Continous Time Finance 2 - Hand-in 3

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Question 1: Variance Swaps

1.a

In this problem we wish to prove the spanning formula, as stated in equation (1) in Carr & Madan (2001), which states

$$f(S) = [f(S_0) - f'(S_0)S_0] + f'(S_0)S + \int_0^{S_0} f''(K)(K-S)^+ dK + \int_{S_0}^{\infty} f''(K)(S-K)^+ dK$$

Where f is a twice continuously differentiable function of S. Thus the spanning formula states that we can replicate a (twice continuously differentiable) function of the terminal stock price, S, with a unique initial position of $[f(S_0) - f'(S_0)S_0]$ units of discount bonds, $f'(S_0)$ units stock and $f''(K)$ OTM calls of all strikes K.

Let f be a twice continuously differentiable function. Then the derivatives f' and f'' are Riemann integrable on the closed interval $[S_0, S]$. Therefore we can use the Fundamental Theorem of Calculus (FTC). Use this to see that $f(S) - f(S_0) = \int_{S_0}^{S} f'(u)du$. Thus we find

$$\begin{aligned}
f(S) &= f(S_0) + \int_{S_0}^{S} f'(u)du \\
\text{Add zero} \qquad &= f(S_0) + \int_{S_0}^{S} f'(u) + f'(S_0) - f'(S_0)\, du \\
&= f(S_0) + \int_{S_0}^{S} f'(S_0)\, du + \int_{S_0}^{S} f'(u) - f'(S_0)\, du \\
\text{Does not depend on u} \qquad &= f(S_0) + f'(S_0)(S - S_0) + \int_{S_0}^{S} f'(u) - f'(S_0)\, du \\
\text{Use FTC} \qquad &= f(S_0) + f'(S_0)(S - S_0) + \int_{S_0}^{S} \int_{S_0}^{u} f''(K)dK\, du
\end{aligned}$$

We now wish to use Fubini's Theorem. The limits for the double Riemann integral is $S_0 < u < S$ and $S_0 < K < u$. By Fubini's Theorem we change the order of integration, where the inner integral limits becomes $K < u < S$ and the outer becomes $S_0 < K < S$. We can use Fubini theorem as

$$
\begin{aligned}
&= f(S_0) + f'(S_0)(S - S_0) + \int_{S_0}^{S}\int_{K}^{S} f''(K)du\,dK \\
\text{Does not depend on u} \qquad &= f(S_0) + f'(S_0)(S - S_0) + \int_{S_0}^{S} f''(K)(S-K)\,dK \\
\text{Divide into two scenarios} \qquad &= f(S_0) + f'(S_0)(S - S_0) + \int_{S_0}^{S} (\mathbb{1}_{S_0 \leq S} + \mathbb{1}_{S_0 > S}) f''(K)(S-K)\,dK \\
&= f(S_0) + f'(S_0)(S - S_0) + \mathbb{1}_{S_0 > S}\int_{S_0}^{S} f''(K)(S-K)\,dK + \mathbb{1}_{S_0 \leq S}\int_{S_0}^{S} f''(K)(S-K)\,dK
\end{aligned}
$$

We now use the fact that $\mathbb{1}_{S_0 \leq S} = 1$ only when $S_0 \leq S$ and further that $\mathbb{1}_{S_0 > S} = 1$ only when $S_0 > S$, to note that we can convert the linear parts to max functions in each integrand.

$$
= f(S_0) + f'(S_0)(S - S_0) + \int_{0}^{S_0} f''(K)(K-S)^{+}dK + \int_{S_0}^{\infty} f''(K)(S-K)^{+}dK
$$

We recognize the spanning formula, which was what we wanted to show.

1.b

We first wish to explain how the static/dynamic hedge of a variance swap works.

A variance swap is something which at time-T pays

$$
\text{Payoff}_T = \underbrace{\sum_{i=0}^{n-1}\left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2}_{:=(A)} - K
$$

This payoff is composed of two parts. K is some constant referred to as 'the variance swap rate' and is chosen such that the initial value of the variance swap is zero. (A) is stochastic and is defined as the sum of the squared rates of return, i.e. the realized variance. The intuition behind a long variance swap is that we switch the payment (A) for the payment K. If it turns out that the realized variance is more significant than the strike, then payoff at maturity will be positive. In order to determine the fair variance swap rate, i.e. the K which makes the initial value zero, we will need to know the value of (A) at time zero. We assume zero interest rate and dividends and that the underlying stock follows an Ito process. We consider a discrete timeindex where the value of the stock at time t_i is given by S_{t_i} for $i = 0, ..., n$ and where the length between each timestep is $dt = t_{i+1} - t_i$, fx one day.

We now take a small detour - one which will make sense by the end of it. We wish to Taylor expand $\log(x)$ to the second order around $x_0 = 1$. We note the following

$$\log(1) = 0 \qquad \log'(x) = \frac{1}{x} \qquad \log'(1) = 1 \qquad \log''(x) = -\frac{1}{x^2} \qquad \log''(1) = -1$$

Thus we get that

$$\log(x) \approx \log(1) + \frac{\log'(x_0)}{1!}(x - x_0) + \frac{\log''(x_0)}{2!}(x - x_0)^2 = (x-1) - \frac{1}{2}(x-1)^2$$

Applying this to $x = \frac{S_{t_{i+1}}}{S_{t_i}}$ yields, for all $i = 0, ..., n-1$,

$$\begin{aligned} \log\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right) &\approx \left(\frac{S_{t_{i+1}}}{S_{t_i}} - 1\right) - \frac{1}{2}\left(\frac{S_{t_{i+1}}}{S_{t_i}} - 1\right)^2 \\ &= \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right) - \frac{1}{2}\left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2 \end{aligned}$$

Rearranging and summing over $i = 0, ..., n-1$ yields

$$\sum_{i=0}^{n-1}\left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2 \approx \underbrace{\sum_{i=0}^{n-1}\frac{2}{S_{t_i}}\left(S_{t_{i+1}} - S_{t_i}\right)}_{:=(B)} + \underbrace{2\log\left(\frac{S_0}{S_T}\right)}_{:=(C)}$$

Note that (C) appears as $\sum_{i=0}^{n-1}\log\left(\frac{S_{t_i}}{S_{t_{i+1}}}\right)$ is a telescoping sum and therefore will only depend on the first and last term.

We recognize that the (B) term corresponds to a dynamic trading strategy where we for $i = 0, ..., n-1$ do the following independently

- At time t_i: Buy $\frac{2}{S_{t_i}}$ units of stock using money we borrow. This will cost $\frac{2}{S_{t_i}} \cdot S_{t_i} - 2 = 0$.
- At time t_{i+1}: Liquidate the portfolio. This will yield the payoff: $\frac{2}{S_{t_i}} \cdot S_{t_{i+1}} - 2 = \frac{2}{S_{t_i}}\left(S_{t_{i+1}} - S_{t_i}\right)$

The total value of this strategy is then the sum of the individual independent payoffs, i.e. the (B) term.

We recognize that the (C) term is the payoff of a simple T-claim with payoff function $\Phi(x) = 2\log\left(\frac{S_0}{x}\right)$. If one were to draw this function it is clear that perhaps a portfolio of puts and calls can approximate the (C) payoff. This approximation is useful (and necessary) as the log-contracts cannot be traded in the market but puts and calls can. This strategy is static as it does not need to be adjusted. To approximate the payoff of (C) we need to determine the portfolio weights. Let $K_J = S_0 + J \cdot dK$ for $J = -m, ..., m$.

