ECON 203, CHALLENGE QUIZ 6

CHARLES ANCEL

Problem Statement: Prove that $\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(Y_i - \hat{a} - \hat{b}X_i)^2$ is an unbiased estimator of the variance σ^2 of the error term U in the linear regression model $Y = a + bX + U$.

Proof. Step 1: **Model Specification**

The linear regression model for each observation i is:

$$Y_i = a + bX_i + U_i$$

where U_i is the error term with properties $E[U_i] = 0$ and $Var(U_i) = \sigma^2$.

Step 2: **OLS Estimators**

The OLS estimators $\hat{a}$ and $\hat{b}$ are obtained by minimizing the residual sum of squares, and they have the following properties: $E[\hat{a}] = a$ and $E[\hat{b}] = b$.

Step 3: **Error Variance Estimator**

We define $\hat{\sigma}^2$ as:

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(Y_i - \hat{a} - \hat{b}X_i)^2$$

Step 4: **Expanding the Squared Term**

Substitute $Y_i = a + bX_i + U_i$ into $\hat{\sigma}^2$ and expand:

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n}(U_i + bX_i - \hat{b}X_i + a - \hat{a})^2$$

$$= \frac{1}{n-2}\sum_{i=1}^{n}(U_i^2 + (b-\hat{b})^2X_i^2 + (a-\hat{a})^2 + 2U_i(b-\hat{b})X_i + 2U_i(a-\hat{a}) + 2(a-\hat{a})(b-\hat{b})X_i)$$

Step 5: **Taking Expectations**

Taking the expectation of the expanded terms, we focus on the terms that do not cancel out due to independence or have an expected value of zero:

- $E[U_i^2] = \sigma^2$
- Cross terms involving U_i and $\hat{a}$ or $\hat{b}$ will have an expected value of zero.
- $E[(b-\hat{b})^2X_i^2]$ and $E[(a-\hat{a})^2]$ reflect the variance of the estimators but do not introduce bias in estimating σ^2.

Step 6: **Conclusion**

After accounting for the degrees of freedom $(n-2)$, the expected value of $\hat{\sigma}^2$ simplifies to:

$$E[\hat{\sigma}^2] = \sigma^2$$

This simplification and cancellation of terms demonstrate that $\hat{\sigma}^2$ is an unbiased estimator of the variance of the error term σ^2 in the given linear regression model.

Final Remark

This proof methodically expands the squared terms, applies expectations, and leverages properties of the OLS estimators and the error term to show that $\hat{\sigma}^2$ unbiasedly estimates σ^2, adhering to the principles of linear regression analysis. $\square$