Continous Time Finance 2 - Hand-in 1

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Throughout this assignment the references to 'Björk' is a reference to Tomas Björk, *Arbitrage Theory in Continuous Time*, third edition (2009).

1 Quanto Hedging

In this exercise we consider an arbitrage-free currency model of Black/Scholes-type with the domestic currency set in USD and the foreign currency being JPY. By Björk's Proposition 17.5 we write bank-accounts, exchange-rate and Japanese stock dynamics as

$$\begin{aligned}
dB_{US}(t) &= r_{US}B_{US}(t)dt \\
dX(t) &= X(t)(r_{US} - r_J)dt + X(t)\sigma_X^T dW^{\mathbb{Q}^{US}}(t) \\
dB_J(t) &= r_J B_J(t)dt \\
dS_J(t) &= S_J(t)(r_J - \sigma_X^T\sigma_J)dt + S_J(t)\sigma_J^T dW^{\mathbb{Q}^{US}}(t)
\end{aligned}$$

We consider a quanto put, which is an option that at time T pays

$$Y_0\left(K - S_J(T)\right)^+$$

where Y_0 is some agreed-upon-in-advance exchange-rate; fx. the time-0 exchange rate.

1.a

We now wish to show that the arbitrage-free time-t price of the quanto put is $F^{QP}(t, S_J(t))$ where

$$F^{QP}(t,s) = Y_0 e^{-r_{US}(T-t)}\left(K\Phi(-d_2(t,s)) - se^{(r_J - \sigma_X^T\sigma_J)(T-t)}\Phi(-d_1(t,s))\right)$$

Φ is the standard normal distribution function and

$$d_{1/2}(t,s) = \frac{\log(s/K) + (r_J - \sigma_X^T\sigma_J \pm ||\sigma_J||^2/2)(T-t)}{\sqrt{T-t}||\sigma_J||}$$

We wish to do so looking for patterns to take us back to the Black-Scholes model. We begin by looking at the dynamics of the foreign stock.

$$dS_J(t) = S_J(t)(r_J - \sigma_X^T\sigma_J)dt + S_J(t)\sigma_J^T dW(t)^{\mathbb{Q}^d}$$

We multiply with $1 = \frac{||\sigma_J||}{||\sigma_J||}$ in order to make the σ-term one dimensional. Let $W^*(t) = \frac{\sigma_J}{||\sigma_J||}W^{\mathbb{Q}^d}(t)$ define a new Brownian motion (see Appendix A for proof of this).

$$= S_J(t)(r_J - \sigma_X^T\sigma_J)dt + S_J(t)||\sigma_J||dW^*(t)$$

We add $0 = r_{US} - r_{US}$ to the dt-term.

$$= S_J(t)(r_{US} - (r_{US} - r_J + \sigma_X^T\sigma_J))dt + S_J(t)||\sigma_J||dW^*(t)$$

We now recognize a stock with Black-Scholes dynamics which has drift r_{US} and pays dividend $r_{US} - r_J + \sigma_X^T\sigma_J$. So, by our usual Risk Neutral Valuation, the price of the quanto put is given by

$$\begin{aligned} F^{QP}(t, S_J(t)) &= e^{-r_{US}(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[Y_0(K - S_J(T))^+\right] \\ &= Y_0 e^{-r_{US}(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[(K - S_J(T))^+\right] \end{aligned}$$

This we recognize as the payoff of a put option, which we know how to price in the Black-Scholes model.

$$= Y_0 Put^{BS}(t, S_J(t), ||\sigma_J||, r_{US}, r_{US} - r_J + \sigma_X^T\sigma_J)$$

Using Björk Proposition 16.10 on the pricing function

$$= Y_0 Put^{BS}(t, S_J(t)e^{-(r_{US}-r_J+\sigma_X^T\sigma_J)(T-t)}, ||\sigma_J||, r_{US}, 0)$$

To get the final expression we first need to make the following observation. By the Put-Call parity (Björk Proposition 9.2), the Black-Scholes formula (Björk Proposition 7.10) and the symmetry of the standard normal distribution $(1-\Phi(d)=\Phi(-d))$ we find

$$\begin{aligned}
Put^{BS}(t,s) &= Ke^{-r(T-t)} + Call^{BS}(t,s) - s \\
&= Ke^{-r(T-t)} + s\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) - s \\
&= Ke^{-r(T-t)}(1-\Phi(d_2)) - s(1-\Phi(d_1)) \\
&= Ke^{-r(T-t)}\Phi(-d_2) - s\Phi(-d_1)
\end{aligned}$$

Using $s = se^{-(r_{US}-r_J+\sigma_X^T\sigma_J)(T-t)}$ and $r = r_{US}$ we get the desired expression

$$F^{QP}(t,s) = Y_0 e^{-r_{US}(T-t)}\left(Ke^{-r(T-t)}\Phi(-d_2) - se^{-(r_{US}-r_J+\sigma_X^T\sigma_J)(T-t)}\Phi(-d_1)\right)$$

with

$$\begin{aligned}
d_{1/2}(t,s) = d_{1/2}^{BS}(t, se^{-(r_{US}-r_J+\sigma_X^T\sigma_J)(T-t)}) &= \frac{\log(\frac{s}{K}e^{-(r_{US}-r_J+\sigma_X^T\sigma_J)(T-t)}) + (r_{US} \pm ||\sigma_J||^2/2)(T-t)}{\sqrt{T-t}||\sigma_J||} \\
&= \frac{\log(s/K) + (r_J - \sigma_X^T\sigma_J \pm ||\sigma_J||^2/2)(T-t)}{\sqrt{T-t}||\sigma_J||}
\end{aligned}$$

We now wish to show that

$$\frac{\partial F^{QP}(t,s)}{\partial s} = Y_0 e^{(r_J-\sigma_X^T\sigma_J-r_{US})(T-t)}\left(\Phi(d_1(t,s))-1\right) =: g(t,s)$$

As $F^{QP}(t,s)$ is given by a constant, Y_0, multiplied with a european put option on a stock with dividends, we know the Black-Scholes delta of such option.[1] We therefore get

$$\frac{\partial F^{QP}(t,s)}{\partial s} = Y_0 \cdot e^{-\delta(T-t)}\Delta^{put} = Y_0 e^{-(r_{US}-r_J+\sigma_X^T\sigma_J)(T-t)}\left(\Phi(d_1(t,s))-1\right)$$

Which was what we wanted to show.

1.b

We now wish to perform a discrete hedge experiment, where we sell a quanto put option and delta-hedge this position. The domestic money account is chosen such that the net worth of the portfolio is zero, i.e. the strategy becomes self-financing per construction. We split the time interval $[0,T]$ into n pieces with equidistant time-points.[2] We repeat the following N times

- For $t=0$:

  The initial outlay is calculated and the delta-neutral and self-financing strategy is set up.

[1] As there is no direct reference to this in Björk a derivation can be found in Appendix A.
[2] $T=1$ and $n=252$ means that we update our portfolio daily throughout a year.

$\Pi(0) = B_{US} - F^{QP}(0, S_J(0)) - X_0\Delta^{QP}(0)S_J(0)$ s.t. $\Pi(0) = 0$.