We wish to set up a static put- and call- portfolio with weights w_J where

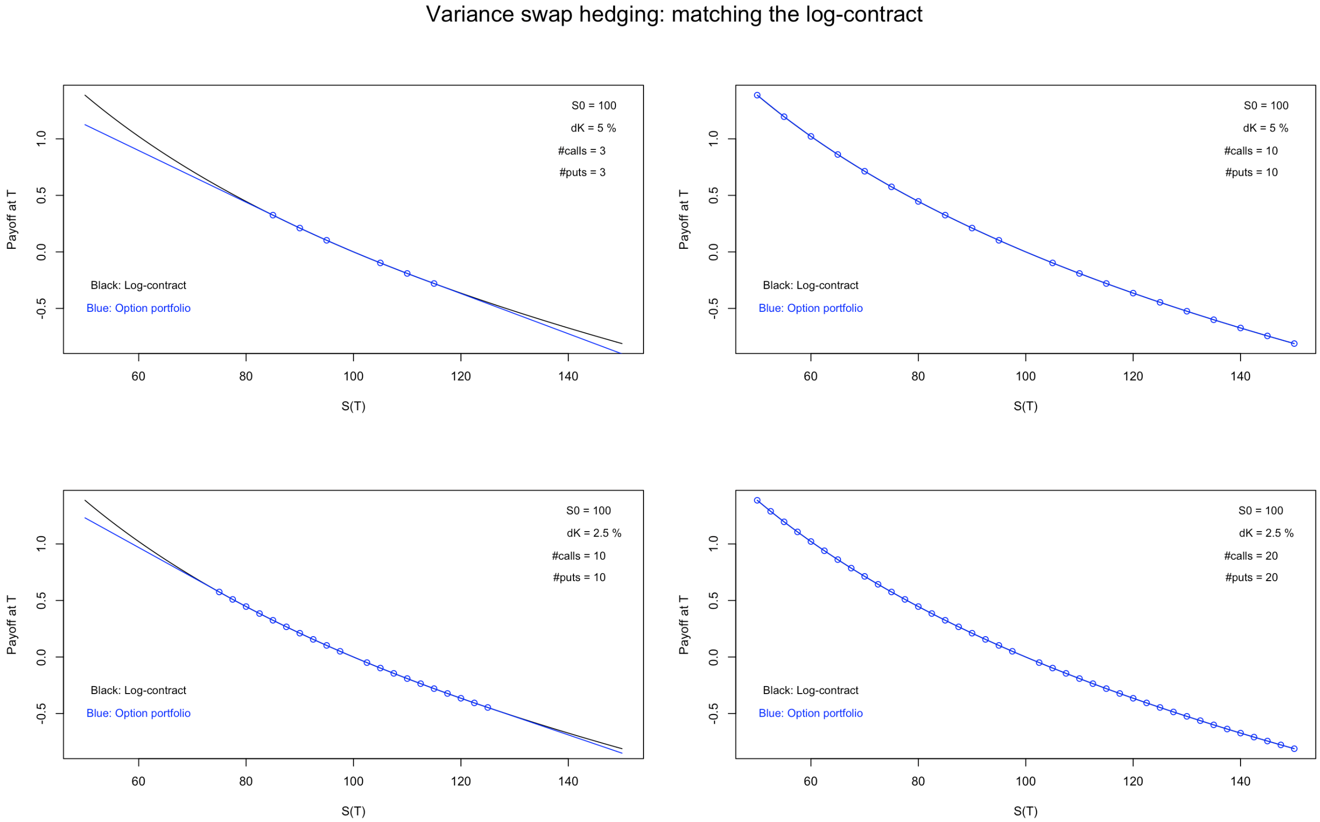
$$w_J = \begin{cases} \#\text{ calls with strike } K_J & J \geq 0 \\ \#\text{ puts with strike } K_J & J < 0 \end{cases}$$

The intuition here is that we wish for our portfolio to include calls with high strikes, i.e. above S_0, and puts with strikes below. Further we want the value of the payoff function evaluated at a given time point, K_l, to match the payoff of the portfolio of puts and calls. Thus the weights, $w_J \in \mathbb{R}^{2m}$, are found by solving for w_J in the following system of linear equations

$$(C) := 2\log\left(\frac{S_0}{K_l}\right) \approx \underbrace{\sum_{J=-(m-1)}^{-1} w_J(K_J - K_l)}_{\text{Put payoffs}} + \underbrace{\sum_{J=1}^{m-1} w_J(K_l - K_J)}_{\text{Call payoffs}} := (D) \tag{1}$$

The sum indexes are defined in this way, as the put options don't contribute to the portfolio value for indices above S_0 and similarly for the calls on the lower indices. To illustrate the approximation, we plot the time-T payoffs (C) and (D) together against the spot at time T. We plot with various distances between options, dK, and various number of options, #call and #puts.

The idea of the plots below is to have some strike points where we match the payoff functions exactly. We first define a vector consisting of the strike points where we want replication to take place. Then we write out the system of equations and solve for the weights as in (1). We then plot the value of the payoff function (the blue curves) together with the value of the log-contract function (the black curves), the latter being what we are trying to replicate.



Figur 1

We note that on the upper left plot the portfolio consisting of 3 call and 3 put options approximates the log-contract well on spots 85 to 115. Changing number of calls and puts to 10 (upper right plot) we notice that the option portfolio fits the log-contract on the whole interval of spots. Decreasing the distance between the options from 5% of initial spot to $2,5\%$ (lower left plot) decreases the interval of fit to spots ranging from 75 to 125, but improves the approximation close to S_0. At last changing the portfolio to consist of 20 call and 20 put options (lower right plot) the fit looks nearly perfect on the observable interval. The blue curves are piece wise linear functions (as we combine puts and calls) which is set up to exactly match the true (black) function in the points. Ideally for replicating we would want a continuum of strike points, however this is not possible. For a finite number of strikes we would never get a perfect approximation, as the approximating function is asymptotic linear and the log function is not. Further for a very broad fit we would need to be able to trade puts and calls on unrealistic high and low strikes in the market.

Before we conclude that (D) approximates (C) we test this statement numerically. What we have derived so far has been model independent, but when we wish to calculate a price we need to make some model assumptions. We price under the risk neutral measure and assume Ito stock dynamics $dS(t) = \sigma S(t) dW^{\mathbb{Q}}(t)$. We use $T = \frac{1}{12}$, $\sigma = 0.15$ and $n = 20$, meaning we have a month-long swap with daily updating. We simulate 10^4 paths and show the Monte Carlo means in Table 1. The hedge error is given by the difference between (C) and (D) and nK denotes the number of calls and puts

nK	3	10	10	20
dK	5%	5%	2.5%	2.5%
(C)	0.0024	0.00079	0.0036	0.0038
(D)	0.0029	0.00007	0.0037	0.0039
|(C)-(D)|	0.0005	0.00072	0.0001	0.0001

Tabel 1

We see from Table 1 that the hedge error is significantly small and thus conclude that (D) approximates (C), however to a degree which depends upon the number of strikes and the distance between these.

The price of the static replicating portfolio, (D), is thus what it costs to replicate the stochastic part, (C), of the variance swap as we found that (B) costs zero. Thus if we put the variance swap rate, K, equal to this price then the time zero value of the variance swap rate will be zero. In conclusion we have a hedging strategy for (A), as the value of (A) is approximated by the payoff of a portfolio which consists of a dynamic part (B) and a static part (D). We examine how this hedge performs numerically using the same scheme as above. The hedge error is given by the difference between (A) and (B)+(D)

nK	3	10	10	20
dK	5%	5%	2.5%	2.5%
(A)	0.00187	0.00187	0.00188	0.00188
(B)+(D)	0.00229	0.00229	0.00198	0.00198
Hedge Error	0.00042	0.00042	0.00011	0.00010

Tabel 2

We note from Table 2 that the hedge errors are significantly small and thus conclude that

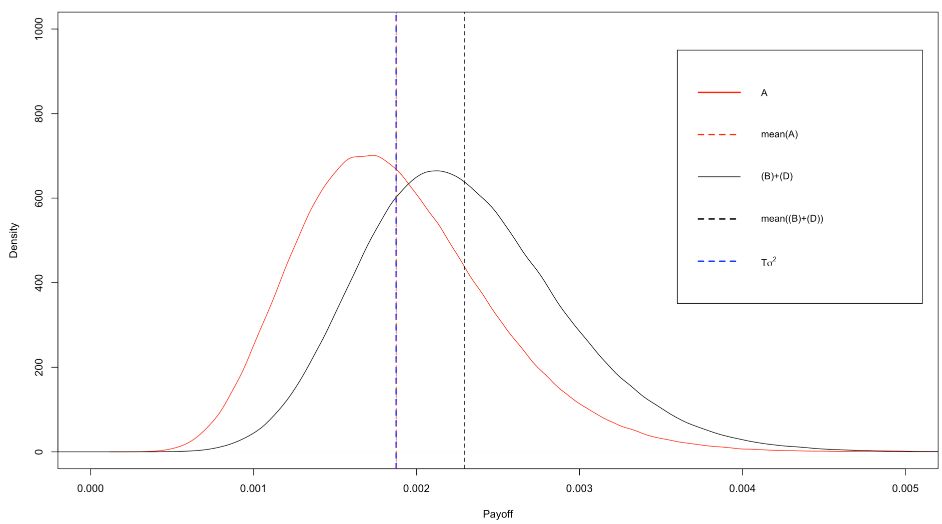
$$(A) \approx (B) + (D)$$

Further we note that the portfolio consisting of 20 calls and 20 puts and with a distance between the options of 2.5% of the initial spot gives the smallest hedge error. However the difference from the strategy where we instead hold 10 of each options is somewhat insignificant. Perhaps it is also more realistic to hold a portfolio with a strike range of 85 to 125, rather than 50 to 150.

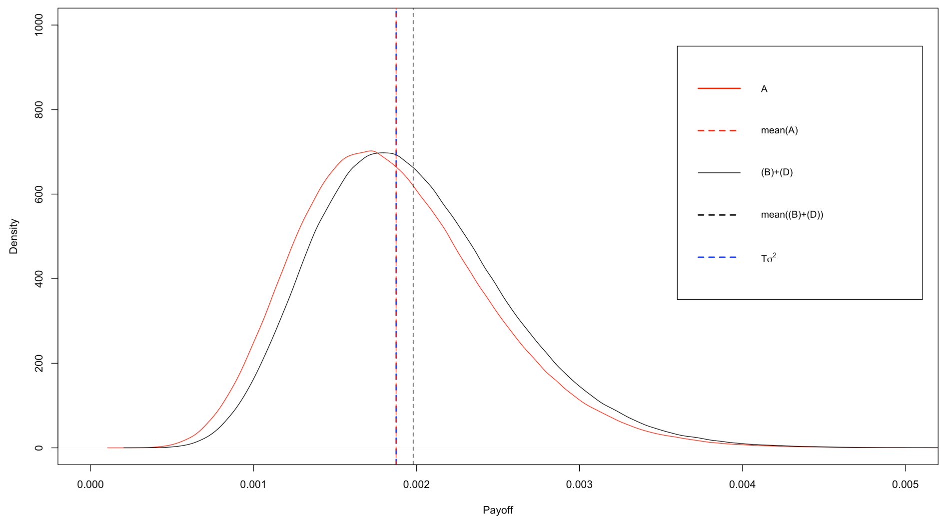
We know, as we have assumed constant volatility and the stock follows an Ito process, that theoretically it should hold that

$$(A) = \sum_{i=0}^{n-1} \left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}} \right)^2 \to T\sigma^2 \quad n \to \infty$$

We now examine whether (A) in fact converges to $T\sigma^2$ by examining density plots for (A) and (B)+(D).



