MATH 417, HOMEWORK 13

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Chapter V.27

Exercise 2. Find all prime ideals and all maximal ideals of $\mathbb{Z}_12$.

Proof. We will examine the proper ideals of $\mathbb{Z}_{12}$ and determine which are prime and which are maximal. The proper ideals are $\langle 0\rangle$, $\langle 2\rangle$, $\langle 3\rangle$, $\langle 4\rangle$, and $\langle 6\rangle$.

- $\langle 0\rangle$: The quotient ring $\mathbb{Z}_{12}/\langle 0\rangle$ is isomorphic to $\mathbb{Z}_{12}$, which is not an integral domain. Thus, $\langle 0\rangle$ is not prime.
- $\langle 2\rangle$: The quotient ring $\mathbb{Z}_{12}/\langle 2\rangle$ is isomorphic to $\mathbb{Z}_2$, which is a field. Therefore, $\langle 2\rangle$ is a maximal ideal, and hence prime.
- $\langle 3\rangle$: The quotient ring $\mathbb{Z}_{12}/\langle 3\rangle$ is isomorphic to $\mathbb{Z}_3$, which is a field. Therefore, $\langle 3\rangle$ is a maximal ideal, and hence prime.
- $\langle 4\rangle$: The quotient ring $\mathbb{Z}_{12}/\langle 4\rangle$ is isomorphic to $\mathbb{Z}_4$, which is not an integral domain. Thus, $\langle 4\rangle$ is not prime.
- $\langle 6\rangle$: The quotient ring $\mathbb{Z}_{12}/\langle 6\rangle$ is isomorphic to $\mathbb{Z}_6$, which is not an integral domain. Thus, $\langle 6\rangle$ is not prime.

In conclusion, the only prime ideals of $\mathbb{Z}_{12}$ are $\langle 2\rangle$ and $\langle 3\rangle$, and they are also maximal. There are no other prime or maximal ideals in $\mathbb{Z}_{12}$. □

Exercise 8. Find all $c \in Z_5$ such that $\mathbb{Z}_5[x]/\langle x2 + x + c\rangle$ is a field.

Proof. We want to find all $c \in \mathbb{Z}_5$ such that the quotient ring $\mathbb{Z}_5[x]/\langle x^2 + x + c\rangle$ is a field. This occurs if and only if the polynomial $x^2 + x + c$ is irreducible over $\mathbb{Z}_5$. A polynomial of degree 2 is irreducible over a field if it does not have any roots in that field.

We check for zeros of $x^2 + x + c$ in $\mathbb{Z}_5$ for each $c \in \mathbb{Z}_5$.

- For $c = 0$, the polynomial $x^2 + x$ has zeros at $x = 0$ and $x = 4$.
- For $c = 1$, the polynomial $x^2 + x + 1$ does not have any zeros in $\mathbb{Z}_5$.
- For $c = 2$, the polynomial $x^2 + x + 2$ does not have any zeros in $\mathbb{Z}_5$.
- For $c = 3$, the polynomial $x^2 + x + 3$ has zeros at $x = 1$ and $x = 3$.
- For $c = 4$, the polynomial $x^2 + x + 4$ has a zero at $x = 2$.

Therefore, $\mathbb{Z}_5[x]/\langle x^2+x+c\rangle$ is a field if and only if $c=1$ or $c=2$. These are the values for which the polynomial x^2+x+c is irreducible over $\mathbb{Z}_5$. □

Exercise 14. Mark each of the following true or false.

a. Every prime ideal of every commutative ring with unity is a maximal ideal.
b. Every maximal ideal of every commutative ring with unity is a prime ideal.
c. $\mathbb{Q}$ is its own prime subfield.
d. The prime subfield of $\mathbb{C}$ is $\mathbb{R}$.
e. Every field contains a subfield isomorphic to a prime field.
f. A ring with zero divisors may contain one of the prime fields as a subring.
g. Every field of characteristic zero contains a subfield isomorphic to $\mathbb{Q}$.
h. Let F be a field. Since F[x] has no divisors of 0, every ideal of F[x] is a prime ideal.
i. Let F be a field. Every ideal of F[x] is a principal ideal.
j. Let F be a field. Every principal ideal of F[x] is a maximal ideal.

Proof. Analyzing each statement for truthfulness:

a. **Every prime ideal of every commutative ring with unity is a maximal ideal.**
   False. Prime ideals are not necessarily maximal in general.
b. **Every maximal ideal of every commutative ring with unity is a prime ideal.**
   True. In commutative rings, maximal ideals are always prime.
c. $\mathbb{Q}$ **is its own prime subfield.**
   True. $\mathbb{Q}$ is a prime field as it has no proper subfields.
d. **The prime subfield of** $\mathbb{C}$ **is** $\mathbb{R}$**.**
   False. The prime subfield of $\mathbb{C}$ is $\mathbb{Q}$.
e. **Every field contains a subfield isomorphic to a prime field.**
   True. Every field contains a smallest subfield, which is the prime field.
f. **A ring with zero divisors may contain one of the prime fields as a subring.**
   True. For example, matrix rings over $\mathbb{Q}$ contain $\mathbb{Q}$ and have zero divisors.
g. **Every field of characteristic zero contains a subfield isomorphic to** $\mathbb{Q}$**.**
   True. Fields of characteristic zero have $\mathbb{Q}$ as their prime subfield.
h. **Let** F **be a field. Since** $F[x]$ **has no divisors of 0, every ideal of** $F[x]$ **is a prime ideal.**
   False. The absence of zero divisors does not imply that every ideal in $F[x]$ is prime.
i. **Let** F **be a field. Every ideal of** $F[x]$ **is a principal ideal.**
   True. $F[x]$ is a principal ideal domain.

j. **Let F be a field. Every principal ideal of $F[x]$ is a maximal ideal.**
False. Not all principal ideals in $F[x]$ are maximal.

