Ex 3.1.1 (1^{st} attempt)

Let a, b, c, d be objects such that $\{a, b\} = \{c, d\}$.
Show that at least one of the two statements "$a = c$ and $b = d$" and "$a = d$ and $b = c$" hold.

1. Suppose none of the statements hold, s.t. given $\{a, b\} = \{c, d\}$.

$$\neg [(a = c) \wedge (b = d)] \vee [(a = d) \wedge (b = c)]$$

$$[(a \neq c) \vee (b \neq d)] \wedge [(a \neq d) \vee (b \neq c)]$$

2. From step 1, $a \in \{a, b\}$, $a \notin \{c, d\}$, since $a \neq c$ and $a \neq d$.
$b \in \{a, b\}$, $b \notin \{c, d\}$. since $b \neq c$ and $b \neq d$.
Similarly for c and d. ※ Contradiction.

At least one statement should hold.

Ex 3.1.2 (1st attempt)

Using only Axiom 3.2, Axiom 3.1, Axiom 3.3, Axiom 3.4.

Prove that the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ are all distinct.

1. $\emptyset \neq \{\emptyset\}$. $\quad \emptyset \in \{\emptyset\}$, $\emptyset \notin \emptyset$

   $\emptyset \neq \{\{\emptyset\}\}$. $\quad \{\emptyset\} \in \{\{\emptyset\}\}$, $\{\emptyset\} \notin \emptyset$.

   $\emptyset \neq \{\emptyset, \{\emptyset\}\}$ $\quad \emptyset \in \{\emptyset, \{\emptyset\}\}$, $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$. But not in $\emptyset$.

2. $\{\emptyset\} \neq \{\{\emptyset\}\}$ $\quad$ By Axiom 3.2 (Equality of sets).

   $\{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$.

3. $\{\{\emptyset\}\} \neq \{\emptyset, \{\emptyset\}\}$.

Ex 3.1.3 (1^{st} attempt)

Prove Lemma 3.1.12

If a and b are objects. Then $\{a,b\} = \{a\} \cup \{b\}$.

If $A, B,$ and C are sets. Then union operation is commutative and associative.

Also, we have $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.

1. $a \in \{a,b\}$, $a \in \{a\} \cup \{b\}$. Vire Versa. if $a \in \{a\}$, $a \in \{a\} \cup \{b\}$.
   $b \in \{a,b\}$, $b \in \{a\} \cup \{b\}$. $a \in \{a,b\}$.
   if $b \in \{b\}$, $b \in \{a\} \cup \{b\}$.
   $b \in \{a,b\}$.

2. Show that $A \cup B = B \cup A$
   If $x \in A \cup B$. Then $x \in A$ or $x \in B$.
   If $x \in A$. Then $x \in B \cup A$
   ...

3. Show that $(A \cup B) \cup C = A \cup (B \cup C)$
   ($\Rightarrow$) If $x \in (A \cup B) \cup C$ Then $x \in A \cup B$ or $x \in C$
   If $x \in C$, Then $x \in B \cup C$.
   If $x \in A \cup B$. Then $x \in A$ or $x \in B$.
   If $x \in A$. Then $x \in A$.
   If $x \in B$. Then $x \in B \cup C$.
   ($\Leftarrow$) If $x \in A \cup (B \cup C)$. Then $x \in A$ or $x \in B \cup C$
   If $x \in A$. Then $x \in A \cup B$
   If $x \in A \cup B$. Then $x \in A$ or $x \in B$.
   If $x \in A$, $x \in A \cup B$
   If $x \in B$, $x \in A \cup B$.
   If $x \in B \cup C$. Then $x \in B$ or $x \in C$
   If $x \in B$. Then $x \in A \cup B$.
   If $x \in C$. Then $x \in C$.

   Thus LHS = RHS.

4. If $x \in A \cup \emptyset$. Then $x \in A$ Thus $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.
   ...

Ex 3.1.4 (1st attempt)

Prove Proposition 3.1.17. (Sets are partially ordered by set inclusion).

Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq C$. Then $A \subseteq C$.

If $A \subseteq B$ and $B \subseteq A$. Then $A = B$.

If $A \subset B$ and $B \subset C$. Then $A \subset C$.

1. $x \in A$, $A \subseteq B$, $\Rightarrow x \in B$.
   $B \subseteq C$, $\Rightarrow x \in C$, thus. $x \in A \Rightarrow x \in C$. $A \subseteq C$.

2. $x \in A$. $A \subseteq B$, $x \in B$.
   $x \in B$, $B \subseteq A$, $x \in A$. Thus. $A = B$. (Axiom 3.2)

3. $x \in A$, $A \subset B$, $x \in B$.
   $B \subset A$. $x \in C$. $A \subset C$. $A \neq C$.

Ex 3.1.5 (1st attempt)

Let A, B be sets. Show that $A \subseteq B$, $A \cup B = B$, $A \cap B = A$ are logically equivalent.

$A \subseteq B$, $\Rightarrow$ $x \in A$, $\Rightarrow$ $x \in B$,

$A \cup B = B$ $\Rightarrow$ $x \in A \cup B$, $x \in A$ or $x \in B$.

$A \cup B = B$, $x \in A$, $\Rightarrow$ $x \in B$.

$A \cap B = A$ $\Rightarrow$ $x \in A \cap B$, $x \in A$ and $x \in B$.

$A \cap B = A$. $A \subseteq B$.

Ex 3.1.6 (1^{st} attempt)
Prove Proposition 3.1.27 (Sets form a boolean algebra).
Let A, B, C be sets. Let X be a set containing A, B, C as subsets.

(a) $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$ Minimal Element.
(b) $A \cup X = X$ and $A \cap X = A$ Maximal Element
(c) $A \cap A = A$ and $A \cup A = A$ Identity
(d) $A \cup B = B \cup A$ and $A \cap B = B \cap A$ Commutativity
(e) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ Associativity
(f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Distributivity.
(g) $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \emptyset$ Partition
(h) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$ DeMorgan's Laws.

1. From Lemma 3.1.12, $A \cup \emptyset = A$. $x \in A$, $x \notin \emptyset$, s.t. $A \cap \emptyset = \emptyset$.

2. Since $A \subseteq X$, s.t. $x \in A$, $x \in X$. $A \cup X = X$.
$A \cap X = A$, $x \in X \Rightarrow x \in A$ or $x \notin A$. Thus $A \cap X = A$.

3. From Lemma 3.1.12, $A \cup A = A$. $x \in A$, $x \in A$, $A \cap A = A$.

4. From Lemma 3.1.12, $A \cup B = B \cup A$.
If $x \in A \cap B$, then $x \in A$ and $x \in B$. $\Rightarrow$ $x \in B \cap A$.

5. From Lemma 3.1.12, $(A \cup B) \cup C = A \cup (B \cup C)$

If $x \in (A \cap B) \cap C$. Then $x \in A \cap B$ and $x \in C$.
If $x \in (A \cap B)$. Then $x \in A$, $x \in B$, and $x \in C$. Then $x \in B \cap C$, $x \in A$.

6. If $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$.
If $x \in B \cup C$. Then $x \in B$ or $x \in C$. $\rightarrow$ $x \in A \cap C$.
Then $x \in A \cap B \rightarrow x \in A \cap B$ or $A \cap C$.

If $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$.
If $x \in A \cap B$. Then $x \in A$ and $x \in B$., $\Rightarrow$ $x \in A \cap (B \cup C)$.
If $x \in A \cap C$. Then $x \in A$ and $x \in C$. $\Rightarrow$ $x \in A \cap (B \cup C)$.

7. If $x \in A \cup (X \setminus A)$. Then $x \in A$ or $x \in (X \setminus A)$.
If $x \in A$, then $x \in X$, since $A \subseteq X$.
If $x \in X \setminus A$, Then $x \notin A$, but $x \in X$.

If $x \in X$. Then ($x \in A$ or $x \notin A$) and $x \in X$.
Thus $A \cup (X \setminus A) = X$.

If $x \in A \cap (X \setminus A)$. Then $x \in A$ and $x \notin A$.
no such x exist.
Thus. $A \cap (X \setminus A) = \emptyset$.

8. If $x \in X \setminus (A \cup B)$. Then $x \notin A \cup B$, $\Rightarrow$ $x \notin A$ and $x \notin B$.
$\Rightarrow$ $x \in (X \setminus A) \cap (X \setminus B)$.
If $x \in (X \setminus A) \cap (X \setminus B)$. Then $x \notin A$ and $x \notin B$.
Thus $x \notin A \cup B$. $\Rightarrow$ $x \in X \setminus (A \cup B)$.

If $x \in X \setminus (A \cap B)$. Then $x \notin A \cap B$ $\Rightarrow$ $x \in (X \setminus A) \cup (X \setminus B)$.

If $x \in (X \setminus A) \cup (X \setminus B)$. Then $x \notin A$ or $x \notin B$.
$\Rightarrow$ $x \notin A \cap B$. $\Rightarrow$ $x \in X \setminus (A \cap B)$.

Ex 3.1.7 (1st attempt)
Let A, B, C be sets. Show that $A \cap B \subseteq A$, $A \cap B \subseteq B$,
Furthermore, show that $C \subseteq A$ and $C \subseteq B$ iff $C \subseteq A \cap B$.

In a similar spirit, show that $A \subseteq A \cup B$, $B \subseteq A \cup B$.
Furthermore, show that $A \subseteq C$ and $B \subseteq C$ iff $A \cup B \subseteq C$.

1. $x \in A \cap B \Rightarrow x \in A$ and $x \in B$. Thus $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

2. ($\Rightarrow$) If $C \subseteq A$ and $C \subseteq B$. Then $x \in C$, $x \in A$ and $x \in B$. Thus $C \subseteq A \cap B$.
($\Leftarrow$) If $C \subseteq A \cap B$. Then $x \in C$. $x \in A \cap B \Rightarrow x \in A$ and $x \in B$. from above $A \cap B \subseteq A$ and $A \cap B \subseteq B$.

3. If $x \in A$. Then $x \in A$ or $x \in B$. Thus $x \in A \cup B$. $A \subseteq A \cup B$.
If $x \in B$. Then $x \in A$ or $x \in B$. Thus $x \in A \cup B$. $B \subseteq A \cup B$.

4. ($\Rightarrow$) If $A \subseteq C$ and $B \subseteq C$. $x \in A$, $x \in C$. $x \in B$, $x \in C$. $\Rightarrow A \cup B \subseteq C$.

($\Leftarrow$) If $A \cup B \subseteq C$. Then $x \in A \cup B$, $x \in C$
If $x \in A$, $x \in C$
If $x \in B$, $x \in C$.
Thus $A \subseteq C$, $B \subseteq C$.

Ex3.1.8 (1st attempt)
Let A, B be sets. Prove that $A \cap (A \cup B) = A$ and $A \cup (A \cap B) = A$. (known as absorption laws).

1. By proposition 3.1.27 (Distributivity), $A \cap (A \cup B) = (A \cap A) \cup (A \cap B)$.
By . . . (Identity), $A \cap A = A$.
Thus. $A \cap (A \cup B) = A \cup (A \cap B)$.
If $x \in A \cup (A \cap B)$. Then $x \in A$ or $x \in A \cap B$
If $x \in A$, $x \in A$.
If $x \in A \cap B$, $x \in A$ and $x \in B$.
$\Rightarrow x \in A$. $A \cap (A \cup B) = A$.

2. By distributivity property. $A \cup (A \cap B) = (A \cup A) \cap (A \cup B)$.
By identity, $A \cup A = A$. Thus $A \cup (A \cap B) = A \cap (A \cup B)$.
$= (A \cap A) \cup (A \cap B)$
$= A \cup (A \cap B)$
$= A$.

Ex 3.1.9 (1st attempt)

Let A, B, X be sets such that $A \cup B = X$ and $A \cap B = \emptyset$.
Show that $A = X \setminus B$ and $B = X \setminus A$.

1. ($\Leftarrow$) If $x \in X \setminus B$, Then $x \in X$ and $x \notin B$.

$x \in X \Rightarrow x \in A \cup B$,

$x \notin B \Rightarrow x \notin A \cap B$.

$x \in (A \cup B) \setminus (A \cap B) \cup B$

$x \in A$.

($\Rightarrow$) If $x \in A$. Then $x \in A \cup B \Rightarrow x \in X \setminus B$.

$x \notin A \cap B$

2. ($\Leftarrow$) If $x \in X \setminus A$. Then $x \in X$ and $x \notin A$.

$x \in A \cup B \setminus A$

$x \in B$

($\Rightarrow$) If $x \in B$. Then $x \in X$, $x \in A \cup B$.

Since $A \cap B = \emptyset$. $x \notin A \cap B$.

$x \in X \setminus A$.

Ex 3.1.10 (1^{st} attempt)

Let A and B be sets.

Show that the three sets $A \setminus B$, $A \cap B$, and $B \setminus A$ are disjoint.

Show that their union is $A \cup B$.

1. If $x \in A \setminus B$. $\Rightarrow x \in A$ and $x \notin B \Rightarrow x \in A \setminus (A \cap B)$ s.t. $x \notin (A \cap B)$.

$\Rightarrow x \notin B \setminus A$.

Thus $(A \setminus B) \cap (A \cap B) = \emptyset$

$(A \setminus B) \cap (B \setminus A) = \emptyset$.

2. If $x \in A \cap B$. Then $x \in A$ and $x \in B$. $\Rightarrow x \notin B \setminus A$.

$(A \cap B) \cap (B \setminus A) = \emptyset$.

3. $(\Rightarrow)$ $(A \setminus B) \cup (B \setminus A) \Rightarrow x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$. $\Rightarrow x \in (A \cup B) \setminus (A \cap B)$.

$((A \setminus B) \cup (B \setminus A)) \cup (A \cap B) \Rightarrow x \in ((A \cup B) \setminus A \cap B) \cup (A \cap B)$.

$x \in A \cup B$.

$(\Leftarrow)$ $x \in A \cup B$, $\Rightarrow x \in A$ or $x \in B$, or $x \in A \cap B$.

$x \in A \setminus B$ or $x \in B \setminus A$ or $x \in A \cap B$.

Ex 3.1.11 (2nd attempt, 1st attempt abort)

Show that the axiom of replacement (Axiom 3.7) implies the axiom of specification (Axiom 3.6).

Axiom 3.7 (Replacement).

Let A be a set. For any object $x \in A$, and any object y, suppose we have a statement $P(x,y)$ pertaining to x and y. s.t. for each $x \in A$, there is at most one y for which $P(x,y)$ is true. Then there exists a set $\{y : P(x,y) \text{ is true for some } x \in A\}$, s.t. for any object z, $z \in \{y : P(x,y) \text{ is true for some } x \in A\}$
$\iff P(x,z)$ is true for some $x \in A$.

Axiom 3.6 (Specification)

Let A be a set, and for each $x \in A$, let $P(x)$ be a property pertaining to x (i.e. for each $x \in A$, $P(x)$ is either true or false). Then there exists a set. $\{x \in A : P(x) \text{ is true}\}$, whose elements are precisely the elements x in A for which $P(x)$ is true. In other words, for any object y.

$$y \in \{x \in A : P(x) \text{ is true}\} \iff (y \in A \text{ and } P(y) \text{ is true}).$$

1. $z \in \{y : P(x,y) \text{ is true for some } x \in A\}$

Suppose we modify the above s.t. $z \in \{y : P(x,y) \text{ is true for any } x \in A\}$.

s.t.

$$\{y : P(x,y) \text{ is true for some } x \in A\} \subseteq \{y : P(x,y) \text{ is true for any } x \in A\}.$$

$$\Updownarrow$$

$$\{y : y \in \{x \in A : P(x)\}\}.$$

2. Thus, from step 1, if $z \in \{y : P(x,y), \exists x \in A\}$.

Then $z \in \{y : y \in \{x \in A : P(x)\}\}$.

s.t. Replacement Implies Specification.

Ex 3.1.12; (1st attempt)

Suppose A, B, A', B' are sets, s.t. $A' \subseteq A$ and $B' \subseteq B$.

(i) Show that $A' \cup B' \subseteq A \cup B$ and $A' \cap B' \subseteq A \cap B$.

1. If $x \in A' \cup B'$. Then $x \in A'$ or $x \in B'$

Since $A' \subseteq A$ and $B' \subseteq B$. If $x \in A'$. Then $x \in A$.

If $x \in B'$. Then $x \in B$.

Thus $x \in A'$ or $x \in B'$ $\Rightarrow$ $x \in A$ or $x \in B$.