Figur 2: Density plot with 10^6 sample paths using dK= 5% and 3 of each option.

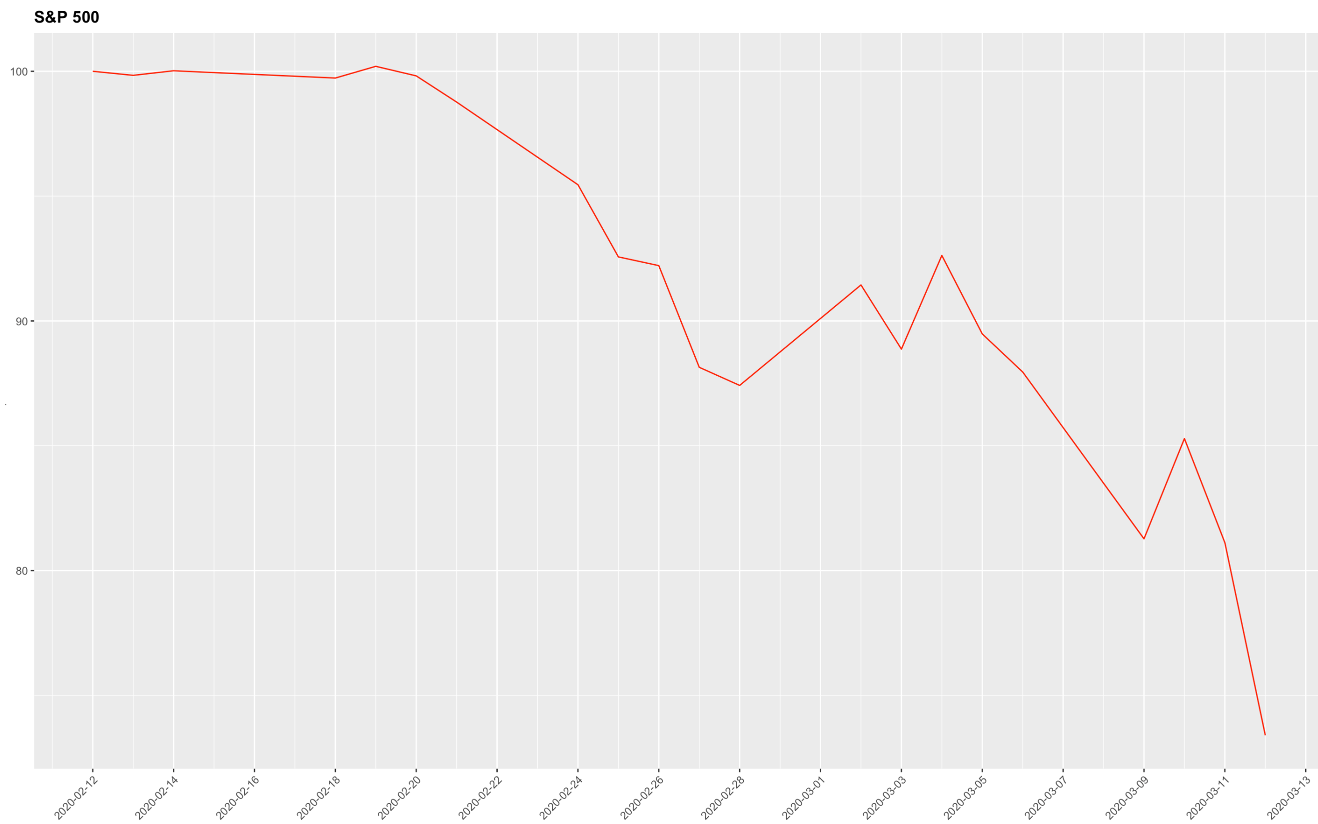


Figur 3: Density plot with 10^6 sample paths using dK= 2.5% and 20 of each option.

We note that in both figure 2 and 3 the mean of (A) seems to be completely on top of the assumed limit $T\sigma^2$, thus it seems that (A) converges to $T\sigma^2$. Further note that with a higher number of options and a lower distance between such, the density of (A) and of ((B)+(D)) is a lot closer to each other, which indicates a better approximation.

1.c

In this problem we wish to try above derived hedge on S&P500 data from mid-February to mid-March 2020.



Figur 4

We consider a month long swap with daily updating ($n = 20$). We again define the strike points where we want the payoff functions to match exactly. We write out the system of equations and solve for the weights as in 1. We calculate (A), (B) and (D) using the S&P500 data and the formulas defined in 1.b. The result is seen in Table 3.

nK dK	3 5%	10 5%	10 2.5%	20 2.5%
(A)	0.03001	0.030012	0.03001	0.030012
(B)+(D)	0.00276	0.032009	0.02983	0.031249
Hedge Error	0.02725	0.001996	0.00018	0.001236

Tabel 3

We notice a significant difference in the hedge error compared to the results from 1.b when the distance between strikes is 5% of the initial spot. We further note a significant difference when comparing ($nK = 10, dK = 2,5\%$) and ($nK = 3, dK = 5\%$) in Table 3. A surprise is that the hedge error of the strategy ($nK = 10, dK = 2,5\%$) seems to perform significantly better, hedge error wise, compared to the strategy where we have doubled the number of options. We note that the strikes surpass the spot during almost the whole period (notice that the red curve in Figure 4 is below $S_0 = 100$ almost everywhere), thus the put options in the hedging portfolio (D) will have a larger impact when using the weights from 1.b. When the portfolio consists of 20 of each options we thereby cover a larger range of strikes. Further note the hedge error when $(nK = 3, dK = 5\%)$ is almost as big as the value of the actual variance swap, (A). In general the hedge errors are larger than the ones from 1.b. In 1.b we assumed that the stock price followed an Ito process, i.e. continuity. The hedge errors are larger in 1.c as the stock price moves very discontinuously and jump-like downwards. The discontinuity and jump-like behaviour is clearly seen on Figure 4.

1.d

In this problem we wish to discuss similarities and differences if we had looked instead at simple variance swaps or normal variance swaps.

A variance swap is an agreement to exchange (2) for some fixed amount at time T. Using the notation from Ian Martin's paper:

$$\left(\log\left(\frac{S_\Delta}{S_0}\right)\right)^2 + \left(\log\left(\frac{S_{2\Delta}}{S_\Delta}\right)\right)^2 + ... + \left(\log\left(\frac{S_T}{S_{T-\Delta}}\right)\right)^2 \tag{2}$$

A simple variance swap is an agreement to exchange (3) for a prearranged strike at time T.

$$\left(\frac{S_\Delta - S_0}{F_{0,0}}\right)^2 + \left(\frac{S_{2\Delta} - S_\Delta}{F_{0,\Delta}}\right)^2 + ... + \left(\frac{S_T - S_{T-\Delta}}{F_{0,T-\Delta}}\right)^2 \tag{3}$$

Where $F_{0,t}$ is the forward price of the underlying asset to time t, which is known at time 0. In our case, where $r = 0$ and option prices are observable at all strikes, then the forward price will be S_0 at all timepoints.

A variance swap, as described by (2), depends on the assumption that prices follow diffusions, and hence cannot jump. This assumptions has been greatly broken in the financial crisis of 2008 and, as we saw in 1.c, in the Corona crisis, where prices did in fact jump. Especially individual stocks tend to jump more dramatically (than indices). As Ian Martin states a consequence of this is that *"the single-name variance swap market was affected particularly severely. It collapsed, and has not recovered."* As Ian Martin continues: a fundamental problem with (2) is if the underlying asset, fx an individual stock, goes bankrupt meaning S_{t_i} hits zero before expiry, then the payoff (2) is infinite. It is amongst others this consideration which led to the definition of the simple variance swap. The simple variance swap can be priced and hedged in the presence of jumps, which the variance swap cannot. The two payoffs are generally similar, but the variance swap payoff (2) is larger than the simple variance swap payoff (3) if fx the S&P500 index drops sharply over the trade horizon, and smaller if it rises sharply. The case of the S&P500 dropping sharply over the trade horizon is what we saw in 1.c.

Further we can compare how the variance swap and simple variance swap are hedged. Ian Martin proves that the hedge for the simple variance swap consists of a static and dynamic part, just as we showed in 1.b is the case for the variance swap. However the dynamic and static parts are constructed differently.