□

Exercise 15. Find a maximal ideal of $\mathbb{Z} \times \mathbb{Z}$.

Proof. Let $p \in \mathbb{Z}_+$ be a prime number. We consider the ideal $\langle (1, p) \rangle$ in $\mathbb{Z} \times \mathbb{Z}$. This ideal consists of all pairs (a, bp) where $a, b \in \mathbb{Z}$.

We show that $\langle (1, p) \rangle$ is a maximal ideal by demonstrating that the quotient ring $\mathbb{Z} \times \mathbb{Z}/\langle (1, p) \rangle$ is a field.

- The quotient ring $\mathbb{Z} \times \mathbb{Z}/\langle (1, p) \rangle$ is isomorphic to $\mathbb{Z}_p$, which is a field. This is because the elements of the quotient ring can be represented as $(a, b) + \langle (1, p) \rangle$, where (a, b) are reduced modulo the ideal $\langle (1, p) \rangle$. In this reduction, the second component becomes $b \mod p$, while the first component can be any integer, thus collapsing the structure to $\mathbb{Z}_p$.
- Since $\mathbb{Z}_p$ is a field (being the integers modulo a prime), the quotient ring $\mathbb{Z} \times \mathbb{Z}/\langle (1, p) \rangle$ is also a field.
- By definition, if the quotient of a ring by an ideal is a field, then that ideal is maximal.

Therefore, the ideal $\langle (1, p) \rangle$ in $\mathbb{Z} \times \mathbb{Z}$ is maximal for any prime $p \in \mathbb{Z}_+$. □

Exercise 16. Find a prime ideal of $\mathbb{Z} \times \mathbb{Z}$ that is not maximal.

Proof. Consider the ideal $\langle (1, 0) \rangle$ in $\mathbb{Z} \times \mathbb{Z}$. This ideal consists of all pairs $(a, 0)$ where $a \in \mathbb{Z}$.

We show that $\langle (1, 0) \rangle$ is a prime ideal but not maximal:

(1) **Prime Ideal:** The quotient ring $\mathbb{Z} \times \mathbb{Z}/\langle (1, 0) \rangle$ is isomorphic to $\mathbb{Z}$. This is because the elements of the quotient ring can be represented as $(a, b) + \langle (1, 0) \rangle$, where (a, b) are reduced modulo the ideal $\langle (1, 0) \rangle$. In this reduction, the first component becomes irrelevant, effectively collapsing the structure to $\mathbb{Z}$. Since $\mathbb{Z}$ is an integral domain (but not a field), the quotient ring is an integral domain, implying that $\langle (1, 0) \rangle$ is a prime ideal.

(2) **Not Maximal:** However, $\langle (1, 0) \rangle$ is not maximal in $\mathbb{Z} \times \mathbb{Z}$ because it is properly contained in larger ideals. For instance, the ideal $\langle (1, 0), (0, 1) \rangle$ contains $\langle (1, 0) \rangle$ but is not the entire ring $\mathbb{Z} \times \mathbb{Z}$. The existence of such an intermediate ideal shows that $\langle (1, 0) \rangle$ is not maximal.

Therefore, the ideal $\langle (1, 0) \rangle$ in $\mathbb{Z} \times \mathbb{Z}$ is an example of a prime ideal that is not maximal.

□

Exercise 24. Let R be a finite commutative ring with unity. Show that every prime ideal in R is a maximal ideal.

Let P be a prime ideal in the finite commutative ring R with unity. We need to show that P is also a maximal ideal.

1. **Quotient Ring is a Field Implies Maximal Ideal:** An ideal I in a ring R is maximal if and only if the quotient ring R/I is a field.

2. **Quotient Ring is an Integral Domain Implies Prime Ideal:** An ideal I in a commutative ring R is prime if and only if the quotient ring R/I is an integral domain.

3. **Finite Integral Domain is a Field:** A key property in algebra is that every finite integral domain is a field. This is because in a finite integral domain, every non-zero element must have a multiplicative inverse (else the ring would have zero divisors due to finiteness, contradicting the integral domain property).

4. **Applying to R/P:** Since P is a prime ideal, R/P is an integral domain. Because R is finite, R/P is also finite. Therefore, R/P, being a finite integral domain, must be a field.

5. **Conclusion:** Since R/P is a field, P is a maximal ideal in R.

Therefore, every prime ideal in a finite commutative ring with unity is a maximal ideal.

Proof. Let P be a prime ideal in a finite commutative ring R with unity. Since P is prime, the quotient ring R/P is an integral domain. As R is finite, so is R/P. By the property that every finite integral domain is a field, R/P is a field. Therefore, P is a maximal ideal in R. $\square$