MATH 417, HOMEWORK 10

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Exercise 1. Given a polyhedron with set V of vertices of size m, consider the evident action by its symmetry group G on the set V, which gives a homomorphism $\phi : G \to \mathrm{Sym}(V) \simeq S_m$. For each of the four cases (cube, octahedron, dodecahedron, icosahedron), describe the cycle type of $\phi(g)$ for each type of element $g \in G$(according to the classification of elements of G that I gave in class).

Solution

The symmetry groups of the polyhedra are as follows:

– Cube and Octahedron: Symmetry group S_4

– Dodecahedron and Icosahedron: Symmetry group A_5

The cycle type of $\phi(g)$ in S_4 and A_5 corresponds to the permutation of vertices induced by the symmetry g.

Cube and Octahedron (Symmetry Group: S_4). The cube has 8 vertices, and the octahedron has 6 vertices.

Cube (8 vertices):

- Rotation by $\pm 2\pi/3$ (order 3): 4-cycle among vertices.
- Rotation by $\pm\pi/2$ (order 4): 4-cycle among vertices on the face.
- Rotation by π (order 2): product of two 2-cycles.

Octahedron (6 vertices):

- Rotation by $\pm 2\pi/3$ (order 3): 3-cycle among vertices.
- Rotation by π (order 2): product of two 2-cycles.
- Rotation by $\pm 2\pi/4$ (order 4): 4-cycle among vertices.

Dodecahedron and Icosahedron (Symmetry Group: A_5). The dodecahedron has 20 vertices, and the icosahedron has 12 vertices.

Dodecahedron (20 vertices):

- Rotation by $\pm 2\pi/3$ (order 3): 5-cycle among vertices.
- Rotation by π (order 2): product of two 2-cycles.
- Rotation by $\pm 2\pi/5$ (order 5): 5-cycle among vertices.

Icosahedron (12 vertices):

- Rotation by $\pm 2\pi/3$ (order 3): 3-cycle among vertices.
- Rotation by π (order 2): product of two 2-cycles.
- Rotation by $\pm 2\pi/5$ (order 5): 5-cycle among vertices.

Exercise 2. Let G act on a set X. Show that for $x \in X$ and $g \in G$ we have $g\text{Stab}(x)g^{-1} = \text{Stab}(gx)$.

Solution

To prove this, we will show that $g\text{Stab}(x)g^{-1} \subseteq \text{Stab}(gx)$ and $\text{Stab}(gx) \subseteq g\text{Stab}(x)g^{-1}$.

Definition 0.1. The stabilizer of an element $x \in X$ under the action of G is defined as $\text{Stab}(x) = \{h \in G \mid h \cdot x = x\}$.

Proof.

- **Step 1: Show that $g\textbf{Stab}(x)g^{-1} \subseteq \textbf{Stab}(gx)$**

  Let $h \in \text{Stab}(x)$. By definition, this means $h \cdot x = x$. Consider an element of $g\text{Stab}(x)g^{-1}$, which is of the form ghg^{-1} for some $h \in \text{Stab}(x)$.

  $$(ghg^{-1}) \cdot (gx) = g(h(g^{-1} \cdot (gx))) = g(h \cdot x) = g \cdot x = gx$$

  Therefore, $ghg^{-1} \in \text{Stab}(gx)$, and hence $g\text{Stab}(x)g^{-1} \subseteq \text{Stab}(gx)$.

- **Step 2: Show that $\textbf{Stab}(gx) \subseteq g\textbf{Stab}(x)g^{-1}$**

  Let $k \in \text{Stab}(gx)$. By definition, this means $k \cdot (gx) = gx$. We need to show that $k \in g\text{Stab}(x)g^{-1}$, i.e., there exists some $h \in \text{Stab}(x)$ such that $k = ghg^{-1}$.

  Consider the element $g^{-1}kg \in G$. We apply it to x:

  $$(g^{-1}kg) \cdot x = g^{-1}(k \cdot (gx)) = g^{-1}(gx) = x$$

  Thus, $g^{-1}kg \in \text{Stab}(x)$, and let $h = g^{-1}kg$. Then $k = ghg^{-1}$. Therefore, $\text{Stab}(gx) \subseteq g\text{Stab}(x)g^{-1}$.