- For $t = 1, ..., n$:

  The value of the portfolio for the previous period is calculated

  $\tilde{\Pi}(t) = e^{r_{US}dt}B_{US}(t-1) + \Delta^{QP}(t-1)S_J(t)X(t)$

  The units are updated such that the strategy is still delta-neutral and self-financing

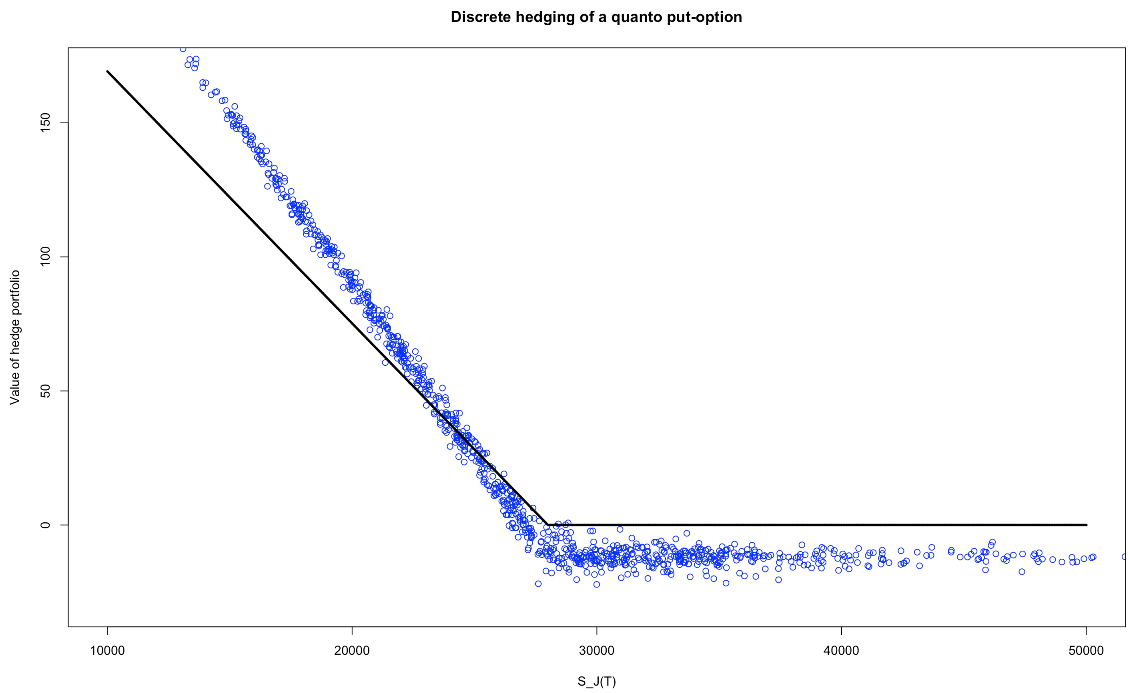
  $\Delta^{QP}(t) = g(t, S_J(t))/X(t)$ and $B_{US}(t) = \tilde{\Pi}(t) - \Delta^{QP}(t)S_J(t)X(t)$

We use the following parameters in the simulation

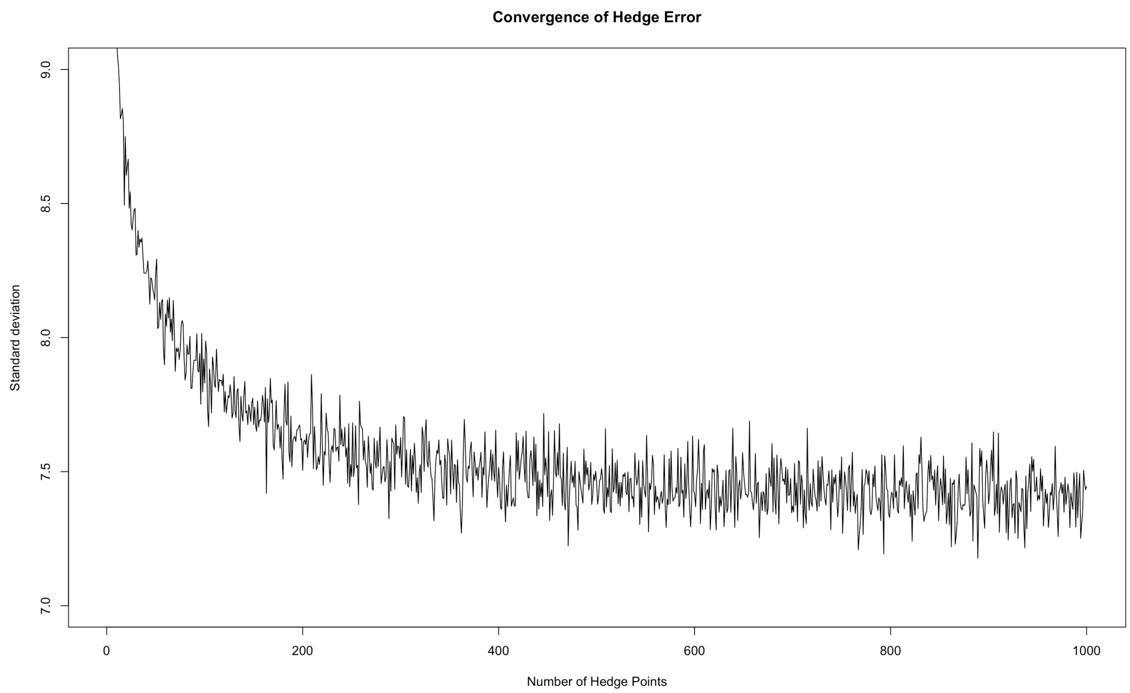
Parameter	Value
S_0	29015
X_0	0.0094
T	1
K	28000
r_{US}	0.015
r_J	0.01
σ_X	$(0.05, 0.1)^T$
σ_J	$(0.1, 0.3)^T$
N	1000
n	252

The parameters are chosen to reflect the set-up. The starting stock price is chosen to be equal to the current Nikkei index, the starting exchange rate is chosen to be equal to the current USD/JPY and the strike is chosen such that the option is barely OTM when initiating. The remaining parameters are assumed to be reasonable values, as they have been used in similar exercises previous in the course.

The value of the hedge portfolio at time T is plotted as a function of the stock price at time T, where the black line denotes the payoff of the quanto put.



We see how the 'replicating' portfolio does not replicate the payoff of the quanto put. Another way to see that the strategy does not replicate the payoff of the quanto put is to look at the convergence of the hedge error. We see in the following plot how the discounted standard deviation of the hedge error converges to 7.5, whereas in a perfect hedging strategy this should converge towards zero.



