MATH 417, HOMEWORK 7

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Exercise 1. Write $Z = \{cI \mid c \in \mathbb{R}^x\}$ for the subgroup of non-zero multiples of the identity matrix in $GL_n(\mathbb{R})$. Show the following:

a. If n is odd, then $SL_n(\mathbb{R})Z = GL_n(\mathbb{R})$ and $SL_n(\mathbb{R}) \cap Z = \{I\}$, and therefore $PGL_n(\mathbb{R}) \simeq SL_n(\mathbb{R})$

b. If n is even, then $SL_n(\mathbb{R}) = GL_n^+(\mathbb{R})$, the matrices A with $\det A > 0$, and that $SL_n(\mathbb{R}) \cap Z = \{\pm I\}$. Conclude that $PGL_n(\mathbb{R})$ contains an index 2 subgroup which is isomorphic to $SL_n(\mathbb{R})/\{\pm I\}$.

Solution

a. $SL_n(\mathbb{R})Z = GL_n(\mathbb{R})$ **for odd** n**.** Let $A \in GL_n(\mathbb{R})$. We need to show that A can be written as a product of an element in $SL_n(\mathbb{R})$ and an element in Z.

Proof.

Consider the matrix $c^{-1}A$, where $c = \det(A)^{1/n}$. Note that:

$$\det(c^{-1}A) = c^{-n} \det(A) = (\det(A)^{1/n})^{-n} \det(A) = 1.$$

Thus, $c^{-1}A \in SL_n(\mathbb{R})$.

Therefore, $A = (c^{-1}A) \cdot cI$, where $c^{-1}A \in SL_n(\mathbb{R})$ and $cI \in Z$. This shows that $A \in SL_n(\mathbb{R})Z$. Since $A \in GL_n(\mathbb{R})$ was arbitrary, we have $GL_n(\mathbb{R}) = SL_n(\mathbb{R})Z$.

Next, we need to show that $SL_n(\mathbb{R}) \cap Z = \{I\}$.

Let $A \in SL_n(\mathbb{R}) \cap Z$. Then $A = cI$ for some $c \in \mathbb{R}^*$, and $\det(A) = 1$. Thus,

$$\det(cI) = c^n = 1.$$

Since n is odd, $c = 1$. Therefore, $A = I$, and $SL_n(\mathbb{R}) \cap Z = \{I\}$.

Hence, $PGL_n(\mathbb{R}) = GL_n(\mathbb{R})/Z \cong SL_n(\mathbb{R})$.

□

b. $SL_n(\mathbb{R}) = GL_n^+(\mathbb{R})$ **for even** n**.** Let n be even. We need to show that $SL_n(\mathbb{R}) = GL_n^+(\mathbb{R})$, the group of matrices $A \in GL_n(\mathbb{R})$ with $\det A > 0$.

Proof.

Consider any matrix $A \in GL_n^+(\mathbb{R})$. We need to show that A can be written as a product of an element in $SL_n(\mathbb{R})$ and an element in Z.

Let $c = \det(A)^{1/n}$. Then $c > 0$, and consider the matrix $c^{-1}A$. We have:

$$\det(c^{-1}A) = c^{-n}\det(A) = (\det(A)^{1/n})^{-n}\det(A) = 1.$$

Thus, $c^{-1}A \in SL_n(\mathbb{R})$.

Therefore, $A = (c^{-1}A) \cdot cI$, where $c^{-1}A \in SL_n(\mathbb{R})$ and $cI \in Z$. This shows that $A \in SL_n(\mathbb{R})Z$. Since $A \in GL_n^+(\mathbb{R})$ was arbitrary, we have $GL_n^+(\mathbb{R}) = SL_n(\mathbb{R})Z$.

Next, we need to show that $SL_n(\mathbb{R}) \cap Z = \{\pm I\}$.

Let $A \in SL_n(\mathbb{R}) \cap Z$. Then $A = cI$ for some $c \in \mathbb{R}^*$, and $\det(A) = 1$. Thus,

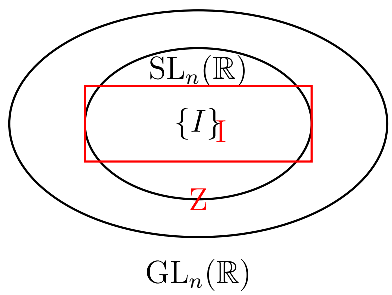
$$\det(cI) = c^n = 1.$$

Since n is even, $c^n = 1$ implies $c = \pm 1$. Therefore, $A = I$ or $A = -I$, and $SL_n(\mathbb{R}) \cap Z = \{\pm I\}$.