Since both inclusions are shown, we conclude that $g\text{Stab}(x)g^{-1} = \text{Stab}(gx)$. □

Exercise 3. Identify D_4 as the group of rotational symmetries of the square in the xy-plane with vertices $\{\pm e_1, \pm e_2\}$. Thus D_4 is a subgroup of $SO(3)$. Determine the orbits of the evident action by D_4 on $\mathbb{R}^3$.

Solution

The dihedral group D_4 consists of 8 elements that represent the symmetries of the square:

- 1 identity element
- 3 rotations by $90°, 180°, 270°$
- 4 reflections (over the x-axis, y-axis, and the two diagonals)

These elements form a subgroup of the special orthogonal group $SO(3)$.

When D_4 acts on $\mathbb{R}^3$, the orbits of the action are determined by how the group elements move points in $\mathbb{R}^3$:

- **Orbit of the origin (0,0,0):** The origin is fixed by all elements of D_4, so its orbit is $\{(0,0,0)\}$.
- **Orbits of points in the xy-plane (except the origin):** Points in the xy-plane are moved around within the plane according to the symmetries of the square. For example, the point $(1,0,0)$ has an orbit of 4 points $\{(1,0,0),(0,1,0),(-1,0,0),(0,-1,0)\}$ under the rotations, and additional reflections double the count to 8 unique points.
- **Orbits of points off the xy-plane:** Points not in the xy-plane are moved to other points not in the xy-plane, but their distance from the xy-plane (the z-coordinate) is preserved. For example, the point $(1,0,1)$ is rotated and reflected within planes parallel to the xy-plane. The orbit of $(1,0,1)$ consists of 8 points:
$\{(1,0,1),(0,1,1),(-1,0,1),(0,-1,1),(1,0,-1),(0,1,-1),(-1,0,-1),(0,-1,-1)\}$.

Exercise 4. Show that for a group G, the function $\tau : G \to \mathrm{Sym}(G)$ defined by $\tau(g)(x) := xg$ is a group action if and only if G is abelian.

Solution

To show that $\tau : G \to \mathrm{Sym}(G)$ defined by $\tau(g)(x) := xg$ is a group action if and only if G is abelian, we need to check the properties of a group action and the condition that G is abelian.

Definition 0.2. A function $\tau : G \to \mathrm{Sym}(G)$ is a group action if it satisfies the following properties:

(1) $\tau(e)(x) = xe = x$ for all $x \in G$, where e is the identity element in G.

(2) $\tau(gh)(x) = \tau(g)(\tau(h)(x))$ for all $x \in G$ and $g, h \in G$.

Proof.

- **If G is abelian:**

  Assume G is abelian. Then for any $g, h \in G$, we have $gh = hg$. Therefore,
  $$\tau(gh)(x) = x(gh) = x(hg) = (xh)g = \tau(g)(\tau(h)(x)).$$
  Thus, τ is a group action.

- **If τ is a group action:**

  Assume τ is a group action. Then for any $x \in G$ and $g, h \in G$,
  $$\tau(gh)(x) = \tau(g)(\tau(h)(x)) \implies x(gh) = (xg)h.$$
  This implies $x(gh) = x(hg)$. For this to hold for all $x \in G$, we must have $gh = hg$. Thus, G must be abelian.

Therefore, τ is a group action if and only if G is abelian. $\square$

Exercise 5. Let $n = 2k+1$ be an odd integer with $n \geq 3$. Describe all the conjugacy classes in D_n (there are $k+2$) and determine their sizes. Pick a representative from each class. For each of these representatives, describe the elements of its centralizer group.

Solution

Let D_n be the dihedral group with $n = 2k + 1$. The elements of D_n can be written as r^i and $r^i s$, where r is a rotation by $\frac{2\pi}{n}$ and s is a reflection. The conjugacy classes are as follows:

- The class containing the identity element e: $\{e\}$.
- The class containing the rotations r^i for $i = 1, \dots, k$: Each class has size $\frac{n}{\gcd(i,n)}$.
- The class containing the reflections s: This class has size n.