$A' \cup B' \in A \cup B$.

2. If $x \in A' \cap B'$. Then $x \in A'$ and $x \in B'$

Similarly. If $x \in A'$ and $x \in B'$. Then $x \in A$ and $x \in B$.

$A' \cap B' \subseteq A \cap B$.

Ex 3.1.12 ii (1st attempt)

Suppose A, B, A', B' are sets, s.t. $A' \subseteq A$ and $B' \subseteq B$.

(ii) Give a counter-example showing $A' \setminus B' \subseteq A \setminus B$ is false.

Can you find a modification of this statement involving the set difference operation $\setminus$ which is true given the stated hypothess?

1. If $x \in A' \setminus B'$. Then $x \in A'$ and $x \notin B'$

If $x \in A'$. Then $x \in A$.

If $x \notin B'$. Then $x \in B$ or $x \notin B$. Since $B' \subseteq B$.

If $x \notin B'$ and $x \notin B$. Then $x \in A \setminus B$

If $x \notin B'$ and $x \in B$. Then $x \in A \setminus B'$.

2. $A' \setminus B' \subseteq A \setminus B'$, The justification is as above.

Ex 3.1.13 (1^{st} attempt)

Define a proper subset of a set A to be a subset B of A with $B \neq A$.
Let A be a non-empty set.
Show that A does not have any non-empty proper subsets iff
A is of the form $A = \{x\}$ for some object x.

1. Such set C is $C = \{x \in B : B \subsetneq A\}$

2. ($\Rightarrow$) $\exists x \in A$, s.t. $x \in C$, (A does not have any non-empty proper subsets).
$\exists y \in C$, $y \notin \emptyset$.
$\exists x \in A$, s.t. $x \in C$, $x \notin \emptyset$. Thus. $A = \{x\}$ for some object x.

($\Leftarrow$) If A is of the form $A = \{x\}$ for some object x.
Then $\exists x \in A$, $x \notin \emptyset$. (A is non-empty set).
$x \neq \emptyset$.

If $x = C$. Then $\exists y \in x = C$, s.t. $C \subsetneq A$.

Ex 3.2.1 (1st attempt)

Show that the universal specification axiom, Axiom 3.9, if assumed to be true, would imply Axioms 3.3, 3.4, 3.5, 3.6, and 3.7 (Axiom 3.8 if we assume all natural numbers are objects).

Axiom 3.3 (Empty set)
Axiom 3.4 (Singleton and pair sets)
Axiom 3.5 (Pairwise Union)
Axiom 3.6 (Specification)
Axiom 3.7 (Replacement)
Axiom 3.8 (Infinity)

Axiom 3.9 (Universal Specification)
Suppose for every object x, we have a property $P(x)$ pertaining to x. Then there exists a set $\{x : P(x)\}$ s.t. for every object y,
$y \in \{x : P(x)\} \iff P(y)$ is true.

1. $x \in \{y : y \in \{\}\} \iff x \in \emptyset$ ($\emptyset$ is empty). $\Rightarrow$ x is not an object.

2. $x \in \{y : y = a\} \iff x = a$ (Singleton set)
   $x \in \{y : (y = a) \vee (y = b)\} \iff (x = a)$ or $(x = b)$ (Pair set)

3. $x \in A \cup B \iff x \in \{y : (y \in A) \vee (y \in B)\} \iff (x \in A)$ or $(x \in B)$ (Pair Union).

4. $x \in \{y : (y \in A) \wedge P(y)\} \iff x \in A \wedge P(x)$ (Specification).

5. $z \in \{y : P(x, y) \text{ for } x \in A\}. \iff P(x, z)$ for some $x \in A$. (Replacement).

6. $x \in \{n{+}{+} : n \in \mathbb{N}\}. \iff x \in \mathbb{N}.$ (Infinity).

Ex 3.2.2 (1st attempt)

Use the axiom of regularity (and the singleton set axiom) to show that if A is a set, then $A \notin A$. Furthermore, show that if A and B are two sets, then either $A \notin B$ or $B \notin A$ (or both).

Then prove the corollary: Given any set A, there exists a mathematical object that is not an element in A, namely A itself. Thus, a larger set can be created by $A \cup \{A\}$.

1. If A is a set. Then by Axiom 3.10 (Regularity), $A = \{x : x \in A\}$

$\exists x \in A,\ x = y$, where $y \in \{x : x \in A\}$. Axiom 3.4 (Singleton sets).

or

$x \cap A = \emptyset$.

Thus $A = \{x : x \in A\} \notin A$.

2. If A and B are two sets. Then $A = \{x : x \in A\}$

$B = \{y : y \in B\}$.

Suppose $A \in B$.

Then $\exists z \in \{v : y \in B\},\ z = \{x : x \in A\}$.

$z \cap B = z$. $\cancel{\cdot}$ Contradict. Axiom 3.10 (Regularity). $z \cap B = \emptyset \Rightarrow z \notin B$.

Thus $A \notin B$. (this also applies to $B \notin A$)

3. $\exists A'$, s.t. $A' = A$ and $A' \notin A$. Then it does not contradict Axiom 3.10 (Regularity).

$A \cup \{A'\}$ is possible.

But not $A \cup A'$, since $A \cup A' = A = A'$

Ex 3.2.3 (1^{st} attempt)

Show that the universal specification axiom, Axiom 3.9, is equivalent to an axiom postulating the existence of a "universal set" Ω consisting of all objects

In other words, show that Axiom 3.9 is true iff a universal set exists.

Furthermore, show that by Axiom 3.1, $\Omega \in \Omega$. contradicting Ex 3.2.2.
Then justify why Axiom 3.9 is excluded from Axiom of regularity.

1. ($\Leftarrow$) $\Omega := \{x : P(x)\} = \{x : x \text{ is a set and } x \notin x\}$. (The universal set).

Suppose if Ω exist. Then if $P(x)$ is true, that is x is a set and $x \notin x$.
Then $x \in \Omega$.

If $x \in \Omega$.
Then x is a set and $x \notin x$, satisfying $P(x)$.

Thus implies Axiom 3.9.

($\Rightarrow$) Suppose Axiom 3.9 is true. Then if $P(y)$ is also y is a set and $y \notin y$.
Then Ω exist, and vice versa.

Thus implies the existence of Ω.

2. Since if Ω is a set, then Ω is also an object, thus, $(\Omega \in \Omega) \vee (\Omega \notin \Omega)$.
This contradicts the Axiom 3.10 (Regularity). as $\Omega \in \Omega$ is possible.

To have Axiom 3.10 (Regularity) working, Axiom 3.9 (Universal specification) is entirely omitted.

Ex 3.3.1 (1st attempt)
Show that the definition of equality in Defn 3.3.8 is reflexive, symmetric, transitive (A.7).

Verify The substitution property: if $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$ are functions such that $f = \tilde{f}$ and $g = \tilde{g}$. Then $g \circ f = \tilde{g} \circ \tilde{f}$

Defn 3.3.8 (Equality of functions)
Given $f: X \to Y$ and $g: X' \to Y'$. $f = g$ if $X = X'$, $Y = Y'$. if $f(x) = g(x)$, $\forall x \in X$

1. Suppose there exist $f: X \to Y$, s.t. $\exists f \in \mathcal{F}$, s.t. $f: X \to Y$. where f and f are identical.
Suppose if $f \neq f$. Then it implies that $\exists x \in X$, $f(x) \neq f(x)$: $\times$ Contradict.
$f = f$ (Reflexive).

2. Suppose $f: X \to Y$, $g: X' \to Y'$. if $f = g$. Then $X = X'$, $Y = Y'$, $f(x) = g(x)$. $\forall x \in X$.
Suppose $g \neq f$, s.t. $\exists x \in X'$, s.t. $f(x) \neq g(x)$: $\times$
$f = g$, $g = f$ (Symmetric).

3. Given $f: X \to Y$, $g: X' \to Y'$, $h: X'' \to Y''$.
Suppose $f = g$, $g = h$, $f \neq h$.

$f(x) = g(x)$, $\forall x \in X$. $X = X'$, $Y = Y'$.

$g(x) = h(x)$, $\forall x \in X'$, $X' = X''$, $Y' = Y''$

$f(x) \neq h(x)$, $\exists x \in X''$. $\times$ $f(x) = g(x) = h(x)$, $\forall x \in X$. $X = X' = X''$.

It is transitive.

4. Suppose $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$, s.t. $f = \tilde{f}$, $g = \tilde{g}$.
Then by Defn 3.3.8, $f(x) = \tilde{f}(x)$, $\forall x \in X$.
$g(y) = \tilde{g}(y)$, $\forall y \in Y$.

By Axiom of Substitution, $(g \circ f)(x) = (g \circ \tilde{f})(x)$ since $g = \tilde{g}$
$(\tilde{g} \circ f)(x) = (\tilde{g} \circ \tilde{f})(x)$

Then the four expressions are equivalent.
Thus. $g \circ f = \tilde{g} \circ \tilde{f}$.

Ex 3.3.2 (1^{st} attempt)

Let $f: X \to Y$ and $g: Y \to Z$ be functions.

Show that if f and g are both injective. Then so is $g \circ f$.

Similarly, show that if f and g are both surjective. Then so is $g \circ f$.

1. Suppose f and g are injective, s.t. $\forall x_1, x_2 \in X$, $\forall y_1, y_2 \in Y$

   $x_1 \neq x_2 \iff f(x_1) \neq f(x_2)$ $\quad$ $x_1 = x_2 \iff f(x_1) = f(x_2)$

   $y_1 \neq y_2 \iff g(y_1) \neq g(y_2)$. $\quad$ $y_1 = y_2 \iff g(y_1) = g(y_2)$

2. From step 1, fix $y_1 = f(x_1)$, $y_2 = f(x_2)$

   and we know that $y_1 = y_2 \iff g(y_1) = (g \circ f)(x_1) = g(y_2) = (g \circ f)(x_2)$

   $y_1 \neq y_2) \iff g(y_1) = (g \circ f)(x_1) \neq g(y_2) = (g \circ f)(x_2)$

   $\Rightarrow g \circ f$ is injective

3. Suppose f and g are surjective, $\forall y \in Y, \exists x \in X$, s.t. $f(x) = y$

   $\forall z \in Z, \exists y \in Y$, s.t. $g(y) = z$

   $\Rightarrow \forall z \in Z, \exists y \in Y, \exists x \in X$, s.t. $z = g(y) = (g \circ f)(x)$

Ex 3.3.3 (1^{st} attempt)

When is the empty function into a given set X injective? surjective? bijective?

1. Given $f: \emptyset \to X$, where f is the empty function.

   f is always injective as the domain is empty.

   f is never surjective, thus never bijective, since $\exists! x \in X$, s.t. $x = \cancel{f}()$.

Ex 3.3.4 (1st attempt)

Let $f: X \to Y$, $\tilde{f}: X \to Y$, $g: Y \to Z$, and $\tilde{g}: Y \to Z$ be functions.
Show that if $g \circ f = g \circ \tilde{f}$ and g is injective. Then $f = \tilde{f}$.

Is the same statement true if g is not injective?

Show that if $g \circ f = \tilde{g} \circ f$ and f is surjective. Then $g = \tilde{g}$.

Is the same statement true if f is not surjective?

1. $g \circ f = g \circ \tilde{f}$ implies. $\forall x \in X$, $(g \circ f)(x) = (g \circ \tilde{f})(x)$.
If g is injective, s.t. $y_1 \neq y_2 \Leftarrow g(y_1) \neq g(y_2)$.
$y_1 = y_2 \Longleftrightarrow g(y_1) = g(y_2)$.
Suppose $f \neq \tilde{f}$, s.t. $\exists x \in X$, $f(x) \neq \tilde{f}(x)$. $\Rightarrow$ $(g \circ f)(x) \neq (g \circ \tilde{f})(x)$. X.
$f = \tilde{f}$. Contradiction.

2. The above statement is false if g is not injective. also $g \circ f \neq g \circ \tilde{f}$, and cannot prove whether $f = \tilde{f}$.

3. $g \circ f = \tilde{g} \circ f$ implies $\forall x \in X$, $(g \circ f)(x) = (\tilde{g} \circ f)(x)$.
If f is surjective. Then $\forall y \in Y$, $\exists x \in X$, s.t. $y = f(x)$.
Since $f = f$ (Identity). For $(g \circ f)(x) = (\tilde{g} \circ f)(x)$, s.t. $Y = Y$, $Z = Z$.

$\forall y \in Y$, $\exists x \in X$, s.t. $g(y) = (g \circ f)(x)$ since $g \circ f = \tilde{g} \circ f$.
$\tilde{g}(y) = (\tilde{g} \circ f)(x)$. Then $g = \tilde{g}$.

4. The statement is true regardless whether or not f is specified.

Ex 3.3.5 (1st attempt)

Let $f: X \to Y$ and $g: Y \to Z$ be functions.
Show that if $g \circ f$ is injective. Then f must be injective.

Is it true that g must also be injective?

Show that if $g \circ f$ is surjective. Then g must be surjective.

Is it true that f must also be surjective?

1. If $x_1 \neq x_2$ $(g \circ f)(x_1) \neq (g \circ f)(x_2)$.
   Suppose f is not injective. Then $x_1 \neq x_2 \iff f(x_1) = f(x_2)$.
   $\to (g \circ f)(x_1) = (g \circ f)(x_2)$ $\therefore$ X
   $\to (g \circ f)(x_2) \neq (g \circ f)(x_2)$ or

   $\Rightarrow$ f must be injective.
   g must also be injective.

2. If $\forall z \in Z, \exists x \in X,$ s.t. $(g \circ f)(x) = z$
   Suppose g is not surjective, s.t. $\exists z \in Z, \forall y \in Y,$ s.t. $g(y) \neq z$ $\therefore$ X
   g must be surjective.

   f does not have to be surjective.

Ex 3.3.6 (1st attempt)

Let $f: X \to Y$ be a bijective function, and let $f^{-1}: Y \to X$ be its inverse.
Verify the cancellation laws
$f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$.

Conclude that f^{-1} is also invertible, and has f as its inverse (thus $(f^{-1})^{-1} = f$)

1. f is bijective, s.t. $\forall y \in Y, \exists x \in X$, s.t. $y = f(x)$ (Surjective)
$\forall x_1, x_2 \in X$, $x_1 \neq x_2 \iff f(x_1) \neq f(x_2)$ (Injective)
$x_1 = x_2 \iff f(x_1) = f(x_2)$

$\Rightarrow$ $\forall y \in Y$, $\exists! x \in X$, s.t. $f(x) = y$. (at least one, surjectivity, at most one injectivity).
and $x = f^{-1}(y)$. (Remark 3.3.27).
Thus $(f \circ f^{-1})(y) = y$.

$\Rightarrow$ $\forall x_1, x_2 \in X$, $f(x_1), f(x_2) \in Y$,
from above $(f^{-1} \circ f)(x_1) = x_1$
$(f^{-1} \circ f)(x_2) = x_2$

2. Show that f^{-1} is bijective
since f is bijective, $x_1 \neq x_2 \iff f(x_1) \neq f(x_2)$; $\forall x_1, x_2 \in X$.
$(f^{-1} \circ f)(x_1) = x_1$
$(f^{-1} \circ f)(x_2) = x_2$
Thus f^{-1} is injective.

Since f is a function. $\forall x \in X$, $\exists! y \in Y$, s.t. $y = f(x)$.
Since f^{-1} is a function, $\forall y \in Y$, $\exists! x \in X$, s.t. $f^{-1}(y) = x$.
Since f is bijective, $\forall y \in Y$, $\exists! x \in X$, s.t. $y = f(x)$.
f^{-1} is f inverse $\Rightarrow$ $\forall x \in X$, $\exists! y \in Y$. s.t. $f^{-1}(y) = (f^{-1} \circ f)(x) = x$.

f^{-1} is bijective.

Ex 3.3.7 (1st attempt)

Let $f: X \to Y$ and $g: Y \to Z$ be functions.

Show that if f and g are bijective. Then $g \circ f$ is also bijective.

and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

1. If f and g are bijective. Then $\forall y \in Y, \exists! x \in X, y = f(x)$.

$\forall z \in Z, \exists! y \in Y, z = g(y)$

$\Rightarrow \forall z \in Z, \exists! x \in X$, s.t. $z = (g \circ f)(x)$. $(g \circ f)$ is bijective.