Hence, $PGL_n(\mathbb{R}) = GL_n(\mathbb{R})/Z$ contains an index 2 subgroup isomorphic to $SL_n(\mathbb{R})/\{\pm I\}$.

$\square$

Visualization.



Exercise 2.4.17. An automorphism of a group G is an isomorphism $G \to G$ from the group to itself. Fix $g \in G$ and show that the function $c_g : G \to G$ defined by $c_g(x) := gxg^{-1}$ is an automorphism of G.

Solution

To show that $c_g : G \to G$ defined by $c_g(x) := gxg^{-1}$ is an automorphism of G, we need to verify two things:

(1) c_g is a homomorphism.

(2) c_g is bijective.

(1) c_g is a homomorphism. We need to show that for all $x, y \in G$:

$$c_g(xy) = c_g(x)c_g(y).$$

Proof.

Let $x, y \in G$. Then:

$$c_g(xy) = g(xy)g^{-1}.$$

Using the associative property of group multiplication:

$$g(xy)g^{-1} = (gx)(yg^{-1}).$$

Note that $yg^{-1} = (g^{-1})^{-1}y = g^{-1}gyg^{-1}$. Thus:

$$(gx)(yg^{-1}) = (gx)(g^{-1}gyg^{-1}) = gxg^{-1}gyg^{-1}.$$

Simplifying further:

$$gxg^{-1}gyg^{-1} = (gxg^{-1})(gyg^{-1}).$$

Therefore:

$$c_g(xy) = c_g(x)c_g(y).$$

Thus, c_g is a homomorphism.

□

(2) c_g is bijective. We need to show that c_g is both injective and surjective.

Injectivity. To show injectivity, we need to show that if $c_g(x) = c_g(y)$, then $x = y$.

Proof.

Suppose $c_g(x) = c_g(y)$. Then:

$$gxg^{-1} = gyg^{-1}.$$

Multiplying both sides on the left by g^{-1} and on the right by g, we get:

$$g^{-1}(gxg^{-1})g = g^{-1}(gyg^{-1})g.$$

Simplifying:

$$x = y.$$

Thus, c_g is injective.

□

Surjectivity. To show surjectivity, we need to show that for every $y \in G$, there exists an $x \in G$ such that $c_g(x) = y$.

Proof.

Let $y \in G$. We need to find $x \in G$ such that:

$$gxg^{-1} = y.$$

Multiplying both sides on the left by g^{-1} and on the right by g, we get:

$$g^{-1}(gxg^{-1})g = g^{-1}yg.$$

Simplifying:

$$x = g^{-1}yg.$$

Let $x = g^{-1}yg$. Then:

$$c_g(x) = c_g(g^{-1}yg) = g(g^{-1}yg)g^{-1} = y.$$

Thus, c_g is surjective.

□

Visualization. Visualization of the conjugation of x by g:

$$\begin{tikzcd} & g \arrow[dr, "g^{-1}"] & \\ x \arrow[ur, "g"] \arrow[rr, "gxg^{-1}"] & & gxg^{-1} \end{tikzcd}$$

$$g^{-1}$$

Conclusion. Since c_g is both a homomorphism and bijective, it is an automorphism of G. Therefore, $c_g : G \to G$ defined by $c_g(x) := gxg^{-1}$ is an automorphism of G.

Exercise 2.5.3. The center of a group G is the set

$$Z(G) := \{a \in G \mid ag = ga \ \forall\ g \in G\}$$

of elements which commute with every element of G. Show that $Z(G)$ is a normal subgroup of G.

Solution

To show that $Z(G)$, the center of a group G, is a normal subgroup of G, we need to verify two things:

(1) $Z(G)$ is a subgroup of G.

(2) $Z(G)$ is normal in G.

