Math551 Practice Midterm

Instructions

Instructions for Main Exam

- There are 5 questions. Organize all work neatly.
- This is a closed-book exam: do not use any book, notes, reference, or any outside help.
- Organize your work in an unambiguous order. Show all necessary steps.
- **Answers given without supporting work may receive 0 credit!**

1. Consider finding a root of $f(x) = \log(x) - 2x + 2$ using Newton's method with initial guess $x_0 = 2$.
   (a) Show that $x = 1$ is a root of f and write down the Newton iteration.
   (b) Does the iteration converge linearly, quadratically, or some other order? If it is linear, what is the rate? Justify your answer by citing a theorem or giving direct proof.

2. Consider finding a root of $f(x) = \int_{-\infty}^{x} e^{-s^2}\,ds - x^2$ using Bisection. Note that $\int_{-\infty}^{\infty} e^{-s^2}\,ds = 2\int_{-\infty}^{0} e^{-s^2}\,ds = \sqrt{\pi}$.
   (a) There is a root $x > 0$. Bracket the root using the intermediate value theorem.
   (b) Using your interval, what is the error after n steps? How many iterations would be required to find a point within 2^{-10} of the actual root?

3. Consider the fixed point iteration defined by $x_{n+1} = g(x_n)$ where $g(x) = \log(x) + 2$.
   (a) Show that there is a fixed point with $x > 2$, for example by bracketing the root using the intermediate value theorem.
   (b) Does the iteration converge? If it does, state how quickly it converges (linearly, quadratically, or otherwise).

4. (a) Give an example of a matrix with no LU decomposition.
   (b) Find the LU decomposition for

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 4 & 7 \end{bmatrix}$$

5. (a) Define the condition number for a matrix A.

   (b) For the exact solution x to $Ax = b$ and an approximate solution x_k, prove

$$\frac{\|x - x_k\|}{\|x\|} \leq \kappa(A) \frac{\|b - Ax_k\|}{\|b\|}.$$

   (c) Compute the condition number for the following matrix, stating the norm used.

$$A = \begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix}$$

6. Consider solving $A\mathbf{x} = \mathbf{b}$ using the Gauss-Seidel Method where

$$A = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   (a) Find the next two terms of the iteration, $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$.

   (b) Will the iteration converge? (Hint: find the infinity norm of the iteration matrix).

7. (a) Compute the Jacobian of the vector-valued function

$$f(x, y) = \begin{bmatrix} x^2 + y^2 - 1 \\ (x-1)^2 + y^2 - 1 \end{bmatrix}$$

   (b) Use the function in the previous part and compute a single step of Newton's Method starting with the point $(x, y) = (1, 1)$.