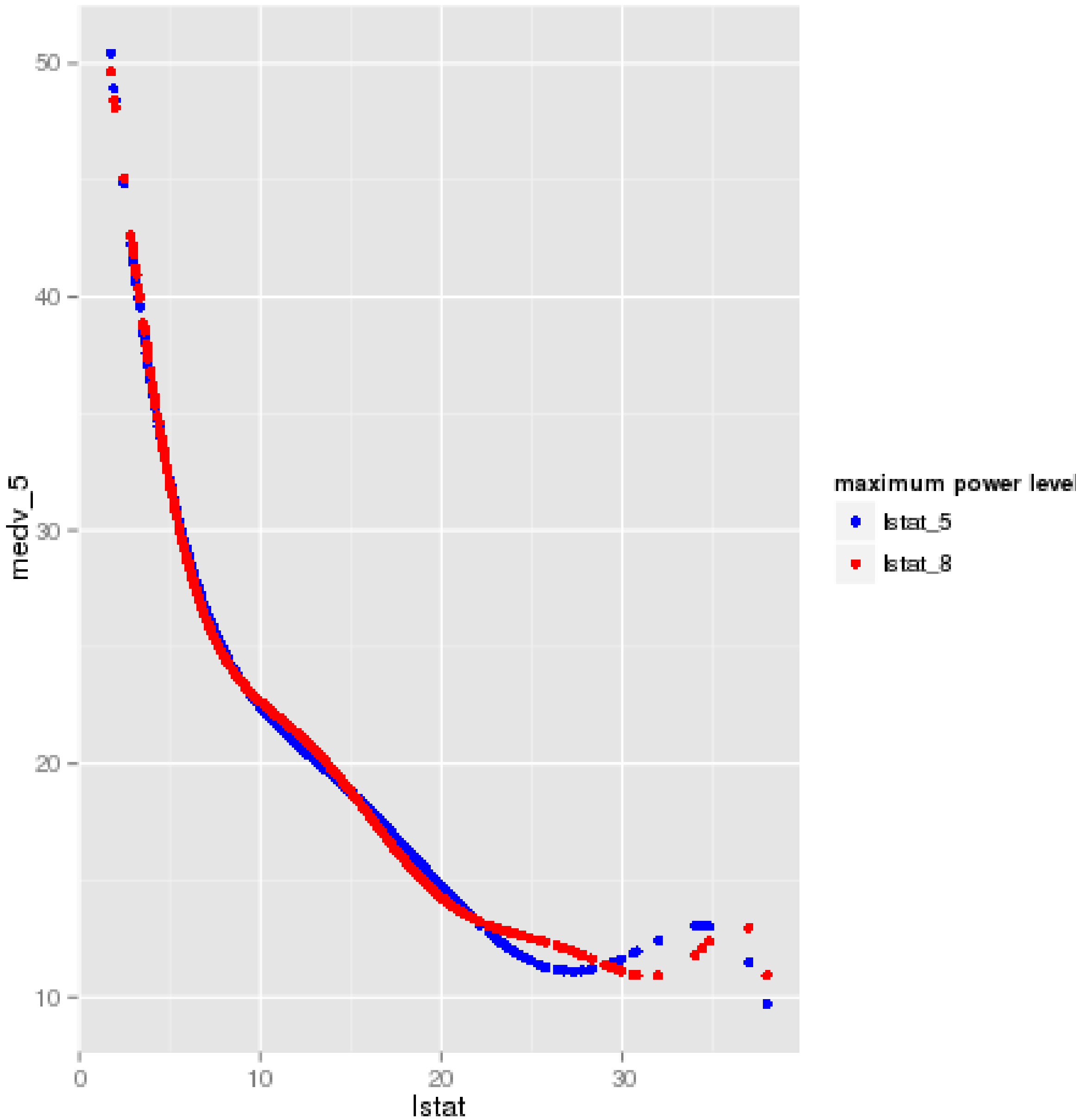
```r
lm_poly_lstat_5 <- lm(medv ~ poly(lstat, 5), data = Boston)
summary(lm_poly_lstat_5)
```

```
Call:
lm(formula = medv ~ poly(lstat, 5), data = Boston)

Residuals:
    Min      1Q  Median      3Q     Max
-13.543  -3.104  -0.705   2.084  27.115
```

```
Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)       22.533      0.232   97.20  < 2e-16 ***
poly(lstat, 5)1 -152.460      5.215  -29.24  < 2e-16 ***
poly(lstat, 5)2   64.227      5.215   12.32  < 2e-16 ***
poly(lstat, 5)3  -27.051      5.215   -5.19  3.1e-07 ***
poly(lstat, 5)4   25.452      5.215    4.88  1.4e-06 ***
poly(lstat, 5)5  -19.252      5.215   -3.69  0.00025 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.21 on 500 degrees of freedom
Multiple R-squared:  0.682,Adjusted R-squared:  0.679
F-statistic:  214 on 5 and 500 DF,  p-value: <2e-16
```

