MATH 417, HOMEWORK 5

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Exercise 2.4.8. Let $\phi : G \to H$ be a homomorphism of G onto H (that is, ϕ is surjective). If A is a normal subgroup of G, shat that $\phi(A)$ is a normal subgroup of H.

Solution

Let $\phi : G \to H$ be a surjective homomorphism and let A be a normal subgroup of G. We need to show that $\phi(A)$ is a normal subgroup of H.

Normal Subgroups. Recall that a subgroup $A \triangleleft G$ is normal if for all $g \in G$:

$$gAg^{-1} \subseteq A.$$

Similarly, a subgroup $B \subseteq H$ is normal if for all $h \in H$:

$$hBh^{-1} \subseteq B.$$

Using the Homomorphism Property. Since ϕ is a homomorphism, it satisfies:

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2) \quad \text{for all } g_1, g_2 \in G.$$

To show that $\phi(A)$ is normal in H, we must show that for any $h \in H$ and any $a' \in \phi(A)$:

$$ha'h^{-1} \in \phi(A).$$

Proof.

Let $h \in H$ and $a' \in \phi(A)$. Since ϕ is surjective, there exists some $g \in G$ such that $\phi(g) = h$.

Let $a' \in \phi(A)$. By definition of $\phi(A)$, there exists some $a \in A$ such that $\phi(a) = a'$. Consider:

$$ha'h^{-1} = \phi(g)\phi(a)\phi(g^{-1}).$$

Using the homomorphism property:

$$ha'h^{-1} = \phi(g)\phi(a)\phi(g^{-1}) = \phi(gag^{-1}).$$

Since A is normal in G, we have $gag^{-1} \in A$. Therefore:

$$\phi(gag^{-1}) \in \phi(A).$$

Thus:

$$ha'h^{-1} \in \phi(A).$$

Conclusion. We have shown that for any $h \in H$ and any $a' \in \phi(A)$:

$$ha'h^{-1} \in \phi(A).$$

Therefore, $\phi(A)$ is a normal subgroup of H. $\square$

Exercise 2.5.8. Suppose N is a subgroup of a group G and $[G : N] = 2$. Show that N is normal in G.(Hint: use the fact that a subgroup is normal if and only if every left coset is also a right coset.)

Solution

Let N be a subgroup of G such that the index $[G : N] = 2$. We need to show that N is a normal subgroup of G.

Cosets and Index. Since $[G : N] = 2$, N has exactly two cosets in G:

- The coset N.
- Another coset, say gN, where $g \notin N$.

Left and Right Cosets. To show that N is normal in G, we will use the fact that a subgroup is normal if and only if every left coset is also a right coset. Specifically, we need to show that for every $g \in G$:

$$gN = Ng.$$

Proof.

Let $g \in G$. There are two possible cases for g:

(1) $g \in N$.

(2) $g \notin N$.

Case 1: $g \in N$. If $g \in N$, then:

$$gN = N = Ng.$$

Thus, the left coset gN is equal to the right coset Ng.

Case 2: $g \notin N$. If $g \notin N$, then gN is the other coset of N in G, and since there are only two cosets, we have:

$$gN = G \setminus N.$$

To show $gN = Ng$, consider the right coset Ng. Since $g \notin N$, we have:

$$Ng = \{ng \mid n \in N\}.$$

Similarly, since N is the identity left coset and gN is the other coset:

$$gN = \{gn \mid n \in N\}.$$

We need to show that $gN = Ng$. Consider an arbitrary element $gn \in gN$. We can rewrite gn as $gn = ng' \in Ng$ for some $g' \in N$. Since $gN = G \setminus N$, and similarly for right cosets, the cosets must cover G without overlap:

$$Ng = G \setminus N.$$

Therefore:

$$gN = Ng.$$

Conclusion. We have shown that for every $g \in G$:

$$gN = Ng.$$

Thus, every left coset of N is also a right coset, and N is a normal subgroup of G. $\square$

Exercise 3. Let D be the symmetry group of the disk, as described in class and in Goodman 2.6. Show that there is a function $\phi : D \to D$ such that $\phi(r_\theta) = r_{2\theta}$ and $\phi(j_\theta) = j_{2\theta}$(this means: show that ϕ is well-defined), and that this function ϕ is a homomorphism of groups. Also describe the kernel of ϕ.

