Box #____ (Name on Back)
Math 65 Section ___
Homework 8
12/4/18

Homework #8: Linear Systems, Phase Diagrams, & Stability

H1: Consider the damped oscillator

$$y'' + 2\alpha y' + \omega^2 y = 0$$

where $\alpha > 0$ and $\omega > 0$ are parameters.

(a) Write this second-order differential equation for $y(t)$ as a first-order linear system.

(b) Find the general solution in two cases, the under-damped case when $\alpha < \omega$ and the over-damped case when $\alpha > \omega$.

(c) Solve the initial value problem with $y(0) = 0$ and $y'(0) = 1$ for both the under-damped and over-damped cases. In each case determine the zero crossings for the oscillator, that is the values of $t > 0$ where $y(t) = 0$.

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H2: For each linear system of DEs given below, **without using WolframAlpha**,

(i) Classify the origin as a node, saddle, center, spiral, degenerate node, or star node;

(ii) Identify it as neutrally stable, unstable or asymptotically stable;

(iii) Sketch a simple phase portrait for the system. Label any features you used to make the sketch.

(a) $\begin{cases} x' = -5x + 4y \\ y' = 4x + y \end{cases}$

(b) $\begin{cases} x' = -x - 4y \\ y' = x - y \end{cases}$

Hint: Check some representative vectors in the direction field.

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H3: For each linear system of DEs given below,

(i) Classify the origin as a node, saddle, center, spiral, degenerate node, or star node;

(ii) Identify it as neutrally stable, unstable or asymptotically stable;

(iii) Use WolframAlpha to draw a phase portrait of the system.

(a) $\begin{cases} x' = 4y \\ y' = -x \end{cases}$

Eigenvalues are $\lambda_1 = 2i$, $\lambda_2 = -2i$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ i \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -i \end{bmatrix}$.

(b) $\begin{cases} x' = x - 2y \\ y' = x + 4y \end{cases}$

Eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 2$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

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H4: Consider the linear system

$$\vec{x}' = \begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix} \vec{x},$$

where $\alpha \in \mathbb{R}$ is a parameter.

(a) Find the general solution.

(b) Classify the equilibrium point at the origin according to whether $\alpha < 0$, $\alpha > 0$, or $\alpha = 0$. In each case give a qualitative sketch of a typical phase portrait.

Note: We call $\alpha = 0$ a *bifurcation point* for this system, as the qualitative nature of the solution changes fundamentally between positive and negative values of α (i. e. the dynamics bifurcate at $\alpha = 0$).