2. From previous. $(g \circ f)^{-1}$, f^{-1}, and g^{-1} are bijective.

$\forall x \in X, \exists! z \in Z$, s.t. $x = (g \circ f)^{-1}(z)$.

$$\left.\begin{array}{ll} \forall x \in X, \exists! y \in Y, \text{s.t.} & x = f^{-1}(y) \\ \forall y \in Y, \exists! z \in Z, \text{s.t.} & y = g^{-1}(z) \end{array}\right] \Rightarrow \begin{array}{l} \forall x \in X, \exists! z \in Z, \text{s.t.} \\ x = (f^{-1} \circ g^{-1})(z) \end{array}$$

Thus $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$.

Ex 3.3.8 (a) (c) (1st attempt)

If X is a subset of Y, let $\iota_{X\to Y} : X \to Y$ be the inclusion map from X to Y.
defined by mapping $x \mapsto x$ for all $x \in X$, i.e., $\iota_{X\to Y}(x) := x$ for all $x \in X$.
The map $\iota_{X\to X}$ is in particular called the identity map on X.

(a) Show that if $X \subseteq Y \subseteq Z$. Then $\iota_{Y\to Z} \circ \iota_{X\to Y} = \iota_{X\to Z}$.

(c) Show that if $f: A \to B$ is bijective. Then $f \circ f^{-1} = \iota_{B\to B}$ and
$f^{-1} \circ f = \iota_{A\to A}$.

1. If $X \subseteq Y \subseteq Z$, then if $x \in X$, $x \in Y$, $x \in Z$,

$\iota_{X\to Y}(x) := x$, $x \in X$, $x \in Y$.

$\iota_{Y\to Z}(x) := x$, $x \in Y$, $x \in Z$, where $x = \iota_{X\to Y}(x)$.

Thus. $\iota_{Y\to Z} \circ \iota_{X\to Y}(x) := x$, $x \in X$, $x \in Z$,
$= \iota_{X\to Z}(x)$.

2. If $f: A \to B$ is bijective. Then we know that $f^{-1}: B \to A$.

$f^{-1} \circ f : A \to A \iff \iota_{A\to A}$
$f \circ f^{-1} : B \to B$. $\quad \iota_{B\to B}$.

Ex 3.3.8 (b) (d)

If X is a subset of Y, let $\iota_{X\to Y}: X \to Y$ be the inclusion map from X to Y. defined by mapping $x \mapsto x$ for all $x \in X$, i.e., $\iota_{X\to Y}(x) := x$ for all $x \in X$. The map $\iota_{X\to X}$ is in particular called the identity map on X.

(b) Show that if $f: A \to B$. Then $f = f \circ \iota_{A\to A} = \iota_{B\to B} \circ f$.

(d) Show that if X and Y are disjoint sets and $f: X \to Z$ and $g: Y \to Z$. Then there is a unique function $h: X \cup Y \to Z$ such that $h \circ \iota_{X\to X\cup Y} = f$ and $h \circ \iota_{Y\to X\cup Y} = g$.

1. If $f: A \to B$. We know that $\iota_{A\to A}: A \to A$
$\iota_{B\to B}: B \to B$

$\iota_{A\to A}(x) := x,\ x \in A \Rightarrow f \circ \iota_{A\to A}: A \to B$

wrong

$\iota_{B\to B}(x) := x,\ x \in B. \Rightarrow \iota_{B\to B} \circ f: A \to B$

as

$X \subseteq Y$

2. If X and Y are disjoint, $X \cap Y = \emptyset$, $f: X \to Z$, $g: Y \to Z$.

Suppose $h: X \cup Y \to Z$ is not unique, s.t. $\exists h_1: X \cup Y \to Z$. $h_1 \neq h$.

$\iota_{X\to X\cup Y}: X \to X \cup Y$

$h \circ \iota_{X\to X\cup Y}: X \to Z$, s.t. $h(x) = f(x), \forall x \in X$.

$h: X \cup Y \to Z$

$h_1: X \cup Y \to Z$

$h \circ \iota_{Y\to X\cup Y}: Y \to Z$., $h(y) = g(y), \forall y \in Y$.

$\iota_{Y\to X\cup Y}: Y \to X \cup Y$

If $h_1 \circ \iota_{X\to X\cup Y}: X \to Z$, s.t. $h_1(x) = f(x), \forall x \in X$.

$h_1 \circ \iota_{Y\to X\cup Y}: Y \to Z$, s.t. $h_1(y) = g(y)\ .\forall y \in Y$.

$\cancel{\because}$, $h(x) = f(x)\ \forall x \in X$ $\quad h(x) \neq h_1(x)$.

$h(y) = g(y)\ \forall y \in Y$.

There is only one unique function h.

This applies as

$X \cap Y = \emptyset$.

$f: \emptyset \to Z$

$g: \emptyset \to Z$,

$f = g$. this is empty function

Ex 3.3.8 (e) (1st attempt)

If X is a subset of Y, let $\iota_{X\to Y} : X \to Y$ be the inclusion map from X to Y. defined by mapping $x \mapsto x$ for all $x \in X$, i.e., $\iota_{X\to Y}(x) := x$ for all $x \in X$. The map $\iota_{X\to X}$ is in particular called the identity map on X.

(e) Show that the hypothesis that X and Y are disjoint can be dropped in (d) if one adds the additional hypothesis that $f(x) = g(x)$ for all $x \in X \cap Y$. $\Leftarrow$ This actually does not change the conclusion. and the hypothesis is interchangeable.

1. Suppose $f = g$, $\forall x \in X \cap Y$, we want to show $\exists! h : X \cup Y \to Z$ s.t. $h \circ \iota_{X\to X\cup Y} = f$
$f : X \to Z$, $g : Y \to Z$. $h \circ \iota_{Y\to X\cup Y} = g$.

2. Suppose $h : X \cup Y \to Z$ is not unique, s.t. $\exists h_1 : X \cup Y \to Z$. $h_1 \neq h$.

$\iota_{X\to X\cup Y} : X \to X \cup Y$ $\quad h \circ \iota_{X\to X\cup Y} : X \to Z$, s.t. $h(x) = f(x), \forall x$.
$h : X \cup Y \to Z$
$h_1 : X \cup Y \to Z$ $\quad h \circ \iota_{Y\to X\cup Y} : Y \to Z$., $h(y) = g(y). \forall y \in Y$.
$\iota_{Y\to X\cup Y} : Y \to X \cup Y$

If $h_1 \circ \iota_{X\to X\cup Y} : X \to Z$, s.t. $h_1(x) = f(x), \forall x \in X$.
$h_1 \circ \iota_{Y\to X\cup Y} : Y \to Z$, s.t. $h_1(y) = g(y)$. $\forall y \in Y$.
$\because$ $h(x) = f(x)$ $\forall x \in X$ $\quad h(x) \neq h_1(x)$.
$h(y) = g(y)$ $\forall y \in Y$.

There is only one unique function h. This applies to $X \cap Y$, as $f = g$. $\forall x \in X \cap Y$.

Exercise from Defn 3.4.1

Given $f: X \to Y$ is a function from X to Y. $S \subseteq X$.

$f(S) := \{f(x) : x \in S\}$, where $f(S)$ is the forward image of S.

Show that $f(S)$ can be defined using Axiom of specification (Axiom 3.6)

Show that $f(S)$ can be defined using Axiom of replacement (Axiom 3.7).

1. By Axiom of specification (Axiom 3.6), $\forall x \in X$, $\exists! y \in Y$, s.t. $y = f(x)$. (Defn of function)

   $y \in \{f(x) : x \in S\} \iff x \in S \subseteq X$ and $y = f(x) \in Y$

2. By Axiom of Replacement (Axiom 3.7),

   $z \in \{y : y = f(x)$ for some $x \in S\} \iff \exists! z \in Y, f(x) = z$, for some $x \in S \subseteq X$.

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Given $f: \mathbb{N} \to \mathbb{N}$, $f(x) = 2x$.

Evaluate whether f is injective.

1. To show that f is injective, we have to show that
$\forall n_1, n_2 \in \mathbb{N}$, if $n_1 = n_2$. Then $f(n_1) = f(n_2)$ By Definition 3.3.17
if $n_1 \neq n_2$. Then $f(n_1) \neq f(n_2)$. (one-to-one functions)

2. Assume $n_1 \times n_2 = m$ (WLOG), $n_1 \neq n_2$.
Fix n_2 and induct on n_1., $n_2 > 0$
Let $n_1 = 0$. $n_1 \times n_2 = 0 \times n_2 = m = 0$.
$m = n_1$, $f(m) = 2 \cdot m = 2 \cdot n_1 = 0$ $f(m) = f(n_1)$.
$f(n_1) = 2 \cdot n_1 = 2 \cdot n_1 = 0$
Let $n_1 = 1$, $n_1 \times n_2 = 1 \times n_2 = n_2 = m$
$m \neq n_1$, $f(m) = 2 \cdot n_2 = 2n_2$ $f(m) \neq f(n_1)$, since $n_2 \neq (n_1 = 1)$
$f(n_1) = 2 \cdot n_1 = 2$ (Base Case).

3. Inductively assume that if $m = n_1$. Then $f(m) = f(n_1)$
if $m \neq n_1$. Then $f(m) \neq f(n_1)$.
Show that if $m = n_1 + 1$. Then $f(m) = f(n_1+1)$
if $m \neq n_1 + 1$. Then $f(m) \neq f(n_1+1)$.

If $m = n_1 + 1$. L.H.S $f(m) = f(n_1+1) = 2(n_1+1) = 2n_1 + 2$
R.H.S $f(n_1+1) = 2(n_1+1) = 2n_1 + 2$ L.H.S = R.H.S.

If $m \neq n_1 + 1$. L.H.S $f(m) = f(n_1 n_2) = 2(n_1 \times n_2)$
R.H.S $f(n_1+1) = 2(n_1+1)$
By Cancellation Law $2(n_1 \times n_2) =? 2(n_1+1)$
$n_1 \times n_2 =? n_1 + 1$
$\exists n_2 \in \mathbb{N}$, $n_1 \neq n_2$, s.t. $n_1 \times n_2 \neq n_1 + 1$.
$\Rightarrow$ L.H.S $\neq$ R.H.S.

Induction closed. f is one-to-one

f is not surjective. as $\exists y \in \mathbb{N}$, s.t. $\nexists x \in \mathbb{N}$, $f(x) = y$.

Ex 3.4.1 (1^{st} attempt)
Let $f: X \to Y$ be bijective function. Let V be any subset of Y.
Let $f^{-1}: Y \to X$ be its inverse. Prove that the forward image of V under f^{-1} is the same set as the inverse image of V under f.

1. Given $f^{-1}: Y \to X$. $V \subseteq Y$.
The forward image $f^{-1}(V) := \{ f^{-1}(y) : y \in V \} \subseteq X$ (Defn 3.4.1)

2. Given $f: X \to Y$, f is bijective function. Let h be its inverse, s.t.
$h(V) := \{ x \in X : f(x) \in V \} \subseteq X$ (Defn 3.4.5)

3. By definition of functions, $\forall z \in f^{-1}(V) \subseteq X$, $\exists! \, y \in Y$, s.t. $f(z) = y$.
Since f^{-1} is the inverse of f. $f(z) = (f \circ f^{-1})(y) = y$, where $y \in V$.

4. By definition of functions, $\forall q \in h(V)$, $\exists! \, y \in Y$. s.t. $h(q) = y$.
and from step 2, we know that $y \in V \subseteq Y$, since h is f's inverse.

5. From steps 3 and 4, we can see that $\forall z \in f^{-1}(V)$, $f(z) \in V$.
$\forall q \in h(V)$, $f(q) \in V$.
Since f is a bijective function. $\forall y \in V$, $\exists! \, z \in f^{-1}(V)$, s.t. $f(z) = y$
$\forall y \in V$, $\exists! \, q \in h(V)$, s.t. $f(q) = y$.

$\forall x_1 \in f^{-1}(V)$, $\forall x_2 \in h(V)$.
If $f(x_1) = f(x_2)$. Then $x_1 = x_2$. Thus $x_1 \in f^{-1}(V)$, $x_1 \in h(V)$.
$x_2 \in h(V)$, $x_2 \in f^{-1}(V)$.
Thus $f^{-1}(V) \subseteq h(V)$, $h(V) \subseteq f^{-1}(V)$. $f^{-1}(V) = h(V)$, $f^{-1} = h$.

Ex 3.4.2 (1st attempt)

Let $f: X \to Y$ be a function. Let S be a subset of X. Let U be a subset of Y.

(i) What can one say about $f^{-1}(f(S))$ and S in general?

(ii) What about $f(f^{-1}(U))$ and U?

(iii) What about $f^{-1}(f(f^{-1}(U)))$ and $f^{-1}(U)$?

1. $(f^{-1} \circ f)(S) := \{x \in X : f(x) \in f(S)\}$, where $f(S) := \{f(x) : x \in S\}$.

   Whether or not $(f^{-1} \circ f)(S) \subseteq S$ depends on f.

   If f is injective. Then $(f^{-1} \circ f)(S) \subseteq S$. (from intuition, not proved).

   If f is surjective. Then $(f^{-1} \circ f)(S) \subseteq S$. (from intuition, not proved).

   If f is bijective. Then $(f^{-1} \circ f)(S) = S$ (Ex 3.4.1 proved)

2. $(f \circ f^{-1})(U) := \{f(x) : x \in f^{-1}(U)\}$, where $f^{-1}(U) := \{x \in X : f(x) \in U\}$.

   Whether or not $(f \circ f^{-1})(U) \subseteq U$ depends on f.

   If f is injective. Then not guarantee.

   If f is surjective. Then $(f \circ f^{-1})(U) \subseteq U$ or not guarantee (not proved).

   If f is bijective. $(f \circ f^{-1})(U) = U$ (Ex 3.4.1 proved)

3. $(f^{-1} \circ f \circ f^{-1})(U) := \{x \in X : f(x) \in (f \circ f^{-1})(U)\}$.

   Whether or not $(f^{-1} \circ f \circ f^{-1})(U) \subseteq f^{-1}(U)$ depends on f.

   If f is injective. Then not guarantee.

   If f is surjective, not guarantee

   If f is bijective. $(f^{-1} \circ f \circ f^{-1})(U) = f^{-1}(U)$ (from intuition not proved)

Ex 3.4.3 (1^{st} attempt)

Let A, B be two subsets of a set X. Let $f: X \to Y$ be a function.

Show that $f(A \cap B) \subseteq f(A) \cap f(B)$ } For these two statements,

Show that $f(A) \setminus f(B) \subseteq f(A \setminus B)$ } is it true that the $\subseteq$ relation can be

Show that $f(A \cup B) = f(A) \cup f(B)$ improved to $=$?

1. ($\Rightarrow$)

If $x \in A \cap B$. Then $(x \in A) \wedge (x \in B)$.

$f(A) := \{ f(x) : x \in A \}$ $f(B) := \{ f(x) : x \in B \}$.

$f(A \cap B) := \{ f(x) : x \in A \cap B \}$.

Thus. $\forall y \in f(A \cap B) \Rightarrow y \in f(A) \cap f(B)$. Otherwise Contradiction.

$f(A \cap B) \subseteq f(A) \cap f(B)$

($\Leftarrow$)

If $x \in A$. Then $f(x) \in f(A)$. If $f(x) \in f(A) \cap f(B)$.

If $x \in B$. Then $f(x) \in f(B)$.

Then by axiom of substitution, $x \in A \cap B$.

$f(x) \in f(A \cap B)$.

$f(A) \cap f(B) \subseteq f(A \cap B)$

Thus $f(A) \cap f(B) = f(A \cap B)$.

2. ($\Rightarrow$)

If $x \in A$. Then $f(x) \in f(A)$. If $x \in B$. Then $f(x) \in f(B)$.

Suppose $f(x) \in f(A) \setminus f(B)$. Then this implies that $f(x) \in f(A)$

$x \notin A \cap B \Leftarrow f(x) \notin f(A \cap B)$

$x \notin B. \Leftarrow f(x) \notin f(B)$,

Thus $x \in A \setminus B$., $f(x) \in f(A \setminus B)$. Thus $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

($\Leftarrow$)

If $x \in A \setminus B$. Then $f(x) \in f(A \setminus B)$.

$x \in A \setminus B \Rightarrow x \in A$, $x \notin A \cap B$, $x \notin B$.

$\Rightarrow f(x) \in f(A)$, $f(x) \notin f(A \cap B)$, $f(x) \notin f(B)$.