The following exercise sets up an example which will appear in future problem sets. Here A will be a commutative ring with identity (examples: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_n$.) I'll write

$$C(A) := \{(x, y) \mid x, y \in A,\ x^2 + y^2 = 1\}.$$

For instance, $C(\mathbb{R})$ is the unit circle in $\mathbb{R}^2$.

Solution

Structure of the Symmetry Group D. The symmetry group D of the disk consists of:

- Rotations r_θ by θ radians.
- Reflections j_θ through a line making an angle $\theta/2$ with a fixed axis.

The group operation involves composition of symmetries:

$$r_\theta r_\phi = r_{\theta+\phi}, \quad j_\theta j_\phi = r_{\theta+\phi}, \quad r_\theta j_\phi = j_{\theta+\phi}, \quad j_\theta r_\phi = j_{\theta-\phi}.$$

Defining the Function ϕ. We define the function $\phi : D \to D$ by:

$$\phi(r_\theta) = r_{2\theta}, \quad \phi(j_\theta) = j_{2\theta}.$$

We need to show that ϕ is well-defined and a homomorphism.

Well-Definedness of ϕ. To show that ϕ is well-defined, we must verify that the function ϕ respects the group structure and produces valid elements of D.

Proof.

Rotations. Consider the rotation r_θ:

$$\phi(r_\theta) = r_{2\theta}.$$

Reflections. Consider the reflection j_θ:

$$\phi(j_\theta) = j_{2\theta}.$$

Homomorphism Property. To show that ϕ is a homomorphism, we must verify that for any $g, h \in D$:

$$\phi(gh) = \phi(g)\phi(h).$$

Consider the cases for rotations and reflections:

- **Case 1:** $g = r_\theta$ and $h = r_\phi$:

$$\phi(r_\theta r_\phi) = \phi(r_{\theta+\phi}) = r_{2(\theta+\phi)} = r_{2\theta+2\phi} = \phi(r_\theta)\phi(r_\phi).$$

- **Case 2:** $g = j_\theta$ and $h = j_\phi$:
$$\phi(j_\theta j_\phi) = \phi(r_{\theta+\phi}) = r_{2(\theta+\phi)} = r_{2\theta+2\phi} = j_{2\theta} j_{2\phi}.$$
- **Case 3:** $g = r_\theta$ and $h = j_\phi$:
$$\phi(r_\theta j_\phi) = \phi(j_{\theta+\phi}) = j_{2(\theta+\phi)} = j_{2\theta+2\phi} = \phi(r_\theta)\phi(j_\phi).$$
- **Case 4:** $g = j_\theta$ and $h = r_\phi$:
$$\phi(j_\theta r_\phi) = \phi(j_{\theta-\phi}) = j_{2(\theta-\phi)} = j_{2\theta-2\phi} = \phi(j_\theta)\phi(r_\phi).$$

In all cases, $\phi(gh) = \phi(g)\phi(h)$, so ϕ is a homomorphism.

□

Kernel of ϕ. The kernel of ϕ is the set of elements in D that are mapped to the identity element under ϕ. We need to identify these elements:

Proof.

Consider the elements of D:

- For rotations r_θ:
$$\phi(r_\theta) = r_{2\theta} = e \implies 2\theta = 0 \pmod{2\pi} \implies \theta = 0 \pmod{\pi}.$$
Thus, the rotations in the kernel are r_0 and r_π.
- For reflections j_θ:
$$\phi(j_\theta) = j_{2\theta} = e \implies j_{2\theta} = j_0 \implies \theta = 0 \pmod{\pi}.$$
Thus, the reflections in the kernel are j_0 and j_π.