(1) $Z(G)$ is a Subgroup of G. We will use the subgroup criteria to show that $Z(G)$ is a subgroup of G:

- The identity element $e \in G$ is in $Z(G)$ since $eg = ge = g$ for all $g \in G$.
- If $a, b \in Z(G)$, then $ab \in Z(G)$ because for any $g \in G$,
  $$(ab)g = a(bg) = a(gb) = (ag)b = (ga)b = g(ab).$$
  Thus, ab commutes with every element of G, so $ab \in Z(G)$.
- If $a \in Z(G)$, then $a^{-1} \in Z(G)$ because for any $g \in G$,
  $$a^{-1}g = a^{-1}(ga^{-1})a = (a^{-1}a)g(a^{-1}a)^{-1} = g(a^{-1}a)^{-1} = g(a^{-1}).$$
  Thus, a^{-1} commutes with every element of G, so $a^{-1} \in Z(G)$.

Since $Z(G)$ contains the identity element, is closed under the group operation, and contains the inverses of its elements, $Z(G)$ is a subgroup of G.

(2) $Z(G)$ is Normal in G. We need to show that $Z(G)$ is invariant under conjugation by any element of G. That is, for any $g \in G$ and $a \in Z(G)$, we have $gag^{-1} \in Z(G)$.

Proof.

Let $a \in Z(G)$. This means a commutes with every element of G. Let $g \in G$. We need to show that $gag^{-1} \in Z(G)$. That is, gag^{-1} commutes with every element of G.

Let $h \in G$. Then:

$$(gag^{-1})h = ga(g^{-1}h) = g(ag^{-1}h) = g(g^{-1}ha) = (gg^{-1})ha = ha.$$

Since a commutes with h, we have:

$$ha = ah.$$

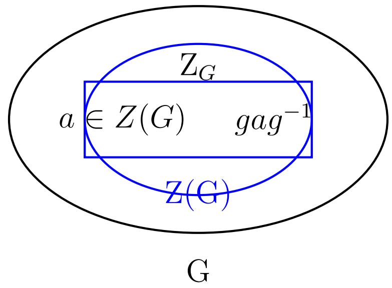
Thus:

$$(gag^{-1})h = ha = ah = h(gag^{-1}).$$

Therefore, gag^{-1} commutes with every element of G, so $gag^{-1} \in Z(G)$.

□

Visualization.



Conclusion. Since $Z(G)$ is a subgroup of G and is invariant under conjugation by any element of G, $Z(G)$ is a normal subgroup of G.

Exercise 2.7.6. Denote the set of all automorphisms of G by $\text{Aut}(G)$.

a. Show that $\text{Aut}(G)$ is a group, with the operation of composition of functions.

b. Show that the function $c : G \to \text{Aut}(G)$ defined by $g \mapsto c_g$ is a homomorphism of groups.

c. Show that the kernel of c is $Z(G)$.

d. The image of c is called the group of inner automorphisms of G and denoted $\text{Inn}(G)$. Show that $\text{Inn}(G) \cong G/Z(G)$.

Solution

a. Aut(G) is a Group. To show that $\text{Aut}(G)$ is a group, we need to verify that it satisfies the group axioms with respect to the operation of composition of functions.

Proof.

- **Closure**: If $\alpha, \beta \in \text{Aut}(G)$, then $\alpha \circ \beta \in \text{Aut}(G)$. This is because the composition of two automorphisms is an automorphism.
- **Associativity**: Function composition is associative, so for any $\alpha, \beta, \gamma \in \text{Aut}(G)$,
$$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma).$$
- **Identity Element**: The identity automorphism $id_G : G \to G$ defined by $id_G(g) = g$ for all $g \in G$ is in $\text{Aut}(G)$ and serves as the identity element for composition:
$$id_G \circ \alpha = \alpha \quad \text{and} \quad \alpha \circ id_G = \alpha \quad \text{for all } \alpha \in \text{Aut}(G).$$
- **Inverse Element**: For each $\alpha \in \text{Aut}(G)$, there exists an inverse automorphism $\alpha^{-1} \in \text{Aut}(G)$ such that
$$\alpha \circ \alpha^{-1} = id_G \quad \text{and} \quad \alpha^{-1} \circ \alpha = id_G.$$

Therefore, $\text{Aut}(G)$ is a group.

□

b. The Function $c : G \to$ Aut(G) is a Homomorphism. Define the function $c : G \to \text{Aut}(G)$ by $c_g(x) = gxg^{-1}$. We need to show that c is a homomorphism of groups.

Proof.