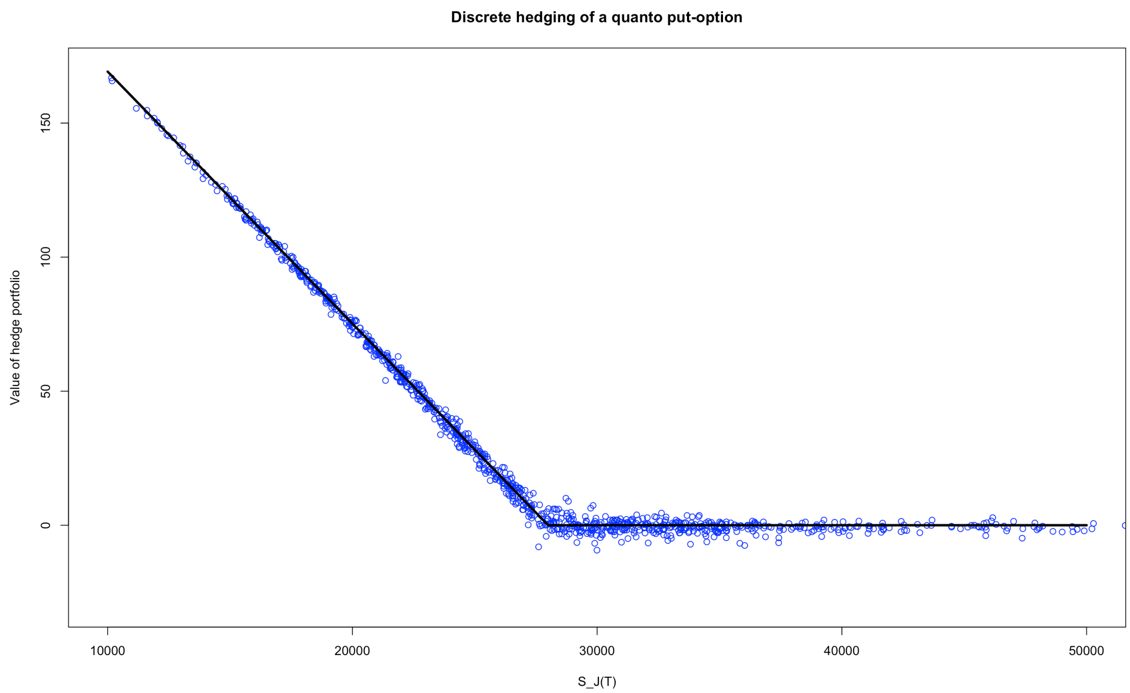
1.c

We now wish to perform another discrete hedge experiment, where we sell a quanto put and hedge this position using the stock, a domestic bank account and currency which are held in a foreign bank account. We again split the time interval $[0, T]$ into n pieces with equidistant timepoints and repeat the following N times

- For $t = 0$:
  The initial outlay is calculated and the self-financing strategy is set up.
  $\Pi(0) = B_{US} - F^{QP}(0, S_J(0)) - X_0\Delta^{QP}(0)S_J(0) - (-\Delta^{QP}S_J(0))X(0)$ s.t. $\Pi(0) = 0$.

- For $t = 1, ..., n$:
  The value of the portfolio for the previous period is calculated
  $\tilde{\Pi}(t) = e^{r_{US}dt}B_{US}(t-1) + \Delta^{QP}(t-1)S_J(t)X(t) + e^{r_J dt}(-\Delta^{QP}(t-1)S_J(t-1))X(t-1)$
  The units are updated and the strategy is kept self-financing
  $\Delta^{QP}(t) = g(t, S_J(t))/X(t)$ and $B_{US}(t) = \tilde{\Pi}(t) - \Delta^{QP}(t)S_J(t)X(t) - (-\Delta^{QP}(t)S_J(t))X(t)$

The value of the hedge portfolio at time T is plotted as a function of the stock price at time T, where the black line denotes the payoff of the quanto put.



1.d

We now wish to explain why the strategy from 1c does and the strategy from 1b does not replicate the payoff of the quanto put. We will do so by using a full-fledged three-holdings continuous-time argument. Let $V^h(t)$ be a self-financing three-holdings portfolio given by

$$V^h(t) = h_0 B_{US}(t) + h_1 S_J(t)X(t) + h_2 B_J(t)X(t)$$

We wish to choose h_0, h_1 and h_2 such that this strategy replicates the payoff of the quanto put. As h is self-financing per construction we know that

$$dV^h(t) = h_0 dB_{US}(t) + h_1 \underbrace{d(S_J(t)X(t))}_{1} + h_2 \underbrace{d(B_J(t)X(t))}_{2}$$

By Ito's product rule we find

$$
\begin{aligned}
1 &= S_J(t)dX(t) + X(t)dS_J(t) + dS_J(t)dX(t) \\
&= S_J(t)X(t)((r_{US} - r_J)dt + \sigma_X^T dW(t)) + X(t)S_J(t)((r_J - \sigma_X^T\sigma_J)dt + \sigma_J^T dW(t)) + S(t)X(t)\sigma_X^T\sigma_J dt \\
&= S_J(t)X(t)r_{US}dt + S_J(t)X(t)(\sigma_X^T + \sigma_J^T)dW(t)
\end{aligned}
$$

$$
\begin{aligned}
2 &= B_J(t)dX(t) + X(t)dB_J(t) + dB_J(t)dX(t) \\
&= B_J(t)X(t)((r_{US} - r_J)dt + \sigma_X^T dW(t)) + X(t)B_J(t)r_J dt \\
&= B_J(t)X(t)r_{US}dt + B_J(t)X(t)\sigma_X^T dW(t)
\end{aligned}
$$

Inserting the now known dynamics into the portfolio dynamic we find

$$
dV^h(t) = r_{US}\underbrace{\left(h_0 B_{US}(t) + h_1 S_J(t)X(t) + h_2 B_J(t)X(t)\right)dt}_{\text{drift term}} + \underbrace{\left(h_1 S_J(t)X(t)(\sigma_X^T + \sigma_J^T) + h_2 B_J(t)X(t)\sigma_X^T\right)}_{\text{diffusion term}} dW(t)
$$

Now that we have these dynamics let's derive the dynamics for the payoff of the quanto put and compare the two. Using Ito, and suppressing some notation for ease, we find

$$
\begin{aligned}
dF^{QP} &= F_t^{QP}dt + F_s^{QP}dS_J(t) + \frac{1}{2}F_{ss}^{QP}(dS_J(t))^2 \\
&= \left(F_t^{QP} + F_s^{QP}S_J(t)(r_J - \sigma_X^T\sigma_J + \frac{1}{2}F_{ss}^{QP}\sigma_J^T\sigma_J)\right)dt + F_s^{QP}S_J(t)\sigma_J^T dW(t)
\end{aligned}
$$

By Björk Proposition 16.7 the dt-term is equal to $r_{US}F^{QP}$. So the dynamics is given by

$$
= r_{US}\underbrace{F^{QP}}_{\text{drift term}} dt + \underbrace{F_s^{QP}S_J(t)\sigma_J^T}_{\text{diffusion term}} dW(t)
$$

For the self-financing portfolio to be replicating, we want the two processes to provide the same result in every possible outcome of the Brownian motion, meaning we want the two diffusion terms to match. This gives us the following equation

$$
F_s^{QP}S_J(t)\sigma_J^T = h_1 S_J(t)X(t)\sigma_X^T + h_1 S_J(t)X(t)\sigma_J^T + h_2 B_J(t)X(t)\sigma_X^T
$$

We 'cleverly' choose $h_1 = \Delta^{QP} = \frac{F_s^{QP}}{X(t)}$, such that $F_s^{QP}S_J(t)\sigma_J^T = h_1 S_J(t)X(t)\sigma_X^T$ and further

$$
F_s^{QP}S_J(t)\sigma_J^T = \frac{F_s^{QP}}{X(t)}S_J(t)X(t)\sigma_X^T + h_1 S_J(t)X(t)\sigma_J^T + h_2 B_J(t)X(t)\sigma_X^T \Leftrightarrow h_2 = -\Delta^{QP}\frac{S_J(t)}{B_J(t)}
$$

Comparing the drift terms we 'luckily' find these to be in agreement with the no arbitrage condition (drift-rate should equal r_{US}) and h_0 with the condition that we put the rest of our money in the domestic bank account

$$h_0 = \frac{F^{QP} - h_1 S_J(t)X(t) - h_2 B_J(t)X(t)}{B_{US}(t)}$$

By these choices of h_0, h_1 and h_2 we see that the value of the self-financing portfolio replicates the payoff of the quanto put, thus the strategy $h = (h_0, h_1, h_2)$ as defined above is a hedging portfolio for the quanto put (Björk Definition 8.1). We note how this strategy is exactly the one we used in problem 1.c where we hold Δ^{QP} units of the foreign stock and a short position of $\Delta^{QP} S_J$ units of the foreign currency (with compounding), while using the domestic bank account to make the portfolio self-financing. The issue with the strategy from 1.b and the reason this strategy does not replicate the payoff of the quanto put is that it does not hedge the exchange rate risk.