Representatives from each class:

- Identity: e
- Rotations: $r, r^2, \dots, r^k$
- Reflections: s

Centralizer groups:

- $C(e) = D_n$
- $C(r^i) = \{e, r^i\}$
- $C(s) = \{e, s\}$

Exercise 6. Let $n = 2k$ be an even integer with $n \geq 4$. Describe all the conjugacy classes in D_n (there are $k+3$) and determine their sizes. Pick a representative from each class. For each of these representatives, describe the elements of its centralizer group.

Solution

Let D_n be the dihedral group with $n = 2k$. The elements of D_n can be written as r^i and $r^i s$, where r is a rotation by $\frac{2\pi}{n}$ and s is a reflection. The conjugacy classes are as follows:

- The class containing the identity element e: $\{e\}$.
- The class containing the rotations r^i for $i = 1, \ldots, k-1$: Each class has size $\frac{n}{\gcd(i,n)}$.
- The class containing the rotation r^k: $\{r^k\}$.
- The class containing the reflections s and sr^k: Each class has size $\frac{n}{2}$.

Representatives from each class:

- Identity: e
- Rotations: $r, r^2, \ldots, r^{k-1}, r^k$
- Reflections: s, sr^k

Centralizer groups:

- $C(e) = D_n$
- $C(r^i) = \{e, r^i\}$
- $C(r^k) = \{e, r^k, s, sr^k\}$
- $C(s) = \{e, s\}$
- $C(sr^k) = \{e, sr^k\}$

Exercise 7. List the conjugacy classes in S_5 (there are 7) and determine their sizes. Pick a representative from each class. For each of these representatives, describe the elements of its centralizer group.

Solution

The conjugacy classes in S_5 (symmetric group on 5 elements) are determined by the cycle types of the permutations:

- (1) (identity): Size 1
- (12) (transposition): Size 10
- (123) (3-cycle): Size 20
- (12345) (5-cycle): Size 24
- (12)(34) (product of two transpositions): Size 15
- (1234) (4-cycle): Size 30
- (123)(45) (product of 3-cycle and transposition): Size 20

Representatives and their centralizers:

- Identity: e - Centralizer: S_5
- Transposition: (12) - Centralizer: $S_3 \times S_2$
- 3-cycle: (123) - Centralizer: $C_3 \times S_2$
- 5-cycle: (12345) - Centralizer: C_5
- Product of two transpositions: (12)(34) - Centralizer: $(S_2)^2$
- 4-cycle: (1234) - Centralizer: C_4
- Product of 3-cycle and transposition: (123)(45) - Centralizer: C_3

Exercise 8. Let G be a group with normal subgroup N. Show that if $\sigma \in N$, then $\mathrm{CL}_G(\sigma) \subseteq N$. Give an example to show that it is possible that $\mathrm{CL}_N(\sigma) \neq \mathrm{CL}_G(\sigma)$.

Solution

- **Show $\mathbf{CL}_G(\sigma) \subseteq N$:**

  Let $\sigma \in N$. Then the conjugacy class of σ in G is defined as $\mathrm{CL}_G(\sigma) = \{g\sigma g^{-1} \mid g \in G\}$.

  Since N is a normal subgroup, for any $g \in G$ and $\sigma \in N$, $g\sigma g^{-1} \in N$. Therefore, $\mathrm{CL}_G(\sigma) \subseteq N$.

- **Example where $\mathbf{CL}_N(\sigma) \neq \mathbf{CL}_G(\sigma)$:**

  Consider $G = S_3$ and $N = A_3$, the alternating group of even permutations. Let $\sigma = (123) \in N$.

  The conjugacy class of σ in N is $\mathrm{CL}_N(\sigma) = \{(123), (132)\}$.

  The conjugacy class of σ in G is $\mathrm{CL}_G(\sigma) = \{(123), (132)\}$, since (123) and (132) are the only 3-cycles in S_3.

  However, consider $\sigma = (12) \in N$. Then $\mathrm{CL}_N((12)) = \{(12)\}$, but $\mathrm{CL}_G((12)) = \{(12), (13), (23)\}$.