Homework Three
Two-Period Consumption-Saving Model

Team Two

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1 The Model

In this assignment, we explore a simple two-period consumption-saving model. This framework is foundational in understanding intertemporal choice and optimal consumption over time. The setup assumes a representative household that makes consumption and saving decisions over two periods with given income and a fixed interest rate.

Setup

- Two periods: Period 1 and Period 2.

- Discrete-time model.

- A single representative household as the decision-maker.

- Income is exogenously given: Y_1 for Period 1 and Y_2 for Period 2.

- In Period 1, the household allocates income Y_1 between consumption C_1 and savings S_1.

- In Period 2, the household has no further savings ($S_2 = 0$) and consumes out of income Y_2 and returns from savings from Period 1: $(1 + r_1)S_1$.

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2 Questions and Solutions

Q1. Period Constraints

Context: To understand the household's decision-making process, we first need to establish the budget constraints for each period. These constraints describe the relationship between income, consumption, and savings within each period.

Solution: The period constraints are derived from the household's budget in each period:

$$\text{Period 1:} \quad Y_1 = C_1 + S_1$$
$$\text{Period 2:} \quad Y_2 + (1 + r_1)S_1 = C_2$$

Q2. Intertemporal Budget Constraint

Context: The intertemporal budget constraint combines the constraints from both periods into a single equation. This constraint ensures that the present value of total consumption equals the present value of total income over the two periods.

  Solution: By solving for savings S_1 in the Period 1 constraint and substituting it into the Period 2 constraint, we derive:

$$S_1 = Y_1 - C_1$$
$$Y_2 + (1 + r_1)(Y_1 - C_1) = C_2$$
$$\implies Y_1 + \frac{Y_2}{1 + r_1} = C_1 + \frac{C_2}{1 + r_1}$$

This represents the intertemporal budget constraint.

Q3. Utility Maximization Problem

Context: The household aims to maximize its intertemporal utility, which depends on consumption in both periods. This involves setting up a utility maximization problem subject to the intertemporal budget constraint.

  Solution: The intertemporal utility function is:

$$U = u(C_1) + \beta u(C_2)$$

where β is the discount factor. The utility maximization problem is:

$$\max_{C_1, C_2} \quad U = u(C_1) + \beta u(C_2)$$
$$\text{subject to} \quad Y_1 + \frac{Y_2}{1 + r_1} = C_1 + \frac{C_2}{1 + r_1}$$

The choice variables are C_1 and C_2.

Q4. Lagrange Function

Context: To solve the optimization problem, we use the method of Lagrange multipliers. This involves forming a Lagrange function that incorporates the utility function and the budget constraint.

  Solution: The Lagrange function is:

$$\mathcal{L} = u(C_1) + \beta u(C_2) + \lambda \left(Y_1 + \frac{Y_2}{1 + r_1} - C_1 - \frac{C_2}{1 + r_1} \right)$$

where λ is the Lagrange multiplier.

Q5. First Order Necessary Conditions (FONCs)

Context: The FONCs are derived by taking the partial derivatives of the Lagrange function with respect to the choice variables and the Lagrange multiplier. These conditions are used to find the optimal consumption levels.

  Solution: The FONCs are:

$$\frac{\partial \mathcal{L}}{\partial C_1} = u'(C_1) - \lambda = 0 \implies u'(C_1) = \lambda$$
$$\frac{\partial \mathcal{L}}{\partial C_2} = \beta u'(C_2) - \frac{\lambda}{1 + r_1} = 0 \implies \beta u'(C_2) = \frac{\lambda}{1 + r_1}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = Y_1 + \frac{Y_2}{1 + r_1} - C_1 - \frac{C_2}{1 + r_1} = 0$$

Q6. Euler Equation

Context: The Euler equation is derived from the FONCs and describes the optimal trade-off between consumption in the two periods. It reflects how the household balances marginal utility across time.

Solution:

$$\frac{u'(C_1)}{\beta u'(C_2)} = 1 + r_1$$

$$\implies \frac{u'(C_1)}{u'(C_2)} = \beta(1 + r_1)$$

Interpretation: The Euler equation ensures that the marginal utility of consumption today, when adjusted for the interest rate and discount factor, equals the marginal utility of consumption tomorrow.

Q7. Explicit Consumption Functions

Context: Assuming a logarithmic utility function, we solve for explicit consumption functions in terms of model parameters. This allows us to express C_1 and C_2 as functions of income, interest rate, and the discount factor.

Solution: Let $u(C_t) = \ln(C_t)$. Then,

$$u'(C_t) = \frac{1}{C_t}$$

Using the Euler equation:

$$\frac{1/C_1}{1/C_2} = \beta(1 + r_1) \implies \frac{C_2}{C_1} = \beta(1 + r_1)$$

Substitute into the budget constraint:

$$Y_1 + \frac{Y_2}{1 + r_1} = C_1 + \frac{\beta(1 + r_1)C_1}{1 + r_1}$$

Solving, we get:

$$C_1 = \frac{Y_1 + \frac{Y_2}{1 + r_1}}{1 + \beta}$$

$$C_2 = \frac{\beta(1 + r_1)(Y_1 + \frac{Y_2}{1 + r_1})}{1 + \beta}$$

Q8. Second Order Sufficient Conditions

Context: To confirm that the solutions represent a maximum, we verify the second order sufficient conditions. This involves checking the concavity of the utility function.

Solution:

- The utility function $u(C_t) = \ln(C_t)$ is concave, as $u''(C_t) = -\frac{1}{C_t^2} < 0$.

- The negative second derivative indicates that the function is concave, satisfying the conditions for a maximum.

Q9. Sensitivity Analysis

Context: We analyze how the optimal consumption choices C_1 and C_2 respond to changes in income and the interest rate. This provides insights into the effects of changes in economic conditions on household consumption.

Solution: Take partial derivatives:

$$\frac{\partial C_1}{\partial Y_1} = \frac{1}{1+\beta}$$

$$\frac{\partial C_1}{\partial Y_2} = \frac{1}{(1+r_1)(1+\beta)}$$

$$\frac{\partial C_1}{\partial r_1} = -\frac{Y_2}{(1+r_1)^2(1+\beta)}$$

$$\frac{\partial C_2}{\partial Y_1} = \frac{\beta(1+r_1)}{1+\beta}$$

$$\frac{\partial C_2}{\partial Y_2} = \frac{\beta}{1+\beta}$$

$$\frac{\partial C_2}{\partial r_1} = \frac{\beta(Y_1 + Y_2/(1+r_1))}{1+\beta}$$

Interpretations:

- $\frac{\partial C_1}{\partial Y_1}$: An increase in initial income Y_1 raises current consumption proportionally.

- $\frac{\partial C_1}{\partial Y_2}$: An increase in future income Y_2 raises current consumption, discounted by the interest rate.

- $\frac{\partial C_1}{\partial r_1}$: An increase in the interest rate r_1 decreases current consumption as the return on savings becomes more attractive.

- $\frac{\partial C_2}{\partial Y_1}$: An increase in initial income Y_1 raises future consumption, adjusted for the return on savings.

- $\frac{\partial C_2}{\partial Y_2}$: An increase in future income Y_2 directly raises future consumption.

- $\frac{\partial C_2}{\partial r_1}$: An increase in the interest rate r_1 raises future consumption as it increases returns on savings.

3 References

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