Let $g, h \in G$. We need to show that $c_{gh} = c_g \circ c_h$. For any $x \in G$,
$$c_{gh}(x) = (gh)x(gh)^{-1}.$$

On the other hand,
$$(c_g \circ c_h)(x) = c_g(c_h(x)) = c_g(hxh^{-1}) = g(hxh^{-1})g^{-1} = (gh)x(h^{-1}g^{-1}) = (gh)x(gh)^{-1}.$$

Thus,

$$c_{gh}(x) = (c_g \circ c_h)(x) \quad \text{for all } x \in G.$$

Therefore, $c_{gh} = c_g \circ c_h$ and c is a homomorphism.

$\square$

c. The Kernel of c is $Z(G)$. The kernel of c consists of all elements $g \in G$ such that c_g is the identity automorphism.

Proof.

We need to show that

$$\ker(c) = \{g \in G \mid c_g(x) = x \text{ for all } x \in G\}.$$

By definition,

$$c_g(x) = gxg^{-1} = x \quad \text{for all } x \in G.$$

This implies that g commutes with every element of G. Hence,

$$\ker(c) = Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}.$$

$\square$

d. $\mathbf{Inn}(G) \cong G/Z(G)$. The image of c is called the group of inner automorphisms of G and denoted $\text{Inn}(G)$. We need to show that $\text{Inn}(G) \cong G/Z(G)$.

Proof.

Consider the homomorphism $c : G \to \text{Inn}(G)$ defined by $c_g(x) = gxg^{-1}$. By the First Isomorphism Theorem,

$$G/\ker(c) \cong \text{Im}(c).$$

From part (c), we know that $\ker(c) = Z(G)$. Thus,

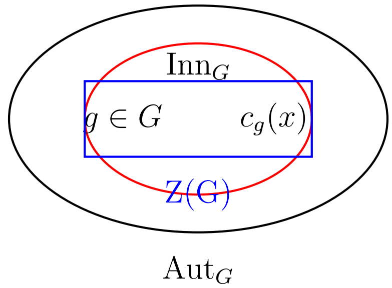
$$G/Z(G) \cong \text{Im}(c) = \text{Inn}(G).$$

Therefore,

$$\text{Inn}(G) \cong G/Z(G).$$

$\square$

Visualization. Visualization of the automorphisms and inner automorphisms:



Exercise 5. The purpose of this exercise is to compute the automorphism group of a finite cyclic group. Let $G = \langle g \rangle$ be a finite cyclic group of order n.

- **a.** Show that for any integer $k \in \mathbb{Z}$, the function $p_k : G \to G$ defined by $p_k(x) := x^k$ defines a homomorphism from G to itself.
- **b.** Show that p_k is an isomorphism if and only if $\gcd(k, n) = 1$.
- **c.** Show that $[k]_n \mapsto p_k$ defines an isomorphism of groups $\Phi(n) \to \mathrm{Aut}(G)$.

Solution

a. p_k **defines a homomorphism from** G **to itself.** Let $G = \langle g \rangle$ be a finite cyclic group of order n. We need to show that for any integer $k \in \mathbb{Z}$, the function $p_k : G \to G$ defined by $p_k(x) := x^k$ is a homomorphism.

Proof.

For any $x, y \in G$, we have:

$$p_k(xy) = (xy)^k.$$

Since G is abelian (all cyclic groups are abelian), we can write:

$$(xy)^k = x^k y^k.$$

Thus,

$$p_k(xy) = x^k y^k = p_k(x)p_k(y).$$

Therefore, p_k is a homomorphism from G to itself.

□

b. p_k **is an isomorphism if and only if** $\gcd(k, n) = 1$**.** We need to show that p_k is an isomorphism if and only if $\gcd(k, n) = 1$.

Proof.

p_k is an isomorphism if and only if it is bijective.

Surjectivity. We first show that p_k is surjective. For any $x \in G$, there exists some $y \in G$ such that:

$$p_k(y) = y^k = x.$$

Since G is a finite cyclic group, this equation has a solution if and only if $\gcd(k, n) = 1$. This is because for each $x \in G$, x can be written as g^m for some $m \in \{0, 1, \ldots, n-1\}$. Therefore, we need $g^{km} = g^m$, which is possible if and only if k is relatively prime to n.

Injectivity. We now show that p_k is injective. Suppose $p_k(x) = p_k(y)$ for $x, y \in G$. This implies:

$$x^k = y^k.$$

Since G is cyclic, x and y can be written as $x = g^a$ and $y = g^b$ for some integers a, b. Thus,

$$(g^a)^k = (g^b)^k \implies g^{ak} = g^{bk} \implies ak \equiv bk \pmod{n}.$$

Since $\gcd(k, n) = 1$, we can divide both sides by k modulo n, yielding:

$$a \equiv b \pmod{n} \implies x = y.$$

Thus, p_k is injective.