$\Rightarrow f(x) \in f(A) \setminus f(B)$. $f(A \setminus B) \subseteq f(A) \setminus f(B)$.

Thus $f(A) \setminus f(B) = f(A \setminus B)$.

3. ($\Rightarrow$)

If $x \in A \cup B$. Then $f(x) \in f(A \cup B)$.

$\Rightarrow x \in A$ or $x \in B$.

$\Rightarrow f(x) \in f(A)$ or $f(x) \in f(B)$. $\Rightarrow f(x) \in f(A) \cup f(B)$. $f(A \cup B) \subseteq f(A) \cup f(B)$.

($\Leftarrow$)

If $x \in A$ or $x \in B$. Then $f(x) \in f(A) \cup f(B)$.

$\Rightarrow x \in A \cup B$. Then $f(x) \in f(A \cup B)$. $f(A) \cup f(B) \subseteq f(A \cup B)$.

$f(A \cup B) = f(A) \cup f(B)$.

Ex 3.4.4 (1st attempt)

Let $f: X \to Y$ be a function from one set X to another set Y.

Let U, V be subsets of Y.

Show that $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$

Show that $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$

Show that $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$

1. ($\Rightarrow$)

$f^{-1}(U \cup V) := \{x \in X : f(x) \in U \cup V\}$

$f(x) \in U \cup V \Leftrightarrow x \in f^{-1}(U \cup V)$

$\Rightarrow (f(x) \in U) \cup (f(x) \in V)$

$\Rightarrow (x \in f^{-1}(U)) \cup (x \in f^{-1}(V))$

$\Rightarrow x \in f^{-1}(U) \cup f^{-1}(V)$

$\Rightarrow f^{-1}(U \cup V) \subseteq f^{-1}(U) \cup f^{-1}(V)$

($\Leftarrow$)

$f^{-1}(U) \cup f^{-1}(V) := \{x \in X : f(x) \in U\}$
$\cup \{x \in X : f(x) \in V\}$.

$f(x) \in U \cup f(x) \in V \Leftrightarrow x \in f^{-1}(U) \cup f^{-1}(V)$.

$\Rightarrow f(x) \in U \cup V \Rightarrow x \in f^{-1}(U \cup V)$

$\Rightarrow f^{-1}(U) \cup f^{-1}(V) \subseteq f^{-1}(U \cup V)$

2. ($\Rightarrow$)

$f^{-1}(U \cap V) := \{x \in X : f(x) \in U \cap V\}$

$f(x) \in U \cap V \Leftrightarrow x \in f^{-1}(U \cap V)$

$\Rightarrow (f(x) \in U) \cap (f(x) \in V)$

$\Leftrightarrow (x \in f^{-1}(U)) \cap (x \in f^{-1}(V))$

$\Leftrightarrow x \in f^{-1}(U) \cap f^{-1}(V)$

$f^{-1}(U \cap V) \subseteq f^{-1}(U) \cap f^{-1}(V)$.

Thus $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$.

($\Leftarrow$)

$f^{-1}(U) \cap f^{-1}(V) := \{x \in X : f(x) \in U\}$
$\cap \{x \in X : f(x) \in V\}$.

$(f(x) \in U) \cap (f(x) \in V) \Leftrightarrow x \in f^{-1}(U) \cap f^{-1}(V)$.

$\Rightarrow f(x) \in U \cap V$

$\Rightarrow x \in f^{-1}(U \cap V)$ (Defn 3.4.5).

$f^{-1}(U) \cap f^{-1}(V) \subseteq f^{-1}(U \cap V)$.

3. ($\Rightarrow$)

$f^{-1}(U \setminus V) := \{x \in X : f(x) \in U \setminus V\}$.

$f(x) \in U \setminus V \Leftrightarrow x \in f^{-1}(U \setminus V)$

$\Rightarrow f(x) \in U$ and $f(x) \notin V$ (Defn 3.4.5)

$\Rightarrow x \in f^{-1}(U)$ and $x \notin f^{-1}(V)$

$\Rightarrow x \in f^{-1}(U) \setminus f^{-1}(V)$.

$f^{-1}(U \setminus V) \subseteq f^{-1}(U) \setminus f^{-1}(V)$.

Thus $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$.

($\Leftarrow$)

$f^{-1}(U) \setminus f^{-1}(V) := \{x \in X : f(x) \in U\}$.
$\setminus \{x \in X : f(x) \in V\}$.

$f(x) \in U$ and $f(x) \notin V$.

$\Leftrightarrow x \in f^{-1}(U) \setminus f^{-1}(V)$.

$f(x) \in U \setminus V$

$\Rightarrow x \in f^{-1}(U \setminus V)$.

$f^{-1}(U) \setminus f^{-1}(V) \subseteq f^{-1}(U \setminus V)$.

Ex 3.4.5 (1^{st} attempt)

Let $f: X \to Y$ be a function from one set X to another set Y.

Show that $f(f^{-1}(S)) = S$ for every $S \subseteq Y$ iff f is surjective.

Show that $f^{-1}(f(S)) = S$ for every $S \subseteq X$ iff f is injective.

1. $(\Rightarrow)$ $f(f^{-1}(S)) := \{f(x) : x \in f^{-1}(S)\}$, where $f^{-1}(S) := \{x \in X : f(x) \in S\}$.

If $f(f^{-1}(S)) = S$. Then $\forall y \in S$, $y \in \{f(x) : x \in f^{-1}(S)\}$.

$\Rightarrow$ $\exists x \in X$, s.t. $x \in f^{-1}(S)$, s.t. $y = f(x)$.

Thus. f is surjective.

$(\Leftarrow)$ If f is surjective. Then $\forall y \in Y$, $\exists x \in X$, s.t. $y = f(x)$.

Thus. $f(f^{-1}(S))$ implies. $\forall y \in Y$. $\exists x \in f^{-1}(S) \subseteq X$, s.t. $y = f(x)$.

Thus $f(f^{-1}(S)) \subseteq S \subseteq Y$ as $f(f^{-1}(S)) \in S$

Suppose $\exists y \in S$, s.t. $y \notin f(f^{-1}(S))$.

Then. $\exists y \in S \subseteq Y$, $\nexists x \in f^{-1}(S) \subseteq X$, s.t. $y = f(x)$. ※ Contradiction

If $y \in S$. Then $y \in f(f^{-1}(S))$ $\Rightarrow$ $S \subseteq f(f^{-1}(S))$

Thus $S = f(f^{-1}(S))$

2. $(\Rightarrow)$ $f^{-1}(f(S)) := \{x \in X : f(x) \in f(S)\}$, where $f(S) := \{f(x) : x \in S\}$.

If $f^{-1}(f(S)) = S$. Then $\forall x \in f^{-1}(f(S))$, $x \in S$.

If $x_1 \neq x_2 \in f^{-1}(f(S))$. Then $f(x_1) \in f(S)$, $f(x_2) \in f(S)$.

If $f(x_1) = f(x_2)$. Then $(f^{-1} \circ f)(x_1)$, $(f^{-1} \circ f)(x_2)$ $\exists x_1$ or x_2 $\in S$, But not in $f^{-1}(f(S))$ ※ Contradiction.

If $x_1 = x_2 \in f^{-1}(f(S))$. Then $f(x_1) \in f(S)$, $f(x_2) \in f(S)$. If $f(x_1) \neq f(x_2)$ ※ Contradiction to definition of function. $\forall x \in S$, $\exists! y \in Y$, s.t. $y = f(x)$.

Thus f is injective.

$(\Leftarrow)$ If f is injective. Suppose $x_1 \neq x_2$, $x_1, x_2 \in f^{-1}(f(S))$.

$f(x_1) \neq f(x_2) \in f(S)$.

$\Rightarrow$ By def, $x_1, x_2 \in S$, $x_1 \neq x_2$.

$f^{-1}(f(S)) \subseteq S$.

Suppose $\exists x_1, x_2 \in S$, s.t. $x_1 \neq x_2$

$f(x_1), f(x_2) \in f(S)$, $f(x_1) \neq f(x_2)$.

$x_1, x_2 \in f^{-1}(f(S))$ $\Rightarrow$ $S \subseteq f^{-1}(f(S))$

Thus $f^{-1}(f(S)) = S$.

Ex 3.4.6 (i) (1st attempt, checked other's answer, then abort)
(2nd attempt)

Prove Lemma 3.4.10

Let X be a set. Then $\{Y : Y$ is a subset of $X\}$ is a set.
In other words, there exists a set Z, s.t. $Y \in Z \iff Y \subseteq X$, for all objects Y.

1. Suppose we have a set $\{0,1\}^X$, s.t. by Axiom 3.11 (Power set Axiom).
   we have $f : X \to \{0,1\}$., $f \in \{0,1\}^X$.

2. From step 1 and Axiom 3.7 (Replacement), suppose we have $P(f,y)$, s.t. for each $f \in \{0,1\}^X$, there is at most one y for which $P(f,y)$ is true.

   Then there exists a set $\{y : P(f,y)$ is true for some $f \in \{0,1\}^X\}$, such that for any object z:

   $z \in \{y : P(f,y)$ is true for some $f \in \{0,1\}^X\}$
   $\iff P(f,z)$ is true for some $f \in \{0,1\}^X$.

3. From step 2. assume that $P(f,y)$ is true if $y = f^{-1}(\{1\}) \subseteq X$. (WLOG)

   Then

   $$z \in \{y : y = f^{-1}(\{1\}),\ f \in \{0,1\}^X\} \iff z = f^{-1}(\{1\}) \subseteq X$$

   Let $z = Y$, $\{y : y = f^{-1}(\{1\}), f \in \{0,1\}^X\} = Z$. Proved the Lemma.

Ex 3.4.6 (ii) (1st attempt)

Show that Axiom 3.11 can be deduced, using the preceding axioms of set theory if one accepts Lemma 3.4.10 as an axiom.

1. We want to show that there exists a set Y^X, where $f \in Y^X$, $f: X \to Y$. (Axiom 3.11)
   Given if Lemma 3.4.10 is an axiom.
   such that $Z = \{Y : Y \subseteq X\}$, if X is a set.

2. From step 1 and Axiom of replacement, suppose we have $P(Y, f)$, s.t.
   for each $Y \in Z$. there is at most one function for which $P(Y, f)$ is true.
   (if $Y \in Z$, then $P(Y, f)$ is true, otherwise false).

   Then there exists a set $\{f : P(Y, f)$ is true for some $Y \in Z\}$, s.t.
   for any function g

   $$g \in \{f : P(Y, f) \text{ is true for some } Y \in Z\}$$
   $$\iff P(Y, g) \text{ is true for some } Y \in Z.$$

   Then since $Y \subseteq X$, $P(Y, g)$ is true for some $Y \in Z$.

   $Y \in Z \Rightarrow Y \subseteq X$, s.t. $\forall g \in \{f : P(Y, f), \exists Y \in Z\}$.

   (Note: if Cardinality is introduced, then easier to prove). $g: X \to Y$, as $\forall x \in X, \exists! y \in Y, g(x) = y$.

3. From step 2, we then know that $\forall g \in \{f : P(Y, f)\} = \{f : f: X \to Y\}$.

   Thus implies Axiom 3.11.

Ex 3.4.7 (1st attempt)

Let X, Y be sets. Define a partial function from X to Y, to be any function $f: X' \to Y'$ whose domain X' is a subset of X, and whose codomain Y' is a subset of Y.

Show that the collection of all partial functions from X to Y is itself a set.

1. By Axiom 3.11, we know that there exist a set Y^X, s.t. $\forall f: X \to Y$, $f \in Y^X$. we also know that for every subset X' of X, and every subset Y' of Y, there exists a partial function h, s.t. $h: X' \to Y'$.

2. By step 1 and Axiom 3.7 (Replacement), for any $h \in Y^X$, and any subsets X' and Y', suppose we have $P(h, Y'^{X'})$ pertaining to h and $Y'^{X'}$, such that for each $h \in Y^X$, there is at most one $Y'^{X'}$, s.t. $P(h, Y'^{X'})$ is true, aka, $h \in Y'^{X'}$.

   Then for any set Q, $Q \in \{Y'^{X'} : h \in Y'^{X'}, h \in Y^X\} = A$
   $\iff h \in Q$, for $h \in Y^X$.

3. From step 2 and Lemma 3.4.10, we know that $\forall Q \in \{Y'^{X'} : h \in Y'^{X'}, h \in Y^X\} = A$
   $Q \subseteq Y^X$,

   and by Axiom 3.12 (Union), we know that all elements in $\{Y'^{X'} : h \in Y'^{X'}, h \in Y^X\} = A$ are sets.

   Then there exist a set $\cup A$ whose elements are elements of elements of A, s.t. $x \in \cup A \iff (x \in Q \text{ for some } Q \in A)$.

   Thus the collection of all partial functions from X to Y is itself a set.

Ex 3.4.8 (1^{st} attempt)

Show that Axiom 3.5 (Pairwise Union) can be deduced from Axiom 3.1 (Sets are objects), Axiom 3.4 (Singleton sets and pair sets), and Axiom 3.12 (Union).

We want to show that $x \in A \cup B \iff (x \in A$ or $x \in B)$. (Axiom 3.5 Pairwise Union)

1. Let A be a set, s.t. all elements of A are sets.
   By Axiom 3.12 (Union), there exists a set $\bigcup A$, s.t.
   $x \in \bigcup A \iff (x \in S$ for some $S \in A)$.

   Similarly, there is another set B, with all the elements are sets.
   By Axiom 3.12, $x \in \bigcup B \iff (x \in S$ for some $S \in B)$.

2. From step 1, there are two sets $\bigcup A$ and $\bigcup B$, apply Axiom 3.1, we would ask:
   If $\bigcup A \in \bigcup B$, Then $x \in \bigcup A \Rightarrow x \in \bigcup B$, $x \in (\bigcup A) \cup (\bigcup B) \iff (x \in A$ or $x \in B)$
   If $\bigcup B \in \bigcup A$, . . . Both imply Axiom 3.5 (Pairwise Union).
   If both $\bigcup A \notin \bigcup B$ and $\bigcup B \notin \bigcup A$:
   Then by Axiom 3.4, there exists a set P, $P = \{\bigcup A, \bigcup B\}$.

3. From step 3, we have a set $P = \{\bigcup A, \bigcup B\}$, where $\bigcup A$, and $\bigcup B$ are sets.
   Applying Axiom 3.11 (Union), there exist a set $\bigcup P$, s.t.
   $x \in \bigcup P \iff (x \in \bigcup A$ or $x \in \bigcup B) \iff x \in (\bigcup A) \cup (\bigcup B)$.
   Implying Axiom 3.5 (Pairwise Union).

Ex 3.4.9 (1st attempt)

Show that if β and β' are two elements of a set I, and to each $\alpha \in I$, we assign a set A_α. Then

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}.$$

Show that definition of $\bigcap_{\alpha \in I} A_\alpha := \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$ doesn't depend on β.

Explain why this is true: $y \in \bigcap_{\alpha \in I} A_\alpha \iff (y \in A_\alpha \text{ for all } \alpha \in I)$.

1. By Axiom of replacement, for each $\alpha \in I$, suppose $P(\alpha, y)$, s.t. for every $\alpha \in I$, there is at most one y, s.t. $P(\alpha, y)$ is true. $P(\alpha, y)$ is true if $y \in A_\alpha$.
   Then there exists a set $\{y : y \in A_\alpha, \alpha \in I\}$, s.t. for any z.

   $$z \in \{y : y \in A_\alpha \text{ for some } \alpha \in I\} \iff z \in A_\alpha, \text{ for some } \alpha \in I.$$

2. From step 1, we know that $z \in \{y : y \in A_\alpha \text{ for some } \alpha \in I\}$
   $\iff z \in A_\alpha$ for some $\alpha \in I$.

   Suppose $I = \{\beta, \beta'\}$, s.t. $\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$.
   $\Rightarrow A_\beta = \{x : x \in A_\alpha \text{ for all } \alpha \in I\}$

   $\{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\}$
   $\Rightarrow A_{\beta'} = \{x : x \in A_\alpha \text{ for all } \alpha \in I\}$
   Thus $A_\beta = A_{\beta'}$.

3. From step 2, we can see that $\bigcap_{\alpha \in I} A_\alpha := \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\}$
   $= \{x : x \in A_\alpha \text{ for all } \alpha \in I\}$ does not depend on β.