A normal variance swap is an agreement to exchange (4) for a prearranged strike at time T.

$$\left(\frac{S_\Delta - S_0}{\sqrt{T}}\right)^2 + \left(\frac{S_{2\Delta} - S_\Delta}{\sqrt{T}}\right)^2 + ... + \left(\frac{S_T - S_{T-\Delta}}{\sqrt{T}}\right)^2 \tag{4}$$

Thus in our set-up, where $r = 0$, we will have that the normal variance swap is equal to the simple variance swap just with a different, also constant, scaling parameter - $\sqrt{T}$ instead of S_0.

Question 2: Rough Bergomi

2.1

We first wish to compute an explicit expression for the forward variances, $\xi_t(u)$, under rough Bergomi. Let $0 \leq t \leq u$.

$$\xi_t(u) = \mathbb{E}_t^{\mathbb{Q}}[V_u] = \mathbb{E}_t^{\mathbb{Q}}\left[\xi_0(u)\exp\left(\eta\sqrt{2H}\int_0^u (u-s)^{H-\frac{1}{2}}dW_s - \frac{\eta^2}{2}u^{2H}\right)\right]$$

As $\xi_0(u)$ is $\mathcal{F}_t$ measurable we find

$$= \xi_0(u)\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(\eta\sqrt{2H}\int_0^u (u-s)^{H-\frac{1}{2}}dW_s - \frac{\eta^2}{2}u^{2H}\right)\right]$$

We split the integral

$$\begin{aligned}
&= \xi_0(u)\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(\underbrace{\eta\sqrt{2H}\int_0^t (u-s)^{H-\frac{1}{2}}dW_s}_{\text{Measurable wrt } \mathcal{F}_t} + \eta\sqrt{2H}\int_t^u (u-s)^{H-\frac{1}{2}}dW_s - \underbrace{\frac{\eta^2}{2}u^{2H}}_{\text{some constant}}\right)\right] \\
&= \xi_0(u)\exp\left(\eta\sqrt{2H}\int_0^t (u-s)^{H-\frac{1}{2}}dW_s - \frac{\eta^2}{2}u^{2H}\right)\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(\underbrace{\eta\sqrt{2H}\int_t^u (u-s)^{H-\frac{1}{2}}dW_s}_{:=M_t(u)}\right)\right]
\end{aligned}$$

We note that $M_t(u)$ is independent of $\mathcal{F}_t$. $M_t(u)$ is a Gaussian stochastic integral on the interval from t to u. The Brownian Motion has independent increments. Thus, the stochastic process which drives $M_t(u)$ will be independent of the stochastic processes that generate $\mathcal{F}_t$ by definition of the filtration.

$$= \xi_0(u)\exp\left(\eta\sqrt{2H}\int_0^t (u-s)^{H-\frac{1}{2}}dW_s - \frac{\eta^2}{2}u^{2H}\right)\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\eta\sqrt{2H}\int_t^u (u-s)^{H-\frac{1}{2}}dW_s\right)\right]$$

We now wish to use the fact that for some $Y \sim \mathcal{N}(\mu, \sigma^2)$ it holds that $\mathbb{E}[e^Y] = e^{\mu+\frac{1}{2}\sigma^2}$. We note by Björk Lemma 4.15, as $\eta\sqrt{2H}(u-s)^{H-\frac{1}{2}}$ is a deterministic function of time, that

$$M_t(u) = \int_t^u \eta\sqrt{2H}(u-s)^{H-\frac{1}{2}}dW_s \sim \mathcal{N}\left(0, \int_t^u \left(\eta\sqrt{2H}(u-s)^{H-\frac{1}{2}}\right)^2 ds\right) = \mathcal{N}\left(0, \eta^2(u-t)^{2H}\right)$$

Thus $\xi_t(u)$ becomes

$$\begin{aligned}\xi_t(u) &= \xi_0(u)\exp\left(\eta\sqrt{2H}\int_0^t (u-s)^{H-\frac{1}{2}}dW_s - \frac{\eta^2}{2}u^{2H}\right)\exp\left(\frac{1}{2}\eta^2(u-t)^{2H}\right)\\ &= \xi_0(u)\exp\left(\eta\sqrt{2H}\int_0^t (u-s)^{H-\frac{1}{2}}dW_s + \frac{\eta^2}{2}\left((u-t)^{2H}-u^{2H}\right)\right)\end{aligned}$$

Note that before we had that the instantaneous variance, V_u, were not an Ito process as the integrating variable and the upper integral limit depended on the same u. This is no longer the case when considering the instantaneous forward variance, $\xi_t(u)$, instead.

For a fixed $u \geq t$, we define the process Z_t by $Z_t = \eta\sqrt{2H}\int_0^t (u-s)^{H-\frac{1}{2}}dW_s$. As $g(s) = \eta\sqrt{2H}(u-s)^{H-\frac{1}{2}} \in \mathcal{L}^2$ and Z_t is a $\mathcal{F}_t$ martingale, by Björk Corollary 4.8 and Lemma 4.9 the dynamics of Z_t is given by

$$dZ_t = g(t)dW_t = \eta\sqrt{2H}(u-t)^{H-\frac{1}{2}}dW_t$$

Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(t, Z_t) = \xi_t(u)$.

$$f(t,z) = \xi_0(u)\exp\left(z + \frac{\eta^2}{2}(u-t)^{2H} - \frac{\eta^2}{2}u^{2H}\right)$$

This function is $C^{1,2}$ and we can therefore use Ito's formula to compute the dynamics of $\xi_t(u)$. We first note that

$$\begin{aligned}f_t :=& \frac{\partial f(t,z)}{\partial t} = (-1)\cdot\frac{\eta^2}{2}\left(2H(u-t)^{2H-1}\right)\xi_0(u)\exp\left(z + \frac{\eta^2}{2}(u-t)^{2H} - \frac{\eta^2}{2}u^{2H}\right)\\ &= -\eta^2 H(u-t)^{2H-1}f(t,z)\\ f_z :=& \frac{\partial f(t,z)}{\partial z} = 1\cdot\xi_0(u)\exp\left(z + \frac{\eta^2}{2}(u-t)^{2H} - \frac{\eta^2}{2}u^{2H}\right) = f(t,z)\\ f_{zz} :=& \frac{\partial^2 f(t,z)}{\partial z^2} = 1\cdot\xi_0(u)\exp\left(z + \frac{\eta^2}{2}(u-t)^{2H} - \frac{\eta^2}{2}u^{2H}\right) = f(t,z)\end{aligned}$$

Thus

$$\begin{aligned}d\xi_t(u) &= df(t,Z_t) = f_t dt + f_z dZ_t + \frac{1}{2}f_{zz}(dZ_t)^2\\ &= -\eta^2 H(u-t)^{2H-1}\xi_t(u)dt + \xi_t(u)\left(\eta\sqrt{2H}(u-t)^{H-\frac{1}{2}}dW_t\right) + \frac{1}{2}\xi_t(u)\left(\eta\sqrt{2H}(u-t)^{H-\frac{1}{2}}dW_t\right)^2\\ &= \xi_t(u)\eta\sqrt{2H}(u-t)^{H-\frac{1}{2}}dW_t + \xi_t(u)\eta^2 H(u-t)^{2H-1}dt - \eta^2 H(u-t)^{2H-1}\xi_t(u)dt\\ &= \xi_t(u)\eta\sqrt{2H}(u-t)^{H-\frac{1}{2}}dW_t\end{aligned}$$

We note that as t moves up towards the fixed u then $\xi_t(u)$ is a Markov $\mathbb{Q}$ martingale. Each of the forward variances, $\xi_t(u)$, for each value of $u \geq t$ is itself a martingale. We further recognize a Geometric Brownian Motion with deterministic and time-dependent diffusion term $\sigma(t) = \eta\sqrt{2H}(u-t)^{H-\frac{1}{2}}$. By exercise 1.3 we know how to find the price of a call option on such. In order to compute the Black-Scholes price, we need to know the mean and variance. As $\xi_T(u)$ is a $\mathbb{Q}$-martingale we know that $\mathbb{E}_t^{\mathbb{Q}}[\xi_T(u)] = \xi_t(u)$ and further as $\xi_T(u)$ is log normally distributed it can be written as the exponential of some Gaussian random variable X. Define the random variable $X = \int_t^T \eta\sqrt{2H}(u-s)^{H-\frac{1}{2}}dW_s$. By lemma 4.15 we have that

$$X \sim \mathcal{N}\left(0, \int_t^T \eta^2 2H(u-s)^{2H-1}ds\right) = \mathcal{N}\left(0, \eta^2\left((u-t)^{2H} - (u-T)^{2H}\right)\right)$$

Thus, as the instantaneous forward variance is a time-inhomogeneous GBM (and thereby Markovian) in its own filtration, the time-t price of a call option expiring at time T, $0 < T \leq u$, on the instantaneous forward variance is given by

$$\mathbb{E}_t^{\mathbb{Q}}\left[(\xi_T(u) - K)^+\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\left(\xi_t(u)e^{X-\frac{1}{2}Var(X)} - K\right)^+\right] \overset{B.S.}{=} \xi_t(u)\Phi\left(d_1(t,\xi_t(u))\right) - K\Phi\left(d_2(t,\xi_t(u))\right)$$

