E1: Consider

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 4 \\ 2 & 5 & 2 & 5 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 5 \\ \alpha \end{bmatrix},$$

where α is a constant we will determine below.

(a) Define a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ where $T(x) = Ax$. Find a basis for $\ker(T)$. Use this basis to find the homogeneous solution, x, which satisfies $Ax = 0$.

(b) Now consider $Ax = b$. For which value of α is b in range(T)? For this value of α find a particular solution, x, to $Ax = b$.

(c) Use parts (a) and (b) to write the general solution for $Ax = b$ using the value of α found in part (b). Is there a solution to $Ax = b$ for other values of α?

E2: In this problem we will solve the *difference equation*

$$\frac{1}{2}\left[p(x+1) - p(x-1)\right] = x.$$

(a) Write down the matrix representation for the linear transformation $T : \mathcal{P}_2 \to \mathcal{P}_2$ where

$$T[p(x)] = \frac{1}{2}\left[p(x+1) - p(x-1)\right],$$

using the standard basis for $\mathcal{P}_2$.

(b) Is the linear transformation you found in part (a) invertible? If yes, find a matrix for the inverse transform. If no, find a basis for $\ker(T)$.

(c) Now, find the general solution to $T[p(x)] = x$ where $p(x)$ is in $\mathcal{P}_2$.

Box #____

E3: In this problem we will solve

$$\frac{1}{2}\left[p(x+1)+p(x-1)\right]=x$$

(a) Write down the matrix representation for the linear transformation $S:\mathcal{P}_2\to\mathcal{P}_2$ where

$$S[p(x)]=\frac{1}{2}\left[p(x+1)+p(x-1)\right],$$

using the standard basis for $\mathcal{P}_2$.

(b) Is the linear transformation you found in part (a) invertible? If yes, find a matrix for the inverse transform. If no, find a basis for $\ker(S)$.

(c) Now, find the general solution to $S[p(x)]=x$ where $p(x)$ is in $\mathcal{P}_2$.

E4: Consider the linear inhomogeneous differential equation $y'' + y = x^3 + 1$.

(a) The linear transformation T is taken to be:

$$T\colon \mathcal{C}^\infty \to \mathcal{C}^\infty \text{ defined by } T[y] = \frac{d^2y}{dx^2} + y$$

Find the kernel and nullity of T.

(b) Now consider linear transformation T defined on polynomials, $p(x)$, of degree 3,

$$T\colon \mathcal{P}_3 \to \mathcal{P}_3 \text{ defined by } T[p] = \frac{d^2p}{dx^2} + p$$

Is T invertible? If so, find a matrix representation for its inverse transformation in terms of the standard basis for $\mathcal{P}_3$. If not, explain why not.

(c) Use part (b) to find all polynomial solutions of

$$T[p] = \frac{d^2p}{dx^2} + p = x^3 + 1.$$

(d) Find the most general solution, $y(x)$, in $\mathcal{C}^\infty$ that satisfies $y'' + y = x^3 + 1$ for all $x \in \mathbb{R}$.

E5: Consider the linear transformation $L\colon \mathcal{C}^\infty \to \mathcal{C}^\infty$ defined by $L[y] = y'' - 3y' - 4y$.

(a) Find all $y(x)$ in $\mathcal{C}^\infty$ that satisfy $y'' - 3y' - 4y = 0$ for all $x \in \mathbb{R}$.
Use this information to compute the nullity of L.

(b) Suppose $\mathcal{V} = \operatorname{span}(\cos x)$. Explain why $L : \mathcal{V} \to \mathcal{V}$ where, as above,

$$L[y] = y'' - 3y' - 4y$$

is **not** a linear transformation.

(c) Suppose $\mathcal{V} = \operatorname{span}(\cos x, \sin x)$. Show $L : \mathcal{V} \to \mathcal{V}$ where, as above,

$$L[y] = y'' - 3y' - 4y$$

is a linear transformation. Compute the matrix representation of $[L]_{\mathcal{V}\leftarrow\mathcal{V}}$, explain why the transformation is invertible, and compute the matrix representation of its inverse.

(d) Use the parts (a) and (c) above to find all $y(x)$ in $\mathcal{C}^\infty$ that satisfy $y'' - 3y' - 4y = \cos x$ for all $x \in \mathbb{R}$.