Since p_k is both injective and surjective, it is an isomorphism if and only if $\gcd(k, n) = 1$.

□

c. $[k]_n \mapsto p_k$ defines an isomorphism of groups $\Phi(n) \to$ Aut(G). We need to show that the map $[k]_n \mapsto p_k$ defines an isomorphism from $\Phi(n)$ to $\mathrm{Aut}(G)$.

Proof.

Define the map $\Phi : \Phi(n) \to \mathrm{Aut}(G)$ by $\Phi([k]_n) = p_k$.

Homomorphism. We first show that Φ is a homomorphism. For any $[k]_n, [m]_n \in \Phi(n)$, we have:

$$\Phi([k]_n[m]_n) = \Phi([km]_n) = p_{km}.$$

Since $p_k \circ p_m = p_{km}$, we have:

$$\Phi([k]_n)\Phi([m]_n) = p_k p_m = p_{km} = \Phi([km]_n).$$

Thus, Φ is a homomorphism.

Injectivity. We now show that Φ is injective. Suppose $\Phi([k]_n) = \Phi([m]_n)$. This implies $p_k = p_m$, meaning $x^k = x^m$ for all $x \in G$.

Since G is cyclic, this implies $k \equiv m \pmod{n}$. Therefore, $[k]_n = [m]_n$ in $\Phi(n)$, showing that Φ is injective.

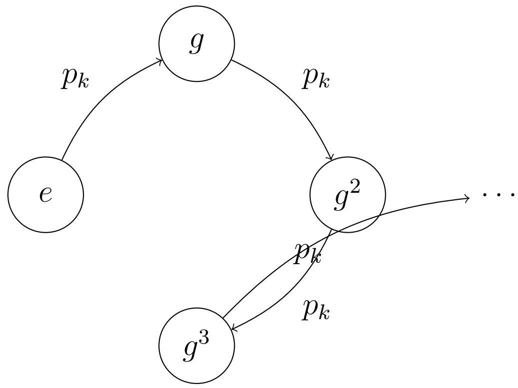
Surjectivity. Finally, we show that Φ is surjective. For any automorphism $\psi \in \mathrm{Aut}(G)$, there exists an integer k such that $\psi = p_k$ because $\psi(g) = g^k$ for some $k \in \{1, 2, \ldots, n-1\}$.

Since ψ is an automorphism, $\gcd(k, n) = 1$. Therefore, $[k]_n \in \Phi(n)$ and $\Phi([k]_n) = p_k = \psi$.

Thus, Φ is surjective.

Since Φ is a bijective homomorphism, it is an isomorphism. Therefore, $[k]_n \mapsto p_k$ defines an isomorphism of groups $\Phi(n) \to \mathrm{Aut}(G)$.

□



$$\Phi(n) \xrightarrow{\quad\Phi\quad} \mathrm{Aut}(G)$$

FIGURE 1. Visualization of Homomorphism and Isomorphism

Exercise 3.1.9. Show that the direct product of $A \times B$ is abelian if and only if A and B are abelian.

SOLUTION

Let A and B be groups, and consider their direct product $A \times B$. We will show that $A \times B$ is abelian if and only if both A and B are abelian.

Necessity: If $A \times B$ is abelian, then A and B are abelian. Suppose $A \times B$ is abelian. This means that for any $(a_1, b_1), (a_2, b_2) \in A \times B$,

$$(a_1, b_1)(a_2, b_2) = (a_2, b_2)(a_1, b_1).$$

The group operation in $A \times B$ is defined component-wise. Therefore,

$$(a_1a_2, b_1b_2) = (a_2a_1, b_2b_1).$$

This implies

$$a_1a_2 = a_2a_1 \quad \text{and} \quad b_1b_2 = b_2b_1 \quad \text{for all } a_1, a_2 \in A \text{ and } b_1, b_2 \in B.$$

Therefore, both A and B are abelian.

