Homework 1. Stats 511
Due Monday Jan 11, by 10:10am

Review of Chapters 1 - 4 of CB

Work out all these problems. You must write down intermediate steps in order to reach the final answer.

1. **(10 pts)** Let X, Y be independent and identically distributed with Uniform$(0,1)$ density.
   (a) Find the density of $X - Y$
   (b) Find the density of X/Y.
2. **(10 pts)** Let X be uniformly distributed on the interval $[-1, 9]$. Let $Y = X^4$. Find the cdf and the pdf for Y.
3. **(0 pts): Exercise** Let X, Y be independent $N(0,1)$ random variables. Consider $U = X + Y$ and $V = X - Y$.
   (a) Obtain the joint density of $f_{U,V}(u, v)$ through Jacobian.
   (b) Show U and V are independent.
4. **(10 pts)** Let X, Y be random variables with finite means. Show that
$$\min_{g(x)} E(Y - g(X))^2 = E(Y - E(Y|X))^2, \tag{1}$$
   where $g(x)$ ranges over all functions. ($E(Y|X)$ is sometimes called the regression of Y on X, the best predictor of Y conditional on X.
5. **(10 pts)** In each of the following, find the pdf of Y.
   (a) $Y = X^2$ and $f_X(x) = 1$, $0 < x < 1$.
   (b) $Y = -\log X$ and X has pdf
$$f_X(x) = \frac{(n+m+1)!}{n!m!} x^n (1-x)^m, \; 0 < x < 1, \; m, n \text{ positive integers} \tag{2}$$
   (c) $Y = e^X$ and X has pdf
$$f_X(x) = \frac{1}{\sigma^2} x e^{-(x/\sigma)^2/2}, \;\; 0 < x < \infty, \;\; \sigma^2 \text{ a positive constant} \tag{3}$$
6. **(10 pts)** Find the moment generating function corresponding to

(a) $f(x) = \frac{1}{c}, \;\; 0 < x < c.$

(b) $f(x) = \frac{2x}{c^2}, \;\; 0 < x < c.$

(c) $f(x) = \frac{1}{2\beta} e^{-|x-\alpha|/\beta}, \;\; -\infty < x < \infty, \;\; -\infty < \alpha < \infty, \;\; \beta > 0.$

7. **(0 pts) exercise** Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2, respectively.

   (a) Compute the joint pmf of $f_{X,Y}(x, y)$.

   (b) Now define $Z = X + Y$ and compute $f_Z(z)$ for $z = 0, 1, 2, \ldots$. What is this distribution?

   (c) Let $z = 0, 1, 2, \ldots$. Show that the conditional distribution of (X, Y) given $Z = z$ is the binomial distribution with parameters z and p_1, p_2 where

   $$p_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2} \quad \text{and} \quad p_2 = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

8. **(10 pts)** Let $X, Y \sim N(0, 1)$ be independent random variables. Let $Z = Y/X$ Show that

   $$f_Z(z) = \frac{1}{\pi(z^2 + 1)}, \quad -\infty < z < \infty.$$

   This density is called **Cauchy density**. The tails of the Cauchy tend to zero very slowly compared to the tails of the normal.

9. **(10 pts)**

   (a) Let X be a continuous, nonnegative random variable, that is $f(x) = 0$ for $x < 0$. Show that

   $$E[X] = \int_0^\infty (1 - F_X(x))dx, \tag{4}$$

   where $F_X(x)$ is the cdf of X.

   (b) Let X be a discrete random variable whose range is the nonnegative integers $0, 1, 2, \ldots$. Show that

   $$E[X] = \sum_0^\infty (1 - F_X(k)), \tag{5}$$

   where $F_X(k) = P(X \le k)$, where $k = 0, 1, 2, \ldots$. Compare this with part 9a.

10. **(15 pts)** Let $X_1, \ldots, X_n \sim \text{Uniform}(0, 1)$ and let $Y_n = \max\{X_1, \ldots, X_n\}$. Find $E(Y_n)$.

11. **(15 pts)** Let $X_1, X_2, \ldots, X_n$ be a random sample from an exponential(β) population (that is, $X_1, \ldots, X_n \sim$ exponential(β) i.i.d).

(a) Write out the joint pdf of the sample $f(x_1, \ldots, x_n|\beta)$.

(b) Let $Y_n = \max\{X_1, \ldots, X_n\}$. Find the PDF for Y. Hint: $Y \leq y$ if and only if $X_i \leq y$ for $i = 1, \ldots, n$.