2 A Reflection Theorem

In this exercise we consider the standard no-dividends Black-Scholes model. Let

$$p = 1 - \frac{2r}{\sigma^2} \tag{1}$$

and consider a simple claim with time-T pay-off specified by a pay-off function g. The arbitrage-free time-t value is

$$\pi^g(t) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[g(S(T))\right] = e^{-r(T-t)}f(t,S(t)) \tag{2}$$

Let $H > 0$ be a constant and define a function $\widehat{g}$ by

$$\widehat{g}(x) = \left(\frac{x}{H}\right)^p g\left(\frac{H^2}{x}\right) \tag{3}$$

2.a

Define the process Z by $Z(t) = \left(\frac{S(t)}{H}\right)^p$. We first wish to show that $\frac{Z(t)}{Z(0)}$ is a positive, mean-1 $\mathbb{Q}$-martingale. Let $Z(t) = f(t, S(t))$ and use Ito's formula to see

$$\begin{aligned} dZ(t) &= f_t dt + f_s dS(t) + \frac{1}{2} f_{ss}(dS(t))^2 \\ &= p\frac{S(t)^{p-1}}{H^p}(rS(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t)) + \frac{1}{2}p(1-p)\frac{S(t)^{p-2}}{H^p}\sigma^2 S(t)^2 dt \\ &= Z(t)(pr + \frac{1}{2}p(p-1)\sigma^2)dt + p\sigma Z(t)dW^{\mathbb{Q}}(t) \end{aligned}$$

We now examine the drift term.

$$pr + \frac{1}{2}p(p-1)\sigma^2 = (1-\frac{2r}{\sigma^2})r + \frac{1}{2}(1-\frac{2r}{\sigma^2})((1-\frac{2r}{\sigma^2})-1)\sigma^2 = r - 2\frac{r^2}{\sigma^2} - r + 2\frac{r^2}{\sigma^2} = 0$$

As $Z(t)$ has no drift term it is a $\mathbb{Q}$-martingale by Björk Lemma 4.9 and further $\frac{Z(t)}{Z(0)}$ must too be a $\mathbb{Q}$-martingale. Now note, by Björk Proposition 5.2, that

$$\frac{Z(t)}{Z(0)} = \left(\frac{S(t)}{S(0)}\right)^p = \left(\frac{S(0)}{S(0)}\right)^p \cdot e^{p((r-\sigma^2/2)t+\sigma W^{\mathbb{Q}}(t))} \sim LN\left(p(r-\sigma^2/2)t, p^2\sigma^2 t\right)$$

Thus $\frac{Z(t)}{Z(0)}$ is positive with probability 1 and has mean $\mathbb{E}\left[\frac{Z(t)}{Z(0)}\right] = e^{p(r-\sigma^2/2)t+\frac{1}{2}p^2\sigma^2 t} = e^0 = 1$. This means that $\frac{d\mathbb{Q}^Z}{d\mathbb{Q}} = \frac{Z(T)}{Z(0)}$ defines a probability measure $\mathbb{Q}^Z \sim \mathbb{Q}$. We now wish to show that

$$\pi^{\widehat{g}}(t) = e^{-r(t-t)}\left(\frac{S(t)}{H}\right)^p \mathbb{E}_t^{\mathbb{Q}^Z}\left[g\left(\frac{H^2}{S(T)}\right)\right]$$

Define $X = \frac{\widehat{g}(S(T))}{Z(T)}$. Then by abstract Bayes Rule we have: $\mathbb{E}_t^{\mathbb{Q}}[X \cdot Z(T)] = \mathbb{E}_t^{\mathbb{Q}^Z}[X] \cdot \underbrace{\mathbb{E}_t^{\mathbb{Q}}[Z(T)]}_{=Z(t)}$.[3] Using this we find

$$
\begin{aligned}
\pi^{\widehat{g}}(t) &= e^{-r(t-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\widehat{g}(S(T))\right] \\
&= e^{-r(t-t)}Z(t)\mathbb{E}_t^{\mathbb{Q}^Z}\left[\frac{\widehat{g}(S(T))}{Z(T)}\right] \\
&= e^{-r(t-t)}\left(\frac{S(t)}{H}\right)^p \mathbb{E}_t^{\mathbb{Q}^Z}\left[\left(\frac{H}{S(T)}\right)^p \left(\frac{S(T)}{H}\right)^p g\left(\frac{H^2}{S(T)}\right)\right] \\
&= e^{-r(t-t)}\left(\frac{S(t)}{H}\right)^p \mathbb{E}_t^{\mathbb{Q}^Z}\left[g\left(\frac{H^2}{S(T)}\right)\right]
\end{aligned}
$$

Which was what we wanted.

2.b

Define the process Y by $Y(t) = \frac{H^2}{S(t)}$. We then wish to show that

$$dY(t) = rY(t)dt + \sigma Y(t)\left(-dW^{\mathbb{Q}^Z}(t)\right)$$

where $W^{\mathbb{Q}^Z}$ is a $\mathbb{Q}^Z$-Brownian motion. Let $Y(t) = f(t, S(t)) = \frac{H^2}{S(t)}$. We use Itô's formula to see

$$
\begin{aligned}
dY(t) &= f_t dt + f_s dS(t) + \frac{1}{2}f_{ss}(dS(t))^2 \\
&= -\frac{H^2}{S(t)^2}(rS(t)dt + \sigma S(t)dW^{\mathbb{Q}}(t)) + \frac{1}{2}2\frac{H^2}{S(t)^3}\sigma^2 S(t)^2 dt \\
&= Y(t)(\sigma^2 - r)dt + \sigma Y(t)\left(-dW^{\mathbb{Q}}(t)\right)
\end{aligned}
$$

We can now use Girsanov (Björk Theorem 11.3), with $\sigma_Z = p\sigma$ by 2.a

$$dW^{\mathbb{Q}}(t) = \sigma_Z dt + dW^{\mathbb{Q}^Z}(t) \Leftrightarrow -dW^{\mathbb{Q}}(t) = (\frac{2r}{\sigma} - \sigma)dt + \left(-dW^{\mathbb{Q}^Z}(t)\right)$$

Using this we get the desired result

$$
\begin{aligned}
dY(t) &= Y(t)(\sigma^2 - r + 2r - \sigma^2)dt + \sigma Y(t)\left(-dW^{\mathbb{Q}^Z}(t)\right) \\
&= rY(t)dt + \sigma Y(t)\left(-dW^{\mathbb{Q}^Z}(t)\right)
\end{aligned}
$$

[3] Björk Proposition B.41

2.c

In this problem we wish to use the result from 2.b to argue that

$$\pi^{\widehat{g}}(t) = e^{-r(t-t)} \left(\frac{S(t)}{H}\right)^p f\left(\frac{H^2}{S(t)}, t\right)$$

We found in 2.b that the distribution of Y under $\mathbb{Q}^Z$ is the same as the distribution of S under $\mathbb{Q}$, i.e.

$$\mathbb{E}_t^{\mathbb{Q}^Z}[g(Y(T)] = \mathbb{E}_t^{\mathbb{Q}}[g(S(T)]$$

Then, for some $s > 0$, it follows that

$$\mathbb{E}_t^{\mathbb{Q}^Z}\left[g(Y(T)|Y(t) = \frac{H^2}{s}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[g(S(T)|S(t) = \frac{H^2}{s}\right] = f\left(\frac{H^2}{s}, t\right) \Rightarrow \mathbb{E}_t^{\mathbb{Q}^Z}[g(Y(T)] = f\left(\frac{H^2}{S(t)}, t\right)$$

Thus, we can conclude what we wished to, namely that

$$\pi^{\widehat{g}}(t) \overset{2.a}{=} e^{-r(t-t)} \left(\frac{S(t)}{H}\right)^p \mathbb{E}_t^{\mathbb{Q}^Z}[g(Y(T)] \overset{2.b}{=} e^{-r(t-t)} \left(\frac{S(t)}{H}\right)^p f\left(\frac{H^2}{S(t)}, t\right)$$

3 Options on Coupon-Bearing Bonds

In this exercise we consider the Vasicek model. Let A and B denote the functions such that $P(t,T) = \exp(A(t,T) - B(t,T)r(t))$. We now look at a coupon bond that makes deterministic positive payments $\alpha_1, ..., \alpha_N$ at dates $T_1, ..., T_N$. The price of this coupon bond is

$$\pi^C(t) = \sum_{i|T_i>t} \alpha_i P(t,T_i)$$

We then consider a European call-option on the coupon bond with positive strike K and expiry T.

3.a

We first wish to show that there exists a unique $r^* \in \mathbb{R}$ such that

$$\pi^C(T) \geq K \Leftrightarrow r(T) \leq r^*$$

By Björk Proposition 24.3 we get an explicit formula for the bond prices

$$\pi^C(T) = \sum_{i|T_i>T} \alpha_i e^{A(T,T_i)-B(T,T_i)r(T)}$$

We now examine how the bond price reacts to changes in $r(T)$. Note that as $a > 0$ and $T_i > T$ then $B(T,T_i) > 0$ and further

$$\lim_{r(T)\to\infty} e^{A(T,T_i)-B(T,T_i)r(T)} = 0$$
$$\lim_{r(T)\to-\infty} e^{A(T,T_i)-B(T,T_i)r(T)} = \infty$$

where we have used that $\lim\limits_{x\to\infty} e^{-x} = 0$ and $\lim\limits_{x\to\infty} e^{x} = \infty$. This leads to the conclusion that the bond price, $P(T,T_i;r(T))$, is a continuous and strictly decreasing function of $r(T)$.

Now define the function $h : \mathbb{R} \to (0,\infty)$ by $h(r(T)) = \pi^C(T;r(T))$. It must hold that this function too is a continuous and strictly decreasing function of $r(T)$ which has inverse $h^{-1} : (0,\infty) \to \mathbb{R}$ such that $h^{-1}(h(r(T))) = r(T)$. By the continuity property there exists some value $K \in (0,\infty)$ such that $h(r(T)) = K$. By the strictly increasing property there exists a unique solution $r^* \in \mathbb{R}$ to the following equation $h^{-1}(h(r(T))) = h^{-1}(K) := r^*$ and further that the following must hold

$$r(T) \leq r^* \Leftrightarrow h(r(T)) = \pi^C(T) \geq K = h(r^*)$$

Define the adjusted strikes by $K_i = e^{A(T,T_i)-B(T,T_i)r^*}$. We then wish to show that

$$\big(\pi^C(T) - K\big)^+ = \sum_{i|T_i>T} \alpha_i (P(T,T_i) - K_i)^+$$

The term $(\pi^C(T) - K)^+$ is non-zero when $\pi^C(T) > K$, which we just derived was the case for $r(T) < r^*$. We further use the fact that r^* is exactly the value such that $K = \pi^C(T; r^*)$.

$$\begin{aligned}
(\pi^C(T) - K)^+ &= \big(\pi^C(T) - K\big) \cdot \mathbb{1}_{(r(T)<r^*)} \\
&= \big(\pi^C(T; r(T)) - \pi^C(T; r^*)\big) \cdot \mathbb{1}_{(r(T)<r^*)} \\
&= \Bigg(\sum_{i|T_i>T} \alpha_i P(T,T_i; r(T)) - \sum_{i|T_i>T} \alpha_i P(T,T_i; r^*) \Bigg) \cdot \mathbb{1}_{(r(T)<r^*)} \\
&= \Bigg(\sum_{i|T_i>T} \alpha_i P(T,T_i; r(T)) - P(T,T_i; r^*) \Bigg) \cdot \mathbb{1}_{(r(T)<r^*)} \\
&= \sum_{i|T_i>T} \alpha_i \Bigg(P(T,T_i; r(T)) - P(T,T_i; r^*) \Bigg) \cdot \mathbb{1}_{(r(T)<r^*)} \\
&= \sum_{i|T_i>T} \alpha_i \Bigg(P(T,T_i; r(T)) - P(T,T_i; r^*) \Bigg)^+ \\
&= \sum_{i|T_i>T} \alpha_i \Bigg(P(T,T_i; r(T)) - K_i \Bigg)^+
\end{aligned}$$

Which was what we wanted to prove. This leads to a closed-form expression for the price of the call on the coupon bond, up to knowledge of r^*. Using the above derived expression we see

$$\pi\Bigg(t, (\pi^C(T) - K)^+\Bigg) = \pi\Bigg(t, \sum_{i|T_i>T} \alpha_i (P(T,T_i; r(T)) - K_i)^+\Bigg)$$

Using linearity of conditional expectations

$$= \sum_{i|T_i>T} \alpha_i \pi\Bigg(t, (P(T,T_i; r(T)) - K_i)^+\Bigg)$$

We have that $\pi\left(t, (P(T,T_i; r(T)) - K_i)^+\right) = Call(t, T, K_i, T_i)$, where $Call()$ is determined by Björk proposition 24.9.

$$= \sum_{i|T_i>T} \alpha_i Call(t, T, K_i, T_i)$$

Here we note that the $Call()$ price function is a closed-form expression which we perform a finite number of standard operations on, thus maintaining a closed-form expression.[4]

3.b

We now wish to calculate the time-0 price of the call on the coupon bond using the following parameters and further the functions derived in 3.a.

Parameter	Value
r_0	0.02
θ^Q	0.05
κ	0.1
σ	0.015
K	4.5
T	1
T_i	$i+1$
N	5
α_i	1

We did the following[5]

- Defined the parameters and functions to be used.
- Used $uniroot()$ to derive $r(T) = r^*$ from the convex function $f(r(T)) = \pi^C(T; r(T)) - K$.
- Defined the adjusted strikes K_i.
- Calculated the time-0 price of the call on the coupon bond: $\pi(0, (\pi^C(T) - K)^+) = 0.1398$.

3.c

In this problem we wish to examine how the call option on the coupon-bearing bond can be hedged. We start by noting that the call option payoff is a finite sum and α_i is a positive constant, so if we can hedge each term, $(P(T, T_i) - K_i)^+$, we can hedge them all. By Björk proposition 24.9 the bond option formula in the Vasicek model is given by

$$call^{ZCB}(t) = \Phi(d)P(t, T_M) - K\Phi(d - \sigma_p)P(t, T_E)$$

[4] This is given that r^* to some extend is known such that we can find the root of $\pi^C(T) - K$.

[5] For R-code see Appendix B. Here we have *transformed* into Björk parameters using $a = \kappa$ and $b = \kappa\theta^Q$.

where

$$
\begin{aligned}
d &= \frac{1}{\sigma_p}\log\left(\frac{P(t,T_M)}{P(t,T_E)K}\right) + \frac{1}{2}\sigma_p \\
\sigma_p &= \frac{1}{a}\left(1 - e^{-a(T_M - T_E)}\right)\cdot\sqrt{\frac{\sigma^2}{2a}(1 - e^{-2a(T_E - t)})}
\end{aligned}
$$

Looking at the above call price formula a qualified guess is that we should hedge using

$$
h = \begin{cases} \Phi(d) & \text{in the } T_M - ZCB \\ -K\Phi(d - \sigma_p) & \text{in the } T_E - ZCB \end{cases}
$$

i.e. the proposed hedging portfolio has value process

$$
V^h(t) = \Phi(d)P(t,T_M) - K\Phi(d-\sigma_p)P(t,T_E)
$$

By Björk Definition 8.1 h is a hedging portfolio if it is self-financing and if the value process of h replicates that of the call option on the zero coupon bond. We check the latter

$$
V^h(T) = \Phi(d_1)P(T,T_M) - K\Phi(d_2)P(T,T_E) = call^{ZCB}(T)
$$

We now wish to show that h is self-financing such that the value process of h equals the call option for the zero coupon bond, i.e. that

$$
dcall^{ZCB}(T) = dV^h(t) = \Phi(d)dP(t,T_M) - K\Phi(d-\sigma_p)dP(t,T_E) \tag{4}
$$

We now wish to derive the LHS and RHS dynamics in this equation and examine whether these are equal.

By Björk Proposition 24.3 and Ito's formula we find the dynamics of $P(t,T)$ to be

$$
\begin{aligned}
dP(t,T) &= P_t dt + P_r dr(t) + \frac{1}{2}P_{rr}(dr(t))^2 \\
&= P(t,T)(A_t - B_t r(t))dt - B(t,T)P(t,T)((b - ar(t))dt + \sigma dW(t)) + \frac{1}{2}B(t,T)^2P(t,T)\sigma^2 dt \\
&= P(t,T)\left(bB(t,T) - \frac{1}{2}\sigma^2B(t,T)^2 - aB(t,T)r(t) + r(t) + \frac{1}{2}B(t,T)^2\sigma^2 - bB(t,T) + aB(t,T)r(t)\right)dt \\
&- P(t,T)B(t,T)\sigma dW(t) \\
&= P(t,T)r(t)dt - P(t,T)B(t,T)\sigma dW(t)
\end{aligned}
$$

Using this we find the RHS dynamics

$$
\begin{aligned}
\Phi(d)dP(t,T_M) - K\Phi(d-\sigma_p)dP(t,T_E) &= \Phi(d)\Big(P(t,T_M)r(t)dt - P(t,T_M)B(t,T_M)\sigma dW(t)\Big) \\
&\quad - K\Phi(d-\sigma_p)\Big(P(t,T_E)r(t)dt - P(t,T_E)B(t,T_E)\sigma dW(t)\Big) \\
&= r(t)\underbrace{\Big(\Phi(d)P(t,T_M) - K\Phi(d-\sigma_p)P(t,T_E)\Big)}_{call^{ZCB}(t)}dt \\
&\quad + \Big(K\Phi(d-\sigma_p)P(t,T_E)B(t,T_E) - \Phi(d)P(t,T_M)B(t,T_M)\Big)\sigma dW(t)
\end{aligned}
$$

We now examine the dynamics of the LHS, i.e. those of the $call^{ZCB}$. We know that a discounted derivative is a martingale and thus must have our known arbitrage free drift term. Considering $call^{ZCB}(t)$ as a stochastic process we must have that the discounted drift rate is $r(t)call^{ZCB}(t)$. By Björk Proposition 4.18 we find the diffusion term to be

$$
\sigma_{call}(t) = \sum_{i=M,E} \sigma_i \frac{\partial call^{ZCB}(t)}{\partial P(t,T_i)} dW_i(t)
$$

Now note that we only have one common Wiener process as our model only have $r(t)$ which generates information. Further we have by the above derivation of $dP(t,T)$ that $\sigma_i = P(t,T_i)B(t,T_i)\sigma$. We now derive $\frac{\partial call^{ZCB}(t)}{\partial P(t,T_i)}$ for $i = M, E$ respectively.[6]

$$
\begin{aligned}
\frac{\partial call^{ZCB}(t)}{\partial P(t,T_M)} &= \Phi(d) + P(t,T_M)\phi(d)\frac{1}{\sigma_p P(t,T_M)} - KP(t,T_E)\phi(d-\sigma_p)\frac{1}{\sigma_p P(t,T_M)} \\
&= \Phi(d) + \frac{\phi(d)}{\sigma_p} - \frac{\phi(d)}{\sigma_p}\frac{P(t,T_M)}{P(t,T_E)K}\frac{P(t,T_E)K}{P(t,T_M)} \\
&= \Phi(d)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial call^{ZCB}(t)}{\partial P(t,T_E)} &= P(t,T_M)\phi(d)\left(-\frac{1}{\sigma_p P(t,T_E)}\right) - K\Phi(d-\sigma_p) - KP(t,T_E)\left(-\frac{1}{\sigma_p P(t,T_E)}\right)\phi(d-\sigma_p) \\
&= -K\Phi(d-\sigma_p) + \frac{\phi(d-\sigma_p)}{\sigma_p}K - \frac{\phi(d)}{\sigma_p}\frac{P(t,T_M)}{P(t,T_E)} \\
&= -K\Phi(d-\sigma_p) + \frac{\phi(d)}{\sigma_p}\frac{K}{K}\frac{P(t,T_M)}{P(t,T_E)} - \frac{\phi(d)}{\sigma_p}\frac{P(t,T_M)}{P(t,T_E)} \\
&= -K\Phi(d-\sigma_p)
\end{aligned}
$$

[6]Throughout these calculations we use that $\phi(d-\sigma_p) = \phi(d)\frac{P(t,T_M)}{P(t,T_E)K}$. See Appendix A for a derivation of this.

We can now compare the two dynamics of equation (4) and see that they clearly are equal

$$\begin{aligned}
&\text{(LHS)} \qquad dcall^{ZCB}(t) &&= r(t)call^{ZCB}(t)dt + \left(K\Phi(d-\sigma_p)B(t,T_E)P(t,T_E) - \Phi(d)B(t,T_M)P(t,T_M)\right)\sigma dW(t) \\
&\text{(RHS)} \qquad dV^h(t) &&= r(t)call^{ZCB}(t)dt + \left(K\Phi(d-\sigma_p)P(t,T_E)B(t,T_E) - \Phi(d)P(t,T_M)B(t,T_M)\right)\sigma dW(t)
\end{aligned}$$

It then follows from Björk Definition 8.1 that h is a self-financing hedging portfolio for the call on a zero coupon bond. Thus the hedging portfolio for the call on the coupon bearing bond is given by the sum of the payments times the hedge, h.

