Box #____ (Name on Back)
Math 65 Section ___
Homework 11
12/14/18

Homework #11: Autonomous Nonlinear Systems

K1: Consider the autonomous first-order system of differential equations

$$\begin{aligned} x' &= y - x \\ y' &= 2xy - 2y \end{aligned}$$

(a) Find all critical points of the system, $\vec{x}_0$.

(b) Compute the Jacobian matrix $J = D\vec{f}(\vec{x}_0)$ at each critical point, and find the eigenvalues and eigenvectors for the linear system.

(c) Assuming the linearization is accurate, use your information from part (b) to draw the phase portrait near the critical points and classify the critical points by type and stability.

(d) Check your work by drawing the entire phase portrait by using Wolfram Alpha.

Box #____

K2: Competing Species. Consider a model for population of rabbits, $x(t) \geq 0$, and hares, $y(t) \geq 0$, (measured in arbitrary units).

$$\begin{aligned} x' &= x(4 - x - 3y) \\ y' &= y(4 - 3x - y) \end{aligned}$$

(a) Find the critical points of the system.

(b) Linearize the system in the neighborhood of each of the critical points, and find the eigenvalues and eigenvectors of the resulting linear system. Classify the critical points by type (i.e., spiral/nodal sink, center, saddle, degenerate node) and stability (i.e., stable or unstable).

(c) Use the results in (b) to graph the phase portrait (recall $x, y \geq 0$).

(d) Assuming that we start out with a non-zero population of both rabbits and hares, what happens to the populations at large time?

K3: Duffing's equation

$$u'' + cu' + u^3 - u = 0,$$

is an important model for phenomena governed by a double-well potential. Here $c \geq 0$ measures the damping in the system. We will consider how the phase portrait for the problem changes from the undamped case ($c = 0$) to the damped case ($c > 0$).

(a) Write this equation as an equivalent system and find all the critical points.

(b) When $c = 0$, the system is undamped and the solution is periodic in time (corresponding to closed orbits in the phase plane) for many trajectories. Use linearization to classify the type and stability of the critical points in this case. Use this information to sketch an accurate phase portrait; be sure to put arrowheads on your orbits to indicate the direction of flow with increasing time.

(c) When $c = 1$, the system is *under-damped*. Use linearization to classify the type and stability of the critical points in this case. Use this information to sketch an accurate phase portrait; be sure to put arrowheads on your orbits to indicate the direction of flow with increasing time.

(d) When $c = 3$, the system is *over-damped*. Use linearization to classify the type and stability of the critical points in this case. Use this information to sketch an accurate phase portrait; be sure to put arrowheads on your orbits to indicate the direction of flow with increasing time.

(e) Suppose you were observing the value of $u(t)$ in a physical system for an initial condition with $u(0) = 1.2$, $u'(0) = 0$. How could you tell if you were in the undamped, under-damped or over-damped regime?