4. And by steps 2 and 3, we can immediately see that

   $$y \in \bigcap_{\alpha \in I} A_\alpha \iff (y \in A_\alpha \text{ for all } \alpha \in I)$$

where the replacement axiom has been applied.

Ex 3.4.10 (1^{st} attempt)

Suppose that I and J are two sets, and for all $\alpha \in I \cup J$, let A_α be a set.

Show that $\left(\bigcup_{\alpha \in I} A_\alpha\right) \cup \left(\bigcup_{\alpha \in J} A_\alpha\right) = \bigcup_{\alpha \in I \cup J} A_\alpha$.

If I and J are non-empty, show that $\left(\bigcap_{\alpha \in I} A_\alpha\right) \cap \left(\bigcap_{\alpha \in J} A_\alpha\right) = \bigcap_{\alpha \in I \cup J} A_\alpha$.

1. By Axiom 3.12 (Union), $\bigcup_{\alpha \in I} A_\alpha := \bigcup \{A_\alpha : \alpha \in I\}$ $\bigcup_{\alpha \in J} A_\alpha := \bigcup \{A_\alpha : \alpha \in J\}$

By Axiom 3.4, there exists a set $P = \{\bigcup_{\alpha \in I} A_\alpha, \bigcup_{\alpha \in J} A_\alpha\}$

By Axiom 3.12, $x \in \bigcup P \iff (x \in \bigcup_{\alpha \in I} A_\alpha$ or $x \in \bigcup_{\alpha \in J} A_\alpha) \iff x \in (\bigcup_{\alpha \in I} A_\alpha) \cup (\bigcup_{\alpha \in J} A_\alpha)$

$\Updownarrow$

$x \in \bigcup_{\alpha \in I \cup J} A_\alpha$. (Also deduced from Ex 3.4.8)

2. Similarly, By Axiom of specification, $\bigcap_{\alpha \in I} A_\alpha := \{x : x \in A_\alpha \text{ for all } \alpha \in I\}$

$\bigcap_{\alpha \in J} A_\alpha := \{x : x \in A_\alpha \text{ for all } \alpha \in J\}$.

By Axiom 3.4, there exists a set $Q = \{\bigcap_{\alpha \in I} A_\alpha, \bigcap_{\alpha \in J} A_\alpha\}$.

By modification of Axiom 3.12 (for intersection),

$x \in \bigcap Q \iff (x \in \bigcap_{\alpha \in I} A_\alpha$ and $x \in \bigcap_{\alpha \in J} A_\alpha) \iff x \in (\bigcap_{\alpha \in I} A_\alpha) \cap (\bigcap_{\alpha \in J} A_\alpha)$

$\Updownarrow$

$x \in \bigcap_{\alpha \in I \cap J} A_\alpha$.

(1st attempt, tired, brain not functioning)

Ex 3.4.11 (2nd attempt)

Let X be a set, let I be a non-empty set, and for all $\alpha \in I$, let A_α be a subset of X.

$\boxed{\text{Show that } X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)}$ and $X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$

Compare this with De Morgan's Laws (Proposition 3.1.27 (h))

1. Recall the definition of $\bigcup_{\alpha \in I} A_\alpha := \bigcup \{A_\alpha : \alpha \in I\}$

$X \setminus \bigcup_{\alpha \in I} A_\alpha := \{x \in X : x \notin \bigcup_{\alpha \in I} A_\alpha\} = \{x \in X : x \notin \bigcup \{A_\alpha : \alpha \in I\}$

2. By replacement axiom, for every $x \notin \bigcup\{A_\alpha : \alpha \in I\}$, and any subset $Q \subseteq X$ suppose we have $P(x, Q)$, s.t. for each $x \notin \bigcup\{A_\alpha : \alpha \in I\}$, there is at most one Q for which $P(x, Q)$ is true, ($P(x, Q)$ is true if $x \in Q \subseteq X$).

Then there exists a set $\{Q : x \in Q \text{ for some } x \notin \bigcup\{A_\alpha : \alpha \in I\}\}$, such that for any set Z,

$$Z \in \{Q : x \in Q \text{ for some } x \notin \bigcup\{A_\alpha : \alpha \in I\}\}$$
$$\iff \quad x \in Z \text{ for some } x \notin \bigcup\{A_\alpha : \alpha \in I\}$$

3. From step 2, denote $\{Q : x \in Q \text{ for some } x \notin \bigcup\{A_\alpha : \alpha \in I\}\}$ as P, s.t. we know that P contains all the subsets which contains at least one $x \notin A_\alpha$, for some $\alpha \in I$.

4. From steps 2 and 3, we can see that $\bigcup P = X \setminus \bigcup_{\alpha \in I} A_\alpha$

5. Consider $y \in \bigcap_{\alpha \in I} (X \setminus A_\alpha) \iff (y \in X \setminus A_\alpha \text{ for all } \alpha \in I)$

And from steps 3 and 4, we can deduce that $\bigcup P = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$. thus $\bigcap_{\alpha \in I} (X \setminus A_\alpha) = X \setminus \bigcup_{\alpha \in I} A_\alpha$.

(1st attempt, tired, brain not functioning)

Ex 3.4.11 (2nd attempt)

Let X be a set, let I be a non-empty set, and for all $\alpha \in I$, let A_α be a subset of X.

Show that $X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$ and $\boxed{X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)}$

Compare this with De Morgan's Laws (Proposition 3.1.27(h))

1. $X \setminus \bigcap_{\alpha \in I} A_\alpha := \{x \in X : x \notin \bigcap_{\alpha \in I} A_\alpha\}$, where $\bigcap_{\alpha \in I} A_\alpha := \{x : x \in A_\alpha \text{ for all } \alpha \in I\}$.

   Since $\forall x \in X \setminus \bigcap_{\alpha \in I} A_\alpha$, $(x \in X)$ and $(x \notin A_\alpha$, for every $\alpha \in I)$

2. By replacement axiom, for every $x \notin \{x : x \in A_\alpha \text{ for all } \alpha \in I\}$, and any subset $Q \subseteq X$ suppose we have $P(x, Q)$, s.t. for each $x \notin \{x : x \in A_\alpha \text{ for all } \alpha \in I\}$, there is at most one Q for which $P(x, Q)$ is true, ($P(x, Q)$ is true if $x \in Q \subseteq X$).

   Then there exists a set $\{Q : x \in Q \text{ for some } x \notin \{x : x \in A_\alpha \text{ for all } \alpha \in I\}\}$, such that for any set Z,

   $$Z \in \{Q : x \in Q \text{ for some } x \notin \{x : x \in A_\alpha \text{ for all } \alpha \in I\}\}$$
   $$\iff \quad x \in Z \text{ for some } x \notin \{x : x \in A_\alpha \text{ for all } \alpha \in I\}$$

3. From step 2, denote $\{Q : x \in Q \text{ for some } x \notin \{x : x \in A_\alpha \text{ for all } \alpha \in I\}\}$ as P, s.t. we know that P contains all the subsets which contains at least one $x \notin A_\alpha$, for all $\alpha \in I$.

4. From steps 2 and 3, we can see that $\bigcup P = X \setminus \bigcap_{\alpha \in I} A_\alpha$

5. Consider $y \in \bigcup_{\alpha \in I} (X \setminus A_\alpha) \iff (y \in X \setminus A_\alpha \text{ for some } \alpha \in I)$

   And from steps 3 and 4, we can deduce that $\bigcup P = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$. thus

   $$X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$$

Ex 3.5.1 (i) (1st attempt)

Suppose we define the ordered pair (x,y) for any object x and y by the formula $(x,y) := \{\{x\}, \{x,y\}\}$ (required serveral applications of Axiom 3.4).

For example: $(1,2)$ is the set $\{\{1\}, \{1,2\}\}$

$(2,1)$ is the set $\{\{2\}, \{2,1\}\}$

$(1,1)$ is the set $\{\{1\}\}$

Show that such a definition (Kuratowski definition of an ordered pair) (Defn 3.5.1) obeys the properties of an ordered pair (3.5). $(x,y) = (x',y') \iff (x=x' \text{ and } y=y')$.

1. ($\Rightarrow$) Suppose $(x,y) = (x',y')$.

Then by Kuratowski's definition of an ordered pair,

$(x,y) = \{\{x\}, \{x,y\}\}$.

$(x',y') = \{\{x'\}, \{x',y'\}\}$.

such that $\{x\}, \{x,y\}$ should be an element of $\{\{x'\}, \{x',y'\}\}$, and

$\{x'\}, \{x',y'\}$ should be an element of $\{\{x\}, \{x,y\}\}$

Comparing singleton sets and pair sets respectively, $\{x\} = \{x'\} \neq \{x',y'\}$.

$\{x,y\} = \{x',y'\} \neq \{x'\}$

Thus $x = x'$, $y = y'$.

($\Leftarrow$) Suppose $x = x'$ and $y = y'$.

Then by Kuratowski's definition of an ordered pair: $(x,y) = \{\{x\}, \{x,y\}\}$.

$= \{\{x'\}, \{x',y'\}\}$. (Axiom 3.4).

Thus, $(x,y) = (x',y')$.

Ex 3.5.1 (i) (2nd attempt)

Suppose we define the ordered pair (x,y) for any object x and y by the formula $(x,y) := \{\{x\}, \{x,y\}\}$ (required serveral applications of Axiom 3.4).

For example: $(1,2)$ is the set $\{\{1\}, \{1,2\}\}$
$(2,1)$ is the set $\{\{2\}, \{2,1\}\}$
$(1,1)$ is the set $\{\{1\}\}$

Show that such a definition (Kuratowski definition of an ordered pair) (Defn 3.5.1) obeys the properties of an ordered pair (3.5). $(x,y) = (x',y') \iff (x=x' \text{ and } y=y')$.

1. $(\Rightarrow)$

If $(x,y) = (x',y')$. Then $\{\{x\}, \{x,y\}\} = \{\{x'\}, \{x',y'\}\}$

This implies that $\{x\} = \{x'\} \vee \{x',y'\} \Rightarrow \{x\} \neq \{x',y'\}$ as $x' \neq y'$

$\{x,y\} = \{x'\} \vee \{x',y'\} \Rightarrow \{x,y\} \neq \{x'\}$ as $x \neq y$

Thus $x = x'$ and $y = y'$

2. $(\Leftarrow)$

If $x = x'$ and $y = y'$. Then $\{x\} = \{x'\}$ and $\{x',y'\} = \{x,y\}$.

Thus in (x,y), $\{x\}$ and $\{x,y\} \in (x,y) \Rightarrow \in (x',y')$ $\quad (x,y) \subseteq (x',y')$

in (x',y'), $\{x'\}$ and $\{x',y'\} \in (x',y') \Rightarrow \in (x,y)$ $\quad (x',y') \subseteq (x,y)$

Thus $(x,y) = (x',y')$

Ex 3.5.1 (ii) (1^{st} attempt)

Suppose we have an alternative definition for ordered pairs. $(x,y) := \{x, \{x,y\}\}$.

Show that this definition (Short definition of an ordered pair)

also verifies the property (3.5), $(x,y) = (x',y') \iff (x = x', y = y')$ of an ordered pair.

($\Rightarrow$) Suppose $(x,y) = (x',y')$, s.t.

by the short definition of an ordered pair,

$\{x, \{x,y\}\} = \{x', \{x',y'\}\}$

s.t. by Axiom 3.2 and Axiom 3.4, $x = x'$, $y = y'$.

($\Leftarrow$) Suppose $x = x'$, $y = y'$.

Then we know that by Axiom 3.4, thus Axiom 3.2,

$\{x, y\} = \{x', y'\}$. Thus $\{x, \{x,y\}\} = \{x', \{x', y'\}\}$.

Thus applying the short definition, $\cdots$, $(x,y) = (x',y')$.

Ex 3.5.1 (ii) (2^{nd} attempt, check solution, then do it)

Suppose we have an alternative definition for ordered pairs. $(x,y) := \{x, \{x,y\}\}$.
Show that this definition (Short definition of an ordered pair)
also verifies the property (3.5), $(x,y) = (x',y') \Leftrightarrow (x=x', y=y')$ of an ordered pair.

1. ($\Rightarrow$) If $(x,y) = (x',y)$ Then $\{x, \{x,y\}\} = \{x', \{x',y'\}\}$
   This implies that $(x = x')$ or $x = \{x',y'\}$ by Axiom of regularity
   If $x = \{x',y'\}$. Then $x \notin \{x',y'\}$, s.t. $x \neq x'$ and $x \neq y'$. $\therefore$ X. Contradiction.
   If $x = x'$. Then $x \in \{x',y'\}$.
   Thus $x = x'$
   Similarly, either $(\{x,y\} = x')$ or $(\{x,y\} = \{x',y'\})$.
   If $\{x,y\} = x'$ Then since $x = x'$, $x \neq \{x,y\}$, $\{x,y\} \notin \{x,y\}$.
   Thus since $(x,y) = (x',y')$, $\{x,y\} \in (x',y') \Rightarrow \{x,y\} = \{x',y'\}$.
   Since $x = x'$, thus $y = y'$.

2. ($\Leftarrow$) If $(x = x')$ and $(y = y')$. Then $x \in \{x,y\}$, $x \in \{x',y'\}$, $y \in \{x,y\}$, $y \in \{x',y'\}$ $\Rightarrow \{x,y\} \subseteq \{x',y'\}$
   $x' \in \{x',y'\}$, $x' \in \{x,y\}$, $y' \in \{x',y'\}$, $y' \in \{x,y\}$ $\Rightarrow \{x',y'\} \subseteq \{x,y\}$.
   $\Rightarrow \{x,y\} = \{x',y'\}$.

Since $x = x'$ and $\{x,y\} = \{x',y'\}$,
Thus $(x,y) \subseteq (x',y')$, $(x',y') \subseteq (x,y)$ Thus $(x,y) = (x',y')$

Ex 3.5.1 (iii) (1st attempt)

Show that regardless of the definition of ordered pair, the Cartesian product $X \times Y$ of any two sets X, Y is again a set.

1. By axiom of replacement, for every $x \in X$, and any object (x,y) suppose $P(x,(x,y))$ is a property pertaining x and (x,y) s.t. for each $x \in X$, there is at most one (x,y), s.t. $P(x,(x,y))$ is true.
Note: $P(x,(x,y))$ is true if $y \in Y$.

There exist a set $\{(x,y) : P(x,(x,y)) \text{ is true for some } x \in X\}$.
for any object z,

$$z \in \{(x,y) : y \in Y \text{ for some } x \in X\} = A$$
$$\iff P(x,z) \text{ is true for some } x \in X.$$

2. From step 1, we know that $z \in A$, z are sets.
for $z = \{x, \{x,y\}\}$, (Short definition of ordered sets).
Thus by Axiom of union, there exist, a set $\bigcup A$, s.t.

$$z \in \bigcup A \iff (z \in (x,y) \text{ for some } (x,y) \in A)$$
$$\iff X \times Y$$

Ex 3.5.2 (1st attempt)

Suppose we define an ordered n-tuple to be a surjective function x,
$x: \{i \in \mathbb{N} : 1 \le i \le n\} \to X$, where X is arbitrary.
Write $x(i)$ as x_i and write x as $(x_i)_{1 \le i \le n}$.

Using this definition, verify that $(x_i)_{1 \le i \le n} = (y_i)_{1 \le i \le n}$ iff $x_i = y_i$ for all $1 \le i \le n$.

Also, show that if $(X_i)_{1 \le i \le n}$ are an ordered n-tuple of sets,
Then the Cartesian Product defined in (Defn 3.5.6) is a set.

1. ($\Rightarrow$) If $(x_i)_{1 \le i \le n} = (y_i)_{1 \le i \le n}$, where $x : \{i \in \mathbb{N} : 1 \le i \le n\} \to X$
$y : \{i \in \mathbb{N} : 1 \le i \le n\} \to X$.
By defn 3.3.8 (Equality of func), $x = y$, iff $x(i) = y(i)$, $\forall i \in \{i \in \mathbb{N} : 1 \le i \le n\}$.
($\Leftarrow$) If $x_i = y_i$ But $(x_i)_{1 \le i \le n} \ne (y_i)_{1 \le i \le n}$.
Then ~~X~~ Contradiction, as $\exists i$, s.t. $x_i \ne y_i$. $\Rightarrow$ thus $x = y$.

2. If $(X_i)_{1 \le i \le n}$ are an ordered n-tuple of sets.
s.t. $(X_1, \dots, X_n)$, which is a collection of sets $X_i, \dots, X_n$.