Sufficiency: If A and B are abelian, then $A \times B$ is abelian. Suppose A and B are abelian. This means that for any $a_1, a_2 \in A$,

$$a_1a_2 = a_2a_1,$$

and for any $b_1, b_2 \in B$,

$$b_1b_2 = b_2b_1.$$

Consider any two elements $(a_1, b_1), (a_2, b_2) \in A \times B$. The group operation in $A \times B$ is defined component-wise, so

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2).$$

Since A and B are abelian,

$$a_1 a_2 = a_2 a_1 \quad \text{and} \quad b_1 b_2 = b_2 b_1.$$

Therefore,

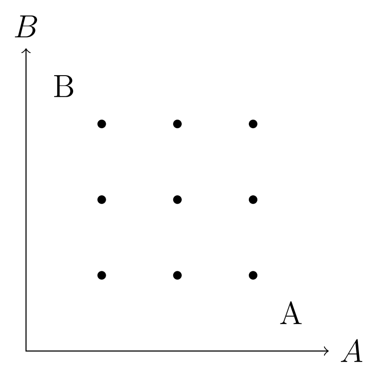
$$(a_1 a_2, b_1 b_2) = (a_2 a_1, b_2 b_1) = (a_2, b_2)(a_1, b_1).$$

This shows that

$$(a_1, b_1)(a_2, b_2) = (a_2, b_2)(a_1, b_1) \quad \text{for all } (a_1, b_1), (a_2, b_2) \in A \times B.$$

Thus, $A \times B$ is abelian.

Visualization. Visualization of the direct product $A \times B$:



Conclusion. The direct product $A \times B$ is abelian if and only if both A and B are abelian.

Exercise 3.1.14. Show that $\mathbb{Z}_4 \times \mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. (See hint in book.)

Solution

To show that $\mathbb{Z}_4 \times \mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we will count the number of elements of order 4 in each group and show that they differ.

Elements of Order 4 in $\mathbb{Z}_4 \times \mathbb{Z}_4$. The group $\mathbb{Z}_4 \times \mathbb{Z}_4$ consists of pairs (a, b) where $a, b \in \mathbb{Z}_4$. An element $(a, b) \in \mathbb{Z}_4 \times \mathbb{Z}_4$ has order 4 if and only if the least common multiple of the orders of a and b is 4. The possible orders of elements in $\mathbb{Z}_4$ are 1, 2, and 4.

- For an element (a, b) to have order 4, either a or b must have order 4, and the other must have order 1 or 4.

Case 1: a has order 4. If a has order 4, then a can be 1 element ($a = 1$ or $a = 3$) because there are 2 elements of order 4 in $\mathbb{Z}_4$.

- b must have order 1 or 4:
  - There are 2 elements of order 4 ($b = 1$ or $b = 3$).
  - There is 1 element of order 1 ($b = 0$).

Therefore, for each a of order 4, there are $2+1 = 3$ choices for b, and there are 2 possible values for a, resulting in $2 \times 3 = 6$ elements.

Case 2: b has order 4. This is similar to Case 1, and it also gives 6 elements.

Therefore, the total number of elements of order 4 in $\mathbb{Z}_4 \times \mathbb{Z}_4$ is $6 + 6 = 12$.

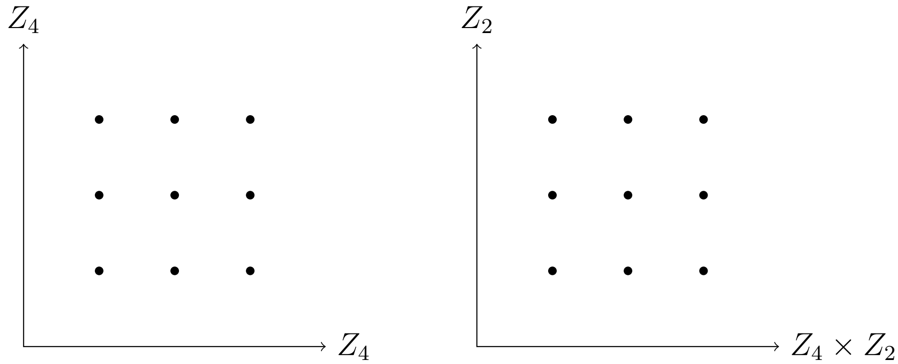
Elements of Order 4 in $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The group $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ consists of triples (a, b, c) where $a \in \mathbb{Z}_4$ and $b, c \in \mathbb{Z}_2$. An element $(a, b, c) \in \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has order 4 if and only if a has order 4.

- There are 2 elements in $\mathbb{Z}_4$ with order 4 ($a = 1$ or $a = 3$).
- Each element $b \in \mathbb{Z}_2$ has order 2 or 1, and similarly for $c \in \mathbb{Z}_2$. There are $2 \times 2 = 4$ combinations of b and c.

