MATH 417, HOMEWORK 2

CHARLES ANCEL

CHAPTER II.8

Exercise 5. In Exercises 1 through 5, compute the indicated product involving the following permutations in S_6:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}, \mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 4 & 3 & 1 & 6 \end{pmatrix}$$

5. $\sigma^{-1}\tau\sigma$

Proof. To find the product $\sigma^{-1}\tau\sigma$, we will compute the action of the composition on each integer from 1 to 6:

1. Starting with 1: $\sigma(1) = 3$, $\tau(3) = 1$, $\sigma^{-1}(1) = 2$. Thus, $\sigma^{-1}\tau\sigma(1) = 2$.

2. For 2: $\sigma(2) = 1$, $\tau(1) = 2$, $\sigma^{-1}(2) = 6$. Thus, $\sigma^{-1}\tau\sigma(2) = 6$.

3. For 3: $\sigma(3) = 4$, $\tau(4) = 3$, $\sigma^{-1}(3) = 1$. Thus, $\sigma^{-1}\tau\sigma(3) = 1$.

4. For 4: $\sigma(4) = 5$, $\tau(5) = 6$, $\sigma^{-1}(6) = 2$. Thus, $\sigma^{-1}\tau\sigma(4) = 5$.

5. For 5: $\sigma(5) = 6$, $\tau(6) = 5$, $\sigma^{-1}(5) = 4$. Thus, $\sigma^{-1}\tau\sigma(5) = 4$.

6. For 6: $\sigma(6) = 2$, $\tau(2) = 4$, $\sigma^{-1}(4) = 3$. Thus, $\sigma^{-1}\tau\sigma(6) = 3$.

Grouping all these results together:

$$\sigma^{-1}\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 5 & 4 & 3 \end{pmatrix}$$

.

$\square$

Exercise 8. In Exercises 6 through 9, compute the expressions shown for the permutations σ, τ and μ defined prior to Exercise 1.

8. Compute σ^{100}

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$$

Proof. We start by determining the cycle decomposition of σ: $\sigma = (1, 3, 4, 5, 6, 2)$.

This 6-cycle implies that after applying σ six times, we achieve the identity permutation, σ_0 for S_6: $\sigma^6 = \sigma_0$.

Given this, for any integer k, the permutation σ^{6k} is equivalent to the identity permutation.

We are also provided with the value of σ^4:

$$\sigma^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}$$

To compute σ^{100}, we decompose 100 as:

$$100 = 6 \times 16 + 4.$$

This means that σ^{100} is equivalent to applying σ^4 after applying σ 96 times (which returns the identity 16 times).

Using the properties of permutations, this is articulated as:

$$\sigma^{100} = \sigma^{6(16)+4} = \sigma^{6(16)}\sigma^4 = (\sigma^6)^{16}\sigma^4 = \sigma_0^{16}\sigma^4 = \sigma_0\sigma^4 = \sigma^4.$$

Thus,

$$\sigma^{100} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}.$$

$\square$

Exercise 12. Let A be a set and let $\sigma \in S_A$. For a fixed $a \in A$, the set

$$\mathcal{O}_{a,\sigma} = \{\sigma^n(a) | n \in \mathbb{Z}\}$$

is the orbit of a under σ. In Exercises 11 through 13, find the orbit of 1 under the permutation defined prior to Exercise 1.

12. τ.

Proof. Consider the permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$$

in S_6.

Direct computation gives:

- $\tau(1) = 2$
- $\tau^2(1) = 4$
- $\tau^3(1) = 3$
- $\tau^4(1) = 1$

Now, using induction, assume that $\tau^{4n}(1) = 1$ for some positive integer n. Then: - $\tau^{4(n+1)}(1) = \tau^{4n}(\tau^4(1)) = \tau^{4n}(1) = 1$

Extending this result to negative integers, we have: - $\tau^0(1) = 1$ - Given any positive integer n, $\tau^{4n}(1) = 1$ and hence, $\tau^{-4n}(1) = 1$

Finally, the orbit for all integer powers is then deduced as: - $\tau^{4n+1}(1) = 2$ - $\tau^{4n+2}(1) = 4$ - $\tau^{4n+3}(1) = 3$

Thus, the orbit of 1 under τ, denoted as $\mathcal{O}_{1,\tau}$, is $\{1, 2, 3, 4\}$.   □

Exercise 16. Find the number of elements in the set $\{\sigma \in S_4 | \sigma(3) = 3\}$.