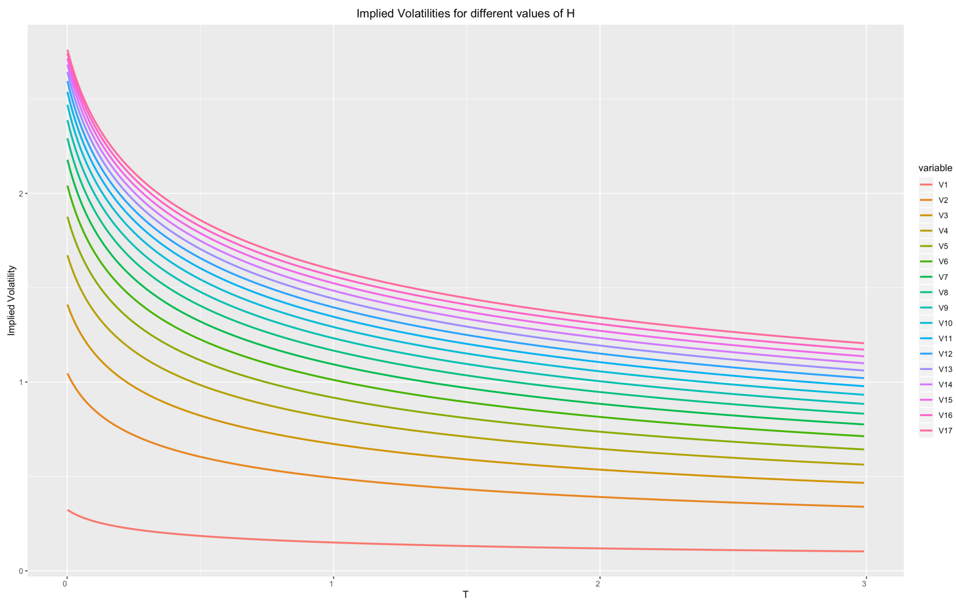
Where

$$d_{1/2}(t,\xi_t(u)) = \frac{\log\left(\frac{\xi_t(u)}{K}\right) \pm \frac{\eta^2}{2}\left((u-t)^{2H} - (u-T)^{2H}\right)}{\sqrt{\eta^2\left((u-t)^{2H} - (u-T)^{2H}\right)}}$$

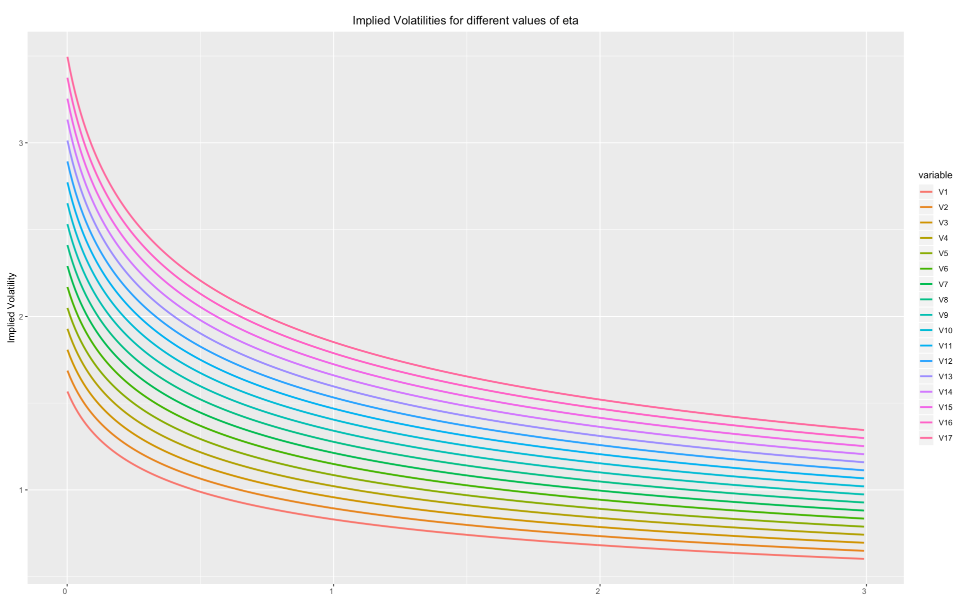
The fact that pricing reduces to the Black-Scholes formula is a very nice, and perhaps surprising, result as we can note from the definition of V_u, that the volatility is non-Markovian. If we consider the process on $u \geq t$ then $(V_u)_{u\geq t}$ depends on the entire forward variance curve which is infinite-dimensional, meaning it depends on more than just the spot value, thus the system (S_t, V_t) is <u>not</u> Markov. However if we instead consider the system $(S_t, \xi_t(u))$ it is indeed Markov. This is very convenient as it allows us to recover our usual martingale, Monte Carlo and pricing techniques.

2.2

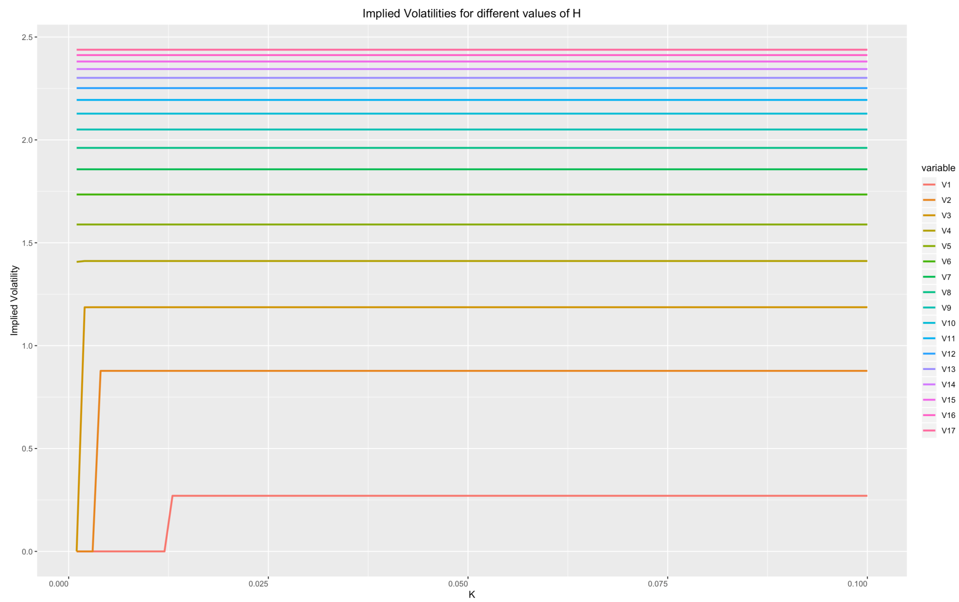
In this problem we wish to compute Black-Scholes implied volatilities for different strikes K and expiries T for options on $\xi_T(T+\Delta)$ as seen from time zero. We use $(\xi_t(T+\Delta))_{t\in[0,T]}$ as the underlying asset when computing implied volatilities. We compute for different values of H and η to examine what effect they have.



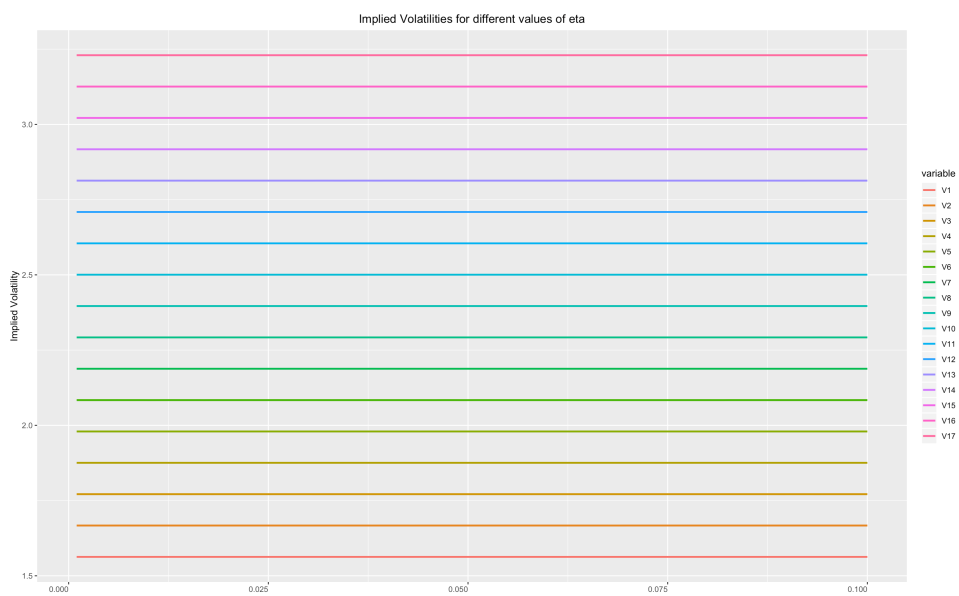
Figur 5: The functions V1 to V17 indicates that different values of H has been used to compute the implied volatilities. H is defined to be a sequence from 0.001 (V1) up to $\frac{1}{6}$ (V17) changing by 0.01 for each function going upwards.



Figur 6: The functions V1 to V17 indicates that different values of η has been used to compute the implied volatilities. η is defined to be a sequence from 1.3 (V1) up to 2.9 (V17) changing by 0.1 for each function going upwards.



Figur 7: The functions V1 to V17 indicates that different values of H has been used to compute the implied volatilities. H is defined to be a sequence from 0.001 (V1) up to $\frac{1}{6}$ (V17) changing by 0.01 for each function going upwards.