The Cartesian product (Defn 3.5.6). $\prod_{1 \le i \le n} X_i$

Induct on n. s.t. Let $n = 2$, $\prod_{1 \le i \le 2} X_i = X_1 \times X_2$
If X_1 and X_2 are sets. Then $X_1 \times X_2$ are sets.

Inductively assume that $\prod_{1 \le i \le n} X_i = X_1 \times X_2 \times \cdots \times X_n$ is a set.
Show that $\prod_{1 \le i \le n+1} X_i$ is also a set.

$\prod_{1 \le i \le n+1} X_i = \left(\prod_{1 \le i \le n} X_i\right) \times X_{n+1}$, since $\left(\prod_{1 \le i \le n} X_i\right)$ and X_{n+1} are sets.

Then $\prod_{1 \le i \le n+1} X_i$ is also a set.

Induction closed.

R $x=x$. S $x=a$, $a=x$. T $a=b$, $b=c$ $a=c$.

Ex 3.5.3 (1st attempt)

Show that the definitions of equality for ordered pair and ordered n-tuple are consistent with the reflexivity, symmetry, and transitivity axioms.

(In the sense that: If these axioms are assumed to hold for the individual components x, y of an ordered pair (x,y). Then they hold for the ordered pair itself).

1. $(x,y) = (x,y)$ iff $(x=x,\ y=y)$ Reflexivity
$(x_i)_{1\le i\le n} = (x_i)_{1\le i\le n}$ iff $(x_1 = x_1,\ x_2 = x_2, \dots, x_n = x_n)$.

2. $(x,y) = (x',y')$ iff $(x=x',\ y=y')$ Symmetry. $\Rightarrow (x',y') = (x,y)$.
$(x_i)_{1\le i\le n} = (x'_i)_{1\le i\le n}$ iff $(x_1 = x'_1,\ x_2 = x'_2, \dots, x_n = x'_n)$.

3. $(x,y) = (x',y')$, $(x',y') = (x'',y'')$ Transitivity.
$(x_i)_{1\le i\le n} = (x'_i)_{1\le i\le n}$, $(x'_i)_{1\le i\le n} = (x''_i)_{1\le i\le n}$

Ex 3.5.4 (1^{st} attempt)
Let A, B, C be sets.
Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$
Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$
Show that $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

1. $A \times (B \cup C) = \{(x, y) : x \in A, y \in B \cup C\}$
$(A \times B) \cup (A \times C) = \{(x, y) : x \in A, y \in B\} \cup \{(x, y) : x \in A, y \in C\}$ since $A = A$.
If $(x, y) \in A \times (B \cup C)$. Then $x \in A$, $y \in B \cup C$
If $(x, y) \in (A \times B) \cup (A \times C)$. Then $x \in A \cup A$, $y \in B \cup C$.

2. $A \times (B \cap C) = \{(x, y) : x \in A, y \in B \cap C\}$
$(A \times B) \cap (A \times C) = \{(x, y) : x \in A, y \in B\} \cap \{(x, y) : x \in A, y \in C\}$.
If $(x, y) \in A \times (B \cap C)$. Then $(x \in A, y \in B \cap C)$
If $(x, y) \in (A \times B) \cap (A \times C)$. Then $(x \in A, y \in B \cap C)$ since $A = A$.

3. $A \times (B \setminus C) = \{(x, y) : x \in A, y \in B \setminus C\}$
$(A \times B) \setminus (A \times C) = \{(x, y) : x \in A, y \in B\} \setminus \{(x, y) : x \in A, y \in C\}$. since $A = A$.
If $(x, y) \in A \times (B \setminus C)$. Then $(x \in A, y \in B \setminus C)$
If $(x, y) \in (A \times B) \setminus (A \times C)$. Then $(x \in A, y \in B)$ and $(x \notin A, y \notin C)$.

Ex 3.5.5 (1^{st} attempt)

Let A, B, C, D be sets.

Show that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Is it true that $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$?

Is it true that $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$?

1. $(A \times B) \cap (C \times D) = \{(x,y) : x \in A, y \in B\} \cap \{(x,y) : x = C, y = D\}.$
$$= \{(x,y) : x \in A \cap C, y \in B \cap D\}$$

$(A \cap C) \times (B \cap D) = \{(x,y) : x \in A \cap C, y \in B \cap D\}.$

2. $(A \times B) \cup (C \times D) = \{(x,y) : x \in A, y \in B\} \cup \{(x,y) : x \in C, y \in D\}.$

$(A \cup C) \times (B \cup D) = \{(x,y) : x \in A \cup C, y \in B \cup D\}.$ $\neq$
$$= (A \times B) \cup (C \times D) \cup (A \times D) \cup (C \times B).$$

3. $(A \times B) \setminus (C \times D) = \{(x,y) : x \in A, y \in B\} \setminus \{(x,y) : x \in C, y \in D\}.$
$$= (x \in A \setminus C, y \in B) \vee (x \in A \setminus C, y \in B \setminus D) \vee (x \in A, y \in B \setminus D)$$
$$\vee (x \in A, y \in B)$$
$\neq$ (if $C \subseteq A$, or $D \subseteq B$).

$(A \setminus C) \times (B \setminus D) = \{(x,y) : x \in A \setminus C, y \in B \setminus D\}$

Ex 3.5.6 (1st attempt)

Let A, B, C, D be non-empty sets.

Show that $A \times B \subseteq C \times D$ iff $A \subseteq C$ and $B \subseteq D$

Show that $A \times B = C \times D$ iff $A = C$ and $B = D$

What happens if some or all of the hypotheses that the A, B, C, D are non-empty, removed?

1. $(\Rightarrow)$

$A \times B \subseteq C \times D \Rightarrow (p, q) \in \{(x,y) : x \in A, y \in B\}$. $(p,q) \in \{(x,y) : x \in C, y \in D\}$.

This implies that $A \subseteq C$ and $B \subseteq D$.

Either $A \not\subseteq C$ or $B \not\subseteq D$. Then contradiction. $\exists (p,q) \in A \times B, \notin C \times D$.

$(\Leftarrow)$ $A \subseteq C$ and $B \subseteq D$.

Then $x \in A$, $x \in C$., $y \in B$, $y \in D$

$z \in \{(x,y) : x \in A, y \in B\} \Rightarrow z \in \{(x,y) : x \in C, y \in D\}$.

$A \times C \subseteq B \times D$.

2. $(\Rightarrow)$

$A \times B = C \times D$ iff $A \times B \subseteq C \times D$ and $C \times D \subseteq A \times B$.

Thus $A \subseteq C$ and $B \subseteq D$ and $C \subseteq A$ and $D \subseteq B$.

$A = C$, $B = D$.

$(\Leftarrow)$

$A = C$. $B = D \Leftrightarrow A \subseteq C$, $C \subseteq A$, $B \subseteq D$, $D \subseteq B$.

$A \times B \subseteq C \times D$ $C \times D \subseteq A \times B$.

$A \times B = C \times D$.

3. Suppose $A = \emptyset$. Then $A \times B \subseteq C \times D$ iff $A \subseteq C$, and $B \subseteq D$ remain true.

But not $A \times B = C \times D$ iff $A = C$, $B = D$.

as $B = C$ or D., and C or $D = \emptyset$.

Ex 3.5.7 (1^{st} attempt)

Let X, Y be sets.

Let $\pi_{X\times Y \to X} : X \times Y \to X$ be map $\pi_{X\times Y\to X}(x,y) := x$

Let $\pi_{X\times Y \to Y} : X \times Y \to Y$ be map $\pi_{X\times Y\to Y}(x,y) := y$.

Co-ordinate functions on $X \times Y$.

Show that for any functions $f: Z \to X$ and $g: Z \to Y$,
there exists a unique function $h: Z \to X \times Y$, s.t. $\pi_{X\times Y\to X} \circ h = f$

$$\pi_{X\times Y \to Y} \circ h = g$$

h is known as the pairing of f and g.
$h = (f, g)$.

1. Given there are two functions h_1 and h_2, s.t. $h_1 : Z \to X \times Y$
where $h_1 \neq h_2$ $\qquad h_2 : Z \to X \times Y$.

$$\pi_{X\times Y\to X} \circ h_1 = f_1$$
$$\pi_{X\times Y\to X} \circ h_2 = f_2$$
$$\pi_{X\times Y\to Y} \circ h_1 = g_1$$
$$\pi_{X\times Y\to Y} \circ h_2 = g_2$$

Given a $z \in Z$, s.t. $f_1(z) = x_1 \qquad g_1(z) = y_1$

$$f_2(z) = x_2 \qquad g_2(z) = y_2$$

By axiom of substitution, $x_1 = x_2$ and $y_1 = y_2$ iff $f_1 = f_2$, $g_1 = g_2$.

2. Assume that $f_1 = f_2$, $g_1 = g_2$ (no unique function h), s.t. $h_1 \neq h_2$.

$\Rightarrow \quad \pi_{X\times Y\to X} \circ h_1 = \pi_{X\times Y\to X} \circ h_2 \quad$ ※ Contradicts $h_1 \neq h_2$.

Thus $f_1 = f_2$, $g_1 = g_2$ iff $h_1 = h_2$. $h_1 = h_2 = h$ is a unique function.

Ex 3.5.8 (1st attempt)

Let $X_1, \dots, X_n$ be sets.

Show that the Cartesian product $\prod_{i=1}^{n} X_i$ is empty iff at least one of the X_i is empty.

1. $(\Rightarrow)$

$$\prod_{i=1}^{n} X_i := \{(x_i)_{1 \le i \le n} : x_i \in X_i \text{ for all } 1 \le i \le n\}$$
$$:= X_1 \times X_2 \times \dots \times X_n.$$

Induction on n. Let $n=2$. If $\prod_{i=1}^{2} X_i = X_1 \times X_2 = \emptyset$.

Suppose $X_2 \neq \emptyset$, s.t.

$$X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\} = \emptyset.$$

$\Rightarrow \nexists x_1 \in X_1 \Rightarrow X_1 = \emptyset$. (Base Case).

2. Inductively assume that if $\prod_{i=1}^{n} X_i = \emptyset$. Then $\exists i$, $1 \le i \le n$, s.t. $X_i = \emptyset$.

(WLOG). Let assume that $\nexists x_i$, s.t. $i = n$, for the above to hold.

Show that if $\prod_{i=1}^{n+1} X_i = \emptyset$. Then $\exists X_i = \emptyset$.

$\prod_{i=1}^{n+1} X_i = \prod_{i=1}^{n} X_i \times X_{n+1} = \emptyset$, We know that if $\prod_{i=1}^{n} X_i = \emptyset$. Then it holds.

Suppose $\prod_{i=1}^{n} X_i \neq \emptyset$, s.t. $\prod_{i=1}^{n} X_i \times X_{n+1} = \{(x_i)_{1 \le i \le n+1} : x_i \in X_i \text{ for all } 1 \le i \le n+1\}$

Then $X_{n+1} = \emptyset$. Otherwise it does not hold.

$$\prod_{i=1}^{n} X_i = \emptyset \Rightarrow \exists X_i, 1 \le i \le n, X_i = \emptyset.$$

3. $(\Leftarrow)$

Suppose $X_1 = \emptyset$, Then similarly by induction $\prod_{i=1}^{n} X_i = \emptyset$.

① Show that $X_1 \times X_2 = \emptyset$

② Assume that $X_n = \emptyset$ Then $\prod_{i=1}^{n} X_i = \emptyset$. others non-empty

③ Show that $X_{n+1} = \emptyset$. Then $\prod_{i=1}^{n+1} X_i = \emptyset$ others non empty.

Ex 3.5.9 (1st attempt)

Suppose that I and J are two sets, and for all $\alpha \in I$, let A_α be a set.
for all $\beta \in I$, let B_β be a set.
Show that $\left(\bigcup_{\alpha \in I} A_\alpha\right) \cap \left(\bigcup_{\beta \in J} B_\beta\right) = \bigcup_{(\alpha,\beta) \in I \times J} (A_\alpha \cap B_\beta)$.

What happens if one interchanges all the union and intersection symbols here?

1. Recall $x \in \bigcup_{\alpha \in I} A_\alpha \Longleftrightarrow (x \in A_\alpha \text{ for some } \alpha \in I)$., $\bigcup_{\alpha \in I} A_\alpha = \bigcup\{A_\alpha : \alpha \in I\}$.

$$P = \left(\bigcup_{\alpha \in I} A_\alpha\right) \cap \left(\bigcup_{\beta \in J} B_\beta\right) = \bigcup\{A_\alpha : \alpha \in I\} \cap \bigcup\{B_\beta : \beta \in J\}$$
$$= \{x \in \bigcup\{A_\alpha : \alpha \in I\} : x \in \bigcup\{B_\beta : \beta \in J\}\}.$$

$$Q = \bigcup_{(\alpha,\beta) \in I \times J} (A_\alpha \cap B_\beta) = \bigcup\{A_\alpha \cap B_\beta : (\alpha,\beta) \in I \times J\}$$
$$= \bigcup\{A_\alpha \cap B_\beta : (\alpha,\beta) \in \{(\alpha,\beta) : \alpha \in I, \beta \in J\}\}$$

2. From step 1, if $x \in P$. Then $x \in A_\alpha \cap B_\beta$ for any $\alpha \in I$, any $\beta \in J$.
$x \in Q$. $\Rightarrow P \subseteq Q$

If $x \in Q$. Then $x \in A_\alpha \cap B_\beta$ for any $(\alpha,\beta) \in I \times J$,
$x \in P$. $\Rightarrow Q \subseteq P$

Thus $P = Q$

3. $\left(\bigcap_{\alpha \in I} A_\alpha\right) \cup \left(\bigcap_{\beta \in J} B_\beta\right) = \{x : x \in A_\alpha \text{ for all } \alpha \in I\} \cup \{x : x \in B_\beta \text{ for all } \beta \in J\}$.

$$\bigcap_{(\alpha,\beta) \in I \times J} (A_\alpha \cup B_\beta) = \{x : x \in A_\alpha \cup B_\beta \text{ for all } (\alpha,\beta) \in I \times J\}$$

They still equal to each other, only if I and J are non-empty.

Ex 3.5.10 (i) (1st attempt)

If $f: X \to Y$ is a function, define the graph of f to be the subset of $X \times Y$ defined by $\{(x, f(x)) : x \in X\}$.

(i) Show that two functions $f: X \to Y$, $\tilde{f}: X \to Y$ are equal iff they have the same graph.

1. $(\Rightarrow)$

If $f = \tilde{f}$. Then $x \in X$, $f(x) = \tilde{f}(x)$.

Thus $(x, f(x)) = (x, \tilde{f}(x))$

$\Rightarrow$ graph of f = graph of $\tilde{f}$, $\{(x, f(x)) : x \in X\} = \{(x, \tilde{f}(x)) : x \in X\}$.

$(\Leftarrow)$

If graph of f and $\tilde{f}$ are the same,

Then $x \in X$, $(x, f(x)) = (x, \tilde{f}(x))$, thus. $\tilde{f}(x) = f(x)$, $\Rightarrow f = \tilde{f}$.

Ex 3.5.10 (ii) (1^{st} attempt)
If $f: X \to Y$ is a function, define the graph of f to be the subset of $X \times Y$ defined by $\{(x, f(x)) : x \in X\}$.

(ii) If G is any subset of $X \times Y$ with the property that for each $x \in X$, the set $\{y \in Y : (x, y) \in G\}$ has exactly one element
Show that there is exactly one function $f: X \to Y$ whose graph is equal to G.

1. Let $P(x, y)$ be the property pertain x and y, s.t. for each $x \in X$, there is at most one y, s.t. $z \in \{y \in Y : (x, y) \in G\}$, where $G \subseteq X \times Y$.

2. By step 1 and axiom of replacement, there exists a set $\{z : P(x, z)$ is true for some $x \in X\}$. s.t. for any object q.
$q \in A = \{z : P(x, z)$ is true for some $x \in X\} \iff P(x, q)$ is true for some $x \in X$.

3. Suppose there are two functions f_1 and f_2, s.t. $f_1, f_2 : X \to Y$, $f_1 \neq f_2$, s.t.
$\{(x, f_1(x)) : x \in X\} = G$
$\{(x, f_2(x)) : x \in X\} = G$

From steps 1 and 2, we know that for every $x \in X$, $f_1(x) \in A$, $f_2(x) \in A$. Thus $f_1(x) = f_2(x)$ -X-

Thus $f_1 = f_2$. There is only one function.