Proof. If $\sigma(3) = 3$, then the number 3 remains fixed. This means we're only considering the permutations of the numbers 1, 2, and 4.

Let's list them out:

(1) 1, 2, 3, 4

(2) 1, 4, 3, 2

(3) 2, 1, 3, 4

(4) 2, 4, 3, 1

(5) 4, 1, 3, 2

(6) 4, 2, 3, 1

Thus, there are 6 permutations in S_4 where $\sigma(3) = 3$.   □

Exercise 21. a. Verify that the six matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

form a group under matrix multiplication. [*Hint:* Don't try to compute all products of these matrices. Instead, think how the column vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is transformed by multiplying it on the left by each of the matrices.]

b. What group discussed in this section is isomorphic to this group of six matrices?

Proof. **a.** The matrices provided are permutation matrices of order 3. Let's verify the properties of the group for these matrices:

(1) **Closure:** Multiplying any two of the provided matrices will result in another matrix from the set. This is because multiplying two permutation matrices results in another permutation matrix, given that they permute the elements of a column vector.

(2) **Associativity:** Matrix multiplication is always associative. That is, for any matrices A, B, and C, $A(BC) = (AB)C$.

Existence of the Identity: The identity matrix, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, is present in the set. When any matrix from the set is multiplied by this matrix, it remains unchanged.

Existence of the Inverse: The inverse of a permutation matrix is the matrix representing the inverse permutation. Since every permutation in S_3 has an inverse in S_3, every matrix in our set has an inverse in our set.

Given that all the properties of a group are satisfied, the matrices form a group under matrix multiplication.

b. The group of matrices is isomorphic to S_3, the symmetric group on 3 elements. Each matrix corresponds to a permutation of the set $\{1,2,3\}$. The multiplication of the matrices reflects the composition of these permutations, and hence it is isomorphic to S_3. $\qquad\square$

Exercise 32. In Exercises 30 through 34, determine whether the given function is a permutation of $\mathbb{R}$.

32. $f_3 : \mathbb{R} \to \mathbb{R}$ defined by $f_3(x) = -x^3$

To determine whether $f_3 : \mathbb{R} \to \mathbb{R}$ defined by $f_3(x) = -x^3$ is a permutation of $\mathbb{R}$, we must check two properties:

(1) **Injectivity (One-to-One)**: A function is injective if every element of the domain maps to a unique element in the codomain.

(2) **Surjectivity (Onto)**: A function is surjective if every element of the codomain has a pre-image in the domain.

Proof.

(1) **Injectivity**: Let $x_1, x_2 \in \mathbb{R}$ such that $f_3(x_1) = f_3(x_2)$. This implies that $-x_1^3 = -x_2^3$. Dividing both sides by -1, we get $x_1^3 = x_2^3$. Now, the cube function is injective over $\mathbb{R}$. Therefore, $x_1 = x_2$. So, f_3 is injective.

(2) **Surjectivity**: Let $y \in \mathbb{R}$. Consider $x = \sqrt[3]{-y}$. Then, $f_3(x) = -x^3 = -(\sqrt[3]{-y})^3 = y$. Hence, every real number y has a pre-image in $\mathbb{R}$ under f_3. Thus, f_3 is surjective.

Since f_3 is both injective and surjective, f_3 is a permutation of $\mathbb{R}$.

$\qquad\square$

Exercise 47. Strengthening Exercise 46, show that if $n \geq 3$, then the only element of σ of S_n satisfying $\sigma\gamma = \gamma\sigma$ for all $\gamma \in S_n$ is $\sigma = \iota$, the identity permutation.

Proof. **Case 1:** $a = c$

This means that $\sigma(a) = b$ and $\sigma(b) = a$. We can choose γ to be the transposition $(a\ b)$.

Then,
$$\sigma \circ \gamma(a) = \sigma(b) = a$$
$$\gamma \circ \sigma(a) = \gamma(b) = a$$

And,
$$\sigma \circ \gamma(b) = \sigma(a) = b$$
$$\gamma \circ \sigma(b) = \gamma(a) = b$$

Thus, for this γ, $\sigma \circ \gamma = \gamma \circ \sigma$.

Case 2: $a \neq c$

For this case, we can choose γ to be the transposition $(a\ c)$.

Then,
$$\sigma \circ \gamma(a) = \sigma(c)$$
$$\gamma \circ \sigma(a) = \gamma(b) = b$$

Since $b \neq c$ (because if $b = c$ then $a = c$ from our assumption which is not the case here), we have $\sigma \circ \gamma(a) \neq \gamma \circ \sigma(a)$. Thus, $\sigma \circ \gamma \neq \gamma \circ \sigma$.