Figur 8: The functions V1 to V17 indicates that different values of η has been used to compute the implied volatilities. η is defined to be a sequence from 1.3 (V1) up to 2.9 (V17) changing by 0.1 for each function going upwards.

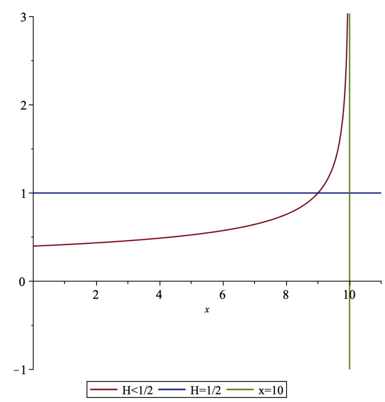
The η parameter is multiplied onto the stochastic part of $\xi_t(u)$ and is a 'volatility of volatility' parameter. As seen on Figure 6 and 8, when η increases so does the implied volatility. η has an effect on the level of the implied volatility curves. We note that η does not changes the shape of the ATM skew, which was expected.

H is both a scaling factor, as it is multiplied onto the stochastic part, but more importantly it is also a rate of mean reversion. On Figure 5 and 7 we note that as H increases so does the implied volatility. Further note from Figure 5 that H controls the decay of the ATM skew for short time-to-maturity. This was expected based on the expression we considered for the rough Bergomi term structure in lectures. Here we saw that the skew(T) amongst others depended on $T^{H-\frac{1}{2}}$.[1]

That H is a rate of mean reversion stems from the fact that H controls how the deterministic function looks, i.e. how much weight is put on the Brownian increments. Consider $H \in (0, \frac{1}{2})$ and

$$Y_{t_i} = \int_0^{t_i} (t_i - s)^{H-\frac{1}{2}} dW_s \approx \sum_{j=0}^{i-1} (t_i - t_j)^{H-\frac{1}{2}} \Delta W_{t_{j+1}}$$

We think of $(t_i - s)^{H-\frac{1}{2}}$ as a weighing function of the Brownian increments.

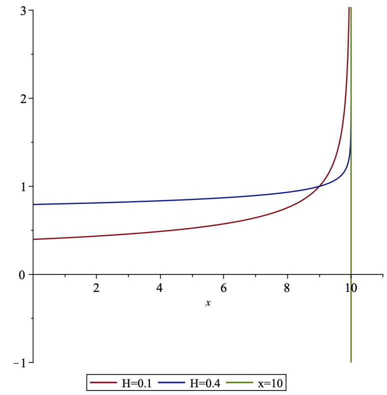


Figur 9: Plot of the weighing function. The values $H = 0.1$, $H = 0.5$ and $t_i = 10$ has been used.

What we note from the shape of the red function in Figure 9 is that a new increment in the Brownian vector has a large impact on the process as it happens and then the impact of the one Brownian increment decays quickly, as the weighting function decreases quickly. Thus a low value of H is what creates the rough 'jump-like'/explosive behavior of the stock sample paths. When H increases the function gets more flat, i.e. less singular[2]. When $H = 0.5$ the effect of the weighing function is completely gone and we are in our standard Brownian Motion case. The change in the weighing function when H increases is seen on Figure 10.

[1]Slideset 'rough$_v olatility_5 - 1.pdf' slide 15$.

[2]Less singular in the sense that it becomes more 'like a constant', i.e. more well-behaved such that the explosive effect vanishes.



Figur 10: Plot of the weighing function. The values $H = 0.1$, $H = 0.4$ and $t_i = 10$ has been used.

We notice in figure 7 and 8 that the implied volatility plotted as a function of strike is flat. We know that if the underlying (fx a stock) follows a log normal distribution with standard deviation σ, then the implied volatility of options priced using this model will be constantly σ. This is exactly what we see in these figures. The VIX is defined as

$$VIX_t^2 = \int_t^{t+\Delta} \xi_t(u)du, \quad t \geq 0, \quad \Delta = \frac{1}{12} \text{ (1 month)} \tag{5}$$

i.e. it is defined as an integral of something log normally distributed over a period of 1 month. This is considered a small period so the process $(\xi_t(u))_{t \leq u \leq t+\Delta}$ consists of very correlated log normal variables. Thus the VIX, as it is defined in (5), is close to being log normally distributed and therefore will have a flat implied volatility smile. If we wish to try a fully rigorous mathematical argument it will be difficult, as one can note in (5) - if we try to evaluate this integral we will have to compute $\xi_t(t)$, which is not an Ito process.

When computing the VIX implied volatility using market data the smile is far from flat, as seen in Figure 1 in the given assignment.

2.3

In this problem we consider the skewed rough Bergomi model

$$\begin{aligned} V_t &= \xi_0(t)\left(\theta X_{1,t} + (1-\theta)X_{2,t}\right) \\ X_{1,t} &= \exp\left(\eta\sqrt{2\alpha+1}\int_0^t (t-s)^\alpha dW_s - \frac{\eta^2}{2}t^{2\alpha+1}\right) \\ X_{2,t} &= \exp\left(\nu\sqrt{2\beta+1}\int_0^t (t-s)^\beta dW_s - \frac{\nu^2}{2}t^{2\beta+1}\right) \end{aligned}$$

We wish to compute implied volatilities for options on $\xi_T(T+\Delta)$ and demonstrate that this model can generate a positive skew.

The instantaneous forward variance is given by, for fixed u and $u \geq t \geq 0$,

$$\begin{aligned}
\xi_t(u) &= \mathbb{E}_t^{\mathbb{Q}}[V_u] = \mathbb{E}_t^{\mathbb{Q}}\left[\xi_0(u)\left(\theta X_{1,u} + (1-\theta)X_{2,u}\right)\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[\xi_0(u)\theta X_{1,u}\right] + \mathbb{E}_t^{\mathbb{Q}}\left[\xi_0(u)(1-\theta)X_{2,u}\right] \\
&= \xi_0(u)\theta \underbrace{E_t^{\mathbb{Q}}\left[\exp\left(\eta\sqrt{2\alpha+1}\int_0^u (u-s)^\alpha dW_s - \frac{\eta^2}{2}u^{2\alpha+1}\right)\right]}_{:=A} \\
&\quad + \xi_0(u)(1-\theta)\underbrace{\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(\nu\sqrt{2\beta+1}\int_0^u (u-s)^\beta dW_s - \frac{\nu^2}{2}u^{2\beta+1}\right)\right]}_{:=B}
\end{aligned}$$

We can now recognize that A and B looks very familiar to both each other and to what we worked with in problem 2.1. Recognize that for $2\alpha+1 = 2H$ it is exactly the same as in 2.1. We can therefore use the same course of action to arrive at

$$\begin{aligned}
A &= \exp\left(\eta\sqrt{2\alpha+1}\int_0^t (u-s)^\alpha dW_s + \frac{\eta^2}{2}\left((u-t)^{2\alpha+1} - u^{2\alpha+1}\right)\right) \\
B &= \exp\left(\nu\sqrt{2\beta+1}\int_0^t (u-s)^\beta dW_s + \frac{\nu^2}{2}\left((u-t)^{2\beta+1} - u^{2\beta+1}\right)\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\xi_t(u) = \xi_0(u)\Bigg[&\theta\exp\left(\eta\sqrt{2\alpha+1}\int_0^t (u-s)^\alpha dW_s + \frac{\eta^2}{2}\left((u-t)^{2\alpha+1} - u^{2\alpha+1}\right)\right) \\
&+ (1-\theta)exp\left(\nu\sqrt{2\beta+1}\int_0^t (u-s)^\beta dW_s + \frac{\nu^2}{2}\left((u-t)^{2\beta+1} - u^{2\beta+1}\right)\right)\Bigg]
\end{aligned}$$

Further note that, by lemma 4.15,

$$\begin{aligned}
Var[\log(A)] &= Var\left[\int_0^t \eta\sqrt{2\alpha+1}(u-s)^\alpha dW_s\right] = \int_0^t \eta^2(2\alpha+1)(u-s)^{2\alpha}ds = \eta^2\left(u^{2\alpha+1} - (u-t)^{2\alpha+1}\right) \\
Var[\log(B)] &= Var\left[\int_0^t \nu\sqrt{2\beta+1}(u-s)^\beta dW_s\right] = \int_0^t \nu^2(2\beta+1)(u-s)^{2\beta}ds = \nu^2\left(u^{2\beta+1} - (u-t)^{2\beta+1}\right)
\end{aligned}$$

Thus the variables A and B are log normally distributed with variance as given above and mean given by, as $g(s) = (u-s)^\alpha$ is $\mathcal{F}_t$ measurable for fixed u and $\int_0^t \mathbb{E}[(u-s)^{2\alpha}]\, ds < \infty$,

$$\mathbb{E}[\log(A)] = \mathbb{E}\left[\eta\sqrt{2\alpha+1}\underbrace{\int_0^t (u-s)^\alpha dW_s}_{=0 \text{ by Björk 4.4}} + \frac{\eta^2}{2}\left((u-t)^{2\alpha+1} - u^{2\alpha+1}\right)\right] = \frac{\eta^2}{2}\left((u-t)^{2\alpha+1} - u^{2\alpha+1}\right)$$