Ex 3.5.10 (iii) (1^{st} attempt)

If $f: X \to Y$ is a function, define the graph of f to be the subset of $X \times Y$ defined by $\{ (x, f(x)) : x \in X \}$.

(iii) Suppose we define a function f to be an ordered triple $f = (X, Y, G)$ where X, Y are sets, and G is a subset of $X \times Y$ that obeys the vertical line test. We then define the domain of such a triple to be X, the co-domain to be Y, and for every $x \in X$, we define $f(x)$ to be the unique $y \in Y$, s.t. $(x, y) \in G$.

Vertical lie test

$\forall x_\circ \in X, \exists ! y \in Y$

$f(x) = Y$.

Show that this definition is compatible with (Defn 3.3.1) and (Defn 3.3.8).

1. We need to show that (Defn 3.3.1) $\forall x \in X, \exists ! y \in Y$, s.t. $f(x) = y$

(Defn 3.3.8) $f = g$, iff $x \in X$, $f(x) = g(x)$.

$f: X \to Y$, $g: X \to Y$.

2. Suppose $\exists y_1, y_2 \in Y$, $y_1 \neq y_2$, s.t. $\exists ! x \in X$, $f(x) = y_1$, $f(x) = y_2$ $\cdot \times \cdot$ Contradict f definite.

for every $x \in X$, $f(x) = y$, y is unique, s.t. $(x, y) \in G$.

Thus, $y_1 = y_2$.

3. Suppose $\exists f, g$, $f = (X_1, Y_1, G_1)$ if $f = g$. Then $X_1 = X_2$, $Y_1 = Y_2$, $G_1 = G_2$.

$g = (X_2, Y_2, G_2)$ Holds.

Ex 3.5.11 (1st attempt)

Show that Axiom 3.11 (Powerset) can in fact be deduced from Lemma 3.4.10

Lemma 3.4.10 ⇒ Spec ⇒ Ex 3.5.10 ⇒ Axiom of replacement.

1. For any two sets X and Y, by definition 3.5.4 (Cartesian product),
$$X \times Y := \{(x,y) : x \in X, y \in Y\}$$
By Lemma 3.4.10, suppose G is a subset of $X \times Y$, s.t. there exists a set Q, s.t. $\{G : G \subseteq X \times Y\} = Q$

2. Let $P((x,y), y)$ be the property pertain x and y s.t. for each $(x,y) \in Q$ there is at most one y, s.t. a set W exists $W = \{y \in Y : P(x,y)$ is true for some $(x,y) \in Q\}$, by axiom of replacement.

3. From step 2, and the axiom of replacement, for every y in W, s.t. suppose $P(y,f)$ be the property pertaining y and f, s.t. for each $y \in W$, there is at most one f, s.t. $f(x) = y$, where $P((x,y), y)$ is true (step 2). Then there exist a set.. s.t. for any object z.
$$z \in \{f : P(y,f) \text{ is true for some } y \in W\}$$

4. From step 3, we can see that such a set should contain all function that satisfies Defn 3.3.1 and Defn 3.3.8. and deduced the powerset Axiom

Ex 3.5.12 (i) (1st attempt)

This will establish Proposition 2.1.16 (Recursive Definitions) rigorously, that avoids circularity.

(i) Let X be a set. Let $f: \mathbb{N} \times X \to X$ be a function.
Let c be an element of X.
Show that there exists a function $a: X \to X$ s.t. $a(0) = c$ and
$a(n{+}{+}) = f(n, a(n))$ for all $n \in \mathbb{N}$, and furthermore this function is unique.

1. Assume that this function a is $a_N : X \to X$, where $X = \{n \in \mathbb{N} : n \leq N\}$.
Fix N, and induct on n.
Let $n = 0$, then $a_N(0) = c$, $a_N(1) = f(0, a_N(0)) = f(0, c)$.

2. Suppose inductively, $a_N(n{+}{+}) = f(n, a_N(n))$ holds.
Show that $a_N((n+1){+}{+}) = f(n+1, a_N(n+1))$ is also valid.

Since $a_N(n{+}{+}) = a_N(n+1) \in X$. Then $f(n+1, a_N(n+1)) \in X$.
Thus $a_N(n+2) \in X$.

3. Suppose $a_N : X \to X$, $a_{N+1} : X \to X$, where $a_N \neq a_{N+1}$.
Suppose $a_N(N{+}{+}) = f(N, a_N(N))$ valids.
and we know that $a_{N+1}(N{+}{+}) = a_N(N{+}{+})$.
But then $a_N(N+2)$ does not exist.
$a_{N+1}(N+2)$ exist, $f(N+1, a_{N+1}(N+1))$
$= f(N+1, f(N, a_{N+1}(N))$
$= f(N+1, f(N, a_N(N))$

Thus it is unique as for each $N \in \mathbb{N}$, N is unique.

Ex 3.5.12 (ii) (1st attempt, check other's solution)
This will establish Proposition 2.1.16 (Recursive Definitions) rigorously.

(ii) Prove (i) without using any properties of the natural numbers other than the Peano Axioms directly.

We want to show that for every natural number $N \in \mathbb{N}$, there exists a unique pair A_N, B_N of subsets of $\mathbb{N}$ which obeys the following properties:

(a) $A_N \cap B_N = \emptyset$
(b) $A_N \cup B_N = \mathbb{N}$
(c) $0 \in A_N$
(d) $N{+}{+} \in B_N$
(e) Whenever $n \in B_N$, we have $n{+}{+} \in B_N$.
(f) Whenever $n \in A_N$ and $n \neq N$, we have $n{+}{+} \in A_N$.

1. Induct on N. Let $N = 0$. Then $A_N = \{0\}$, $B_N = \{1, 2, 3, 4, \dots\} = \mathbb{N} \setminus \{0\}$
All conditions are satisfied.

2. Suppose inductively, that $\exists N_1 \in \mathbb{N}$, s.t. $N = N_1$ satisfies all conditions.
Show that $N_1{+}{+}$ also satisfies the conditions.

$$A_{N++} = A_N \cup \{N{+}{+}\} \qquad B_{N++} = B_N \setminus \{N{+}{+}\}$$

$A_{N++} \cap B_{N++} = \emptyset$ $\quad A_{N++} \cup B_{N++} = \mathbb{N}$, $\quad 0 \in A_{N++}$ as $0 \in A_N$
$(N{+}{+}){+}{+} \in B_{N++}$, etc. All conditions are satisfied.

3. From steps 1 and 2, for every natural number $N \in \mathbb{N}$, there exists a unique pair A_N, B_N of subsets of $\mathbb{N}$ which obeys the following properties:

(a) $A_N \cap B_N = \emptyset$
(b) $A_N \cup B_N = \mathbb{N}$
(c) $0 \in A_N$
(d) $N{+}{+} \in B_N$
(e) Whenever $n \in B_N$, we have $n{+}{+} \in B_N$.
(f) Whenever $n \in A_N$ and $n \neq N$, we have $n{+}{+} \in A_N$.

4. We can now substitute A_N as $\{n \in \mathbb{N} : n \leq N\}$ in (i).

Ex 3.5.12 (i) (2nd attempt)

This will establish Proposition 2.1.16 (Recursive Definitions) rigorously, that avoids circularity.

(i) Let X be a set. Let $f: \mathbb{N} \times X \to X$ be a function.

Let c be an element of X.

Show that there exists a function $a: X \to X$ s.t. $a(0) = c$ and

$a(n{+}{+}) = f(n, a(n))$ for all $n \in \mathbb{N}$, and furthermore this function is unique.

Ex 3.5.12 (ii) (2^{nd} attempt)

This will establish Proposition 2.1.16 (Recursive Definitions) rigorously.

(ii) Prove (i) without using any properties of the natural numbers other than the Peano Axioms directly.

Ex 3.5.13 (1^{st} attempt)

Purpose of this is to show that there is exactly one version of natural number system in set theory.

Suppose we have a set $\mathbb{N}'$ of "alternative natural numbers", an "alternative zero", $0'$, and an "alternative increment operation" which takes any alternative natural numbers $n' \in \mathbb{N}'$ and returns another alternative natural number $n'{+}{+}' \in \mathbb{N}'$, s.t. the Peano Axiom (Axioms 2.1 - 2.5). all hold with the natural numbers, zero, and increment replaced by their alternative counterparts.

Show that there exists a bijection $f: \mathbb{N} \to \mathbb{N}'$ from the natural numbers to the alternative natural numbers, s.t. $f(0) = 0'$, and s.t. for any $n \in \mathbb{N}$ and $n' \in \mathbb{N}'$, we have $f(n) = n'$ iff $f(n{+}{+}) = n'{+}{+}'$.

1. Suppose there exist a function f, s.t. $f: \mathbb{N} \to \mathbb{N}'$, $f(0) = 0'$
   We want to show that if $f(n) = n'$ Then $f(n{+}{+}) = n'{+}{+}'$ and the converse.

2. ($\Rightarrow$) If $f(n) = n'$, for any $n \in \mathbb{N}$ and any $n' \in \mathbb{N}'$,
   Induct on n. Let $n = 0$, Then we know that it holds $f(0) = 0'$
   Suppose inductively that $\exists N \in \mathbb{N}$, s.t. $f(N) = N'$. Then $f(N{+}{+}) = N'{+}{+}'$.
   Show that $f(N{+}{+}) = N'{+}{+}'$. Then $f((N{+}{+}){+}{+}) = (N'{+}{+}'){+}{+}'$.
   By Axiom 2.2, if $n \in \mathbb{N}$, then $n{+}{+} \in \mathbb{N}$, s.t.
   $f(N{+}{+}) = N'{+}{+}' \Rightarrow f((N{+}{+}){+}{+}) = M$, where $M = (N'{+}{+}'){+}{+}'$
   Induction closed.

3. ($\Leftarrow$) If $f(n{+}{+}) = n'{+}{+}'$. Then $f(n) = n'$
   Let $n = 0$. Then it holds
   Inductively assume that it is true for $n = N$.
   If $(N{+}{+}){+}{+}$ is true. and if $(N{+}{+})$ is not true. ※ Contradict.
   $(N{+}{+})$ is true. Backwards induction.
   Closed induction

Ex 3.5.13 (2^{nd} attempt)

Purpose of this is to show that there is exactly one version of natural number system in set theory.

Suppose we have a set $\mathbb{N}'$ of "alternative natural numbers", an "alternative zero", $0'$, and an "alternative increment operation" which takes any alternative natural numbers $n' \in \mathbb{N}'$ and returns another alternative natural number $n'{+}{+}' \in \mathbb{N}'$, s.t. the Peano Axiom (Axioms 2.1 - 2.5). all hold with the natural numbers, zero, and increment replaced by their alternative counterparts.

Show that there exists a bijection $f: \mathbb{N} \to \mathbb{N}'$ from the natural numbers to the alternative natural numbers, s.t. $f(0) = 0'$, and s.t. for any $n \in \mathbb{N}$ and $n' \in \mathbb{N}'$, we have $f(n) = n'$ iff $f(n{+}{+}) = n'{+}{+}'$.

use bijection

injective

surjective

properties

Ex 3.6.1 (1^{st} attempt)

Prove Proposition 3.6.4

Let X, Y, Z be sets. Then X has equal cardinality with X.

If X has equal cardinality with Y. Then Y has equal cardinality with X.

If X has equal cardinality with Y and Y has equal cardinality with Z, then X has equal cardinality with Z.

1. By defn 3.6.1 (Equal Cardinality). $f: X \to X$, for every $x \in X$, $\exists! \, x \in X$, s.t. $f(x) = x$.

   for $x, y \in X$, $x \neq y$, $f(x) = x$, $f(y) = y$, $x \neq y$. Injective.

   Suppose $\exists x \in X$, s.t. $\nexists! \, x \in X$, s.t. $f(x) = x$. $\cancel{\text{i}X}$

   Contradiction, such a x does not exist. (Surjection) Thus f is bijective $\Rightarrow$ X has equal cardinality with X.

2. If X has equal cardinality with Y. Then by defn 3.6.1,

   $f: X \to Y$. f is a bijection from X to Y.

   Thus f^{-1} exists as f's inverse and we know that f^{-1} is bijective from Y to X.

   thus Y has equal cardinality with X.

3. If X has equal cardinality with Y. $f: X \to Y$, f is bijective.

   If Y has equal cardinality with Z. $g: Y \to Z$, g is bijective.

   Then $(f \circ g): X \to Z$ is also bijective $\Rightarrow$ X has equal cardinality with Z.

Ex 3.6.2 (1st attempt)

Show that a set X has cardinality 0 iff X is the empty set.

1. ($\Rightarrow$) X has cardinality 0 if X has equal cardinality with $\{i \in \mathbb{N} : 1 \le i \le 0\}$ (Defn 3.6.5).
   we know that $\{i \in \mathbb{N} : 1 \le i \le 0\} = \emptyset$.

($\Leftarrow$) If X is the empty set. Then X has equal cardinality with $\emptyset = \{i \in \mathbb{N} : 1 \le i \le 0\}$.

$$\#(X) = 0,$$

Ex 3.6.3 (1st attempt)

Let n be a natural number. Let $f: \{i \in \mathbb{N} : 1 \leq i \leq n\} \to \mathbb{N}$ be a function.

Show that there exists a natural number M, s.t. $f(i) \leq M$ for all $1 \leq i \leq n$.

1. We know that $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ for any $n \in \mathbb{N}$ is a finite set (Defn 3.6.10).

   For $\mathbb{N}$, we can see that the set is infinite as suppose if it is finite. Then there is always an additional element in $\mathbb{N}$, that does not have a one-to-one correspondence in the finite set, such a bijection does not exist. So no equal cardinality between finite sets and $\mathbb{N}$.

2. Suppose $G \subseteq \mathbb{N}$, s.t. $f(\{i \in \mathbb{N} : 1 \leq i \leq n\}) = G$, s.t. $\exists N \in G$, s.t. $N \in \mathbb{N}$, where N has the largest value.
   $\forall y \in G$, $y \leq N$. But since $N \in \mathbb{N}$, then $\exists N'$, s.t. $N{+}{+} = N'$, $N \leq N'$.
   Thus $\forall y \in G$, $\exists N' \in \mathbb{N}$, s.t. $y \leq N'$.

Ex 3.6.4 (1st attempt)

Prove Proposition 3.6.14 (Cardinal Arithmetic)

(a) Let X be a finite set, and $x \notin X$. Then $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X) + 1$.

(b) Let X and Y be finite sets. Then $X \cup Y$ is finite, $\#(X \cup Y) \leq \#(X) + \#(Y)$

(c) Let X be a finite set. Let $Y \subseteq X$. Then Y is finite and $\#(Y) \leq \#(X)$.

(d) If X is a finite set, and if $f: X \to Y$ is a function. Then $f(X)$ is a finite set, with $\#(f(X)) \leq \#(X)$.

(e) Let X and Y be finite sets. Then the Cartesian product $X \times Y$ is finite and $\#(X \times Y) = \#(X) \times \#(Y)$.

(f) Let X and Y be finite sets. Then the set Y^X is finite and $\#(Y^X) = \#(Y)^{\#(X)}$

1. If X is a finite set. By defn 3.6.10 (finite sets), $\#(X) = n$, $\exists n \in \mathbb{N}$.
Suppose $X \cup \{x\}$, $\#(X \cup \{x\}) \neq \#(X) + 1$.
Then by Lemma 3.6.9, $\#(X) \neq \#(X \cup \{x\}) - 1$. ※ Contradiction.

2. If X and Y are finite. Then $f: X \to \{i \in \mathbb{N} : 1 \leq i \leq n\}$. $\exists h$, $h: X \to X \cup Y$
$g: Y \to \{i \in \mathbb{N} : 1 \leq i \leq n\}$. $\exists k$, $k: Y \to X \cup Y$.
where h, k are identity maps.
Since $\forall x, y \in X, Y$, $x, y \in X \cup Y$. Then $X \cup Y$ is finite, as if it is infinite, it does not have cardinality for any natural number n. ※ contradict X, and Y are finite.

If $X \cap Y = \emptyset$. Then $\#(X \cup Y) = \#(X) + \#(Y)$.
If $X \cap Y \neq \emptyset$. Then $\exists x \in X \cup Y$. $x \in X$, $x \in Y$. $x = x$. s.t.
by axiom of specification $\{x : x \in X \cap Y\}$.
$X \cup Y = X \setminus \{x : x \in X \cap Y\} \cup Y$
$= \#(X) - \#(\{x : x \in X \cap Y\}) + \#Y$.