From the two cases, we see that for the second case where $a \neq c$, we can find a permutation γ such that $\sigma \circ \gamma \neq \gamma \circ \sigma$. Therefore, if σ is not an identity permutation in S_n, there exists γ in S_n such that $\sigma \circ \gamma \neq \gamma \circ \sigma$.

Let σ be an element in S_n not equal to the identity. If there exists some a such that $\sigma(a) \neq a$, let $\sigma(a)$ be b and $\sigma(b)$ be c. If $a = c$, choosing γ as the transposition $(a\ b)$ satisfies $\sigma \circ \gamma = \gamma \circ \sigma$. If $a \neq c$, choosing γ as the transposition $(a\ c)$ ensures $\sigma \circ \gamma \neq \gamma \circ \sigma$. Hence, for any non-identity permutation σ, there exists a permutation γ such that $\sigma \circ \gamma \neq \gamma \circ \sigma$. $\square$

Exercise 52. Let G be a group. Prove that the permutations $\rho a : G \to G$, where $\rho a(x) = xa$ for $a \in G$ and $x \in G$, do form a group isomorphic to G.

Before solving, let's first show that the set of all such permutations forms a group and then that this group is isomorphic to G.

Closure:

Consider two such permutations: ρa and ρb where $\rho a(x) = xa$ and $\rho b(x) = xb$. The composition $\rho a \circ \rho b$ is given by:

$$\rho a \circ \rho b(x) = \rho a(xb) = xba$$

This is of the same form as the original permutations, xa, for some element a in G. Hence, the set of permutations is closed under composition.

Existence of an Identity:

We need to find an element a in G such that $xa = x$ for all x in G. This is true if a is the identity element, e, of G. Thus, ρe acts as the identity in the set of permutations.

Existence of Inverses:

For each permutation ρa, we need to find a permutation ρb such that $\rho a \circ \rho b(x) = x$ for all x in G. Since $\rho a \circ \rho b(x) = xba$, this is satisfied if $ba = e$ or $b = a^{-1}$, the inverse of a in G. Thus, ρa^{-1} is the inverse of ρa in the set of permutations.

Associativity:

This property is inherited from the group G, since the operations are essentially the same.

Now, to show that the group of permutations is isomorphic to G, consider the map:

$$\phi : G \to \text{Group of Permutations}$$
$$a \mapsto \rho a$$

This map is clearly bijective. It is also a homomorphism since:

$$\phi(ab) = \rho ab$$
$$\phi(a) \circ \phi(b) = \rho a \circ \rho b$$
$$\rho a \circ \rho b(x) = xba = \rho ab(x)$$
$$\phi(ab) = \phi(a) \circ \phi(b)$$

Thus, G is isomorphic to the group of permutations by the map ϕ.

Proof. Consider the set of permutations $\rho a : G \to G$ defined by $\rho a(x) = xa$ for each $a \in G$. These permutations are closed under composition, have an identity permutation ρe where e is the identity of G, and every permutation ρa has an inverse ρa^{-1}. Thus, they form a group.

Further, the mapping $\phi : G \to \text{Group of Permutations}$ defined by $a \mapsto \rho a$ is a bijective homomorphism. Hence, G is isomorphic to the group of permutations.

$\square$

Chapter II.9

Exercise 3. In Exercises 1 through 6, find all orbits of the given permutation.

3. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 5 & 1 & 4 & 6 & 8 & 7 \end{pmatrix}$

Proof. To find the orbits of the given permutation, we'll see where each number gets mapped to under successive applications of the permutation until the sequence begins to repeat itself.

The permutation is given by:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 5 & 1 & 4 & 6 & 8 & 7 \end{pmatrix}$$

(1) Start with 1:

$\sigma(1) = 2, \sigma^2(1) = \sigma(2) = 3, \sigma^3(1) = \sigma(3) = 5, \sigma^4(1) = \sigma(5) = 4, \sigma^5(1) = \sigma(4) = 1.$

So the orbit of 1 is $\{1, 2, 3, 5, 4\}$.

(2) Since $2, 3, 5$, and 4 are in the same orbit as 1, we don't need to calculate their orbits separately.

(3) For 6:

$\sigma(6) = 6$

So the orbit of 6 is $\{6\}$.

(4) For 7:

$\sigma(7) = 8, \sigma^2(7) = \sigma(8) = 7$

So the orbit of 7 is $\{7, 8\}$.

(5) We have already covered 8 since it's in the orbit of 7.