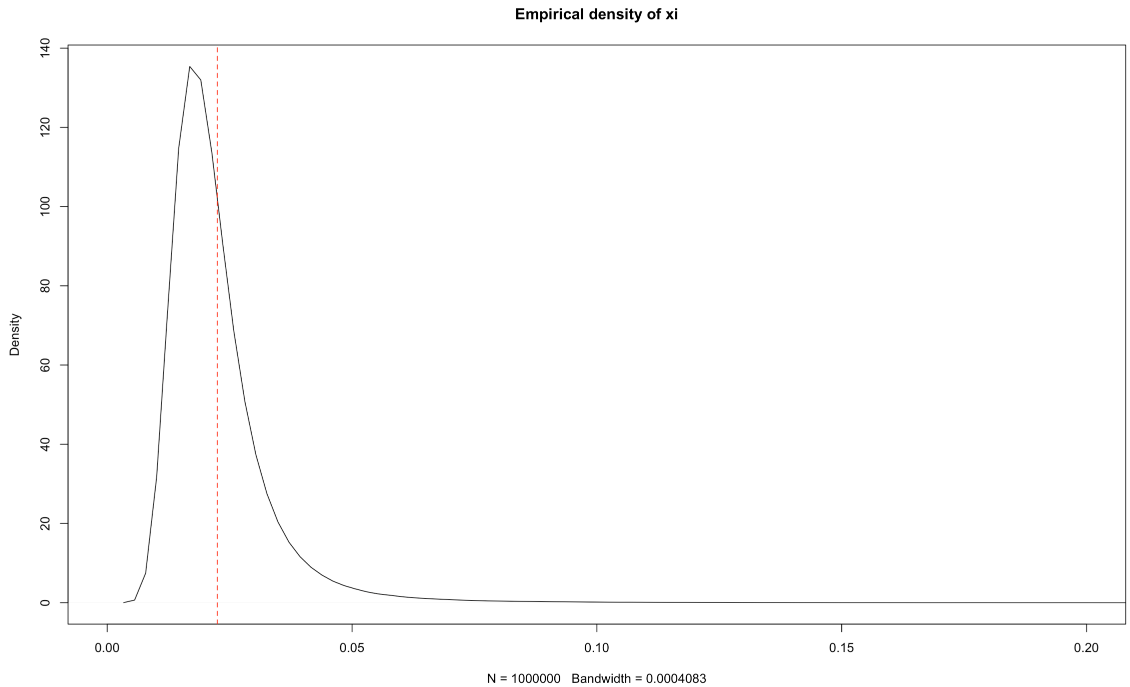
$$\mathbb{E}[\log(B)] = \mathbb{E}\left[\nu\sqrt{2\beta+1}\underbrace{\int_0^t (u-s)^\beta dWs}_{=0 \text{ by Björk 4.4}} + \frac{\nu^2}{2}\left((u-t)^{2\beta+1} - u^{2\beta+1}\right)\right] = \frac{\nu^2}{2}\left((u-t)^{2\beta+1} - u^{2\beta+1}\right)$$

We will use Monte Carlo to estimate the call prices. With the above derivations we are capable of doing so. $\xi_T(T+\Delta)$ can be simulated by sampling from a standard normal distribution and using the moments derived above. Using moment-matching we can sample $\xi_T(T+\Delta)$ without distributional error. We use the parameters given in the assignment and $n = 10^6$ sample paths. We sample n paths of $Z \sim \mathcal{N}(0,1)$ and then do the following for expiry of 1 month, 2 months and 3 months

- Set $A = \exp\left(\sqrt{Var[\log(A)]}\cdot Z + \mathbb{E}[\log(A)]\right)$ and $B = \exp\left(\sqrt{Var[\log(B)]}\cdot Z + \mathbb{E}[\log(B)]\right)$.
- Set
  $\xi_T(T+\Delta) = \xi_0(\theta A + (1-\theta)B))$.
- For each strike value K we calculate the Monte Carlo estimator of the call price, where $\xi[i]$ denotes the i'th element of the n dimensional vector, $\xi_T(T+\Delta)$,
  $\hat{\pi} = \frac{1}{n}\sum_{i=1}^n (\xi[i] - K)^+$.

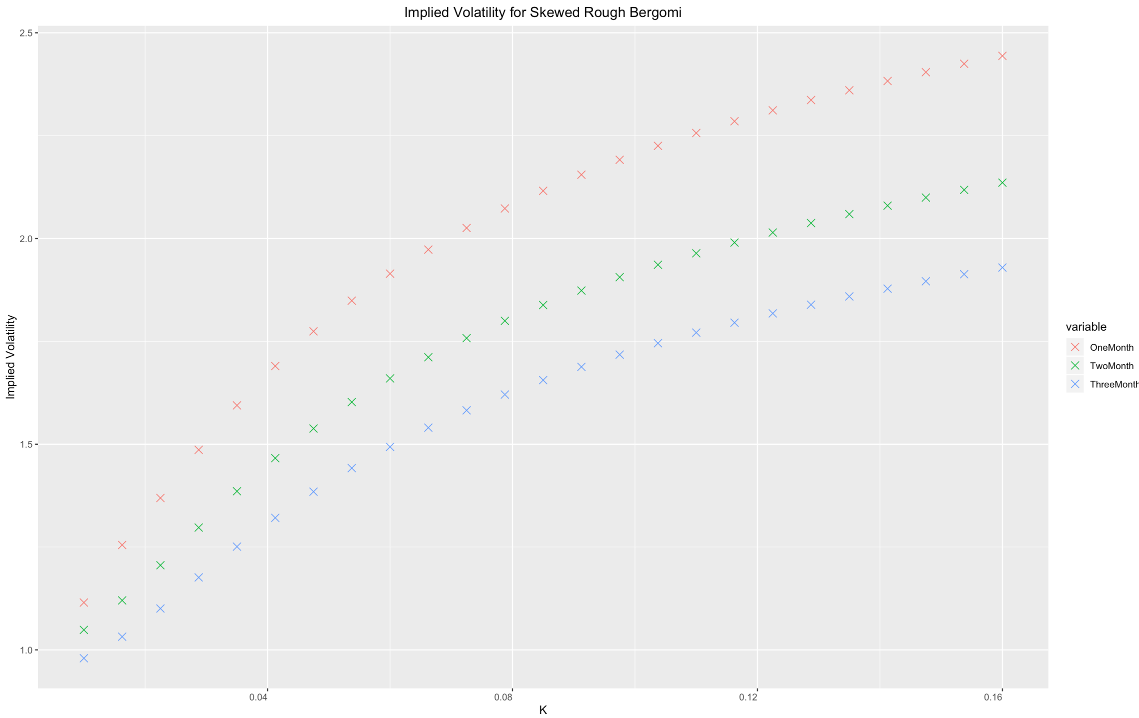
As we have $\alpha = \beta$ one could be tempted to decrease computational time by sampling one variable and multiplying with η^2 and ν^2 respectively afterwards. But what if $\alpha \neq \beta$. In this case it will be necessary to sample A and B separately, as we have done. When doing so it is important to have in mind that it should be the same realization of the Brownian Motion which drives A and B. Hence $\alpha \neq \beta$ is not an issue as long as we are careful.

To chose reasonable strike values we examine the empirical distribution of $\xi_T(T+\Delta)$.



Figur 11: The black line denotes the empirical density of $\xi_T(T+\Delta)$ and the red dotted line denotes the empirical mean of $\xi_T(T+\Delta)$.

We note at Figure 11 that the empirical mean is to the right of the peak of observations. We chose a strike range of $(0.1)^2$ to $(0.4)^2$.



Figur 12: Black Scholes implied volatility for options on instantanous forward variance computed from skewed rBergomi for different strikes K and expiries T. The strikes range from $(0.1)^2$ to $(0.4)^2$. The parameters used are $\alpha = \beta = -0.4$, $\xi_0(t) = (0.15)^2$, $\theta = 0.1$, $\eta = 5$, $\nu = 1$ and $n = 10^6$ sample paths.

We find that skewing the rough Bergomi model allows us to generate a positive skew, which fits market data better.