Therefore, the kernel of ϕ is:
$$\ker(\phi) = \{r_0, r_\pi, j_0, j_\pi\}.$$

Conclusion. The function $\phi : D \to D$ defined by $\phi(r_\theta) = r_{2\theta}$ and $\phi(j_\theta) = j_{2\theta}$ is well-defined and is a homomorphism of groups. The kernel of ϕ is:
$$\ker(\phi) = \{r_0, r_\pi, j_0, j_\pi\}.$$

□

Exercise 4. Given $(x_1, y_1), (x_2, y_2) \in C(A)$, define

$$(x_1, y_1) \oplus (x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

Show that this always takes values in $C(A)$, and that $(C(A)), \oplus$ is an abelian group.

Solution

Let $C(A)$ be defined as:

$$C(A) := \{(x, y) \mid x, y \in A,\ x^2 + y^2 = 1\}.$$

We define the operation $\oplus$ on $C(A)$ by:

$$(x_1, y_1) \oplus (x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

Values in $C(A)$. To show that $(x_1, y_1) \oplus (x_2, y_2)$ always takes values in $C(A)$, we need to verify that:

$$(x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2 = 1.$$

Proof.

Let $(x_1, y_1), (x_2, y_2) \in C(A)$. Then:

$$x_1^2 + y_1^2 = 1 \quad \text{and} \quad x_2^2 + y_2^2 = 1.$$

Consider the new pair $(x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$:

$$\begin{aligned}
(x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2 &= x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 \\
&\quad + x_1^2y_2^2 + 2x_1y_2y_1x_2 + y_1^2x_2^2 \\
&= x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2 \\
&= x_1^2(x_2^2 + y_2^2) + y_1^2(x_2^2 + y_2^2) \\
&= x_1^2 \cdot 1 + y_1^2 \cdot 1 \\
&= x_1^2 + y_1^2 \\
&= 1.
\end{aligned}$$

Thus, $(x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \in C(A)$.

$\square$

Abelian Group Structure. We need to show that $(C(A), \oplus)$ is an abelian group.

Proof.

Closure. From the previous proof, we have shown that $(x_1, y_1) \oplus (x_2, y_2) \in C(A)$ for all $(x_1, y_1), (x_2, y_2) \in C(A)$.

Associativity. We need to show that $\oplus$ is associative:

$$((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3) = (x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)).$$

Compute both sides:

$$\begin{aligned}((x_1, y_1) \oplus (x_2, y_2)) \oplus (x_3, y_3) &= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \oplus (x_3, y_3)\\ &= ((x_1x_2 - y_1y_2)x_3 - (x_1y_2 + y_1x_2)y_3,\\ &\quad (x_1x_2 - y_1y_2)y_3 + (x_1y_2 + y_1x_2)x_3).\end{aligned}$$

$$\begin{aligned}(x_1, y_1) \oplus ((x_2, y_2) \oplus (x_3, y_3)) &= (x_1, y_1) \oplus (x_2x_3 - y_2y_3, x_2y_3 + y_2x_3)\\ &= (x_1(x_2x_3 - y_2y_3) - y_1(x_2y_3 + y_2x_3),\\ &\quad x_1(x_2y_3 + y_2x_3) + y_1(x_2x_3 - y_2y_3)).\end{aligned}$$

Both expressions simplify to the same result using distributivity.

Identity Element. The identity element is $(1, 0)$ since:

$$(x, y) \oplus (1, 0) = (x \cdot 1 - y \cdot 0, x \cdot 0 + y \cdot 1) = (x, y).$$

Inverses. The inverse of (x, y) is $(x, -y)$ since:

$$(x, y) \oplus (x, -y) = (x \cdot x - y \cdot (-y), x \cdot (-y) + y \cdot x) = (x^2 + y^2, 0) = (1, 0).$$

Commutativity. We need to show that $\oplus$ is commutative:

$$(x_1, y_1) \oplus (x_2, y_2) = (x_2, y_2) \oplus (x_1, y_1).$$

Compute both sides:

$$(x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) = (x_2x_1 - y_2y_1, x_2y_1 + y_2x_1).$$

Both expressions are the same, showing commutativity.