So, the orbits are:

$$\{1, 2, 3, 5, 4\}, \{6\}, \{7, 8\}$$

Given the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 5 & 1 & 4 & 6 & 8 & 7 \end{pmatrix}$, we have the following orbits:

(1) The orbit of 1 is $\{1, 2, 3, 5, 4\}$.

(2) The orbit of 6 is $\{6\}$.

(3) The orbit of 7 is $\{7, 8\}$.

Hence, the orbits of the permutation are $\{1, 2, 3, 5, 4\}, \{6\}$, and $\{7, 8\}$. $\qquad\square$

Exercise 10. In Exercises 10 through 12, express the permutation of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ as a product of disjoint cycles, and then as a product of transpositions.

10. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$

To express the permutation as a product of disjoint cycles, we follow the mapping until we return to the starting number.

The permutation is given by:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$$

(1) Starting with 1:

$\sigma(1) = 8, \sigma^2(1) = \sigma(8) = 1.$

This gives us the cycle $(1, 8)$.

(2) For 2:

$\sigma(2) = 2.$

Since 2 is mapped to itself, it doesn't form a cycle other than the trivial cycle.

(3) For 3:

$\sigma(3) = 6, \sigma^2(3) = \sigma(6) = 4, \sigma^3(3) = \sigma(4) = 3.$

This gives us the cycle $(3, 6, 4)$.

(4) 4 and 6 are part of the cycle we've already identified.

(5) For 5:

$\sigma(5) = 7, \sigma^2(5) = \sigma(7) = 5.$

This gives us the cycle $(5, 7)$.

(6) 7 is part of the cycle we've already identified.

(7) 8 is part of the cycle we've already identified.

So, the permutation can be expressed as the product of disjoint cycles:

$$\sigma = (1, 8)(3, 6, 4)(5, 7)$$

.

To express the permutation as a product of transpositions (2-cycles):

(1) The cycle $(1, 8)$ is already a transposition.

(2) The cycle $(3, 6, 4)$ can be expressed as $(3, 4)(3, 6)$.

(3) The cycle $(5, 7)$ is already a transposition.

So, the permutation as a product of transpositions is:

$$\sigma = (1,8)(3,4)(3,6)(5,7)$$

.

Proof. Given the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$, we can express it as the product of disjoint cycles:

$$\sigma = (1,8)(3,6,4)(5,7)$$

. This permutation can further be expressed as a product of transpositions:

$$\sigma = (1,8)(3,4)(3,6)(5,7)$$

. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Exercise 13a. Recall that element a of a group G with identity element e has order $r > 0$ if $a^r = e$ and no smaller positive power of a is the identity. Consider the group S_8.

a. What is the order of the cycle (1, 4, 5, 7)?

Given the cycle $(1,4,5,7)$, it will have the following mappings after repeated applications:

1. The first application:

$$1 \rightarrow 4, 4 \rightarrow 5, 5 \rightarrow 7, \text{ and } 7 \rightarrow 1$$

2. The second application:

$$1 \rightarrow 5, 4 \rightarrow 7, 5 \rightarrow 1, \text{ and } 7 \rightarrow 4$$

3. The third application:

$$1 \rightarrow 7, 4 \rightarrow 1, 5 \rightarrow 4, \text{ and } 7 \rightarrow 5$$

4. The fourth application:

$$1 \rightarrow 1, 4 \rightarrow 4, 5 \rightarrow 5, \text{ and } 7 \rightarrow 7$$

Here, after the fourth application, all elements return to their initial positions. Thus, the order of the cycle $(1,4,5,7)$ is 4.

Proof. The order of a cycle in a permutation group is the length of the cycle since an element returns to its original position after the length of the cycle has been applied. Thus, for the cycle $(1,4,5,7)$ in S_8, the order is 4. $\qquad\square$

Exercise 23g. Mark each of the following true or false.

g. A_3 is a commutative group.

To determine whether A_3 (the alternating group on 3 elements) is commutative, we'll need to first identify its elements and then check if the group operation (in this case, function composition or permutation multiplication) is commutative.

A_3 is a subgroup of S_3 (the symmetric group on 3 elements) and is composed of all even permutations in S_3. An even permutation is one that can be expressed as a product of an even number of 2-cycles (transpositions).

The elements of S_3 are:

(1) $e = (1)(2)(3)$

(2) (12)

(3) (13)

(4) (23)

(5) (123)

(6) (132)

Of these, e, (123), and (132) are even permutations. Therefore, A_3 consists of the elements e, (123), and (132).