Therefore, for each a of order 4, there are 4 choices for (b, c), resulting in $2 \times 4 = 8$ elements of order 4.

Conclusion. The number of elements of order 4 in $\mathbb{Z}_4 \times \mathbb{Z}_4$ is 12, while the number of elements of order 4 in $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is 8. Since the number of elements of order 4 differs in the two groups, they cannot be isomorphic.

Visualization. Visualization of the elements of order 4:



Exercise 3.1.15. Let K_1 be a normal subgroup of G_1, and K_2 a normal subgroup of G_2. Show that $K_1 \times K_2$ is a normal subgroup of $G_1 \times G_2$, and that

$$(G_1 \times G_2)/(K_1 \times K_2) \cong (G_1/K_1) \times (G_2/K_2)$$

Solution

Normality of $K_1 \times K_2$ in $G_1 \times G_2$. Let K_1 be a normal subgroup of G_1, and K_2 be a normal subgroup of G_2. We want to show that $K_1 \times K_2$ is a normal subgroup of $G_1 \times G_2$.

Proof.

Consider any element $(g_1, g_2) \in G_1 \times G_2$ and any element $(k_1, k_2) \in K_1 \times K_2$. We need to show that:

$$(g_1, g_2)(k_1, k_2)(g_1, g_2)^{-1} \in K_1 \times K_2.$$

Calculate the product:

$$(g_1, g_2)(k_1, k_2)(g_1, g_2)^{-1} = (g_1k_1, g_2k_2)(g_1^{-1}, g_2^{-1}) = (g_1k_1g_1^{-1}, g_2k_2g_2^{-1}).$$

Since K_1 is normal in G_1, we have $g_1k_1g_1^{-1} \in K_1$. Similarly, since K_2 is normal in G_2, we have $g_2k_2g_2^{-1} \in K_2$.

Therefore:

$$(g_1k_1g_1^{-1}, g_2k_2g_2^{-1}) \in K_1 \times K_2.$$

Thus, $K_1 \times K_2$ is a normal subgroup of $G_1 \times G_2$.

□

Isomorphism $(G_1 \times G_2)/(K_1 \times K_2) \cong (G_1/K_1) \times (G_2/K_2)$. We want to show that:

$$(G_1 \times G_2)/(K_1 \times K_2) \cong (G_1/K_1) \times (G_2/K_2).$$

Define the map $\phi : (G_1 \times G_2) \to (G_1/K_1) \times (G_2/K_2)$ by:

$$\phi(g_1, g_2) = (g_1K_1, g_2K_2).$$

Proof.

First, we show that ϕ is a homomorphism. For any $(g_1, g_2), (h_1, h_2) \in G_1 \times G_2$:

$$\phi((g_1, g_2)(h_1, h_2)) = \phi(g_1h_1, g_2h_2) = (g_1h_1K_1, g_2h_2K_2).$$

Since $g_1h_1K_1 = (g_1K_1)(h_1K_1)$ and $g_2h_2K_2 = (g_2K_2)(h_2K_2)$, we have:

$$(g_1h_1K_1, g_2h_2K_2) = (g_1K_1, g_2K_2)(h_1K_1, h_2K_2) = \phi(g_1, g_2)\phi(h_1, h_2).$$

Thus, ϕ is a homomorphism.

Next, we show that ϕ is surjective. For any $(g_1K_1, g_2K_2) \in (G_1/K_1) \times (G_2/K_2)$, there exist $g_1 \in G_1$ and $g_2 \in G_2$ such that:

$$\phi(g_1, g_2) = (g_1K_1, g_2K_2).$$

Therefore, ϕ is surjective.

Finally, we show that $\ker(\phi) = K_1 \times K_2$. For any $(g_1, g_2) \in G_1 \times G_2$:

$$\phi(g_1, g_2) = (K_1, K_2) \implies g_1K_1 = K_1 \text{ and } g_2K_2 = K_2 \implies g_1 \in K_1 \text{ and } g_2 \in K_2.$$

Thus, $\ker(\phi) = K_1 \times K_2$.

By the First Isomorphism Theorem, we have:

$$(G_1 \times G_2)/(K_1 \times K_2) \cong \phi(G_1 \times G_2) = (G_1/K_1) \times (G_2/K_2).$$

□

Visualization. Visualization of the isomorphism: