Ex 3.1.1 (Mctfenpt) Let a,b¢,d be objects such thet. 5a,b% = Fe, 2. Show thet. af Jeost one ft The tio statements "Q=€ and fed . and. “acd and bec ” hold . /. Suppose none of The statements held, st. yen 3a,b 3 = 3,4). 9 [laze) (b-d)JVE Ged) Albee) — [G#9 vibed) (AL Gee) v (bec) — 2. From step l, ae 3a,b3 , ag Ec,d3, Smee a#C and akd. be Fabs bf §c,d3_ since BFC aod SFA. Similedy fe Cc and d. oe CorFadletion . At fest one sfalemert shoald hold

Ex 3.1.3 (]"atlengt) Frove Lemma 3.1.12 ya a and b are object. Then Sab = ja3U0 $b3. 7 A,B, and C are sels. Then union operation is commutative. and associative . Abo, we have AVA = Avg = DUA =A . ae £a,b3, ae €a3U $b3 Vire Vesa. f ae F233, ae §23V0 $b3, /. be fabs, be $03U %b3, LE €a-b3, if be ¢b3, be 4a3VGb3. be Fa,b3. 2. Sow that AUB = BUA tt ee AUB. Then xe A eo xe B. Tt xe A. Th, x© BUA 3. Show thet (AuB)uc = AU (Buc) @)tf x € (AUB)UC Than xe AUB ow xe C Tf x€C, Thn x € BUC. Zt xe AUB. Tha xeAow x eb. TH cA. Tle wc A. Ths LHS +RH.S. af xe B. The x @ BUC. Tf xE AU (vc). Tr xe Aor xe BUC TExe A. Than xe AUB Hx é€ AvB, Tle xe A of x € 8. TfxeA, xa AvB ThxeB, xe AB. Tye BUC. Tm xe B wo XE C TH x zB. Jew xe AVP. Zt xe Cc. Thr KEC. (& we 4 Tf xe AUD. Th veh Ts AvA = Aud: GA =A.

Ex 3,14 Clattenpt) Prove Js poston 3/. /7, (Sets are. portially ordered by set inclusion) . Let A,B,C be sets. Ff ASB amd Bec. Tn ASC. Tf AcR ad BSA: The A=B. af AcB and Bel. Ten Ac CC. — xehk, ASB = xeB, Bel, 2 xe Cl, ts. xeAa xec, ASC, 2. xeA ASB, xe B. xEB SA, eA, Ths, A=B. (Axiom 3.2) 3. xe AcB, xe B BeA. xe, Acc. AAC.

Ex 3.).] (attempt) Let A,B,C be sets. So that ANB SA, ANB CB, Ficthermore, Show that C£A and CEB H Cc SZ ANB. Tho similar spirit, show That Ac AUR, Be AUB. Ficthermore , show that. A € Cand Be Cc a AVB EC. [xe ANB = €A od xe B. Ths ANB © A md ANB EB. 2. @) vf cfA and CER The xe C, xeAand xeB Ths CEAR, (¡7) Tt CCAAB. Then xe C. xe ANB ¿ xe A and xe BB. from above ANBSA ad AnBe B. 3. Zf xe A. Then xe A oe xe B. Ths xe AUB. ASAUB. Gf xe B. Then xe Ace xe 8B Ths xe AUB Be AUB. 4. (¿) DEAS Cad BEC. web xe l =¿ AUBS C. AE B 7xé C. le) Tf AvB s C. Thn xe AVB xe C Ii xcA, eC 4. AEC, BLC. af xe B, KE C.

Ex 3.Llo C(athimgt) Let Aand B be sets, Show That the Three sel A , AAB and Bare dis om, Sho thet Tele union is AUB. LAKEA= eA md xf BD xe A (ANB) ., =¿ X¢ Bx ¢é (ANB) Tes @)n AaB) = 9 (A) A B= ©. 2 Tf xe ANB. Then ve A ad xeB = xg B. (ANB) a (B) = @. c=) 3. (A) VU (B) ¿ xe A and x¢B on =) x¢ Gus)). xeBSad xgA. (tare) (8xA) U had) ¿ xe (Av9) An) LU An) x E AVE. &) x € AVB, =) “¡¢ Ao £61 ee XE AN B. x EC ANR of xe Bof xe AA,

nd st Ex 2.1-U (2 a thept, — offen ¢é abet} Show that The axjo ot re Jecement implies te AKiom f specifeaton. CA') ixtem3- inm 3.6 Axiom 3.71 Replacemect) Let A ben set for any Het x eA, and. any object Yr Sppose_we have stifemen{ Plx,y) pring 4 x andl y st for enh x eA, Bere is of most one Y fre hick Pl x,y) is Tree. Then there existe a sel Fy: Ply) is re por some xeA$, 8b, Aor any object z, ZeE dy: Plx,y) is True Soe Some Xe AS ¿) Plx2) is Tee fer some xe A. Axiom 3-6 Gprerfiatin) Let A be o set, and Fir each xe A, ket Pi) leu poopy prbing Ta x Cie, Ae cack xe A, Ple) is etther Five or tbe). Then there exits a set. SxeA: P&S) is treed whose element ave recisely the okmeits x in A soe which Pl) is Troe. Za otter works, or any object y, ye gxeA: P&) is Tre 3 ¡=¿ lyeA andl PG) ws Tue). ZE By: Px y) is Tme foc sume x € A3 Suppose we modity the ahve s-T DE Z yi Puy) +s Five f- any x€ At. suf. 3 Y- Py) ss Tue fir some XC A3 c gy: Plx,y) is re fir ony xe A' a Ey:y e 3xeA: Pe)$ y 2. ths, fam step f 2E Sy: Ply) Laxeh3, Then 26 Sy: ye §xe A: PGBS. s.-T. Replacemestt Implees SpoctheLion.

Ex 31. 1Qii Cl atfengt) Sup pose A, 8, A', 3° are sets Lst. A'sA and B'¡ B. (ii) Give a Coviler — excample shoumg A'¡ A is falta. Gun yeu fad a modifies fon of th slefement imelving the set d:Hlerence gperoctem which is Tine ye the shied bypothess?, [— Zf xe A'. Then xeA and x¥ B Jf xe AN. Than xe A The g Bl. Then xe Ro xf B, Sire BEB. If x€B' ond Kx¢B. Ten xe ATf x ¢ Bl and xc i3. The xe A'. 2. A ' cS A BL "The jest ticthn is aS oh e —

E3113 Clettengt) DeFine a Pups wbset fa seb A b fe a sbset Bf Avi BEA. Le€ A be a non-empty sel Show That A does aot have any pon- empty Pepe shee he As The Poem A: 5x3 x Some beet x, —, Sach et Cis C= FxeB: BEAS 2. (=?) aK eA, st. KE C , (A does rot hove any hon - empty proper subsets). Aye C, y FB, dee A, st xe C, x¢ D Thess. Az €%3 Xr some abject x, (¡=) zt Ais d te fon Az {x3 fe some odject x, Tien FxeA, x¢ DB (Ais non-empty $0) x#D, Be- C . Ton ay € x= C, 3st CSA.

Ex 3.2.— (("athmgt] Shows tot He. anivecse— spe ifmtiom axiom Ailes 3.9, if assumed t le true, would imphy Axioms 2.3, 3.4,3.5, 3.6, ard 3.) (Asjom 3% if we assume alfn fom number$ ave ects). Axiom 3.3. (Eupty set) Axiom 3-9 (Usinecsel Sperfictvs) on 3 Geyttn edwin) Sipe for ery abject x, we hae popely PG) patinig fe x Ain 35S) Then Tere exis a set Ex: Pl)' sé we every abject a Adon 37 (Reylwznent y€ §x: PG)' ¡=¿ PY) is tree. Axion 3.8 (Tetoit [xe § y: 76935 ¡9 xe D Ch os empty). =) x is al an object. 2. xe 3 y: yeas ¡=¿ X=a C Sime set) KE gy: Cy= a) v (y=8)3 a) (= a) oF (+4) Pair st) 3. x€ AUB es xe Sy: VeA)v (yeB)5 ¡=¿ eA) or eB) (Ric Vinn), xeSy: YeAAPG)3 ¡= eA A Pe) pens tati.) as Na 2¢€ by: Py) fe xe A 3. ¡- PX, 2) fr Some xeA. Repluerct) LZ. xe sntt: neIN3 ced x é WN. (Tort, J,

Ex 3.2.2 ((''attenpt— Use the axiom of cephoty Cand the singleton set axtom) tb thw thet f Ais aset, Hen A @A. Fectheroace show thet tA ad Bae fuo se, Tien er Ther A€B8 or BEA Ge oh). Then pee the. corollary ; briven any get A, there exist a malhemefical object Fiat is not an emot jn A, nanely A ite Ths. # biger sel com de erected by AV GAS. — BA a sot, Then by Arm 3-10 (Kepke), Ax $x: xed} AxecA, x= Y where ye gx: xe A}. Axiom 3.4 (Salt sc} or xNA= Z. "Thos A: 3x: xe A$ ¢ A. 2. Bt A ad B ore feo sels. Thn Az fx: xe A} B: Sy: ye B3. Suppose A&B. Then deze iv: y 6 83, Zz: gx: xeA3, Z2NB= 2. yb Greil Axiom 3.10 (Regslenty). anB-PVadgB Ths A€B. (this also applies BEA) 3. Ay sf, A'=A and A' €A, Then it ders mt cotediel Axism3.(0 (Regulasty) AU SA'S is possible . But not AUA', sine AUA'=A= A'

Fx3.22 (I "atlenpt) Show thet the universal speerFieaTion axiom , Axiom 3. 7, 6 equivalent b an axiom post ling He existence tf a "wniversel set" SL consisting Af all objects Tp other wos, chev thet Axion 3.9 is Tue itt a unnersn/ sel exish. Farthermore, show that b Axlin 3.1, SU e Q. cotbnd ith Be?.2.2. Then pustily hy Axle 3-7 Js exe[vded fim Wim T recabelly. — (¡) SL = $x: Pe), = § xs xisaset and x¢x§ (The unnersal set) Suppose if SL exist. Then # 44] By Trae , Het is X ts set and x fx. Then x GS, HY xen. Jen xX is a sel and X é x, setubty 7), Thus implies Axiom 3.9. (=) Swprse Axrem 3.4 1s Tre. Then if Fly) ws tbe y tS oa st end yer Ten exist , and vie verse. Thes implies He exis/emnce - S2. 2. Since if SL is a sel Gen SL is alco an object, tes (22 €52) v(Q¢éQ). This earhodicls the Axiom 3.10C Reg lacity) as f2€ RQ is possible . Te have Axiom 310 (Reo lon ty) porkirg AKrom BY CUnisercal specific.) 4 ortiely omitted .

Ex 3.3.2 (—atlempt) Let P:X 2 Y and 3 DE be fraslons, Show thet it Ff ond g are both iyective . Then 50 is per Similary Show thot it Ford 9 are doth sugjective. . Men 50 5 J of. L Suppose Food g are inyectme tt Wx, x2 ¢ X, Vy Ye ev Xx. ¿ An) # Ae) x =x, —¿ #l4) =SAln) YAW ¡¿ 9G) # Oh). Yaw ¡= gn) = 7 lv) 2. From skp I, Six yi = 4.) 2 Yrn¿ Slr) and we knew that. Yi = Ye =¿ IM) = Gf)G:) = Gh) = (g°F 0) y # %) ¡=¿ Fly) = (oF) &) # ol) = Gof)&) = zt "s injective. 3. Suppose SF and J are surjective , Vy é Y, — x € XxX, s-T, He) = ¥ Vee2z, dye ¥, st, gz ¿ VeeZ. dye ¥, dxeX, t 2s gly) = Gofl&)

Ex 3.3.4 (Matlongt — Let A:X ¿ ¥. fiX YX, F Y¿ Zz, and G2 Yo ZS be Prctone. Show thet if get = got and 4 is inject hve — Then f:f. Ts fhe some theme True ff JS is wel crjecThe 2 Show thet if got = Gof ard f is Suvejective Then org: Zs the same stifemext toe #£ Lic not Susective ® }. get = 5oF ingles. Vee x, Gof) &) Gpf)O), Ft 5 is njeethe, 60. yt PO Gln) # gl). ˍ yep ¿ Fl)2 gl). ae Suppose ff, st. qxex, Ae) 4 FZ), = G-f)() # (gf) &) f-L[. Qifradeton. 2. Tle above stvbement 's fle if 4 Is not injecthe also ger F pet, and conn an pove wheter fs f. 3. get * 5 ot implies Vee X, (5°) &) 2(Fof) (x). 74 Ff is segective. "Thea Wy é y, Jxe x, 3G y? 76). Since As Pf (Dbotty). Fer Gf)G) + Gef)O), 7%. FY. Bz. Wye, Bee, st Gy) = Got)&) sme gf = 5-8, 54) = Gefl&). Then gF9 - 4 The stilemest. is The cegodless wheter or nf fF 's spreitbed .

Ex 3.3.5 (I*atlempt) Let S:X¿Y aod p¿ Z be tactons. Show that if gof is injective Men S omc de injective . TJs it Tue Thect. 4 mut abo de inective 2 Sow thet if get is Surjective. Then g mst be svejective . Ts it Toe thet SF mst abe be. svjective * — tf yee @-£)&,) # (et (Can Suppsxe fis ast injective . Then X#X, SS = ti) = = Sh). = Gef166) = gph - Cif) (6) AGN) ¿ Xx mst te injective . v" g mst obo je inject . 2. tf Vee Z, deex, st Ge f)&)= 2 Suppse 4 is not "sgecthe. sf B2e%, Wei st GQ) #z a J mst be Sunective ; f does nol hove t be Seetive .

Ex 3.3-6 (1% tfeapt) Let ZX ¿Y bea bryecte fametion , and let fo: YX be it inverse. Verity the cancel [ton laws SCH) =x fre all xe X ond LEG) = Fe all ye. Grelude thet ft" ie abo invertible , and has f as its inverse (thus (f7') ls +) [. Fis bijecthe , st. We Y, dee X, 5. y= 76) (Svijecthe) Vex eX, 1 #m% & AX) + 7K) (Lijectre) Rem ¡7 Hi) = Axe) ¿ We y, Fl xeX, st. S&) = y. Cat len one, svcectivity, at mest one injeeteity), ad x= Sy). (Remark 3.321). Ts (fof ')ly)= ¥. = Vx,,4 € X, lk)) Sl.) € Y, from above. (Sef) &:)

=x, (fFsF) G) = % 2. Show Tat fois bijective. since. f ic bietie, X#X & Hx.) # Hs) 7 Wem EX, (Jef)O) =X pe (F'of) (.) = xX Thes f is injective . Smee fis a Smation . Vee %, al ye Y, 7, Sf YF SE). Since Dis e Sanction, Vye ¥ axe x st. LC) =¥ Since f 1s bivective , Vye ¥ =! xe X, st Y= SC Sis SF imese ¿ Vee X Alye % 54 SG) f " is bijective . M0

Fx 3.3.¥ (@) Ce) Ci atterpt) rf X és a subsel of Y hE best i XDY be He inchision map Fam XAT deTred by mapping XH? X for all xéeX, ne, Lx sy 0) := xX ge all xe x. The. map Ixox 'Ss in porticelae called The jdeitity map on X, (+) Shoy tt fF XESS DZ. Tim Tys2 e nor = Un22. (c) Show thet fLA ¿ Bis bijecthe ˍ Then fof 2 larg od fof = lacy. — TE XE YS 2B, thn df xeXx, xe¥, xe 2, bevy &) i= x, KEX, KET. Lyo2 (X): =x, xe Y €€2Z, whee x = Teor). The. Tr32° Vxar (x) := xX, xKEX, *€ a = Zxo2). 2. If FAB is bijective. Then we brow thd foB¿ A. fvof: fA 35A ¡= LArA fef':B¿ B. Ve 8B.

Fx 3.3.5 Ce) (latent) Tf X is a svbsel of TV, bt bese i XY be He inchision map Fam XBT deTned by mapping XX foe all xEX, fe, Levy XK) i= xX Pe allxe X, The. map Ix-+x 'sin porticela- called The jdeitity map on x, (e) Show thet The Aypsthests that Xed Tare disemt f& L "tly can be dropped in td) if one odds He addifona(hypothesis change the Conclsien fot SE) "fo jor all xe€é xn Y. and the Appethesis 's Inferchary e able. /. Suppose Fg, Kee XO, we vat bh show LA: XVY ¿ 2st. he Zxoxvr = 7 Pi XVZ Ge Xe. he Dry xey = 9. 2. Suppose A: XUY ¿ ZS iis nek unrque , ot. Dh Xvr¿ 2. heh. Zx¿ xox x? xuY hoa Lx 2 Kur : x— Z , st. AG) =f), vx. h: xu¥Y ¿a 2 hii XU? & he Dror ¥3B., Aly) = gly. Wyte Craxer : Y¿ xUuYyY Tf he Pome XD B, st h&)= SG), YueX hie Zyaxuy 2 YO B, sth = 7G) Wye ¥. "he Alx) = LE) Vr e X Al) £ hb. &). oa Aly) fal I) Wyre a. Thee is oly ono migne PuncBeon L This sgpher Te XY, 6 Ag. Woy

Ex 3.4. ((I attengt) Let f: X¿Tt be bijective farction lel Vk any subse of y. Let F': YX be its inverse. fave Hot The frrnedl image a Vref! is the same set as Te imerse image of V under . LL Giwen fF: Yo X, Vey. The forward image fv) := Ef "(y): yeV3 2X (Den 3.4.1) 2. Given F:X DY, Fis bisectivc frction, Leth be its verse, ssf. Alv):= $xeX: flelevh SX (Deh 3.45) 3. By definition of finctions, V2 ig (V) = x, 21! ye Y% st. Al2) = Y. Omee fis the invege of f. S12) = Gof) fy) =y, where yeV. + By def nition of Five fions, Vee ACV), al ye Y st AG) = y. and. from Step 2, we know that ye VEY, sine his #3 inverse. 5 From steps dard 4 we con see tet Vee fv), fa)ev, Vege hlv), FG) ev. Since Lis a biectwe fuctin. Wye V, Flee f),st fz) 7 Vy eV, !ge Alv), 57. 7G) zy Vx, € fv), Vr ACV), A SCX.) = tn). Then X, =, Thvs xe £'), Xx, € AO). X2EACV) , xref WV). Tos Fv) shiv), Av) se flv). £7) =A, Sieh.

Ex 3.4.2 (1etfemt') let f:X Y be a function. Let S be sebsetof X. Leé U be astet of ¥ (5) hat com one say abst PCL) od S ingarem(? Ci) whet oboat A(LCU)) md U2 Gis) whet aboxt F'FG AU) and Cu)? IL (P74) (s) 2= 8xE Xs AK) E LG)}, where Hs) 12 $F): xe SB Whefhee or not (F".f)Cs) 2 S depends en f. Tf F ic injective, Ten (f'ef)(S & S. (frm inbvitionr, not proved). Tf F i sujecthe. Thee (fof) (5) £ S$. (fram inbrition, nat proved). HF is diet. Then (fF 'of)CS) = § (Ex 3.4.1 proved) 2. Pef")(v)i= $f): xe fW)3 here $“(u) = 3 xeX: 4G) ¢ U3. Whether or mt (fof ')(V) © U depends on f Ht fis injective . Then rot qrarmtee . uti swjective Then (fof ")lV) 2U & ast quaronTce (at proves J. Tf Ff is biectee. (fef'')(u) = VU (Ex 3.4. — proved) 3B (fle fo f')(u) = BxeX: Ase (fof)(U)§ Whe ther or not. (f'efef')lv) S f"W) depends ont. xf f s inecBve Then net qeoconee . Lf Fis sujectie not quarartee zt fF is diectrr . (fitofof“')(V) = f 'lv) (form inh iter net proved]

(1 "et fempt, checked others answer, Then abort) Ex3.4. 6 (i) (2 attempt) Ruwe Lemme 34 lo Let X be a Sel, Then 3Y: T is c subset x3 1s a sel, Tn othe. words, thero oxxts sel 2,st. Ye F¡ YEX ' Be all objects ¥. L Suppose we have a set FO, on sl. p. Axiem 3.1! (Prue set Axiom) . we have f:X¿ 50,13. , fe £0,13". ?. From step — and Axiom 3.7 (Replacemert) , Suppose we have PE, ") , 6.t fe each Se 30, 13% There ts at most one y jor which PG, y) is Trac. Then tere ents set $y: Phy) is Tae fe some Fé $0,13°3, euch that fe any abject Zz: ZE By: PY y) is True fr Some fe $0,13%3 ¡2 PY 2) is Tee rv some fe fo, 13", 3. fom step 2. assona thet Py) te Tue tf y= f(813) £X, (w LOG?) Then zeSy: y= f (513), Fe $2 13"5 ¡=) 2-f &3)¡ x bet 22%, $y: y= fC), fe 801373 = 2. Roved te Lemma.

23.46 Ci) (atlapt) Show that Axiom 2. [— can be dedwed , 1 using the preceding aXions of set Theor, if one access Lemma 3.4.10 oS on axiom. [We wart % shou thet there extils a set ¥% where LEV f2X7¥.(Aiom3.l) Lriven it Lemma 3.4.10 is an axiom, Such tek Ze ¥: ¥Ox3, if Ks asl. 2. From step! and Axion of renlacemerl , sppose ne hove PCY), ot, for each Ye 2. there is af most on@ nc len Sar which PCYF) “ Tre (if YES, Tm 1x74) is Tre _ ofherv-lye. Sabe) ; Then There cxisls x sek S42 AULA) is The foc some Ye 23 st fer any Ainction ¥ LESH: POLL) is True fo some Ye %S =? UY, gy) is Tre far some YeS . Then Since YEX, PULZ) is Tue for some FEZ YeZ 7 TEX, ot. Wy 6 84: PTF), Ther} ote : ard nally ¿ as Wye &)+ CNet? Gobind "16 easier + pod, x £ vxe x, ve%, 8 ve 7 From dep 2. me the lowe thee Vy 2 $f PIA 2 Bf) AX}. Ths implies Axion 3.11,

3 Ex 3.4.7 (Iathapt) Let xX, ¥ be sets. Define a partial fonction tam X BY, f be. any function Si X'—3 Y' whose domain X is a subset af X, and whose co domain Yo is a sbset of Y. Show that the callection of all partial finations srom X ts Y is itl set. by Axiom 3.11, we know thet there est a set y* 7. VE: X79 Y, fe y* we alse know That tr every sbse€ xX' of xX, and every subset Y of Y, There exists a peetial AncTion h, st. Ar X'9 1. 8 step — and Axiom 3.7 (Replacemert) ; for any he ¥*, and any gubsels Xx' and ¥' suppose we have PCh, ¥%') peelain'ns BA andl ¥'X — geod thet fe each he Y%, There is at most one YX st. PLA, yx) is Ive, aka, hé yx" They foe any et Q Qe STR: he, het hr A ¿ heQ , fhe Y%. Feem step 2. and Lemma 3.4:(0, re knw thet WRe SY: he Y*, he ¥*$ =A Qs" and by Axiom 3.12.(Usion), we know fat all elmats in FY* she ¥*, he VS+A ave sets. Then there exist. a set UA whe ehmente ave elements f ekmenk fA, st. xe VA ¡=¿ (xe Q fe some QeA). Ths Tle collection of qll partial factors tom Xt Y is itself a _set.

Ex 3.48 ([*atlempt) Show tet Ar¡ien 3-5 CRibinise Union) Con he deduced Srom Axiom 3. ((Sets ore ebjects), Arlem 3.4 (Singleton sets and pai sets), avd Axiom 3.12 (Union) -

We wort te shew that xE AUB ¡=? (xeA or xe B). (Ariem 3.5 Peiewige Usion) Let A be a sel, sf. all elements of A are sels. By Axiom 3./2 (Union) , there exists a sel VA ,s. xe UA =¿ (xeS for Some SecA). Sinilody , Phe iS anotle- sel B, with all the elements are sols, By AMiem3.12, 6 UB ¡¿ (xES for some Se 8). . From step —, thete are Tw sets VA and UB, apply Axiom 3,—, ve vould ask - tt UAEUB, Tm xe VA ¿ xe UB, xeUA)UWUB) 9 & EA o xe B) 5 UVBEVA, – - Both imply Axiom 3.5 (fairrise Union) . Sf bith VUAEUB ond UB EVA: “Then by Axiom 34, There exislh au set pB Pe SUA, UBS. 3. Fem step 3, we hwe a get P= £UA, VB3, ere VA od UB are sels, Applying Aer 3.1 (Unin), thera exst o sot UP, 2. xe VP c= (xe VA or xe UB) ce xe (VA)U (UB). Tophing Axton 3.5 (Riise Union)

E344 (—Tethenpt) Show thet if B amd B’ are Tio elements of set Z, and To euch A € 1, we ass{y¿ a set Ax. Men F xe Ag: xe Ag Fr all he TS = § xeAge: xe Ag foe al LETS. Stew thet defusClon of (Aa SxKe Ag ¿ xé Ag Sor oll & ez’ dosen’t depend on B (a4 Explain why this is Tet yé OO Ax ¡=? (ye Aa oe all he Z). ékEL I. By Axiom oplocemel, Ac each LET, suppese PCot, y) . sf foc every Ae LT, there & of ms one x st Roy) & Tee Pld. y) is Toe if yé Ag. Then Were orice 2 gat. Bye yeAe, kOTY, st Lh any 2 Ze Sys ye Au He sme he I$ ¡=) 26 An he some he L 2. From step—, we kim thet 2 Fy: yeAe foc Some KEL 4 ¡¿ 26€AL K some KEL, Suppose. Te 5p,B°3, st SxeAg: xeAg foc all he £3 3 Ape 8x: xe Ache allde TS $xe Ag: xe Aw fe all Ke ZB =? Ap’ = 3x: xe Aa fr all he Z3 “Thes Ag = Ag. 3, From skp 2, we con see thE (1) Ay: AGT gxeAg: xeAn fr all he L$ 3x : xeAg toc alke Ls does nof depend on P. u 4 And ly Steps Zand 3, ne can immvediade by see thet yé OAs c=? (ye Ag he all LeL) where the reglacement OXinr has heen applied .

Citottemt, tired , brain not Anctioning) Ex 3.411 Co” at fempt) Let X bea sel, kt Lbeaw non - enply se€, and Ser all Xe tT, let Ax bea sbset of X. Slow tet XAg = O (X) fend X= U (KV Aa) ACI ACL aet det Compare this mith De Megan $ Laws Chapsition 3.1.27 0)) —, Recall th cbhiition of UAg iz UiAgi tel} MEL X VAs :2 3xeX: xX¢ VAs 3 = jxeX i xd UfAg i re T$ er ET 2. RB feplacement. axiom , yor every x ¢ U s Ax : ke ry and. omy svbset Q EX Suppose we have Plx, 8). 5.7, for each xd U $Aq:KeL} Tore i, at myt one Q fm whlch Pa) ‘3 Trae, (PlA) i Te of xEQS x). Then ‘there exists a set JQ: x €Q Ar some x¢ UgAa de 3). such Cut Loo any set 2, Ze JQ: xe OA for some x USAg heT3$ ¡=) xe BZ Ae some xf USAg: d€Z} 3, Fromstep2, gence SQ: xe Q the some xg USha: herd os P, ot we knew that Proteins all the sbseks ubich contains ot least one x f Ax, for Some eT 4. From steps 2 and 3, we Can cee thet UP = Xez 5 Gosider ye OV (X) ¡=? Lye Xfr all Ke TZ) del And thom steps Bond 4, we can dedee that UP = $X\ Ax).TheAVA) = X.olen.$

Citattemyt, tired, brain not Aactimirg) Ex 3.4. O”™ at fempt) Let X hea set, ht Tbeaz non - emply set, and Sor all X¢ Z, let An be a sehset of X. Slow tet, XAg = O (X) and [XVO Ag = U (KN Aa) ACI ACL aet det Grnpere This with De Nor gen § Lows Ce esi tiem 3.1.27) X= SKEX! KE NALS, where NAg = EX: xEAa fe allde LT} lé2 Kez lez Since Vx € X NA - feeXjamd PAu, for every deEL) - By feplacement. axiom, yor every x fix xeAg fe alld an any abet Qe X Suppose we hove Plx,&), s.7, foe Caeh xd EK xCAG fe alldel Lore a awl myt one Q foe ehtch Pl.B) x rae, (Pl) is tre sf xeQ€X). Then ‘there exists a set 3Q: xER fc some X f SK: xeAg fralldeB? , such Gol joc any set Z, Ze 5Q :

£€ A for some ¿ AS Ki xeAy fealldе Bd ¡=) xe BZ Ae some x AS xixe Ay beled ' From step 2, gerele EQ 7 xe Q ofr some x $x: xeAgto-allde TY as P, st. we knew That P a alecins all the mbseke ubich contains orf lest one x ¢ Ad, fr all KE ZL. From steps Zand 3, we can see thet UP = XeZ ¿ Consider yé U (x) ¡=? (ye XAr some KL) A€T And tom steps Bons! 4, ve. can dedew that UP = U(X) , Thes x = UOUK Aa ve)

Ex 3.5.1G) (attempt) Suppose. we define. the ordered pair Gey) fr any object x and y by the formu (a, (x,y) is § §x% , axyd} (required servers — eapplioctims BI Axiom 3.4). for exemple; (1,2) 1s the set $513, $1,233 Q,1) is the set £ 52% 2,13 Cit) is the seé F534 Show thet such cx definition (Kerclouske definition a an ordered pair) (defn 3.5: ') obeys the properties of an ordered pote G5). (xy) = (xy) ¡=¿ (x=x' and y sy'). —. (=) Soper (xy) = (ery). Then by Koreloski $ defn: 'Team a an ore red. pair, (ez) = $543, fx, y33. C4 ¥) = 33x, $733, sech thal $43, Ex.y3 should fe an elemeat f $403, 1x5 y/23, and 8} £4Ly3 shoal be an element f £423, 4x,73$ Compatng Siuglklon sel, and. pric sels respectively, 5X3 = gxn°3 # $x) 7'3. exy3 = $x, y's # 3x3 Thess x 2x', yey' (¡) Suppese x=x" and yey: The by Kerahusks i defhifen on ortlered porns (x. y) = £ $x3 $x 2733, = $ 3x3, be 7 33. Aro 3. 4) Ths. Car) (ery).

Ex 3.5.16) (Dattengt) Suppose. we define. the ordered pale Gy) Far any object x and y by the formula (x,y) t= § fh, axyd} (required servers — eapplioctoms a] Axiom 3.4), for exemple, (1,2) 1s the set $913, 51,233 Qt) is Be set § 52h, 82,133 Cit) is te seé F534 Show thet suck a definition (KercTouske detwmition 4 an ordered pair) Cdefn 3S. ') obeys the properties of an ordered pair G.5), (xy) = (x, y') c= (=x! and y -y). 1 (¿) A (xy) = (x) 7') Ten S83, Spy th= LE, Ixy "This implies thet 2x2 = Sx VEX ys = sx ZF sx, y'$ as x '#y' Sxyh = 3x3 VEX. DD Fx ySH Ex' os x#Y Ths x =x' and yey! 2. (¡=) TE K=x' and 777" The Sxh = $x'3 ond Sx, yh= Sx,y3. Ths in (wy) : $x} and Sxy3 et, y) ¿ ¢€ ily') (y) £ (x'y') im Ky), §x3ed EtyRe ly) o¿ ey) Kye & y) Ths ly) = (2, y')

Fx 3.5.1 Ci) ([Matfopt) Show that regacd less of fhe det nition of ordered pare, the Cacte sian podet xX * Y of any Too sel xX, Y is ayein a set, [. By oxiem f replace men, or every xe xX, onal any object &,y) Suppose PA, Gay) 1 Or pryety pertuirny x end Gy) st. per euch x& x, there he aC ma€ one ty) st. Pl bey) is Tre. Nite: Axi) is Tae if ye ¥ Tan exirtia set. Llry) : Pll) is tue fowme xe XS, tr any object 2, ZeE Sty): ye ¥ for ome xé X3=A —¿ PG,2) is Free fir ema xek 2. Fiom ep l, ve kmnw tet 2Ee A, E ae sls. fir B= 3x, $,y33, CShut detivition of ordered sefs), Ths ly Aram cf nisa , thre ent, a sf UA 5. ze UA ¿ (ze (x.y) fr some (uve A) ¡=¿ XxY

aN Ss ij x2%, Xom, ts ¿ Ex 3.5.3 (—"etlenpt) mee, Show thet, fhe detinitions of equalily for ordered pale and ordered. a- Tpke are consig tent with the ceflerivity, symmeley, and Transitivity axioms . (z the sense thet ¿ Lf these axioms ore assumed fi hold for He individual components xy of an ordered pac (x.y). Then they hold for the ordered paic helt). (x.y) = (uy) Hf (42%, yey) Retest (Xz) eign = CX:) ies en ff (x, HX, Xr eX), X= Xn) . l+,y) ¿ (x' 7') At (x=x' 2 y') Symmetry = (¡'y7') = (x,y) . (XN ie = (Ki) isin H (K=K, Kis Xf ye Xe 2x). (xy) = (x, v'), (x4, y') = (x"y") Teensitialy, (Xion = OX2) sien, CX iseen = (XD tecen

Ex 3.5.4 (1" atlenpt) Let A,B,C be sets Show Det. Ax (BUC) = (AxB) U (Axc) Show that, Ax (Bac) = (AxB) a (Axc) Show thet Ax (B)-= LAxB)$Ax6)Ax(BUC) = Sty) : xeA, yeBuck(AxB)uv(Axc) = S(x, y) : xeA, yeBsuf(xy) : eA, yerysimceA = Aaf(xr)Ax(BYO), ThenxcAyeBUCya(x.y)é(AxB)v(Axc).ThenxeAVA, yeBUC.2.AxBnc) = Slxy) : eA, yeBnch(A*B)9(AxC) = Fy) : xeA, yeBSaSlay) : xeAyech, Aky)Ax(BNC).Then(xeA, yeB SGy)1XEA, yeBAxB)) = Sy) : KeA, yeBS Sry)|eA, yeCh.SinceA : AFFlay)Ax(BVO).Then(xeA, y$

E355 [—"ctfengt) Let A,B,C, D be sets Show Det (AxB)n (CxD) = CAac)x (Bnd) Ts it te tet (AxB) u (cxd) = (Au) x(BuDd)? Is it True tet (AxB) (xd) = CA) x (BD)? —, (AxB)a (cxD) = Slay): xeA,ye BS nSGey)i x2 ¢ yD. = Jay 2 x€ ANC, yeBnD} (Anc) (BAD) = Sixy); xe ANC, yeBADS, 2. (AxB) U (cx D) = day): xe A, ye BS VU $y) :xEC, yeds. (AvC) (BUD) = SG y): xe AUC, yeBUDS = (AxB) UCCxD) U(A¥D) v (Cx B) 3. (AxB) (Cx D) = 3 Gay) sxe A, yeB3 Sy): xeC, yeds. = (xe A, ye B) v (KE ANG, ye B) v Cxe A, ye 810) v (xeA, yeB) x (f ceA, ow DEB) (A) (B) = SGy): Xe A, ye B3

Ex 3.5.6 (etfengt) Let A,B,C, D be non- empty sets . Show tht AxBe cxD iff ACC and BED Show thet. AxB= CxD if A=C and B=D What happens if some or all of the AypsTieses that the A,B,C, D ave nan empty , removed % /. (=) AxBéeEcxD 3 (pg) e Flay): xe A, ye Bh . (pg)¢ ZG, y) + xeC, yeD3, This implies thst Ak Cad BED. Fifer A Zg C or B g D. Then conlradl Linn - AG.9) AB, é cxD. f) AsCaed BCD, Tn xe A, xe GC, , ye B ye D ze3lxy): x GA, ye35 =¿ Be flay: xEC, yeD3, AxC = B&D, 2. &) AxB= cD ff AxBE CxD oa CxD SAG, Ts ASC and BeDad CeAaw DEB, A: C. BD, (¡=) A=C. BeD & Rec, ced, BEN, DEB. AXBE CXD cud ZAK. AXBs C&D. ¿. Suppose A= PD Ten AxBS CxD ff ASC, md BED remin Tre, Bot mt AxB= GO HAC, BD, os BCD, wad CoD sPW.

Fx 3.5.§ (latempt) Let Xi, Xn be sets, 5 Show thet the Grlestan product They Xe is emply iff at least one tf the Xi is enply, —, () TTX: = J XK) igcen x €X fr all leign$ OO te XK eX eo Xn. Tidectin on n. Let n=2, Ff IX: X*X% ZB Suppose Xr #D, s-F. " XX% = Jum): EX, mw eX. } = D. =¿ AXE X, 9 K+ DB (Base Gre). 2. Fads tively assume thet_ if IX bal x Then Ai, Ise fn. st Xx = oa (WLoG). Let assvee that PRK, st es n, for The abe f held. nt] Sheu tee if TIX = GB, Tha AK = GW, mt " , JX. = TIX x Xo = DB, Wwe kmv that if TX =f. Then it holds Supp ese Mie F D, st, TTX; Xr = 5 (XD) ¢ 5 enay : XL E X {- all Ise Smt} . Then Xn = DP. Offense it dees ast Ill, TIX, =P ¿ WK jecer, Xi 2S 4 3 &) Suppose X = D, Then Similedy by inde tin TTX = (4 . (She tet Xi 2OG = D @ Assvme thet. Xr : p Ten Tx. c Z . Oflers Hor enty ® Sho that Xati = g Te Tx. :D othes aan ot.

Ex 3.5.4 (Mattengt) Suppsse thet Land Tare tho sets, and fr all Le LT, kt Agden set, for all BE LT kt Bp be a sel. Ow Ja = U CA a8 . Show That (VY Aa) a (Be) aarti 5p) What happens if one infirchanget all the union and. itfersection symbols here. ® [. Recall xe VAg ¡=7 (xe Ante some KET), UA = USA: de Th, 7+ (YA) 0 (U8) + UArterh

0 UPBp per = $xe UIA del} : xe USBe peT33. Q= U (AcaBp) = USAnBp + Gp)e Tes} Cpe Ixy : U SAa a Bp : (ple f@.p): der, pes} $ 2. From step —, of xe P Then xe An NB for any AE I, any Bed. x¥E Q. =¿ PEQ Hhexe€Q. Thr xe Aun Be ze ony (A,p) € Ix 5, xéEP. 2 Acf Ths P28 2. (OA)¥ (1 Bs) ¿ $x: xe A fer allke I U 2x: x é Bp f- ell pe TS. 1 (Adu Bp) = $x: xe AL UBs & all (KPETxT} G.p)éeIky Tey still open fF each other, oly f Lond J ove non enply

Ex 3.5.10 Gi) Cl athengt) Tf SiX¿Y is a tnction, define fhe argh of f 4 be The sbset of xx ¥ defined by § (x, 4@) ' xeXh Gi) Es avy obsel af XXY with the prpeely fet tor each xé X, the ef Lye Ys lu) € GS has cxnetly ome elemert Show thet Hoe is exastly one frelon AX Y whose gryh is egal f G.) kt Pt, y) be fhe Propels pitbin xandy, sT. por each x EX, There is of ms€ one y , 5b. ze Syek: lye Gk, where Ge Xx Y. 2. 4b skp — and axon & replacement . Hee exfia sf S2:P(x.2) a Tee te Some XE XS. st te ony ore g. ge A = sz: P.2) s Free tee seme xe Xk ¡¿ Plx-9) ce as I some xe xX, 3. Sup pete flee ave te functions fF and Ay pst. 4. 4x9 x, Arh, ne § (x. AG) ixeX$ = G S$ KAO): xex = G Frm steps land 2, we know et. for very xeX tk) eA Ths fle) = hb) AEA. -X. Thes f: fy The “s onhy one fonction

Ex3.5.10 (ii) Cl atferpt) wie SiX¿ Y is a Sirction define The gregh aft b de the shet of XY defied by § (x, 1&)) sxe X$. (it) Svppsse we define x function £ t be am ortlered tiple f= (X,Y G) Vetial lie test where X, Yara sels, and Gris a sebset sf Xx Y thet obeys the VeX, Zz! yer vertiod le til. Le then detne the demain af such a Tiple f be X, filet the co-domom fa be Y, nd for every x & X, we dethe Ax!) Like I. He usioe ye Vist, lee G. Show thet. this defnithes Gmpetille ttn Oh 33. i) and (Defe 3.3.6). /, bie reed To chow that (Def. 2.3.[) Vee X Tye7 s-1, Lk)= y Liefn3.3.8) Ay Af xeX, &) = 96). PK, X27. 2. Supyse Dy, yes y# yr st Bl xe X, Ax): y, , A) = ya -K Collet fre every KEX, fle)=y, y is vriqee te y) EG? f detrste , Ths, Y= 2. jr CX, kk, Gr) 3. Spprse TH 4 fs: (Xi, ¥,@.) it £29, Then X= Xe, 2h G+ @¿ Holds .

Fx 3.5.1 (Iethengt) Show that Axiom 3.1(com mtuct he deduced fom Lemma 3.4(0) CRwerret / Fee any Tho sels X ond Y, by defwitl 3S (CoAesinr prc), Common 3. &= Ke Viz ¥& 7): xex, ye ¥ Jb by Lemma 24.10, Suppose G is = shel of Xx Ys. There exists a sel QO, st. FO: GEXxFF=Q £3.50 Y & 2. Let P(),y) de fhe Propels pibin a andy Py an peat, por cach ky) €Q There is o€ mes€ one y ST. co seh WV enls We $ ye Y: Plxy) is Teve For some Cx.) € QK — by axiom of replacemert . 3. From step 2, and the axiom f raplacemenl, oe every y in W, gt, Suppose PC yf) be The prpety perTani y and Ff, st. or each y € Wy, tee is af most one £, 11. PA zy, were Ply), x) 6 True (Step 2), Then thre exist a sel. st, Soc any objeclh 2. BELL: Ply A) is Tra te sme ye Wk 4 Fron skep 3, he Can see that sch a seb gheald contain all fomctlen that. stisfies Defr 3.3. nd Defr 23.¥, and dedved fia presse Bron

Fx 3.5.12 () (* o tlemgt) Tis will estellish Frapositlon 2.].16 (Recrsive Detniti-s) figeronsly , thet avoids cicevlacity. (:) Let X Je a set. let fiNX FX be a tnction. Let c bean ekmet of xX. Show Get Here ens a faction a:X 2X st. ale)= © and alntt) = fn, aln Ar all ne IN, , wd frllemse bis fncth 4 Urge. / Assume het This fonction a is an: XX, whee X= 3neiIN:in¡NS Fix N, and induc€ on n. let n=O, Then ay (0) =¡ , an C1) = #(0, we)

40, ¡). 2. 5 ppese indvetivehy, ay fort) =A, ava) holds . Stow thet Qn (@+) +t) = A (0+l, An (+1)) is abe valid. Since Quln+s) z an (otieX, Then Fae, an Gt) €exX. Thes Qn (nt2) EX. 3. Suppose an: KX? X , Any -X¿ x, whardln # Aurl, Super An (NH) = ACN, an, (V)) valbids. ard we know thet Gna, (N++) = a. Nee) — GC then an (N+ 2) does ns€ exis Ane (N42) exist, AL N#1, Are (N#1)) = HUN+l, FUN, ave LW) ¿ AN, HIN, av lN)) Ths it is migue as feo each Ne NV, N is wnique .

Ex 3.5.12 G:) (etlenpt , check other 5 selvtion) This vill establish Iropositlon 2.).16 (Rearsive Dedinitie-s) figgronsly (ii) Frove G) without using any properties F the rcleral numbers other than the Peano Axioms directly, We wert FS show thet yor every ache nmbe- A/€ IN, there exist a Unique pai- An, Bn At shsets of IN which obeys the Following prpeties : La) An 1 BY =D (e) O€An Ce) Whenever né€ Bn ,we hve Art C Bn. (b) Aw U Br =/N G) N+ € &y (Ff) Whenever ne An and. az¢nNn, we have. avec An [. Trdvet on N. Left Nz O, Then An= $0% Bus $1.2,3,.4...$ =N
$03 Allcanditfersavesatisfell. 2. Supposeindedively, thatSIN, IN, stNIN, soctisfasallcondrtars. Tetet$
$AnUINttHSBrat = Bu\ SNeAnaBun = PAntUBytt = IN, OfAvy\&OFAN(N+t)+$
$tBuse.ete.Allconditionsaire.satistied.3.Fromsteps|and2, goreveryacknmbe-MWIN, thereexistavni$
$An, Bnf subsetsofINwhichobeysThePlowingprpeties: la)ANNBN?DB\&)oeAnCe)MenevernBu, w$
$/NG)Nrny(f)WheneverneAnandatN, wehaveatteAn.4WecomrowsebsthileAwaspneIN:$
$néN3nn\backslash).$

Ex 3.5, 13 (—"atlengt) Fipote of fhis "s 4 show, Got Thre is excell, ont verspn ot ne Lal number syslem in sel Theory, Suppose we have a set IN of Hemalie aukel pumbers ; an "Herrnalhe Zen 0", and an "atlernetve incremel opersLin * dich Tbec any alenalive roferal ambers n'e IN' and relins ansther aHornetive nolera(nember nt' & IN "st The Rano Axiom (Axioms 2.— - 2.5), all Asld vith the nealora number, Zero, and increment replaced Ju, theie affencLive. conlerpats ; Shows thot There existe a bisection fi iN 2 IN from the nalora[nembers vA The. a/fematіve naliral mmbecs ot H(0) = 0 p and 3.0, Fr any né WN ard ne IN" , Wwe have Sb) pn' Af S (att) ante' J. Suppese there exist function fist fin IN', £0) =o he want b Show That it LO} =p" Then Stars) =hte' and the convene. 2. () af Ah)an', tor ary ane IN ard any well'. Tiducl on rn. Let hz O, Then ne kaw Tet it holds Hc) =O' Seppe inducthely that SNe IN, st. FON) = NV'. Then A (N44) = V4" Show thet He) = Nit' Thn LW +4) ++) = Weeder! By Arion 2.2, if nelN, t. net EWN, oF. F(N): Nie" ¿ Allvedes) = M, rce Mx (Nas) 44° Traduction chsed . 2 (¡=) of Het) = p'at Thr Als) = nr! LC 2+ ©, Than st hel Tdsctely atsme tat it i Tee Por n¿/V. TE }yt)4t is Fe and if Wt) spot Fre SL Grobe. (V7) is Fee. Closed Ju fucten Backim4s ad von,

Ex 3.5.13 (2% theme') Fipole of fhis "és 14 show Hol Thre is excell, ont versen ef ne bral number system in sel Theory, Suppose we have o set. IN of lenatve nutkenl numbers ; an "aHersatte Zea" 0", and an alernetve incremeL oper hich tebee any olemalive nefera— anders n'e IN" and relirns ansther aHprnetive nolera(nember t+' & IN "st. The Rano Axiom (Axioms 2.— - 2.5), all Asld vith the nafora number, Zero, and increment replaced Ju, thetic alfereLive. conlerpats ; Show thet. there exists a bisection fi iN IN from The nadir embers 1A he. alfemtive naleral mombers ot flo) exe} , and s-C. for any né IN ard ne (N" we have flan! Af LOt+) = nae'. injec sgeeta_ ports

Ec3. 63 Yetfengt) Let Aa bea nolan! number. let pf: fieN sls égn id IN be = Sanction . Sew that here excts a nafiral nvmder M st. fli)Sm yor all KeSn. L We know teat ANE le égn3 fir any nein soe Jule get (Debs 3.6.)¿). For (IN, we com see thet the st is ifoke AS supprte. if it is Au, Then flee is always on adllitire/ elomot in IN, Hal does nt hove are- e- on Correspmderee in he forte ST, Sve ow bi jectfin do2e mat aout , So WW egal codrelty befnren Inthe sels and JN, 2. Seppe Gr sin, st Ahem ieises) = & is. ANEé CG, st. NeIN, waa / hes fie lnayert val , Vye G, ye. Rt Sieg (NEN, ten TN" 6 Net = Ne NsNn Is Wye Gr, AN EIN. st. yen'

F3.6.8 C)Mottengt) Let A and B be sets such Het there exis an injeclinn Z A ¿ B tom A 6B. Assome A is nen: enphy, Show That here ouisls ox surjection gh ¿ Atm Bo A, bie wart thw tot Woe A, She Bot pa)¿ /, Sinee. ve buen thet PAB . San inject, 6F. Wa, a6 A, H#RS fla) f Hla.) eB. = A has n lester or ee cob ly tf. (Fe 36.7). = #(A)¡ #8). 2. Fesm Slep [, Vie can SEE thet_ tore. exbt o Soalir- 4, 3.T. eB, st. POarA. a bigettn. Ths U b ako a sugecton ,

Ex3.6.9 (Patlengt) Let Aad B be Tite sels. Show tint. AUB and ANB are alco frile sels, and that #(A) + #(B) = ACAUB) + # (ANB). /. DE A md 1S we finite sels. Then LA ¿ Siew: [¡ign¥ fir some ne IN'. sl. #A)=n. This holds sill, £- B, st. kt 4B) =m. 2. TH AVB ond ANB we not fuse st, To Lee exist a bjyecton betrem AUB amd IN -X- Corhediction. , smibly ith An@. 3. Finn, Drobec on m, Hf # (A) =n. #8) =m: O. Then #A) +4 (B) con #(B)=0 =7 B= GD AND=-DB AvB= AVP = A. Tes # (Anf)= 2 #lAvg)= a, Ts Base Coe is Tre _ 4. Suppose atoticl, tot #0) + #(B)= HAVO + # A086). f #@) =m. Show that fF #CB) = met - Thr te above still halds. # 6B) = mt imolies thet. B= By 3x3, Lemma 3.6). Lf xed. "Thea AUB = AUB , ANB' = AnB vu sxk Ff x¢d A. Tn AUB'= AVBUSK3 , AN8'= ANB. Ties. :fce A. # (AUB): #¢ (AUB), #CANB) = #(AnB) +— Bef A. €CAB)= #00) +] , #ANBD = BCANPD. Thes, CA) + #03) = # (AUB) + €(ANB) + — Lits - #CA)+ €(8) ¿ #A)t+ #(0O*— RItS Liducton closed -

E23. Z./0 LI Hep Let Ay, ..., An be file sets och fot #(U Az) ¿ 0. (figerbebe Principle) Tho yn Show tt there wists 3 € 3). } such Hot #OA:) 22 [. Recall U Ag: = 3Az 2 ce S004 Atm 312 Union). éeY),.. Cerote. UA as RQ, smee. Avs An ore file ob. pe kw that Qs alse a fife set. by Ex 3.6.4 and. 2 HA) - : #(A) + # (Ww), lee Wie OA p= 3x: xe A fr ie SI,.., 83 CES, m3, 2. From step —, Suppose thet Ale... An Gre all nen - empty sb, vith one az: E A: : for each LE Jd] nf, Ts FECA) + #(m) = ZH) an Suppose #( ) = oO. st. we @. #(0) = ZFA) = 3. From step 2 Tadnel on fn. Let n+ [, Q= Ar. #@) = 240A) = #4.) 2 — Fer 2051 =¿ HA)2 2 bere A= AUS*S = Sa,,x%4 Suppae inductuely thet n= N, holds €, #EC N =) #(Q) VIF =? Tee S[u4n3, stACZAU ERS, Show that +his Jelds her n= Net, = Sa; xd. #(6) + N+t =¿ #(B) = Wei)ir ¿ NO, - 24h) OX =

QU Ant Tf #Q) = WV. Then Awe i+ Anas USX*3 =¿ HCAwn) = [+t H #(@) =Nrt Tl from asi mation Some FZ! 7é $] 08, 30 AH ALU BOL Th iadoctgen closed.

E3612) (tempt) fir any neler (mmbor n, hé Sr te the sel a all kjecTions EX ¿ X where X= SEEN: [e005 Gi) detwe. the pederial al fn paloral pum ber n reowsively by ol r= / and (n+)! = (att) x nl fr all nakral num bers n Show that ## (Sn) = al fe all nileru—, numbers rn. As tronn ls), # (Snr) = (att) x #(S). Te show #(S) = al por all net, we reed show flere evist such Soe morphten . —. det SWIM, ot Flo)anl, Lhe o frcbrel tntr - Sanpete Plo) te th propty pert Phlis te f Fl = AGS). 2 Tet mn Let no, She #@Sc) =/. 3. Suppose inductively tht TNE ox. HN) = AG.) . Show thet F(N+t) = #(Suer) . we Ino Tht AiN+) = Oe)! = AN) ver). rum previews “), ve fenow thet Hu.) = Wer) x #6.) . st. SN++) = €CSr:), Ta tL cto, deed, 4. From ¢ 3, be lene thet Bere adsl a bije Sh. hefren Hr) crm BCG, for all nafrul humdes p.

Ex 3.1.2 (1st attempt)

Using only Axiom 3.2, Axiom 3.1, Axiom 3.3, Axiom 3.4,
Prove that the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ are all distinct.

1. $\emptyset \neq \{\emptyset\}$. $\emptyset \in \{\emptyset\}$. $\emptyset \notin \emptyset$
   $\emptyset \neq \{\{\emptyset\}\}$. $\{\emptyset\} \in \{\{\emptyset\}\}$, $\{\emptyset\} \notin \emptyset$.
   $\emptyset \neq \{\emptyset, \{\emptyset\}\}$ $\emptyset \in \{\emptyset, \{\emptyset\}\}$. $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$. But not in $\emptyset$.

2. $\{\emptyset\} \neq \{\{\emptyset\}\}$ By Axiom 3.2 (Equality of sets).
   $\{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$.

3. $\{\{\emptyset\}\} \neq \{\emptyset, \{\emptyset\}\}$.

Ex 3.1.5 (1st attempt)

Let A, B be sets. Show that $A \subseteq B$, $A \cup B = B$, $A \cap B = A$ are logically equivalent.

$A \subseteq B, \Rightarrow x \in A, \Rightarrow x \in B,$

$A \cup B = B \Rightarrow x \in A \cup B, \quad x \in A$ or $x \in B.$

$A \cup B = B, \quad x \in A, \Rightarrow x \in B.$

$A \cap B = A \Rightarrow x \in A \cap B, \quad x \in A$ and $x \in B.$

$A \cap B = A. \quad A \subseteq B.$

Ex 3.1.6 (1^{st} attempt)

Prove Proposition 3.1.27 (Sets form a boolean algebra).

Let A, B, C be sets. Let X be a set containing A, B, C as subsets.

(a) $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$ Minimal Element.

(b) $A \cup X = X$ and $A \cap X = A$ Maximal Element

(c) $A \cap A = A$ and $A \cup A = A$ Identity

(d) $A \cup B = B \cup A$ and $A \cap B = B \cap A$ Commutativity

(e) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ Associativity

(f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Distributivity.

(g) $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \emptyset$ Partition

(h) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$ DeMorgan's Laws.

1. From Lemma 3.1.12, $A \cup \emptyset = A$. $x \in A$, $x \notin \emptyset$, s.t. $A \cap \emptyset = \emptyset$.

2. Since $A \subseteq X$, s.t. $x \in A$, $x \in X$. $A \cup X = X$.
   $A \cap X = A$, $x \in X \Rightarrow x \in A$ or $x \notin A$. Thus $A \cap X = A$.

3. From Lemma 3.1.12, $A \cup A = A$. $x \in A$, $x \in A$, $A \cap A = A$.

4. From Lemma 3.1.12, $A \cup B = B \cup A$.
   If $x \in A \cap B$, then $x \in A$ and $x \in B$. $\Rightarrow x \in B \cap A$.

5. From Lemma 3.1.12, $(A \cup B) \cup C = A \cup (B \cup C)$

   If $x \in (A \cap B) \cap C$. Then $x \in A \cap B$ and $x \in C$.
   If $x \in (A \cap B)$. Then $x \in A$, $x \in B$, and $x \in C$. Then $x \in B \cap C$, $x \in A$.

6. If $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$.
   If $x \in B \cup C$. Then $x \in B$ or $x \in C$. $\rightarrow x \in A \cap C$
   $x \in A \cap B$ or $A \cap C$.
   ↳ Then $x \in A \cap B$

   If $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$.
   If $x \in A \cap B$. Then $x \in A$ and $x \in B$., $\Rightarrow x \in A \cap (B \cup C)$.
   If $x \in A \cap C$. Then $x \in A$ and $x \in C$. $\Rightarrow x \in A \cap (B \cup C)$.

7. If $x \in A \cup (X \setminus A)$. Then $x \in A$ or $x \in (X \setminus A)$.
If $x \in A$, then $x \in X$, since $A \subseteq X$.
If $x \in X \setminus A$, Then $x \notin A$, but $x \in X$.

If $x \in X$. Then ($x \in A$ or $x \notin A$) and $x \in X$.
Thus $A \cup (X \setminus A) = X$.

If $x \in A \cap (X \setminus A)$. Then $x \in A$ and $x \notin A$.
No such x exist.
Thus. $A \cap (X \setminus A) = \emptyset$.

8. If $x \in X \setminus (A \cup B)$. Then $x \notin A \cup B$, $\Rightarrow$ $x \notin A$ and $x \notin B$.
$\Rightarrow x \in (X \setminus A) \cap (X \setminus B)$.
If $x \in (X \setminus A) \cap (X \setminus B)$. Then $x \notin A$ and $x \notin B$.
Thus $x \notin A \cup B$. $\Rightarrow x \in X \setminus (A \cup B)$.

If $x \in X \setminus (A \cap B)$. Then $x \notin A \cap B \Rightarrow x \in (X \setminus A) \cup (X \setminus B)$.

If $x \in (X \setminus A) \cup (X \setminus B)$. Then $x \notin A$ or $x \notin B$.
$\Rightarrow x \notin A \cap B$. $\Rightarrow x \in X \setminus (A \cap B)$.

Ex 3.1.8 (1st attempt)
Let A, B be sets. Prove that $A \cap (A \cup B) = A$ and $A \cup (A \cap B) = A$. (known as absorption laws).

1. By proposition 3.1.27 (Distributivity), $A \cap (A \cup B) = (A \cap A) \cup (A \cap B)$.
   By . . . (Identity), $A \cap A = A$.
   Thus, $A \cap (A \cup B) = A \cup (A \cap B)$.
   If $x \in A \cup (A \cap B)$. Then $x \in A$ or $x \in A \cap B$
   If $x \in A$, $x \in A$.
   If $x \in A \cap B$, $x \in A$ and $x \in B$. $\Rightarrow$ $x \in A$. $A \cap (A \cup B) = A$.

2. By distributivity property. $A \cup (A \cap B) = (A \cup A) \cap (A \cup B)$.
   By identity, $A \cup A = A$. Thus $A \cup (A \cap B) = A \cap (A \cup B)$.

$$= (A \cap A) \cup (A \cap B)$$
$$= A \cup (A \cap B)$$
$$= A.$$

Ex 3.1.9 (1^{st} attempt)

Let A, B, X be sets such that $A \cup B = X$ and $A \cap B = \emptyset$.
Show that $A = X \setminus B$ and $B = X \setminus A$.

1. ($\Leftarrow$) If $x \in X \setminus B$. Then $x \in X$ and $x \notin B$.

$x \in X \Rightarrow x \in A \cup B$.

$x \notin B \Rightarrow x \notin A \cap B$.

$x \in (A \cup B) \setminus (A \cap B) \cup B$

$x \in A$.

($\Rightarrow$) If $x \in A$. Then $x \in A \cup B$ $\Rightarrow$ $x \in X \setminus B$.

$x \notin A \cap B$

2. ($\Leftarrow$) If $x \in X \setminus A$. Then $x \in X$ and $x \notin A$.

$x \in A \cup B \setminus A$

$x \in B$

($\Rightarrow$) If $x \in B$. Then $x \in X$, $x \in A \cup B$.

Since $A \cap B = \emptyset$. $x \notin A \cap B$.

$x \in X \setminus A$.

Ex 3.1.12; (1st attempt)

Suppose A, B, A', B' are sets, s.t. $A' \subseteq A$ and $B' \subseteq B$.

(i) Show that $A' \cup B' \subseteq A \cup B$ and $A' \cap B' \subseteq A \cap B$.

1. If $x \in A' \cup B'$. Then $x \in A'$ or $x \in B'$

Since $A' \subseteq A$ and $B' \subseteq B$. If $x \in A'$. Then $x \in A$.

If $x \in B'$. Then $x \in B$.

Thus $x \in A'$ or $x \in B'$ $\Rightarrow$ $x \in A$ or $x \in B$.

$A' \cup B' \subseteq A \cup B$.

2. If $x \in A' \cap B'$. Then $x \in A'$ and $x \in B'$

Similarly. If $x \in A'$ and $x \in B'$. Then $x \in A$ and $x \in B$.

$A' \cap B' \subseteq A \cap B$.

Ex 3.3.1 (1st attempt)

Show that the definition of equality in Defn 3.3.8 is reflexive, symmetric, transitive (A.7).

Verify the substitution property: if $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$ are functions such that $f = \tilde{f}$ and $g = \tilde{g}$. Then $g \circ f = \tilde{g} \circ \tilde{f}$

Defn 3.3.8 (Equality of functions)
Given $f: X \to Y$ and $g: X' \to Y'$. $f = g$ if $X = X'$, $Y = Y'$.
if $f(x) = g(x)$, $\forall x \in X$

1. Suppose there exist $f: X \to Y$, s.t. $\exists f \in \mathcal{F}$, s.t. $f: X \to Y$.
where f and f are identical.
Suppose if $f \neq f$. Then it implies that $\exists x \in X$, $f(x) \neq f(x)$ $\therefore$ Contradict.
$f = f$ (Reflexive).

2. Suppose $f: X \to Y$, $g: X' \to Y'$. if $f = g$. Then $X = X'$, $Y = Y'$, $f(x) = g(x)$. $\forall x \in X$.
Suppose $g \neq f$, s.t. $\exists x \in X'$, s.t. $f(x) \neq g(x)$ $\therefore$ Contradict
$f = g$, $g = f$ (Symmetric).

3. Given $f: X \to Y$, $g: X' \to Y'$, $h: X'' \to Y''$.
Suppose $f = g$, $g = h$, $f \neq h$.

$f(x) = g(x)$, $\forall x \in X$. $X = X'$, $Y = Y'$.

$g(x) = h(x)$, $\forall x \in X'$, $X' = X''$, $Y' = Y''$

$f(x) \neq h(x)$, $\exists x \in X''$. $\therefore$ Contradict $f(x) = g(x) = h(x)$, $\forall x \in X$. $X = X' = X''$.

It is transitive.

4. Suppose $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$, s.t. $f = \tilde{f}$, $g = \tilde{g}$.
Then by Defn 3.3.8, $f(x) = \tilde{f}(x)$, $\forall x \in X$.
$g(y) = \tilde{g}(y)$, $\forall y \in Y$.

By Axiom of Substitution, $(g \circ f)(x) = (g \circ \tilde{f})(x)$ since $g = \tilde{g}$
$(\tilde{g} \circ f)(x) = (\tilde{g} \circ \tilde{f})(x)$
Then the four expressions are equivent.
Thus. $g \circ f = \tilde{g} \circ \tilde{f}$.

Ex 3.3.3 (1^{st} attempt)

When is the empty function into a given set X injective? surjective? bijective?

1. Given $f: \emptyset \to X$, where f is the empty function.

   f is always injective as the domain is empty.

   f is never surjective, thus never bijective, since $\exists! x \in X$, s.t. $x = f()$.

Ex 3.3.7 (1^{st} attempt)

Let $f: X \to Y$ and $g: Y \to Z$ be functions.

Show that if f and g are bijective. Then $g \circ f$ is also bijective.

and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

1. If f and g are bijective. Then $\forall y \in Y, \exists! x \in X, \ y = f(x)$.

   $\forall z \in Z, \exists! y \in Y, \ z = g(y)$

$\Rightarrow \ \forall z \in Z, \ \exists! x \in X, \text{ s.t. } z = (g \circ f)(x)$. $(g \circ f)$ is bijective.

2. From previous. $(g \circ f)^{-1}$, f^{-1}, and g^{-1} are bijective.

   $\forall x \in X, \ \exists! z \in Z, \text{ s.t. } x = (g \circ f)^{-1}(z)$.

$$\left.\begin{array}{l} \forall x \in X, \ \exists! y \in Y, \text{ s.t. } x = f^{-1}(y) \\ \forall y \in Y, \ \exists! z \in Z, \text{ s.t. } y = g^{-1}(z) \end{array}\right] \Rightarrow \begin{array}{l} \forall x \in X, \ \exists! z \in Z, \text{ s.t. } \\ x = (f^{-1} \circ g^{-1})(z) \end{array}$$

Thus $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$.

Ex 3.3.8 (b) (d)

If X is a subset of Y, let $\iota_{X\to Y} : X \to Y$ be the inclusion map from X to Y.
defined by mapping $x \mapsto x$ for all $x \in X$, i.e., $\iota_{X\to Y}(x) := x$ for all $x \in X$.
The map $\iota_{X\to X}$ is in particular called the identity map on X.

(b) Show that if $f : A \to B$. Then $f = f \circ \iota_{A\to A} = \iota_{B\to B} \circ f$.

(d) Show that if X and Y are disjoint sets and $f: X \to Z$ and $g: Y \to Z$
Then there is a unique function $h : X \cup Y \to Z$ such that $h \circ \iota_{X\to X\cup Y} = f$
and $h \circ \iota_{Y \to X\cup Y} = g$.

1. If $f : A \to B$. We know that $\iota_{A\to A} : A \to A$
$\iota_{B\to B} : B \to B$

$\iota_{A\to A}(x) := x$, $x \in A$ $\Rightarrow$ $f \circ \iota_{A\to A} : A \to B$
$\iota_{B\to B}(x) := x$, $x \in B$. $\Rightarrow$ $\iota_{B\to B} \circ f : A \to B$

wrong as $X \subseteq Y$

2. If X and Y are disjoint, $X \cap Y = \emptyset$, $f: X \to Z$, $g : Y \to Z$.
Suppose $h : X \cup Y \to Z$ is not unique, s.t. $\exists h_1 : X \cup Y \to Z$. $h_1 \neq h$.

$\iota_{X\to X\cup Y} : X \to X \cup Y$
$h : X \cup Y \to Z$
$h_1 : X \cup Y \to Z$
$\iota_{Y\to X\cup Y} : Y \to X \cup Y$

$h \circ \iota_{X\to X\cup Y} : X \to Z$, s.t. $h(x) = f(x), \forall x \in X$.
$h \circ \iota_{Y\to X\cup Y} : Y \to Z$., $h(y) = g(y), \forall y \in Y$.

If $h_1 \circ \iota_{X\to X\cup Y} : X \to Z$, s.t. $h_1(x) = f(x), \forall x \in X$.
$h_1 \circ \iota_{Y\to X\cup Y} : Y \to Z$, s.t. $h_1(y) = g(y)$. $\forall y \in Y$.
$\therefore$ $h(x) = f(x)$ $\forall x \in X$ $\quad h(x) \neq h_1(x)$.
$h(y) = g(y)$ $\forall y \in Y$.

There is only one unique function h.

This applies as
$X \cap Y = \emptyset$.
$f : \emptyset \to Z$ $\quad f = g$. this is empty function
$g : \emptyset \to Z$,

Exercise from Defn 3.4.1
Given $f: X \to Y$ is a function from X to Y. $S \subseteq X$.
$f(S) := \{f(x) : x \in S\}$, where $f(S)$ is the forward image of S.

Show that $f(S)$ can be defined using Axiom of specification (Axiom 3.6)
Show that $f(S)$ can be defined using Axiom of replacement (Axiom 3.7).

1. By Axiom of specification (Axiom 3.6), $\forall x \in X$, $\exists! y \in Y$, s.t. $y = f(x)$. (Defn of function)
$y \in \{f(x) : x \in S\} \iff x \in S \subseteq X$ and $y = f(x) \in Y$

2. By Axiom of Replacement (Axiom 3.7),
$z \in \{y : y = f(x)$ for some $x \in S\} \iff \exists! z \in Y, f(x) = z$, for some $x \in S \subseteq X$.

P.57

Given $f: \mathbb{N} \to \mathbb{N}$, $f(x) = 2x$.

Evaluate whether f is injective.

1. To show that f is injective, we have to show that

$\forall n_1, n_2 \in \mathbb{N}$, if $n_1 = n_2$. Then $f(n_1) = f(n_2)$ By Definition 3.3.17

if $n_1 \neq n_2$. Then $f(n_1) \neq f(n_2)$. (one-to-one functions)

2. Assume $n_1 \times n_2 = m$ (WLOG), $n_1 \neq n_2$

Fix n_2 and induct on n_1. , $n_2 > 0$

Let $n_1 = 0$. $n_1 \times n_2 = 0 \times n_2 = m = 0$.

$m = n_1$, $f(m) = 2 \cdot m = 2 \cdot n_1 = 0$ $f(m) = f(n_1)$.

$f(n_1) = 2 \cdot n_1 = 2 \cdot n_1 = 0$

Let $n_1 = 1$, $n_1 \times n_2 = 1 \times n_2 = n_2 = m$

$m \neq n_1$, $f(m) = 2 \cdot n_2 = 2n_2$ $f(m) \neq f(n_1)$, since $n_2 \neq (n_1 = 1)$

$f(n_1) = 2 \cdot n_1 = 2$ (Base Case).

3. Inductively assume that if $m = n_1$. Then $f(m) = f(n_1)$

if $m \neq n_1$. Then $f(m) \neq f(n_1)$.

Show that if $m = n_1 + 1$. Then $f(m) = f(n_1 + 1)$

if $m \neq n_1 + 1$. Then $f(m) \neq f(n_1 + 1)$.

If $m = n_1 + 1$. L.H.S $f(m) = f(n_1 + 1) = 2(n_1 + 1) = 2n_1 + 2$

R.H.S $f(n_1 + 1) = 2(n_1 + 1) = 2n_1 + 2$ L.H.S = R.H.S.

If $m \neq n_1 + 1$. L.H.S $f(m) = f(n_1 n_2) = 2(n_1 \times n_2)$

R.H.S $f(n_1 + 1) = 2(n_1 + 1)$

By Cancellation Law $2(n_1 \times n_2) \overset{?}{=} 2(n_1 + 1)$

$n_1 \times n_2 \overset{?}{=} n_1 + 1$

$\exists n_2 \in \mathbb{N}, n_1 \neq n_2$, s.t. $n_1 \times n_2 \neq n_1 + 1$.

$\Rightarrow$ L.H.S $\neq$ R.H.S.

Induction closed. f is one-to-one

f is not surjective as $\exists y \in \mathbb{N}$, s.t. $\nexists x \in \mathbb{N}$, $f(x) = y$.

Ex 3.4.3 (1^{st} attempt)

Let A, B be two subsets of a set X. Let $f: X \to Y$ be a function.

Show that $f(A \cap B) \subseteq f(A) \cap f(B)$ } For these two statements,

Show that $f(A) \setminus f(B) \subseteq f(A \setminus B)$ } is it true that the $\subseteq$ relation can be

Show that $f(A \cup B) = f(A) \cup f(B)$ improved to $=$?

1. ($\Rightarrow$)

If $x \in A \cap B$. Then $(x \in A) \wedge (x \in B)$.

$f(A) := \{f(x) : x \in A\}$ $f(B) := \{f(x) : x \in B\}$.

$f(A \cap B) := \{f(x) : x \in A \cap B\}$.

Thus. $\forall y \in f(A \cap B) \Rightarrow y \in f(A) \cap f(B)$. Otherwise Contradiction.

$f(A \cap B) \subseteq f(A) \cap f(B)$

($\Leftarrow$)

If $x \in A$. Then $f(x) \in f(A)$. If $f(x) \in f(A) \cap f(B)$.

If $x \in B$. Then $f(x) \in f(B)$.

Then by axiom of substitution, $x \in A \cap B$.

$f(x) \in f(A \cap B)$.

$f(A) \cap f(B) \subseteq f(A \cap B)$

Thus $f(A) \cap f(B) = f(A \cap B)$.

2. ($\Rightarrow$)

If $x \in A$. Then $f(x) \in f(A)$. If $x \in B$. Then $f(x) \in f(B)$.

Suppose $f(x) \in f(A) \setminus f(B)$. Then this implies that $f(x) \in f(A)$

$x \notin A \cap B \Leftarrow f(x) \notin f(A \cap B)$

$x \notin B. \Leftarrow f(x) \notin f(B)$.

Thus $x \in A \setminus B$. , $f(x) \in f(A \setminus B)$. Thus $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

($\Leftarrow$)

If $x \in A \setminus B$. Then $f(x) \in f(A \setminus B)$.

$x \in A \setminus B \Rightarrow x \in A$, $x \notin A \cap B$, $x \notin B$.

$\Rightarrow f(x) \in f(A)$, $f(x) \notin f(A \cap B)$, $f(x) \notin f(B)$.

$\Rightarrow f(x) \in f(A) \setminus f(B)$. $f(A \setminus B) \subseteq f(A) \setminus f(B)$.

Thus $f(A) \setminus f(B) = f(A \setminus B)$.

3. ($\Rightarrow$)

If $x \in A \cup B$. Then $f(x) \in f(A \cup B)$.

$\Rightarrow x \in A$ or $x \in B$.

$\Rightarrow f(x) \in f(A)$ or $f(x) \in f(B)$. $\Rightarrow f(x) \in f(A) \cup f(B)$. $f(A \cup B) \subseteq f(A) \cup f(B)$.

($\Leftarrow$)

If $x \in A$ or $x \in B$. Then $f(x) \in f(A) \cup f(B)$.

$\Rightarrow x \in A \cup B$. Then $f(x) \in f(A \cup B)$. $f(A) \cup f(B) \subseteq f(A \cup B)$.

$f(A \cup B) = f(A) \cup f(B)$.

Ex 3.4.4 (1^{st} attempt)
Let $f: X \to Y$ be a function from one set X to another set Y.
Let U, V be subsets of Y.
Show that $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$
Show that $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$
Show that $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$

1. ($\Rightarrow$)
$f^{-1}(U \cup V) := \{x \in X : f(x) \in U \cup V\}$
$f(x) \in U \cup V \iff x \in f^{-1}(U \cup V)$
$\Rightarrow (f(x) \in U) \cup (f(x) \in V)$
$\Rightarrow (x \in f^{-1}(U)) \cup (x \in f^{-1}(V))$
$\Rightarrow x \in f^{-1}(U) \cup f^{-1}(V)$
$\Rightarrow f^{-1}(U \cup V) \subseteq f^{-1}(U) \cup f^{-1}(V)$

($\Leftarrow$)
$f^{-1}(U) \cup f^{-1}(V) := \{x \in X : f(x) \in U\} \cup \{x \in X : f(x) \in V\}$.
$f(x) \in U \cup f(x) \in V \iff x \in f^{-1}(U) \cup f^{-1}(V)$.
$\Rightarrow f(x) \in U \cup V \Rightarrow x \in f^{-1}(U \cup V)$
$\Rightarrow f^{-1}(U) \cup f^{-1}(V) \subseteq f^{-1}(U \cup V)$

2. ($\Rightarrow$)
$f^{-1}(U \cap V) := \{x \in X : f(x) \in U \cap V\}$
$f(x) \in U \cap V \iff x \in f^{-1}(U \cap V)$
$\Rightarrow (f(x) \in U) \cap (f(x) \in V)$
$\iff (x \in f^{-1}(U)) \cap (x \in f^{-1}(V))$
$\iff x \in f^{-1}(U) \cap f^{-1}(V)$
$f^{-1}(U \cap V) \subseteq f^{-1}(U) \cap f^{-1}(V)$.

Thus $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$.

($\Leftarrow$)
$f^{-1}(U) \cap f^{-1}(V) := \{x \in X : f(x) \in U\} \cap \{x \in X : f(x) \in V\}$.
$(f(x) \in U) \cap (f(x) \in V) \iff x \in f^{-1}(U) \cap f^{-1}(V)$.
$\Rightarrow f(x) \in U \cap V$
$\Rightarrow x \in f^{-1}(U \cap V)$ (Defn 3.4.5).
$f^{-1}(U) \cap f^{-1}(V) \subseteq f^{-1}(U \cap V)$.

3. ($\Rightarrow$)
$f^{-1}(U \setminus V) := \{x \in X : f(x) \in U \setminus V\}$.
$f(x) \in U \setminus V \iff x \in f^{-1}(U \setminus V)$
$\Rightarrow f(x) \in U$ and $f(x) \notin V$ (Defn 3.4.5)
$\Rightarrow x \in f^{-1}(U)$ and $x \notin f^{-1}(V)$
$\Rightarrow x \in f^{-1}(U) \setminus f^{-1}(V)$.
$f^{-1}(U \setminus V) \subseteq f^{-1}(U) \setminus f^{-1}(V)$.

Thus $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$.

($\Leftarrow$)
$f^{-1}(U) \setminus f^{-1}(V) := \{x \in X : f(x) \in U\} \setminus \{x \in X : f(x) \in V\}$.
$f(x) \in U$ and $f(x) \notin V$.
$\iff x \in f^{-1}(U) \setminus f^{-1}(V)$.
$f(x) \in U \setminus V$
$\Rightarrow x \in f^{-1}(U \setminus V)$.
$f^{-1}(U) \setminus f^{-1}(V) \subseteq f^{-1}(U \setminus V)$.

Ex 3.4.5 (1^{st} attempt)

Let $f: X \to Y$ be a function from one set X to another set Y.

Show that $f(f^{-1}(S)) = S$ for every $S \subseteq Y$ iff f is surjective.

Show that $f^{-1}(f(S)) = S$ for every $S \subseteq X$ iff f is injective.

1. ($\Rightarrow$) $f(f^{-1}(S)) := \{f(x) : x \in f^{-1}(S)\}$, where $f^{-1}(S) := \{x \in X : f(x) \in S\}$.

If $f(f^{-1}(S)) = S$. Then $\forall y \in S$, $y \in \{f(x) : x \in f^{-1}(S)\}$.

$\Rightarrow \exists x \in X$, s.t. $x \in f^{-1}(S)$, s.t. $y = f(x)$.

Thus. f is surjective.

($\Leftarrow$) If f is surjective. Then $\forall y \in Y$, $\exists x \in X$, s.t. $y = f(x)$.

Thus. $f(f^{-1}(S))$ implies. $\forall y \in Y$. $\exists x \in f^{-1}(S) \subseteq X$, s.t. $y = f(x)$.

Thus $f(f^{-1}(S)) \subseteq S \subseteq Y$ as $f(f^{-1}(S)) \in S$

Suppose $\exists y \in S$, s.t. $y \notin f(f^{-1}(S))$.

Then. $\exists y \in S \subseteq Y$, $\nexists x \in f^{-1}(S) \subseteq X$, s.t. $y = f(x)$ ∵※ Contradiction

$\nexists$ $y \in S$. Then $y \in f(f^{-1}(S)) \Rightarrow S \subseteq f(f^{-1}(S))$

Thus $S = f(f^{-1}(S))$

2. ($\Rightarrow$) $f^{-1}(f(S)) := \{x \in X : f(x) \in f(S)\}$, where $f(S) := \{f(x) : x \in S\}$.

If $f^{-1}(f(S)) = S$. Then $\forall x \in f^{-1}(f(S))$, $x \in S$.

If $x_1 \neq x_2 \in f^{-1}(f(S))$. Then $f(x_1) \in f(S)$, $f(x_2) \in f(S)$. If $f(x_1) = f(x_2)$. Then $(f^{-1} \circ f)(x_1)$, $(f^{-1} \circ f)(x_2)$ $\exists x_1$ or x_2 $\in S$, But not in $f^{-1}(f(S))$ ∵※ Contradiction.

If $x_1 = x_2 \in f^{-1}(f(S))$. Then $f(x_1) \in f(S)$, $f(x_2) \in f(S)$. If $f(x_1) \neq f(x_2)$ ∵※ Contradiction to definition of function. $\forall x \in S$, $\exists! y \in Y$, s.t. $y = f(x)$.

Thus f is injective.

($\Leftarrow$) If f is injective. Suppose $x_1 \neq x_2$, $x_1, x_2 \in f^{-1}(f(S))$,

$f(x_1) \neq f(x_2) \in f(S)$.

$\Rightarrow$ By def, $x_1, x_2 \in S$, $x_1 \neq x_2$.

$f^{-1}(f(S)) \subseteq S$.

Suppose $\exists x_1, x_2 \in S$, s.t. $x_1 \neq x_2$

$f(x_1), f(x_2) \in f(S)$, $f(x_1) \neq f(x_2)$.

$x_1, x_2 \in f^{-1}(f(S)) \Rightarrow S \subseteq f^{-1}(f(S))$

Thus $f^{-1}(f(S)) = S$.

Ex 3.4.10 (1^{st} attempt)

Suppose that I and J are two sets, and for all $\alpha \in I \cup J$, let A_α be a set.

Show that $\left(\bigcup_{\alpha\in I} A_\alpha\right) \cup \left(\bigcup_{\alpha\in J} A_\alpha\right) = \bigcup_{\alpha\in I\cup J} A_\alpha$.

If I and J are non-empty, show that $\left(\bigcap_{\alpha\in I} A_\alpha\right) \cap \left(\bigcap_{\alpha\in J} A_\alpha\right) = \bigcap_{\alpha\in I\cup J} A_\alpha$.

1. By Axiom 3.12 (Union), $\bigcup_{\alpha\in I} A_\alpha := \bigcup\{A_\alpha : \alpha\in I\}$ $\bigcup_{\alpha\in J} A_\alpha := \bigcup\{A_\alpha : \alpha\in J\}$

By Axiom 3.4, there exists a set $P = \{\bigcup_{\alpha\in I} A_\alpha, \bigcup_{\alpha\in J} A_\alpha\}$

By Axiom 3.12, $x \in \bigcup P \iff (x\in \bigcup_{\alpha\in I} A_\alpha \text{ or } x\in \bigcup_{\alpha\in J} A_\alpha) \iff x\in (\bigcup_{\alpha\in I} A_\alpha)\cup(\bigcup_{\alpha\in J} A_\alpha)$

$\Updownarrow$

$x\in \bigcup_{\alpha\in I\cup J} A_\alpha$. (Also deduced from Ex 3.4.8)

2. Similarly, By Axiom of specification, $\bigcap_{\alpha\in I} A_\alpha := \{x : x\in A_\alpha \text{ for all } \alpha\in I\}$

$\bigcap_{\alpha\in J} A_\alpha := \{x : x\in A_\alpha \text{ for all } \alpha\in J\}$.

By Axiom 3.4, there exists a set $Q = \{\bigcap_{\alpha\in I} A_\alpha, \bigcap_{\alpha\in J} A_\alpha\}$.

By modification of Axiom 3.12 (for intersection),

$x\in \bigcap Q \iff (x\in \bigcap_{\alpha\in I} A_\alpha \text{ and } x\in \bigcap_{\alpha\in J} A_\alpha) \iff x\in (\bigcap_{\alpha\in I} A_\alpha)\cap(\bigcap_{\alpha\in J} A_\alpha)$

$\Updownarrow$

$x\in \bigcap_{\alpha\in I\cap J} A_\alpha$.

Ex 3.5.1 (ii) (1st attempt)

Suppose we have an alternative definition for ordered pairs. $(x,y) := \{x, \{x,y\}\}$.
Show that this definition (Short definition of an ordered pair)
also verifies the property (3.5), $(x,y) = (x',y') \iff (x=x',\ y=y')$ of an ordered pair.

($\Rightarrow$) Suppose $(x,y) = (x',y')$, s.t.
by the short definition of an ordered pair,
$\{x, \{x,y\}\} = \{x', \{x',y'\}\}$
s.t. by Axiom 3.2 and Axiom 3.4, $x = x'$, $y = y'$.

($\Leftarrow$) Suppose $x = x'$, $y = y'$.
Then we know that by Axiom 3.4, thus Axiom 3.2,
$\{x,y\} = \{x',y'\}$. Thus $\{x, \{x,y\}\} = \{x', \{x',y'\}\}$.
Thus applying the short definition, $\cdots$, $(x,y) = (x',y')$.

Ex 3.5.1 (ii) (2^{nd} attempt, check solution, then do it)
Suppose we have an alternative definition for ordered pairs. $(x,y) := \{x, \{x,y\}\}$.
Show that this definition (Short definition of an ordered pair)
also verifies the property (3.5), $(x,y) = (x',y') \Leftrightarrow (x = x', y = y')$ of an ordered pair.

1. ($\Rightarrow$) If $(x,y) = (x',y)$ Then $\{x, \{x,y\}\} = \{x', \{x',y'\}\}$
   This implies that $(x = x')$ or $x = \{x',y'\}$ by Axiom of regularity
   If $x = \{x',y'\}$. Then $x \notin \{x',y'\}$, s.t. $x \neq x'$ and $x \neq y'$. $\cdot\times\cdot$ Contradiction.
   If $x = x'$. Then $x \in \{x',y'\}$.
   Thus $x = x'$
   Similarly, either $(\{x,y\} = x')$ or $(\{x,y\} = \{x',y'\})$.
   If $\{x,y\} = x'$ Then since $x = x'$, $x \neq \{x,y\}$, $\{x,y\} \notin \{x,y\}$.
   Thus since $(x,y) = (x',y')$, $\{x,y\} \in (x',y') \Rightarrow \{x,y\} = \{x',y'\}$.
   Since $x = x'$, thus $y = y'$.

2. ($\Leftarrow$) If $(x = x')$ and $(y = y')$. Then $x \in \{x,y\}$, $x \in \{x',y'\}$, $y \in \{x,y\}$, $y \in \{x',y'\}$ $\Rightarrow \{x,y\} \subseteq \{x',y'\}$
   $x' \in \{x',y'\}$, $x' \in \{x,y\}$, $y' \in \{x',y'\}$, $y' \in \{x,y\}$ $\Rightarrow \{x',y'\} \subseteq \{x,y\}$.
   $\Rightarrow \{x,y\} = \{x',y'\}$.
   Since $x = x'$ and $\{x,y\} = \{x',y'\}$,
   Thus $(x,y) \subseteq (x',y')$, $(x',y') \subseteq (x,y)$ Thus $(x,y) = (x',y')$

Ex 3.5.2 (1^{st} attempt)

Suppose we define an ordered n-tuple to be a surjective function x,
$x : \{i \in \mathbb{N} : 1 \le i \le n\} \to X$, where X is arbitrary.
Write $x(i)$ as x_i and write x as $(x_i)_{1 \le i \le n}$.

Using this definition, verify that $(x_i)_{1\le i\le n} = (y_i)_{1\le i\le n}$ iff $x_i = y_i$ for all $1 \le i \le n$.

Also, show that if $(X_i)_{1\le i\le n}$ are an ordered n-tuple of sets.
Then the Cartesian Product defined in (Defn 3.5.6) is a set.

1. ($\Rightarrow$) If $(x_i)_{1\le i\le n} = (y_i)_{1\le i\le n}$, where $x : \{i \in \mathbb{N} : 1 \le i \le n\} \to X$
$y : \{i \in \mathbb{N} : 1 \le i \le n\} \to X$.
By defn 3.3.8 (Equility of fnc), $x = y$. iff $x(i) = y(i)$, $\forall i \in \{i \in \mathbb{N} : 1 \le i \le n\}$.
($\Leftarrow$) If $x_i = y_i$ But $(x_i)_{1\le i\le n} \ne (y_i)_{1\le i\le n}$.
Then $\cancel{X}$ Contradiction, as $\exists i$, s.t. $x_i \ne y_i$. $\Rightarrow$ thus $X = Y$.

2. If $(X_i)_{1\le i\le n}$ are an ordered n-tuple of sets.
s.t. $(X_1, \ldots, X_n)$, which is a collection of sets $X_1, \ldots, X_n$.

The Cartesian product (Defn 3.5.6). $\prod_{1\le i\le n} X_i$

Induct on n. s.t. Let $n = 2$, $\prod_{1\le i\le 2} X_i = X_1 \times X_2$
If X_1 and X_2 are sets. Then $X_1 \times X_2$ are sets.

Inductively assume that $\prod_{1\le i\le n} X_i = X_1 \times X_2 \times \cdots \times X_n$ is a set.
Show that $\prod_{1\le i\le n+1} X_i$ is also a set.

$\prod_{1\le i\le n+1} X_i = \left(\prod_{1\le i\le n} X_i\right) \times X_{n+1}$, since $\left(\prod_{1\le i\le n} X_i\right)$ and X_{n+1} are sets.

Then $\prod_{1\le i\le n+1} X_i$ is also a set.

Induction closed.

Ex 3.5.7 (1^{st} attempt)

Let X, Y be sets.

Let $\pi_{X\times Y \to X} : X\times Y \to X$ be map $\pi_{X\times Y\to X}(x,y) := x$ — Co-ordinate functions on $X\times Y$.

Let $\pi_{X\times Y \to Y} : X\times Y \to Y$ be map $\pi_{X\times Y\to Y}(x,y) := y$.

Show that for any functions $f: Z \to X$ and $g: Z \to Y$, there exists a unique function $h: Z \to X\times Y$, s.t. $\pi_{X\times Y\to X} \circ h = f$, $\pi_{X\times Y\to Y} \circ h = g$

h is known as the pairing of f and g. $h = (f,g)$.

1. Given there are two functions h_1 and h_2, s.t. $h_1 : Z \to X\times Y$, $h_2 : Z \to X\times Y$, where $h_1 \neq h_2$

$$\pi_{X\times Y\to X} \circ h_1 = f_1$$
$$\pi_{X\times Y\to X} \circ h_2 = f_2$$
$$\pi_{X\times Y\to Y} \circ h_1 = g_1$$
$$\pi_{X\times Y\to Y} \circ h_2 = g_2$$

Given a $z \in Z$, s.t. $f_1(z) = x_1$, $g_1(z) = y_1$,
$f_2(z) = x_2$, $g_2(z) = y_2$

By axiom of substitution, $x_1 = x_2$ and $y_1 = y_2$ iff $f_1 = f_2$, $g_1 = g_2$.

2. Assume that $f_1 = f_2$, $g_1 = g_2$ (no unique function h), s.t. $h_1 \neq h_2$.

$\Rightarrow$ $\pi_{X\times Y\to X} \circ h_1 = \pi_{X\times Y\to X} \circ h_2$ ※ Contradict $h_1 \neq h_2$.

Thus $f_1 = f_2$, $g_1 = g_2$ iff $h_1 = h_2$. $h_1 = h_2 = h$ is a unique function.

Ex 3.5.10 (i) (1st attempt)

If $f: X \to Y$ is a function, define the graph of f to be the subset of $X \times Y$ defined by $\{(x, f(x)) : x \in X\}$.

(i) Show that two functions $f: X \to Y$, $\tilde{f}: X \to Y$ are equal iff they have the same graph.

1. ($\Rightarrow$)

If $f = \tilde{f}$. Then $x \in X$ $f(x) = \tilde{f}(x)$.

Thus $(x, f(x)) = (x, \tilde{f}(x))$

$\Rightarrow$ graph of f = graph of $\tilde{f}$, $\{(x, f(x)) : x \in X\} = \{(x, \tilde{f}(x)) : x \in X\}$.

($\Leftarrow$)

If graph of f and $\tilde{f}$ are the same,

Then $x \in X$, $(x, f(x)) = (x, \tilde{f}(x))$, thus. $\tilde{f}(x) = f(x)$, $\Rightarrow$ $f = \tilde{f}$.

Ex 3.5.12 (i) (2nd attempt)

This will establish Proposition 2.1.16 (Recursive Definitions) rigorously, that avoids circularity.

(i) Let X be a set. Let $f : \mathbb{N} \times X \to X$ be a function.

Let c be an element of X.

Show that there exists a function $a : X \to X$ s.t. $a(0) = c$ and

$a(n{+}{+}) = f(n, a(n))$ for all $n \in \mathbb{N}$., and furthermore this function is unique.

Ex 3.5.12 (ii) (2^{nd} attempt)

This will establish Proposition 2.1.16 (Recursive Definitions) rigorously.

(ii) Prove (i) without using any properties of the natural numbers other than the Peano Axioms directly.

Ex 3.6.1 (1st attempt)
Prove Proposition 3.6.4
Let X, Y, Z be sets. Then X has equal cardinality with X.
If X has equal cardinality with Y. Then Y has equal cardinality with X.
If X has equal cardinality with Y and Y has equal cardinality with Z,
Then X has equal cardinality with Z.