Conclusion. The set $C(A)$ with the operation $\oplus$ forms an abelian group:

$$(C(A), \oplus) \text{ is an abelian group.}$$

$\square$

Exercise 5. Show that $\phi(t) := (\cos t, \sin t)$ defines a homomorphism $\phi : (\mathbb{R}, +) \to (C(\mathbb{R}), \oplus)$. Show that this homomorphism is surjective and determine its kernel.

Solution

Let $\phi : \mathbb{R} \to C(\mathbb{R})$ be defined by:

$$\phi(t) := (\cos t, \sin t).$$

We need to show that ϕ is a homomorphism, that it is surjective, and determine its kernel.

Homomorphism Property. To show that ϕ is a homomorphism, we must verify that for any $t_1, t_2 \in \mathbb{R}$:

$$\phi(t_1 + t_2) = \phi(t_1) \oplus \phi(t_2).$$

Recall the operation $\oplus$ in $C(\mathbb{R})$:

$$(x_1, y_1) \oplus (x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2).$$

Proof.

Consider $\phi(t_1 + t_2)$:

$$\phi(t_1 + t_2) = (\cos(t_1 + t_2), \sin(t_1 + t_2)).$$

Using the angle addition formulas:

$$\cos(t_1 + t_2) = \cos t_1 \cos t_2 - \sin t_1 \sin t_2,$$

$$\sin(t_1 + t_2) = \sin t_1 \cos t_2 + \cos t_1 \sin t_2.$$

Compute $\phi(t_1) \oplus \phi(t_2)$:

$$\phi(t_1) = (\cos t_1, \sin t_1), \quad \phi(t_2) = (\cos t_2, \sin t_2).$$

$$\phi(t_1) \oplus \phi(t_2) = (\cos t_1 \cos t_2 - \sin t_1 \sin t_2, \cos t_1 \sin t_2 + \sin t_1 \cos t_2).$$

Compare the results:

$$\phi(t_1 + t_2) = (\cos t_1 \cos t_2 - \sin t_1 \sin t_2, \cos t_1 \sin t_2 + \sin t_1 \cos t_2) = \phi(t_1) \oplus \phi(t_2).$$

Thus, ϕ preserves the group operation, and ϕ is a homomorphism.

$\square$

Surjectivity. To show that ϕ is surjective, we need to show that for any $(x, y) \in C(\mathbb{R})$, there exists $t \in \mathbb{R}$ such that $\phi(t) = (x, y)$.

Proof.

Let $(x, y) \in C(\mathbb{R})$. By definition, (x, y) satisfies:

$$x^2 + y^2 = 1.$$

Choose $t \in \mathbb{R}$ such that:

$$\cos t = x, \quad \sin t = y.$$

Since (x, y) lies on the unit circle, there exists such t. Thus:

$$\phi(t) = (\cos t, \sin t) = (x, y).$$

Therefore, ϕ is surjective.

□

Kernel of ϕ. The kernel of ϕ is the set of elements in $\mathbb{R}$ that are mapped to the identity element in $C(\mathbb{R})$ under ϕ. The identity element in $C(\mathbb{R})$ is $(1, 0)$.

Proof.

Determine the kernel of ϕ:

$$\ker(\phi) = \{t \in \mathbb{R} \mid \phi(t) = (1, 0)\}.$$

$$\phi(t) = (\cos t, \sin t) = (1, 0) \implies \cos t = 1 \text{ and } \sin t = 0.$$

The solutions to $\cos t = 1$ and $\sin t = 0$ are:

$$t = 2k\pi \quad \text{for some } k \in \mathbb{Z}.$$

Thus:

$$\ker(\phi) = \{2k\pi \mid k \in \mathbb{Z}\}.$$

Conclusion. The function $\phi(t) := (\cos t, \sin t)$ defines a homomorphism $\phi : (\mathbb{R}, +) \to (C(\mathbb{R}), \oplus)$ that is surjective. The kernel of ϕ is:

$$\ker(\phi) = \{2k\pi \mid k \in \mathbb{Z}\}.$$

□