Now, let's check if the group operation is commutative:

For e, since it's the identity, it commutes with all elements in the group. So:

$$e \circ (123) = (123) \circ e = (123)$$
$$e \circ (132) = (132) \circ e = (132)$$

For (123):
$$(123) \circ (123) = (132)$$
$$(123) \circ (132) = e$$

For (132):
$$(132) \circ (123) = e$$
$$(132) \circ (132) = (123)$$

As we can see, for every pair of elements in A_3, the operation (in this case permutation composition) is commutative.

Proof. The group A_3, consisting of the even permutations of S_3, is commutative, as demonstrated by the fact that for all pairs of elements in A_3, their product (composition) is the same regardless of the order in which they are multiplied. **Thus, A_3 is a commutative group.** $\square$

Exercise 23h. Mark each of the following true or false.

  h. S_7 is isomorphic to the subgroup of all those elements of S_8 that leave the number 8 fixed.

  Any permutation in S_8 that leaves 8 fixed can be thought of as a permutation of the numbers $\{1, 2, 3, 4, 5, 6, 7\}$. This is because the 8th position remains unchanged. Such permutations in S_8 can be identified with the permutations in S_7 since S_7 is the group of permutations of 7 elements.

  Given a permutation in S_7, we can map it to a permutation in S_8 that fixes 8, simply by adding 8 in the last position of the permutation in S_7. This gives us an isomorphism between S_7 and the subgroup of S_8 that fixes 8.

  For instance, consider the permutation (123) in S_7. This maps to the permutation $(123)(8)$ in S_8, which is a permutation that fixes 8.

  This mapping is bijective and respects the group operation, establishing an isomorphism.

Proof. The statement is **true**.

  The subgroup of S_8 that leaves the number 8 fixed can be seen as the group of all permutations of the numbers $\{1, 2, 3, 4, 5, 6, 7\}$. This subgroup is isomorphic to S_7 as every permutation in S_7 can be identified with a permutation in this subgroup of S_8 that keeps 8 fixed. Therefore, S_7 is isomorphic to the subgroup of all those elements of S_8 that leave the number 8 fixed. $\square$

Exercise 23j. Mark each of the following true or false.

  j. The odd permutations in S_8 form a subgroup of S_8.

Proof. The statement is **false**.

  To see why, we need to recall a few things about odd permutations and the definition of a subgroup.

  An odd permutation is one that can be expressed as a product of an odd number of transpositions. A transposition is a permutation that swaps two numbers and leaves all others in their original positions.

  For a set to form a subgroup, it needs to satisfy three criteria:

(1) The identity element of the main group should be present in the subset.

(2) If an element is present in the subset, its inverse should also be there.

(3) The subset should be closed under the group operation.

  Now, let's analyze the set of odd permutations in S_8:

(1) The identity permutation is even, as it can be seen as the result of 0 transpositions (and 0 is even).

(2) The inverse of an odd permutation is odd.

(3) The product of two odd permutations is even. (Because an odd number of swaps combined with another odd number of swaps results in an even number of total swaps.)

From the third property, we see that the set of odd permutations in S_8 is not closed under the group operation. Therefore, the odd permutations in S_8 do not form a subgroup of S_8. $\square$

Exercise 29. Show that for every subgroup H of S_n for $n \geq 2$, either all the permutations in H are even or exactly half of them are even.

Proof. Let's consider any subgroup H of S_n for $n \geq 2$.

(1) **All permutations in H are even:** If the identity is in H (which it must be since H is a subgroup), and if all the products of elements in H are even (i.e., H is closed under the operation with respect to even permutations), then all elements in H must be even. This is because the identity is even and the product of two even permutations is even.

(2) **Exactly half of the permutations in H are even:** Let's assume there is at least one odd permutation in H. We'll denote this permutation by σ.

Now, for any even permutation τ in H, the product $\sigma\tau$ is odd since the product of an even and an odd permutation is odd. Also, since H is a subgroup and is closed under the group operation, $\sigma\tau$ must be in H.

Now, consider the mapping $f : H \to H$ where $f(\tau) = \sigma\tau$ for every τ in H. Since σ is odd, f maps even permutations to odd permutations and vice versa. Moreover, f is a bijection: its inverse is $f^{-1}(\tau) = \sigma\tau$, so every element in H has a unique image under f and vice versa.

From the properties of f, we can deduce that the number of even and odd permutations in H are the same.

In conclusion, for any subgroup H of S_n, either all its permutations are even or exactly half of them are. $\square$