4 Appendix A - minor less important calculations

(1.a) $W^*(t) = \frac{\sigma_J}{||\sigma_J||}W(t)$ is a martingale

By Björk Definition 4.1 we find, using that $W(t)$ is a martingale

1 Starting at zero

$W(0) = 0 \Rightarrow W^*(0) = \dfrac{\sigma_J}{||\sigma_J||} \cdot 0 = 0$

2 Independent increments ($r < s \leq t < u$)

$W(u) - W(t) \perp\!\!\!\perp W(s) - W(r) \Rightarrow \dfrac{\sigma_J}{||\sigma_J||}(W(u) - W(t)) \perp\!\!\!\perp \dfrac{\sigma_J}{||\sigma_J||}(W(s) - W(r))$

3 Normal increments ($s < t$)

$W(t) - W(s) \sim \mathcal{N}(0, t-s) \Rightarrow W^*(t) - W^*(s) = \dfrac{\sigma_J}{||\sigma_J||}(W(t) - W(s)) \sim N\left(0, \dfrac{||\sigma_J||^2}{||\sigma_J||^2}(t-s)\right) = \mathcal{N}(0, t-s)$

4 Continuous paths

$t \mapsto W(t)$ continuous $\Rightarrow t \mapsto \dfrac{\sigma_J}{||\sigma_J||}W(t)$ continuous

(1.a) Delta of BS put with dividends

The price, $Put(t, S(t))$, of a European put option on a dividend-paying stock is given by

$$Put(t, S(t)) = Ke^{-r(T-t)}\Phi(-d_2) - S(t)e^{-\delta(T-t)}\Phi(-d_1)$$

Then the delta is given by

$$\begin{aligned}
\Delta_\delta^{put} &= \frac{\partial Put(t, S(t))}{\partial S(t)} = Ke^{-r(T-t)}\frac{\partial \Phi(-d_2)}{\partial S(t)} - e^{-\delta(T-t)}\Phi(-d_1) - S(t)e^{-\delta(T-t)}\frac{\partial \Phi(-d_1)}{\partial S(t)} \\
&= Ke^{-r(T-t)}\frac{\partial(1-\Phi(d_2))}{\partial S(t)}\frac{\partial d_2}{\partial S(t)} - e^{-\delta(T-t)}(1-\Phi(d_1)) - S(t)e^{-\delta(T-t)}\frac{\partial(1-\Phi(d_1))}{\partial S(t)}\frac{\partial d_1}{\partial S(t)} \\
&= -Ke^{-r(t-t)}\frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}\frac{S(t)}{K}e^{(r-\delta)(T-t)}\frac{1}{S(t)\sigma\sqrt{T-t}} - e^{-\delta(T-t)}(1-\Phi(d_1)) + S(t)e^{-\delta(T-t)}\frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}\frac{1}{S(t)\sigma\sqrt{T-t}} \\
&= e^{-\delta(T-t)}(\Phi(d_1) - 1) + e^{-\delta(T-t)}\frac{1}{\sigma\sqrt{2\pi(T-t)}}e^{-d_1^2/2} - e^{-\delta(T-t)}\frac{1}{\sigma\sqrt{2\pi(T-t)}}e^{-d_1^2/2} \\
&= e^{-\delta(T-t)}(\Phi(d_1) - 1)
\end{aligned}$$

(3.c) $\phi(d-\sigma_p) = \phi(d)\frac{P(t,T_M)}{P(t,T_E)K}$

$$\begin{aligned}
\phi(d-\sigma_p) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d-\sigma_p)^2} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d^2+\sigma_p^2-2d\sigma_p)} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d^2}e^{-\frac{1}{2}(\sigma_p^2-2d\sigma_p)} \\
&= \phi(d)e^{-\frac{1}{2}\sigma_p^2+(\frac{1}{\sigma_p}\log\left(\frac{P(t,T_M)}{P(t,T_E)K}\right)+\frac{1}{2}\sigma_p^2)\sigma_p} \\
&= \phi(d)e^{-\frac{1}{2}\sigma_p^2+\frac{1}{2}\sigma_p^2+\log\left(\frac{P(t,T_M)}{P(t,T_E)K}\right)} \\
&= \phi(d)\log\left(\frac{P(t,T_M)}{P(t,T_E)K}\right)
\end{aligned}$$

5 Appendix B - R code

1.b

```
## Parameters
# ——————————————
r_US <- 0.015
r_J <- 0.01
S_J_start <- 29015
X_0 <- 0.0094
strike <- 28000
sigma_X <- as.matrix(c(0.05, 0.1))
sigma_J <- as.matrix(c(0.1, 0.3))
Y_0 <- 0.0094
T <- 1
Nhedges <- 252
Nreps <- 1000
dt <- T/Nhedges

## Functions
# ——————————————
d <- function(spot,strike,r,q,sigma,timetomat,type){

  if (type == 1) result <- (log(spot/strike)+(r-q)*timetomat
  + (1/2)*(sigma^2)*timetomat)/(sigma*sqrt(timetomat))

  if (type == 2) result <- (log(spot/strike)+(r-q)*timetomat
  - (1/2)*(sigma^2)*timetomat)/(sigma*sqrt(timetomat))

  d <- result
}

BlackScholesFormula <- function(spot,strike,timetomat,r,q,sigma,option){

  d1 <- d(spot,strike,r,q,sigma,timetomat,type=1)
  d2 <- d(spot,strike,r,q,sigma,timetomat,type=2)
```

```
 if (option=="Call"){result<-exp(-q*timetomat)*spot*pnorm(d1)
 - strike*exp(-r*timetomat)*pnorm(d2)}

 if (option=="Put"){result<- exp(-q*timetomat)*spot*pnorm(d1)
 - strike*exp(-r*timetomat)*pnorm(d2) -exp(-q*timetomat)*spot
 + strike*exp(-r*timetomat)}

BlackScholesFormula <- result
}

g <- function(Y_0,spot,strike,q,timetomat,r,sigma){
d1 <- d(spot,strike,r,q,sigma,timetomat,type=1)
g <- Y_0*exp(-q*timetomat)*(pnorm(d1)-1)
}

## Hedge Experiment
set.seed(5)
initialoutlay <- Y_0*exp(-r_US*T)*BlackScholesFormula(S_J_start,strike,T,r_US,
r_US-r_J+as.vector(t(sigma_X)%*%sigma_J),norm(sigma_J,"F"),"Put")

pf_value <- rep(initialoutlay,Nreps)
S_J <- rep(S_J_start, Nreps)
X <- rep(X_0, Nreps)

a <- g(Y_0,S_J,strike,as.vector(r_US-r_J+as.vector(t(sigma_X)%*%sigma_J)),T,r_US,
norm(sigma_J,"F"))/X_0
b <- pf_value - X_0*a*S_J

for (i in 2:Nhedges){
Z <- cbind(rnorm(Nreps), rnorm(Nreps))
S_J <- S_J*exp((r_J - as.vector(t(sigma_X)%*%sigma_J)
    -(1/2)*norm(sigma_J,"F")^2)*dt + Z%*%sigma_J*sqrt(dt))
X <- X*exp((r_US- r_J - (1/2)*norm(sigma_X,"F")^2)*dt '+ Z%*%sigma_X*sqrt(dt))
pf_value <- a*X*S_J + b*exp(r_US*dt)
```

```
a <- g(Y_0,S_J,strike,as.vector(r_US-r_J+t(sigma_X)%*%sigma_J),T-(i-1)*dt,r_US,
norm(sigma_J,"F"))/X
b <- pf_value - X*a*S_J
}

plot(S_J,pf_value,col="blue",xlab="S_J(T)",ylab="Value of hedge portfolio",
ylim=c(-30,170),xlim=c(10000,50000),main="Discrete hedging of a quanto put-option")
points(10000:50000,Y_0*pmax(strike - 10000:50000,0),type='l',lwd=3)

## Convergence of hedge error
r_US <- 0.015
r_J <- 0.01
S_J_start <- 29015
X_0 <- 0.0094
strike <- 28000
sigma_X <- as.matrix(c(0.05, 0.1))
sigma_J <- as.matrix(c(0.1, 0.3))
Y_0 <- 0.0094
T <- 1

HedgeExperiment <- function(Nhedges, Nreps){
dt <- T/Nhedges

d <- function(spot,strike,r,q,sigma,timetomat,type){

  if (type == 1) result <- (log(spot/strike)+(r-q)*timetomat
  + (1/2)*(sigma^2)*timetomat)/(sigma*sqrt(timetomat))

  if (type == 2) result <- (log(spot/strike)+(r-q)*timetomat
  - (1/2)*(sigma^2)*timetomat)/(sigma*sqrt(timetomat))

  d <- result
}
BlackScholesFormula <- function(spot,strike,timetomat,r,q,sigma,option){
```

```
  d1 <- d(spot, strike, r, q, sigma, timetomat, type=1)
  d2 <- d(spot, strike, r, q, sigma, timetomat, type=2)



  if (option=="Call"){result<-exp(-q*timetomat)*spot*pnorm(d1)
  - strike*exp(-r*timetomat)*pnorm(d2)}

  if (option=="Put"){result<- exp(-q*timetomat)*spot*pnorm(d1)
  - strike*exp(-r*timetomat)*pnorm(d2) -exp(-q*timetomat)*spot
  + strike*exp(-r*timetomat)}

  BlackScholesFormula <- result

}
g <- function(Y_0,spot, strike, q, timetomat, r, sigma){

  d1 <- d(spot, strike, r, q, sigma, timetomat, type=1)

  g <- Y_0*exp(-q*timetomat)*(pnorm(d1)-1)

}

initialoutlay <- Y_0*exp(-r_US*T)*BlackScholesFormula(S_J_start, strike, T, r_US,
r_US-r_J+as.vector(t(sigma_X)%*%sigma_J), norm(sigma_J, "F"), "Put")
pf_value <- rep(initialoutlay, Nreps)
S_J <- rep(S_J_start, Nreps)
X <- rep(X_0, Nreps)
a <- g(Y_0,S_J, strike, as.vector(r_US-r_J+as.vector(t(sigma_X)%*%sigma_J)), T, r_US,
norm(sigma_J, "F"))/X_0
b <- pf_value - X_0*a*S_J

for (i in 2:Nhedges){
Z <- cbind(rnorm(Nreps), rnorm(Nreps))
S_J <- S_J*exp((r_J - as.vector(t(sigma_X)%*%sigma_J)
- (1/2)*norm(sigma_J, "F")^2)*dt + Z%*%sigma_J*sqrt(dt))
```

```
X <- X*exp((r_US- r_J - (1/2)*norm(sigma_X,"F")^2)*dt+ Z%*%sigma_X*sqrt(dt))
pf_value <- a*X*S_J + b*exp(r_US*dt)
a <- g(Y_0,S_J,strike,as.vector(r_US-r_J+t(sigma_X)%*%sigma_J),T-(i-1)*dt,r_US,
norm(sigma_J,"F"))/X
b <- pf_value - X*a*S_J
}

Payoff <- Y_0*pmax(strike-S_J,0)
hedgeError <- abs(pf_value - Payoff)

HedgeExperiment <- sd(exp(-r_US*T)*hedgeError)
}

numberOfPaths <- 10000
testHedgePts <- NA
for (j in 1:1000){testHedgePts[j] <- j}
se <- numeric(length(testHedgePts))

for (i in 1:length(testHedgePts)){se[i] <- HedgeExperiment(testHedgePts[i],
numberOfPaths)}

plot(testHedgePts, se, xlab = "Number of Hedge Points", ylim=c(7,9),
     type="l", ylab = "Standard deviation", main = "Convergence of Hedge Error")
```

