Box #____ (Name on Back)
Math 65 Section ___
Homework 4
11/9/18

Problems for Section 6.5: The Kernel and Range of a Linear Transformation

D1: Determine whether the linear transformation T is (a) one-to-one and (b) onto.

$$T\colon \mathcal{P}_2 \to M_{22} \text{ defined by } T(a + bx + cx^2) = \begin{bmatrix} a+b & a+2c \\ 2a+c & b-c \end{bmatrix}$$

D2: (Poole, p. 496) Section 6.5 #34.

Let $S\colon V \to W$ be and $T\colon U \to V$ be linear transformations.

(a) Prove that if $S \circ T$ is one-to-one, so is T.

(b) Prove that if $S \circ T$ is onto, so is S.

Problems for Section 6.6: The Matrix of a Linear Transformation

The next three problems us the following theorem from Section 6.6:

Theorem 6.26: Let V and W be two finite-dimensional vector spaces of dimension n and m respectively. Suppose $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis for V and $\mathcal{C}$ is a basis for W.
If $T : V \to W$ is a linear transformation, then the $m \times n$ matrix A defined by

$$A = [[T(v_1)]_{\mathcal{C}}\ [T(v_2)]_{\mathcal{C}} \cdots [T(v_n)]_{\mathcal{C}}]$$

has the property that

$$A[v]_{\mathcal{B}} = [T(v)]_{\mathcal{C}}$$

for all $v \in V$.

D3: (Poole, p. 512) Section 6.6 #2.

Find the matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ of the linear transformation $T\colon V \to W$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, respectively. Verify Theorem 6.26 for the vector v by computing $T(v)$ first directly and then by using the theorem.

$$T\colon \mathcal{P}_1 \to \mathcal{P}_1 \text{ defined by } T(a + bx) = b - ax,$$
$$\mathcal{B} = \{1 + x, 1 - x\}, \quad \mathcal{C} = \{1, x\}, \quad v = p(x) = 4 + 2x$$

Problem #3 Continued:

Box #____

D4: (Poole, p. 513) Section 6.6 #12.

Find the matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ of the linear transformation $T\colon V \to W$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, respectively. Verify Theorem 6.26 for the vector $\vec{v}$ by computing $T(\vec{v})$ first directly and then by using the theorem.

$$T\colon M_{22} \to M_{22} \text{ defined by } T(A) = A - A^T,$$

$$\mathcal{B} = \mathcal{C} = \{E_{11}, E_{12}, E_{21}, E_{22}\}, \quad \vec{v} = A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

D5: (Poole, p. 513) Section 6.6 #14.

Let $\mathscr{D}$ be the set of differentiable real-valued functions $f(x)$ defined for all $x \in \mathbb{R}$.
The set $W = \text{span}(e^{2x}, e^{-2x})$ is a subspace of $\mathscr{D}$.

(a) Show that the differential operator D, where $D[f(x)] = \frac{d}{dx}f(x)$, maps W into itself.
(b) Find the matrix of D with respect to $\mathcal{B} = \{e^{2x}, e^{-2x}\}$.
(c) Compute the derivative of $f(x) = e^{2x} - 3e^{-2x}$ indirectly using Theorem 6.26, and verify that it agrees with $f'(x)$ as computed directly.

Box #____

The next two problems may use the following theorem from Section 6.6:

> **Theorem 6.28:** Let $T : V \to W$ be a linear transformation between two n-dimensional vector spaces V and W, and let $\mathcal{B}$ and $\mathcal{C}$ be bases for V and W, respectively. Then T is invertible if and only if $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible. In this case,
>
> $$([T]_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = [T^{-1}]_{\mathcal{B}\leftarrow\mathcal{C}} \ .$$

D6: (Poole, p. 513) Section 6.6 #22.

Determine whether the linear transformation T is invertible by considering its matrix with respect to the standard bases, $[T]_{\mathcal{E}'\leftarrow\mathcal{E}}$. If T is invertible, use Theorem 6.28 to find T^{-1} by finding $[T^{-1}]_{\mathcal{E}\leftarrow\mathcal{E}'}$.

$$T\colon \mathcal{P}_2 \to \mathcal{P}_2 \text{ defined by } T(p(x)) = p'(x)$$

D7: (Poole, p. 514) Section 6.6 #24.

Determine whether the linear transformation T is invertible by considering its matrix with respect to the standard bases, $[T]_{\mathcal{E}' \leftarrow \mathcal{E}}$. If T is invertible, use Theorem 6.28 to find T^{-1} by finding $[T^{-1}]_{\mathcal{E} \leftarrow \mathcal{E}'}$.

$$T\colon M_{22} \to M_{22} \text{ defined by } T(A) = AB, \text{ where } B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$