Appendix - code

Problem 1.b

```
S0<-100
dK<-0.05*S0
nK<-3
Kj<-S0+dK*c(((-(nK-1)):(nK-1)))
Ki<-S0+dK*c((-nK):(-1),1:nK)
C_fun <- function(x){ 2*log(S0/x)}
A<-matrix(0, nrow=2*nK, ncol=(2*nK))
for(i in 1:nK){
  A[i,1:i] <- dK*i:1
  A[2*nK-(i-1), (2*nK-(i-1)):(2*nK)] <- rev(A[i, 1:i])
}
w<-solve(t(A), C_fun(Ki))
strikes<-50:150
D <- C(strikes)
for (i in 1:length(strikes)){
  D[i] <- sum(w[1:nK]*pmax(Kj[1:nK]-strikes[i],0))+
    sum(w[(nK+1):(nK*2)]*pmax(strikes[i]-Kj[nK:(2*nK-1)],0))
}

mtext("Variance swap hedging: matching the log-contract", outer = TRUE, cex = 1.5)
par(mfrow=c(2,2), oma=c(0,0,2,0))

#For n=3 dK=5\% og n=10 dK=5\%
plot(strikes, C(strikes), type='l',
     xlab="S(T)", ylab="Payoff at T")
points(Ki, C(Ki), col='blue')
points(strikes, D, col='blue', type = 'l')
text(60, -0.3, "Black: Log-contract",col='black', lty=1, cex=0.9)
text(60, -0.5, "Blue: Option portfolio",col='blue', lty=1, cex=0.9)
text(149,1.3,paste("S0 =",S0),adj=1, cex=0.9)
text(149,1.1,paste("dK =",dK,"\%"),adj=1, cex=0.9)
text(147,0.9,paste(" #calls =",nK),adj=1, cex=0.9)
```

```
text(147,0.7,paste(" #puts =",nK),adj=1, cex=0.9)

#For n=20,dK=2,5\% og n=10 dK=2,5\%
plot(strikes, C(strikes), type='l',
     xlab="S(T)", ylab="Payoff at T")
points(Ki, C(Ki), col='blue')
points(strikes, D, col='blue', type = 'l')
text(60, -0.3, "Black: Log-contract",col='black', lty=1, cex=0.9)
text(60, -0.5, "Blue: Option portfolio",col='blue', lty=1, cex=0.9)
text(148,1.3,paste("S0 =",S0),adj=1, cex=0.9)
text(150,1.1,paste("dK =",dK,"\%"),adj=1, cex=0.9)
text(147,0.9,paste(" #calls =",nK),adj=1, cex=0.9)
text(147,0.7,paste(" #puts =",nK),adj=1, cex=0.9)

## Hedge error tables
capT<-1/12
n<-20
dt<-capT/n; dt05<-sqrt(dt)
time<-seq(0, capT, by=dt)
S <- rep(S0, (n+1))
sigma <- 0.15

npath <- 10^6
A <- rep(NA, npath)
B <- rep(NA, npath)
C <- rep(NA, npath)
D <- rep(NA, npath)
BD <- rep(NA, npath)
BC <- rep(NA, npath)
HedgeError <- rep(NA, npath)

for (i in 1:npath){
  W <- c(0, cumsum(rnorm(n, 0, dt05)))
  S <- S0 * exp(-(1/2)*sigma^2*time+sigma*W)
  A[i] <- sum((diff(S,1)/S[1:n])^2)
  B[i] <- 2*sum(diff(S,1)/S[1:n])
```

```
  C[i] <- 2*log(S[1]/S[n+1])
  D[i] <- sum(w[1:nK]*pmax(Kj[1:nK]-S[n+1],0))+sum(w[(nK+1):(nK*2)]*pmax(S[n+1]-Kj[nK:(2*nK-1)],0))
  BD[i] <- B[i]+D[i]
  BC[i] <- B[i]+C[i]
  HedgeError[i] <- A[i]-(B[i]+D[i])
}

## Density plots
plot(density(A), col='red', xlim=c(0,0.005), ylim=c(0,1000), main="", xlab="Payoff")
lines(density(BD))
abline(v=sigma^2*capT, col='blue', lwd=2, lty=2)
abline(v=mean(A), col='red', lty=2)
abline(v=mean(BD), lty=2)
legend (x = 0.0036, y = 950,
        legend = c('A', 'mean(A)', '(B)+(D)', 'mean((B)+(D))', expression(T*sigma^2)),
        col=c('red', 'red', 'black', 'black', 'blue'),
        lty=c(1,2,1,2,2), lwd=c(2,2,1), cex=0.9)
```

Problem 1.c

```
SP500 <- read.csv("SP500.csv", header=FALSE, sep=";")
colnames(SP500) <- c("Date", "Spot")
SP500$Date <- as.factor(SP500$Date)
SP500$Date <- as.Date(SP500$Date,format = "\%Y\%m\%d")

ggplot(SP500, aes(Date, Spot)) +
  geom_line(col='red') +
  scale_x_date(date_breaks = "2 days") +
  theme(axis.text.x=element_text(angle=60, hjust=1)) + ggtitle("S&P 500") +
  theme(plot.title = element_text(lineheight=0.8, face="bold"),
        ,axis.text.x = element_text(angle = 45, hjust = 1,vjust=1))

n <- 20
S <- SP500$Spot

dK<-0.025*S[1]
```

```
nK<-20
Kj<-S[1]+dK*c(((-(nK-1)):(nK-1)))
Ki<-S[1]+dK*c((-nK):(-1),1:nK)

C_func <- function(x){ 2*log(S[1]/x)}
AA <- matrix(0, nrow=2*nK, ncol=(2*nK))
for(i in 1:nK){
  AA[i,1:i] <- dK*i:1
  AA[2*nK-(i-1), (2*nK-(i-1)):(2*nK)] <- rev(AA[i, 1:i])}
w<-solve(t(AA), C_func(Ki))
D <- sum(w[1:nK]*pmax(Kj[1:nK]-S[n+1],0))+sum(w[(nK+1):(nK*2)]*pmax(S[n+1]-Kj[nK:(2*nK-1)],0))
A <- sum((diff(S,1)/S[1:n])^2)
B <- 2*sum(diff(S,1)/S[1:n])
HedgeError <- A-(B+D)

print(A)
print(B+D)
print(HedgeError)
```

Problem 2.2

```
source("BlackScholesFormula.R")
Call_bergomi <- function(xi,CapT,eta,H,u,K){
  var <- eta^2*(u^(2*H)-(u-CapT)^(2*H))
  d1 <- (log(xi/K)+(1/2)*var)/sqrt(var)
  d2 <- (log(xi/K)-(1/2)*var)/sqrt(var)
  return(xi*pnorm(d1)-K*pnorm(d2))
}
Call_bergomi <- Vectorize(Call_bergomi)

xi0 <- 0.15^2
CapT <- 1/12 #seq(0.001,3,by=0.01)
K <-  seq(0.001,0.1,0.001) #xi0
u <- CapT + 1/12
H <- seq(0.001,1/6,0.01) #0.1
eta <- 2.1 #seq(1.3,2.9,by=0.1)
```

```
# Change to length(eta) or length(CapT) if these are the seq()
imp_vols <- matrix(ncol=length(H), nrow = length(K))

for (i in 1:length(H)){
  imp_vols[,i] <- BlackScholesImpVol(Call_bergomi(xi0,CapT,eta,H[i],u,K),xi0,CapT,K,r=0, q=0)
}

library(ggplot2)
library(reshape2)

df <- as.data.frame(imp_vols)
df$time <- K
df_melt <- melt(df, id = 'time',  measure.vars = c("V1", "V2", "V3","V4","V5","V6","V7","V8","V9",
    "V10","V11","V12","V13","V14","V15","V16","V17"))

ggplot(data=df_melt, aes(x=time, y=value, color=variable)) +
  geom_line(size=1) +
  xlab("K") +
  ylab("Implied Volatility") +
  ggtitle("Implied Volatilities for different values of H")+
  theme(plot.title = element_text(hjust = 0.5)) +
  theme(plot.title = element_text(lineheight=0.8) # face="bold"
        ,axis.text.x = element_text(angle = 0, hjust = 1,vjust=1))
```

Problem 2.3

```
source("BlackScholesFormula.R")
library(ggplot2)
library(reshape2)

nPaths <- 10^6
Z <- rnorm(nPaths, mean = 0, sd = 1)
#------
## Do the following for CapT = 1/12, 2/12, 3/12
xi_0 <- 0.15^2
```

```
CapT <- 3/12
u <- CapT + 1/12
H <- 0.1
alpha <- -0.4
beta <- -0.4
theta <- 0.1
eta <- 5
nu <- 1
Var_A <- eta^2*(u^(2*alpha+1)-(u-CapT)^(2*alpha+1))
Var_B <- nu^2*(u^(2*beta+1)-(u-CapT)^(2*beta+1))

xi_t <- rep(xi_0, nPaths)
MC_Price <- rep(NA, length(K))

A <- exp(sqrt(Var_A)*Z-(1/2)*Var_A)
B <- exp(sqrt(Var_B)*Z-(1/2)*Var_B)
xi_t <- xi_0*(theta*A+(1-theta)*B)

plot(density(xi_t), col='black',xlim=c(0,0.2), main="Empirical density of xi")
abline(v=mean(xi_t), col='red', lty=2)

K <- seq((0.1)^2,(0.4)^2,length.out = 25)

for (i in 1:length(K)){
MC_Price[i] <- mean(pmax(xi_t-K[i],0))
}

imp_vols_3 <- BlackScholesImpVol(MC_Price,xi_0,CapT,K,r=0)

df <- data.frame(time=K, OneMonth = imp_vols_1, TwoMonth = imp_vols_2, ThreeMonth = imp_vols_3)
df_melt <- melt(df, id = "time", measure.vars = c("OneMonth", "TwoMonth","ThreeMonth"))

ggplot(data=df_melt, aes(x=time, y=value, color=variable)) +
  geom_point(shape=4, size = 3,show.legend = TRUE) +
  xlab("K") +
  ylab("Implied Volatility") +
```

```
theme(plot.title = element_text(hjust = 0.5)) +
ggtitle("Implied Volatility for Skewed Rough Bergomi")+
theme(plot.title = element_text(lineheight=0.8) # face="bold"
      ,axis.text.x = element_text(angle = 0, hjust = 1,vjust=1))
```