The fitted curve using $5th$ and $8th$ power are almost the same.

```r
library(ggplot2)
ggplot(data=data.frame(lstat = Boston$lstat,
                       medv_8 = predict(lm_poly_lstat_8, data = Boston),
                       medv_5=predict(lm_poly_lstat_5, data = Boston) ),
       aes(x = lstat, y = medv_5)) + geom_point(aes(color = "lstat_5")) +
           geom_point(aes(y = medv_8, color = "lstat_8")) +
               scale_color_manual("maximum power level", values = c("lstat_5" = "blue", "lstat_8" =
```



We can also use other non-linear transformation, such as *log*.

```
lm_logrm <- lm(medv ~ log(rm), data = Boston)
```

Qualitative Predictors (Categracial Variable)

Categorical variable are coded into *dummy* variables:

```
library(ISLR)
str(Carseats)

```

```r
lm_model <- lm(Sales ~ . + Income:Advertising + Price:Age, data = Carseats)
summary(lm_model)
contrasts(Carseats$ShelveLoc)
```

```
'data.frame': 400 obs. of  11 variables:
 $ Sales      : num  9.5 11.22 10.06 7.4 4.15 ...
 $ CompPrice  : num  138 111 113 117 141 124 115 136 132 132 ...
 $ Income     : num  73 48 35 100 64 113 105 81 110 113 ...
 $ Advertising: num  11 16 10 4 3 13 0 15 0 0 ...
 $ Population : num  276 260 269 466 340 501 45 425 108 131 ...
 $ Price      : num  120 83 80 97 128 72 108 120 124 124 ...
 $ ShelveLoc  : Factor w/ 3 levels "Bad","Good","Medium": 1 2 3 3 1 1 3 2 3 3 ...
 $ Age        : num  42 65 59 55 38 78 71 67 76 76 ...
 $ Education  : num  17 10 12 14 13 16 15 10 10 17 ...
 $ Urban      : Factor w/ 2 levels "No","Yes": 2 2 2 2 2 1 2 2 1 1 ...
 $ US         : Factor w/ 2 levels "No","Yes": 2 2 2 2 1 2 1 2 1 2 ...

Call:
lm(formula = Sales ~ . + Income:Advertising + Price:Age, data = Carseats)

Residuals:
   Min     1Q Median     3Q    Max
-2.921 -0.750  0.018  0.675  3.341

Coefficients:
                    Estimate Std. Error t value Pr(>|t|)
(Intercept)         6.575565   1.008747    6.52  2.2e-10 ***
CompPrice           0.092937   0.004118   22.57  < 2e-16 ***
Income              0.010894   0.002604    4.18  3.6e-05 ***
Advertising         0.070246   0.022609    3.11  0.00203 **
Population          0.000159   0.000368    0.43  0.66533
Price              -0.100806   0.007440  -13.55  < 2e-16 ***
ShelveLocGood       4.848676   0.152838   31.72  < 2e-16 ***
ShelveLocMedium     1.953262   0.125768   15.53  < 2e-16 ***
Age                -0.057947   0.015951   -3.63  0.00032 ***
Education          -0.020852   0.019613   -1.06  0.28836
UrbanYes            0.140160   0.112402    1.25  0.21317
USYes              -0.157557   0.148923   -1.06  0.29073
Income:Advertising  0.000751   0.000278    2.70  0.00729 **
Price:Age           0.000107   0.000133    0.80  0.42381
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.01 on 386 degrees of freedom
Multiple R-squared:  0.876,Adjusted R-squared:  0.872
F-statistic:  210 on 13 and 386 DF,  p-value: <2e-16
       Good Medium
Bad       0      0
```

```
Good      1       0
Medium    0       1
```