1.c

```
set.seed(5)
initialoutlay <- Y_0*exp(-r_US*T)*BlackScholesFormula(S_J_start,strike,T,r_US,
r_US-r_J+as.vector(t(sigma_X)%*%sigma_J),norm(sigma_J,"F"),"Put")

pf_value <- rep(initialoutlay,Nreps)
S_J <- rep(S_J_start, Nreps)
X <- rep(X_0, Nreps)

a <- g(Y_0,S_J,strike,as.vector(r_US-r_J+as.vector(t(sigma_X)%*%sigma_J)),T,r_US,
norm(sigma_J,"F"))/X_0
```

```
c <- -a*S_J
b <- pf_value - X_0*a*S_J - c*X_0

for (i in 2:Nhedges){
  Z <- cbind(rnorm(Nreps), rnorm(Nreps))
  S_J <- S_J*exp((r_J - as.vector(t(sigma_X)%*%sigma_J)
    - (1/2)*norm(sigma_J,"F")^2)*dt + Z%*%sigma_J*sqrt(dt))
  X <- X*exp((r_US- r_J - (1/2)*(norm(sigma_X,"2"))^2)*dt
    + Z%*%sigma_X*sqrt(dt))
  pf_value <- a*S_J*X + b*exp(r_US*dt) + X*c*exp(r_J*dt)
  a <- g(Y_0,S_J,strike,r_US-r_J+as.vector(t(sigma_X)%*%sigma_J),T-(i-1)*dt,r_US,
  norm(sigma_J,"F"))/X
  c <- -a*S_J
  b <- pf_value - a*S_J*X - c*X
}

plot(S_J,pf_value,col="blue",xlab="S_J(T)",ylab="Value of hedge portfolio",
ylim=c(-30,170),xlim=c(10000,50000),main="Discrete hedging of a quanto put-option")
points(10000:50000,Y_0*pmax(strike - 10000:50000,0),type='l',lwd=3)
```

3.b

```
## Parameters
r_0 <- 0.02
theta_Q <- 0.05
kappa <- 0.1
sigma <- 0.015
K <- 4.5
T_expiry <- 1
N <- 5
T_maturities <- c(2:6)
alpha <- rep(1,5)
a <- kappa
b <- kappa*theta_Q

## Functions
```

```
P <- function(r, t, T, a, b, sigma){

  B <- 1/a*(1-exp(-a*(T-t)))

  A <- ((B-T+t)*(a*b-(1/2)*sigma^2))/(a^2)-((sigma^2)*(B^2))/(4*a)

 P <- exp(A-B*r)
}
pi_C <- function(r, t, T, a, b, sigma, alpha){
sum(alpha[T>t]*P(r=r, t=t, T=T[T>t], a=a, b=b, sigma=sigma))
}
Call_Vasicek <- function(r, t, T, a, b, sigma, alpha, S, K){

sigma_p <- (1/a)*(1-exp(-a*(S-T)))*sqrt(((sigma^2)/(2*a))*(1-exp(-2*a*(T-t))))

d <- (1/sigma_p)*log((P(r, t, S, a, b, sigma))/(P(r, t, T, a, b, sigma)*K))
     +(1/2)*sigma_p

Call_Vasicek <- P(r=r, t=t, T=S, a=a, b=b, sigma=sigma)*pnorm(d)
     -P(r=r, t=t, T=T, a=a, b=b, sigma=sigma)*K*pnorm(d-sigma_p)
}

## Calculations
r_star <- uniroot( function(r){ pi_C(r=r , t=T_expiry , T=T_maturities, a=a, b=b,
sigma=sigma, alpha=alpha) -K},c(-1,1))$root
K_i <- P(r=r_star, t=T_expiry, T=T_maturities, a=a, b=b,
     sigma=sigma)
Price_i <- Call_Vasicek(r=r_0, t=0, T=T_expiry, a=a, b=b, sigma=sigma, alpha=alpha,
     S=T_maturities, K=K_i)
Price <- sum(alpha*Price_i)
round(Price_i,4)
```