3. If $Y \subseteq X$, X is a finite set. Then there exist a set $\{x : x \in X \setminus Y\} = S$, s.t. $Y \cup \{x : x \in X \setminus Y\} = X$.
by induction if S is empty. $\#(S) = 0$. Then $\#Y = \#X$.
But for all $n > 0$, $\#(S) = n$. $\#Y < \#X$.

4. If $f: X \to Y$. X is finite. Then since $\forall x \in X$, $\exists! y \in Y$, s.t. $y = f(x)$.
s.t. If X is finite. Then $f(X)$ cannot be infinite, as it would contradict the definition of functions.

5. If X and Y are finite. Then $X \times Y := \{(x, y) : x \in X, y \in Y\}$.
Suppose $\{x_1, \ldots, x_n\} = X$, $\#(X) = n$, $\{y_1, \ldots y_m\} = Y$. $\#(Y) = m$.
Then for each x_i, $1 \leq i \leq n$, there are m elements in Y, s.t. forming m ordered pairs in $X \times Y$.
$\sum_{i=1}^{n} m = \underset{\in \mathbb{N}}{n \times m}$. ordered pairs. It cannot be infinite

6. If X and Y are finite. Then $\#(Y^X) = \#(Y)^{\#(X)}$. Prove by induction.
① Fix $\#(Y) = n$. Induct on $\#(X) = m$.
② Induct on $\#(Y) = n$.

Ex 3.6.5 (1st attempt)

Let A and B be sets.

Show that $A \times B$ and $B \times A$ have equal cardinality by constructing an explicit bijection between the two sets.

Then we Proposition 3.6.14 (Cardinal Arithmetic) to conclude an alternative proof of Lemma 2.3.2 (Multiplication is commutative)

1. Suppose $f : A \times B \to B \times A$, $f(a, b) = (b, a)$, f is a bijection.

Since $\#(A \times B) = \#(B \times A)$

$\#(A \times B) = \#(A) \times \#(B)$

$\#(B \times A) = \#(B) \times \#(A)$. Thus $\#(B) \times \#(A) = \#(A) \times \#(B)$.

Ex 3.6.6 (1st attempt)

Let A, B and C are sets

Show that the sets $(A^B)^C$ and $A^{B\times C}$ have equal cardinality by constructing an explicit bijection between the two sets.

Conclude that $(a^b)^c = a^{bc}$ for any $a, b, c \in \mathbb{N}$.

and $a^b \times a^c = a^{b+c}$

1. $\forall f \in (A^B)^C \iff f: C \to A^B$

$\forall f \in A^{B\times C} \iff f: B\times C \to A$

If $f \in (A^B)^C$. Then $\forall c \in C$, $f(c) \in A^B$, which imply $f(c): B \to A$.

Define a map from $(A^B)^C$ to $A^{B\times C}$ as follows

$h: (A^B)^C \to A^{B\times C}$, $\forall f \in (A^B)^C$, $h(f) = g \in A^{B\times C}$

where $g(b,c) = (f\circ c)(b)$

Suppose h is a bijection, s.t.

Both sets have the same cardinality.

$\#(A^B)^C = \#(A)^{\#(B)\times\#(C)} \Rightarrow (a^b)^c = a^{b\times c}$

2. Show that $A^B \times A^C$ have the same cardinality as $A^{B\cup C}$.

Define a map from $A^B \times A^C$ to $A^{B\cup C}$

$h: A^B \times A^C \to A^{B\cup C}$. $\forall f \in A^B \times A^C$, $h(f) = g \in A^{B\cup C}$

$g(x) \in A$.

$x \in B\cup C$.

h is a bijection.

$\#(A^B \times A^C) = \#(A^{B\cup C})$

$= \#(A)^{\#(B\cup C)} = \#(A)^{\#(B)+\#(C)}$ if $B\cap C = \emptyset$.

Ex 3.6.7 (1st attempt)

Let A and B be sets.

Let us say that A has lesser or equal cardinality to B if there exists an injection $f: A \to B$ from A to B.

Show that if A and B are finite sets. Then A has lesser or equal cardinality to B iff $\#(A) \leq \#(B)$.

1. If A and B are finite sets.

($\Rightarrow$) If A has lesser or equal cardinality to B. (using Proposition 3.6.14(c))

Then there exist a subset G of B, s.t.

G has a cardinality equal with A. as $\exists y \in B$, s.t. $\nexists x \in A$, $f(x) = y$.

Since $G \subseteq B$. Then $\#(G) \leq \#(B)$, $\#(G) = \#(A)$.

Thus $\#(A) \leq \#(B)$.

2. ($\Leftarrow$) If $\#(A) \leq \#(B)$

Suppose there exist a subset $G, \subseteq B$. (if not, contradicting Proposition 3.6.14(c))

s.t. $\#(G) \leq \#(B)$. s.t. $\forall x_1, x_2, x_1 \neq x_2 \in A$. $f(x_1) \neq f(x_2) \in B$.

Suppose $\#(A) = N$, for $N \in \mathbb{N}$, $N \leq \#(B)$.

s.t. $\#(G) = N'$, s.t. $N' \in \{ i \in \mathbb{N} : 0 \leq i \leq \#(B) \}$.

$\#(G) = \#(A)$, by definition 3.6.10 and 3.6.1

G and A have the same Cardinality.

Thus A has a lesser or equal Cardinality to B.

Ex 3.6.8 (1st attempt)

Let A and B be sets such that there exists an injection $f: A \to B$ from A to B. Assume A is non-empty. Show that there exists a surjection $g: B \to A$ from B to A.

We want to show that $\forall a \in A, \exists b \in B$, s.t. $g(b) = a$.

1. Since we know that $f: A \to B$, is an injection, s.t.

$\forall a_1, a_2 \in A, \quad a_1 \neq a_2 \Leftrightarrow f(a_1) \neq f(a_2) \in B.$

$\Rightarrow$ A has a lesser or equal cardinality to B. (Ex 3.6.7).

$\Rightarrow \#(A) \leq \#(B)$.

2. From step 1, we can see that there exist a function g, s.t.

$G \subseteq B$, s.t. $g: G \to A$. is a bijection.

Thus g is also a surjection.

Ex 3.6.9 (1^{st} attempt)

Let A and B be finite sets.

Show that $A \cup B$ and $A \cap B$ are also finite sets.

and that $\#(A) + \#(B) = \#(A \cup B) + \#(A \cap B)$.

1. If A and B are finite sets. Then $f: A \to \{i \in \mathbb{N} : 1 \leq i \leq n\}$, for some $n \in \mathbb{N}$. s.t. $\#(A) = n$. This holds similarly for B, s.t. let $\#(B) = m$.

2. If $A \cup B$ and $A \cap B$ are not finite sets, then there exist a bijection between $A \cup B$ and $\mathbb{N}$ -X- Contradiction., similarly with $A \cap B$.

3. Fix n. Induct on m. If $\#(A) = n$. $\#(B) = m = 0$.

   Then $\#(A) + \#(B) = n$

   $\#(B) = 0 \Rightarrow B = \emptyset$. $A \cap \emptyset = \emptyset$. $A \cup B = A \cup \emptyset = A$.

   Thus $\#(A \cap \emptyset) = 0$

   $\#(A \cup \emptyset) = n$.

   Thus Base Case is true.

4. Suppose inductively that $\#(A) + \#(B) = \#(A \cup B) + \#(A \cap B)$.

   if $\#(B) = m$.

   Show that if $\#(B') = m+1$. Then the above still holds.

   $\#(B') = m+1$ implies that $B' = B \cup \{x\}$, (Lemma 3.6.9).

   If $x \in A$. Then $A \cup B' = A \cup B$, $A \cap B' = (A \cap B) \cup \{x\}$

   If $x \notin A$. Then $A \cup B' = A \cup B \cup \{x\}$, $A \cap B' = A \cap B$.

   Thus. if $x \in A$. $\#(A \cup B') = \#(A \cup B)$. $\#(A \cap B') = \#(A \cap B) + 1$

   if $x \notin A$. $\#(A \cup B') = \#(A \cup B) + 1$, $\#(A \cap B') = \#(A \cap B)$.

   Thus, $\#(A) + \#(B') = \#(A \cup B) + \#(A \cap B) + 1$

   $\#(A) + \#(B') = \#(A) + \#(B) + 1$ L.H.S = R.H.S

   Induction closed.

Ex 3.6.10 (1st attempt)

Let $A_1, \dots, A_n$ be finite sets such that $\#(\bigcup_{i \in \{1,\dots,n\}} A_i) > n$. (Pigeonhole Principle)

Show that there exists $i \in \{1,\dots,n\}$ such that $\#(A_i) \geq 2$.

1. Recall $\bigcup_{i \in \{1,\dots,n\}} A_i := \bigcup\{A_i : i \in \{1,\dots,n\}\}$ (Axiom 3.12 Union).

   Denote $\bigcup_{i \in \{1,\dots,n\}} A_i$ as Q, since $A_1,\dots,A_n$ are finite sets.

   We know that Q is also a finite set by Ex 3.6.9, and.

   $\sum_{i=1}^{n} \#(A_i) = \#(Q) + \#(W)$, where $W := \bigcap_{i \in \{1,\dots,n\}} A_i := \{x : x \in A_i \text{ for } i \in \{1,\dots,n\}\}$

2. From step 1, suppose that $A_1,\dots,A_n$ are all non-empty sets, with one $a_i \in A_i$, for each $i \in \{1,\dots,n\}$.

   Thus $\#(Q) + \#(W) = \sum_{i=1}^{n} \#(A_i) \geq n$

   Suppose $\#(W) = 0$, s.t. $W = \emptyset$. $\#(Q) = \sum_{i=1}^{n} \#(A_i) = n$.

3. From step 2.

   Induct on n. Let $n = 1$, s.t. $Q = A_1$, $\#(Q) = \sum_{i=1}^{1} \#(A_i) = \#(A_1) = 1$

   For $\#(Q) > 1 \Rightarrow \#(A_1) \geq 2$, where $A_1 := A_1 \cup \{x^2\} = \{a_1, x^2\}$

   Suppose inductively that $n = N$, holds, s.t. $\#(Q) > N \Rightarrow \#(Q) = N{+}{+}$.

   $\Rightarrow \exists! \, i \in \{1,\dots,n\}$, s.t. $A_i := A_i \cup \{x^2\} = \{a_i, x^2\}$.

   Show that this holds for $n = N{+}{+}$,

   $\#(Q') = N{+}{+} = \sum_{i=1}^{N++} \#(A_i) \Rightarrow \#(Q') = (N{+}{+}){+}{+} > N{+}{+}$.

   $Q' = Q \cup A_{N++}$

   If $\#(Q) = N$. Then $A_{N++} := A_{N++} \cup \{x^2\} \Rightarrow \#(A_{N++}) = 1{+}{+}$

   If $\#(Q) = N{+}{+}$. Then from assumption some $\exists! \, i \in \{1,\dots,n\}$, s.t. $A_i := A_i \cup \{x^2\}$.

   Thus induction closed.

Ex 3.6.11 (1^{st} attempt)

Let $f: X \to Y$ be a function.

Show that (a) and (b) are equivalent.

(a) : f is injective (b) Whenever $E \subseteq X$, has cardinality $\#(E) = 2$

Then the image $f(E)$ also has cardinality $\#(f(E)) = 2$.

1. (a) $\Rightarrow$ (b)

If f is injective. Suppose $E \subseteq X$, with $\#(E) = 2$.

Then since $\forall x_1, x_2 \in X$, $f(x_1) = f(x_2)$ iff $x_1 = x_2$

$f(x_1) \neq f(x_2)$ iff $x_1 \neq x_2$.

$\#(E) = 2 \Rightarrow E = \{x_1, x_2\}$, s.t. $x_1 \neq x_2$.

$\Rightarrow f(E) = \{f(x_1), f(x_2)\}$. $\Rightarrow \#(f(E)) = 2$.

2. (b) $\Rightarrow$ (a)

If $E \subseteq X$. $\#(E) = 2$. Then there exsit a map f. $f: X \to Y$,

s.t. $\#(f(E)) = 2$.

Suppose f is not injective. s.t. $f(x_1) = f(x_2)$ for $\exists x_1, x_2 \in X$, $x_1 \neq x_2$.

$f(E) = \{f(x_1), f(x_2)\}$, s.t. $f(x_1) = f(x_2)\}$

$= \{f(x_1)\}$ $\times$. Contradiction, $\#(f(E)) = 2$.

$\Rightarrow f$ must be injective.

Ex 3.6.12(i) (1^{st} attempt)

For any natural number n, let S_n be the set of all bijections $\phi : X \to X$. where $X := \{i \in \mathbb{N} : 1 \le i \le n\}$, such bijections also known as permutations of X.

(i) For any natural number n, show that S_n is finite, and $\#(S_{n++}) = (n++) \times \#(S_n)$.

1. Given $\phi : X \to X$, from the power set Axiom. $\phi \in X^X$, where by Cardinal Arithmetic, X^X is finite, we can see that $S_n \subseteq X^X$, s.t. S_n is finite.

2. Given $S_{n++} := \{\phi : \phi : \{i \in \mathbb{N} : 1 \le i \le n++\} \to \{i \in \mathbb{N} : 1 \le i \le n++\}, \phi \text{ is a bijection}\}$.

Induct on n. Let $n := 0$. $f : \phi \to \phi$. $\Rightarrow$ only 1 f.

Thus. $(0++) \times 1 = 1$

$\#(S_1) = 1$ $f : \{x\} = \{x\}$.

only 1 f.

Suppose inductively that $\#(S_{N++}) = (N++) \times \#(S_N)$.

Show that $\#(S_{(N++)++}) = [(N++)++] \times \#(S_{N++})$

3. (WLOG) assume in S_{N+2}, there exist $N+2$ subsets, aka, $A_1, \dots A_{N+2}$.

$A_n := \{\phi : \phi(i) = i \text{ for all } 1 \le i \le n\}$, s.t. we can see that $A_1 \subseteq A_2 \subseteq \dots \subseteq A_{N+2}$.

We know that $\prod_{i=1}^{N+2} \#(A_i) = \#(\bigcup \{x \in A_i : i \in \{1, \dots, N+2\}\}) + \#(\{x : x \in A_i \text{ for all } 1 \le i \le N+2\})$

$$\prod_{i=1}^{N+2} \#(A_i) = \#(S_{N+2}) + \#(\{x : x \in A_i \text{ for all } 1 \le i \le N+2\})$$

$$\prod_{i=1}^{N+2} i = \#(S_{N+2}) + 0$$

Similarly we know that $\prod_{i=1}^{N+1} i = \#(S_{N+1})$, $\prod_{i=1}^{N+2} i = (\prod_{i=1}^{N+1} i) \times (N+2)$

$$\#(S_{N+2}) = \prod_{i=1}^{N+2} i = \#(S_{N+1}) \times (N+2)$$

Induction closed.

Ex 3.6.12(ii) (1st attempt)

For any natural number n, let S_n be the set of all bijections $\phi: X \to X$. where $X := \{i \in \mathbb{N} : 1 \le i \le n\}$

(ii) Define the factorial $n!$ of a natural number n recursively by $0! := 1$ and $(n{+}{+})! := (n{+}{+}) \times n!$ for all natural numbers n.

Show that $\#(S_n) = n!$ for all natural numbers n.

As from (i). $\#(S_{n++}) = (n{+}{+}) \times \#(S_n)$.
To show $\#(S_n) = n!$ for all $n \in \mathbb{N}$, we need to show there exist such a isomorphism.

1. Let $f: \mathbb{N} \to \mathbb{N}$, s.t. $f(n) = n!$, f be a factorial function.
Suppose $P(n)$ be the property pertaining n, $P(n)$ is true if $f(n) = \#(S_n)$.

2. Induct on n. Let $n = 0$, $f(n) = 1$
$\#(S_0) = 1$.

3. Suppose inductively that $\exists N \in \mathbb{N}$, s.t. $f(N) = \#(S_N)$.
Show that $f(N{+}{+}) = \#(S_{N++})$.
we know that $f(N{+}{+}) = (N+1)! = f(N) \times (N+1)$.

From previous (i), we know that $\#(S_{N++}) = (N{+}{+}) \times \#(S_n)$.
s.t. $f(N{+}{+}) = \#(S_{N++})$,

Induction closed.

4. From step 3, we know that there exist a bijection between $f(n)$ and $\#(S_n)$.
for all natural numbers n.