Box #____ (Name on Back)
Math 65 Section ___
Homework 10
12/11/18

Homework #10: Driven Systems &Variation of Parameters

J1: **Review of the Method of Integrating Factors.** Use the method of integrating factors to solve the following three initial value problems for $y(t)$.

(a) $y' + 2y = e^t, \quad y(0) = 0.$

(b) $y' + 2y = e^{2t}, \quad y(0) = 1.$

(c) $y' + y = \sin t, \quad y(0) = 1.$

J2: Solve the initial value problem for $\vec{x}(t) \in \mathbb{R}^2$,

$$\vec{x}' = A\vec{x} + \vec{f}(t) \qquad \vec{x}(0) = \vec{x}_0$$

where

$$A = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}, \qquad \vec{f}(t) = \begin{bmatrix} 1 \\ e^t \end{bmatrix}, \qquad \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

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J3: Solve the initial value problem for $\vec{x}(t) \in \mathbb{R}^2$,

$$\vec{x}\,' = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} \cos \omega t \\ \sin \omega t \end{bmatrix}, \qquad \vec{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix}$$

where $\omega > 0$. You may assume that $\omega \neq 2$.

Hint: The following formulas will be of use:

$$\cos(p+q) = \cos(p)\cos(q) - \sin(p)\sin(q)$$
$$\sin(p+q) = \sin(p)\cos(q) + \cos(p)\sin(q)$$

J4: In Lecture 12 we introduced a compartment model for the absorption of medicine from the GI tract into the bloodstream and the subsequent excretion of the medicine from the body. Consider now a similar model for the amount of caffeine in the GI tract, $x(t)$, and in the bloodstream, $y(t)$,

$$\begin{aligned} x' &= -kx + I(t), \\ y' &= kx - ky, \end{aligned}$$

where we have chosen the two rate constants to be equal ($k_1 = k_2 = k$) and $k > 0$. Here $I(t)$ is the caffeine input rate and $x(0) = x_0, y(0) = y_0$, are the initial concentration of caffeine in the GI tract and bloodstream. We revisit the problem from the systems perspective (and in the deficient case where $k_1 = k_2$, for fun; the same approach can be used for arbitrary constants).

(a) Write the system in matrix form $\vec{x}\,' = A\vec{x} + \vec{F}$.

(b) Solve the system with $I(t) = 0$ as a *cascade* by solving first for $x(t)$, and then using your result for $x(t)$ to solve for $y(t)$ via the method of integrating factors.

(c) Use your work from part (b) to find e^{At}.
Hint: You have done almost all of the work already in part (b).

(d) Use Variation of Parameters to solve for $x(t)$ and $y(t)$ when $I(t) = 1$ (steady drip) and $x_0 = y_0 = 0$ (initially no caffeine). Graph your solutions when $k = 2$ (although it should not be necessary, using a calculator/computer is fine). What happens as $t \to \infty$?

Note: One advantage of the systems perspective is for more realistic $I(t)$ we can easily numerically integrate the expression for your solution. For example, you might consider what happens when $I(t) = 1$ for $0 < t < 1$, then $I(t) = 0$ for $t \geq 1$ (when you stop drinking coffee or the coffee drip runs out), or $I(t) = e^{-0.5t}$, and so on. Once you have computed the fundamental matrix, you can use it to study any forcing term, at least numerically.