1. By defn 3.6.1 (Equal Cardinality). $f: X \to X$, for every $x \in X$, $\exists!\, x \in X$, s.t. $f(x) = x$.
for $x, y \in X$, $x \neq y$, $f(x) = x$, $f(y) = y$, $x \neq y$. Injective.
Suppose $\exists x \in X$, s.t. $\nexists!\, x \in X$, s.t. $f(x) = x$. ⨳
Contradiction, such a x does not exist. (Surjection) Thus f is bijective $\Rightarrow$ X has equal cardinality with X.

2. If X has equal cardinality with Y. Then by defn 3.6.1,
$f: X \to Y$. f is a bijection from X to Y.
Thus f^{-1} exists as f's inverse and we know that f^{-1} is bijective from Y to X.
thus Y has equal cardinality with X.

3. If X has equal cardinality with Y. $f: X \to Y$, f is bijective.
If Y has equal cardinality with Z. $g: Y \to Z$, g is bijective.
Then $(f \circ g): X \to Z$ is also bijective $\Rightarrow$ X has equal cardinality with Z.

Ex 3.6.2 (1st attempt)

Show that a set X has cardinality 0 iff X is the empty set.

1. ($\Rightarrow$) X has cardinality 0 if X has equal cardinality with $\{i \in \mathbb{N} : 1 \leq i \leq 0\}$ (Defn 3.6.5).
   we know that $\{i \in \mathbb{N} : 1 \leq i \leq 0\} = \emptyset$.

   ($\Leftarrow$) If X is the empty set. Then X has equal cardinality with $\emptyset = \{i \in \mathbb{N} : 1 \leq i \leq 0\}$.
   $\#(X) = 0$,

Ex 3.6.4 (1st attempt)

Prove Proposition 3.6.14 (Cardinal Arithmetic)

(a) Let X be a finite set, and $x \notin X$. Then $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X) + 1$.

(b) Let X and Y be finite sets. Then $X \cup Y$ is finite, $\#(X \cup Y) \leq \#(X) + \#(Y)$

(c) Let X be a finite set. Let $Y \subseteq X$. Then Y is finite and $\#(Y) \leq \#(X)$.

(d) If X is a finite set, and if $f: X \to Y$ is a function. Then $f(X)$ is a finite set, with $\#(f(X)) \leq \#(X)$.

(e) Let X and Y be finite sets. Then the Cartesian product $X \times Y$ is finite and $\#(X \times Y) = \#(X) \times \#(Y)$.

(f) Let X and Y be finite sets. Then the set Y^X is finite and $\#(Y^X) = \#(Y)^{\#(X)}$.

1. If X is a finite set. By defn 3.6.10 (finite set), $\#(X) = n$, $\exists n \in \mathbb{N}$.
   Suppose $X \cup \{x\}$, $\#(X \cup \{x\}) \neq \#(X) + 1$.
   Then by Lemma 3.6.9, $\#(X) \neq \#(X \cup \{x\}) - 1$. $\times$ Contradiction.

2. If X and Y are finite. Then $f: X \to \{i \in \mathbb{N} : 1 \leq i \leq n\}$. $\exists h$, $h: X \to X \cup Y$
   $g: Y \to \{i \in \mathbb{N} : 1 \leq i \leq n\}$. $\exists k$, $k: Y \to X \cup Y$.
   where h, k are identity maps.
   Since $\forall x, y \in X, Y$, $x, y \in X \cup Y$. Then $X \cup Y$ is finite, as if it is infinite, it does not have cardinality for any natural number n. $\times$ contradict X, and Y are finite.

   If $X \cap Y = \emptyset$. Then $\#(X \cup Y) = \#(X) + \#(Y)$.
   If $X \cap Y \neq \emptyset$. Then $\exists x \in X \cup Y$. $x \in X$, $x \in Y$. $x = x$. s.t.
   by axiom of specification $\{x : x \in X \cap Y\}$.
   $X \cup Y = X \setminus \{x : x \in X \cap Y\} \cup Y$
   $= \#(X) - \#(\{x : x \in X \cap Y\}) + \#Y$.

3. If $Y \subseteq X$, X is a finite set. Then there exist a set $\{x : x \in X \setminus Y\} = S$, s.t. $Y \cup \{x : x \in X \setminus Y\} = X$.
   by induction if S is empty. $\#(S) = 0$. Then $\#Y = \#X$.
   But for all $n > 0$, $\#(S) = n$. $\#Y < \#X$.

4. If $f: X \to Y$. X is finite. Then since $\forall x \in X$, $\exists! y \in Y$, s.t. $y = f(x)$.
   s.t. If X is finite. Then $f(X)$ cannot be infinite, as it would contradict the definition of functions.

5. If X and Y are finite. Then $X \times Y := \{(x, y) : x \in X, y \in Y\}$.
   Suppose $\{x_1, \ldots, x_n\} = X$, $\#(X) = n$, $\{y_1 \ldots y_m\} = Y$. $\#(Y) = m$.
   Then for each x_i, $1 \leq i \leq n$, there are m elements in Y, s.t. forming m ordered pairs in $X \times Y$.
   $\sum_{i=1}^{n} m = \underset{\in \mathbb{N}}{n \times m}$. ordered pairs. It cannot be infinite

6. If X and Y are finite. Then $\#(Y^X) = \#(Y)^{\#(X)}$. Prove by induction.
   ① Fix $\#(Y) = n$. Induct on $\#(X) = m$.
   ② Induct on $\#(Y) = n$.

Ex 3.6.5 (1st attempt)
Let A and B be sets.
Show that $A \times B$ and $B \times A$ have equal cardinality by constructing an explicit bijection between the two sets.

Then we Proposition 3.6.14 (Cardinal Arithmetic) to conclude an alternative proof of Lemma 2.3.2 (Multiplication is commutative)

1. Suppose $f: A \times B \to B \times A$, $f(a,b) = (b,a)$, f is a bijection.

Since $\#(A \times B) = \#(B \times A)$

$\#(A \times B) = \#(A) \times \#(B)$

$\#(B \times A) = \#(B) \times \#(A)$. Thus $\#(B) \times \#(A) = \#(A) \times \#(B)$.

Ex 3.6.6 (1st attempt)

Let A, B and C are sets

Show that the sets $(A^B)^C$ and $A^{B\times C}$ have equal cardinality by constructing an explicit bijection between the two sets.

Conclude that $(a^b)^c = a^{bc}$ for any $a, b, c \in \mathbb{N}$.

and $a^b \times a^c = a^{b+c}$

1. $\forall f \in (A^B)^C \iff f : C \to A^B$

$\forall f \in A^{B\times C} \iff f : B\times C \to A$

If $f \in (A^B)^C$. Then $\forall c \in C$, $f(c) \in A^B$, which implys $f(c) : B \to A$.

Define a map from $(A^B)^C$ to $A^{B\times C}$ as follows.

$$h : (A^B)^C \to A^{B\times C}, \quad \forall f \in (A^B)^C, \; h(f) = g \in A^{B\times C}$$

where $g(b,c) = (f\circ c)(b)$

Suppose h is a bijection, s.t.

Both sets have the same cardinality.

$$\#(A^B)^C = \#(A)^{\#(B)\times\#(C)} \Rightarrow (a^b)^c = a^{b\times c}$$

2. Show that $A^B \times A^C$ have the same cardinality as $A^{B\cup C}$.

Define a map from $A^B \times A^C$ to $A^{B\cup C}$

$$h : A^B \times A^C \to A^{B\cup C}. \quad \forall f \in A^B \times A^C, \; h(f) = g \in A^{B\cup C}$$

h is a bijection.

$g(x) \in A$.

$x \in B\cup C$.

$$\#(A^B \times A^C) = \#(A^{B\cup C})$$

$$= \#(A)^{\#(B\cup C)} = \#(A)^{\#(B)+\#(C)} \text{ if } B\cap C = \emptyset.$$

Ex 3.6.7 (1st attempt)

Let A and B be sets.

Let us say that A has lesser or equal cardinality to B if there exists an injection $f: A \to B$ from A to B.

Show that if A and B are finite sets. Then A has lesser or equal cardinality to B iff $\#(A) \leq \#(B)$.

1. If A and B are finite sets.

($\Rightarrow$) If A has lesser or equal cardinality to B. (using Proposition 3.6.14(c))

Then there exist a subset G of B, s.t.

G has a cardinality equal with A. as $\exists y \in B$, s.t. $\nexists x \in A$, $f(x) = y$.

Since $G \subseteq B$. Then $\#(G) \leq \#(B)$, $\#(G) = \#(A)$.

Thus $\#(A) \leq \#(B)$.

2. ($\Leftarrow$) If $\#(A) \leq \#(B)$

Suppose the exist a subset $G, \subseteq B$. (if not, contradicting Proposition 3.6.14(c))

s.t. $\#(G) \leq \#(B)$. s.t. $\forall x_1, x_2, x_1 \neq x_2 \in A$. $f(x_1) \neq f(x_2) \in B$.

Suppose $\#(A) = N$, for $N \in \mathbb{N}$, $N \leq \#(B)$.

s.t. $\#(G) = N'$, s.t. $N' \in \{ i \in \mathbb{N} : 0 \leq i \leq \#(B) \}$.

$\#(G) = \#(A)$, by definition 3.6.10 and 3.6.1

G and A have the same cardinality.

Thus A has a lesser or equal cardinality to B.

Ex 3.6.11 (1st attempt)
Let $f: X \to Y$ be a function.
Show that (a) and (b) are equivalent.
(a): f is injective (b) Whenever $E \subseteq X$, has cardinality $\#(E) = 2$
Then the image $f(E)$ also has cardinality $\#(f(E)) = 2$.

1. (a) $\Rightarrow$ (b)
If f is injective. Suppose $E \subseteq X$, with $\#(E) = 2$.
Then since $\forall x_1, x_2 \in X$, $f(x_1) = f(x_2)$ iff $x_1 = x_2$
$f(x_1) \neq f(x_2)$ iff $x_1 \neq x_2$.
$\#(E) = 2 \Rightarrow E = \{x_1, x_2\}$, s.t. $x_1 \neq x_2$.
$\Rightarrow f(E) = \{f(x_1), f(x_2)\}$. $\Rightarrow \#(f(E)) = 2$.

2. (b) $\Rightarrow$ (a)
If $E \subseteq X$. $\#(E) = 2$. Then there exist a map f, $f: X \to Y$,
s.t. $\#(f(E)) = 2$.

Suppose f is not injective. s.t. $f(x_1) = f(x_2)$ for $\exists x_1, x_2 \in X$, $x_1 \neq x_2$.
$f(E) = \{f(x_1), f(x_2)\}$, s.t. $f(x_1) = f(x_2)\}$
$= \{f(x_1)\}$ ✗. Contradiction, $\#(f(E)) = 2$.
$\Rightarrow$ f must be injective.

Ex 3.6.12(i) (1st attempt)

For any natural number n, let S_n be the set of all bijections $\phi : X \to X$. where $X := \{i \in \mathbb{N} : 1 \le i \le n\}$, such bijections also known as permutations of X.

(i) For any natural number n, show that S_n is finite, and $\#(S_{n++}) = (n{+}{+}) \times \#(S_n)$.

1. Given $\phi : X \to X$, from the power set Axiom. $\phi \in X^X$, where by Cardinal Arithmetic, X^X is finite, we can see that $S_n \subseteq X^X$, s.t. S_n is finite.

2. Given $S_{n++} := \{\phi : \phi : \{i \in \mathbb{N} : 1 \le i \le n{+}{+}\} \to \{i \in \mathbb{N} : 1 \le i \le n{+}{+}\}, \phi \text{ is a bijection}\}$.

Induct on n. Let $n = 0$. $f : \emptyset \to \emptyset$. $\Rightarrow$ only 1 f.

Thus. $(0{+}{+}) \times 1 = 1$

$\#(S_1) = 1$ $\quad f : \{x\} = \{x\}$. only 1 f.

Suppose inductively that $\#(S_{N++}) = (N{+}{+}) \times \#(S_N)$.

Show that $\#(S_{(N++)++}) = [(N{+}{+}){+}{+}] \times \#(S_{N++})$

3. (WLOG) assume in S_{N+2}, there exist $N+2$ subsets, aka, $A_1, \dots A_{N+2}$.

$A_n := \{\phi : \phi(i) = i \text{ for all } 1 \le i \le n\}$, s.t. we can see that $A_1 \subseteq A_2 \subseteq \dots \subseteq A_{N+2}$.

We know that $\prod_{i=1}^{N+2} \#(A_i) = \#(\bigcup\{x \in A_i : i \in \{1, \dots, N+2\}\}) + \#(\{x : x \in A_i \text{ for all } 1 \le i \le N+2\})$

$$\prod_{i=1}^{N+2} \#(A_i) = \#(S_{N+2}) + \#(\{x : x \in A_i \text{ for all } 1 \le i \le N+2\})$$

$$\prod_{i=1}^{N+2} i = \#(S_{N+2}) + 0$$

Similarly we know that $\prod_{i=1}^{N+1} i = \#(S_{N+1})$, $\prod_{i=1}^{N+2} i = (\prod_{i=1}^{N+1} i) \times (N+2)$

$$\#(S_{N+2}) = \prod_{i=1}^{N+2} i = \#(S_{N+1}) \times (N+2)$$

Induction closed.