A Simple Iterative Method for the Valuation of American Options*

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Abstract

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Abstract

We introduce a simple iterative method to determine the optimal exercise boundary for American options, allowing us to compute the values of American options and their Greeks quickly as well as accurately. Our method is to determine the whole optimal exercise boundary of American options iteratively, and is computationally superior to the conventional ones that compute the optimal exercise boundary time-recursively. Our analysis indicates that the iteration method is computationally more efficient that the methods currently available.

Key Words: option pricing, American option, exercise boundary, numerical approach, derivative

JEL Classification: C63, G13

I Introduction

Since the seminal papers of Black and Scholes (1973) and Merton (1973), it is well-known that European options have closed-form valuation formulas, while the majority of American-style options are not valued in closed form. This is mainly due to the fact that possibility of early exercise of American options leads to complications for analytic calculation. Therefore, financial theorists and researchers have given much attention to developing either analytical approximation formulas such as Parkinson (1977), Johnson (1983), Geske and Johnson (1984), MacMillan (1986), Barone-Adesi and Whaley (1987), Omberg (1987), Bjerksund and Stensland (1993), Broadie and Detemple (1996), Ju (1998), or numerical methods such as Brennan and Schwartz (1976, 1978), Boyle (1977), Hull and White (1990), and Longstaff and Schwartz (2001).

McKean (1965) and Van Moerbeke (1976) lay foundation for American option valuation and they think of it as a free boundary problem looking for a boundary changing in time to maturity (generally it is called a *optimal exercise boundary*). Following their works, there have been various attempts to value American options more accurately and quickly. Historical reviews of such attempts are found in Karatzas and Shreve (1998) or Barone-Adesi (2005).

As pointed out by Barone-Adesi (2005), there are some remarkable works which give new impetus to the quest for the valuation of American-style options. Kim (1990), Jacka (1991), Jamshidian (1990), and Carr *et al.* (1992) provide us with a valuation formula for American options in integral form as a function of the optimal exercise boundary, postulating that the underlying asset prices follow a lognormal diffusion process. Moreover, Kim (1990) finds an implicit form of integral equation for the optimal exercise boundary. He shows that, once the optimal exercise boundary is determined, it is simple to compute the value of American options. Therefore, the important task of valuing American options is how to determine the optimal exercise boundary efficiently. Kim also suggests a numerical method solving as many integral equations as the number of discretized time intervals.

We focus on solving the integral equation which suggested by Kim (1990). More specifically, we develop an simple iteration method to solve it. We show that a modification of the integral equation yields a numerical functional form for the iteration method, and develop a iteration method to calculate the optimal exercise boundary as a fixed point of the functional. On the other hand, the method suggested by Kim (1990) is to repeat a numerical method solving the implicit integral equation at each point in time recursively. Self-evidently, his method gets relatively huge computation time as well as cumulative numerical errors hard to control, as mentioned by Barone-Adesi (2005). Kim (1990) also confesses that his method needs an improvement of computational efficiency.

Most numerical methods for valuing American options, including the binomial method by Cox *et al.* (1979) and the finite difference method (FDM) by Brennan and Schwartz (1976), are time-recursive, namely, their basic idea is to discretize the lifetime of an option and to evaluate the optimal exercise boundary as well as its value backwardly in time. Since the time-recursive methods execute calculations repeatedly for every (discretized) time step, they require large computing time not to mention cumulative (thus relatively large) pricing errors, for long-lived options in particular.

One may think that large computing time can be reduced if we use the approximation methods. However, the methods by Geske and Johnson (1984) and Huang *et al.* (1996) are unlikely to overcome the limitation of time-recursive methods. They must be used repeatedly whenever we calculate the value of American options with different time to expiration even though all other conditions remain the same. Inevitably, it takes much computing time for valuing a bunch of such options. On the other hand, the approximation methods by Barone-Adesi and Whaley (1987) and Bjerksund and Stensland (1993) provide us with relatively small computing time but low accuracy. The randomization method by Carr (1998) can give us a relatively fast algorithm for valuing an American put option without sacrificing accuracy. However, it seems that it is not easy to reduce computing error, especially in calculating sensitivity coefficients (Greeks), since it calls for an extrapolation method to obtain a limit point of a series of approximate option values. The results by Sullivan (2000), who developed a Gaussian quadrature method approximating the value of an American option given by a time-recursive function of its payoff, talks some drawbacks of Carr's method. At last, Ju (1998) proposes the method to calculate the value of American option by approximating its early exercise boundary with connecting pieces of exponential functions. It is necessary to figure out several coefficients whenever American options with different time to maturity considered, even though it provide us with relatively accurate results.

Recently, Zhu (2006) find an exact and explicit solution of the Black-Scholes equation for the valuation of American put options. He presented the solution as a Taylor's series expansion with infinitely many terms, where each term contains three single integrals and two double integrals. His work is an excellent outcome in option pricing theory, however, it seems difficult to implement his solution numerically. The infinite sum with two double integrations is likely to produce much computing errors, so valuing American options by Zhu's solution doesn't seem to be superior to Carr's approximation method.

The recent financial crisis makes people realize that it might be very important for institutional market participants such as investment banks and hedge funds to calculate the values of lots of financial products containing lots of American options instantly and accurately in order to hedge them and manage their market risks.

The main contribution of this paper is to introduce an *iterative* method for calculating the optimal exercise boundary in non-time-recursive way. We employ the idea of Little *et al.* (2000), who suggests a new representation of the early exercise boundary containing only a single(or one-dimensional) integral. We show that a modification of their representation can lead us to develop a simple but powerful iteration method for pricing American options.

The method in this paper provides us with a relatively fast and accurate algorithm for calculating the optimal exercise boundary for some time interval (not a time spot) simultaneously. After then, the value of the American options and their hedge ratios (Greeks) can be calculated directly by the closed-form formulae containing only one summation. Therefore, if the optimal exercise boundary for an American option is once calculated, the method gives relatively quicker and more accurate results of the values of American options with the same strike price and underlying asset, for any time to expirations and any underlying asset prices, compared with currently available methods.

This paper is organized as follows. First, we review *early exercise premium representation* and the corresponding valuation formula for an American put option. Subsequently, the iteration method is explained, and its convergence, computing speed and accuracy are explored. After then we apply the method to valuation of other American-style options, and conclude in the final section.

II A Valuation Formula for American Puts

In this section, we present the valuation formula for American put options introduced by Kim (1990). His early exercise premium representation has important implications for our iteration method.

Consider an American put option on an asset (stock) with exercise price K and maturity T, and denote its value at time $t = T - \tau$ by $P(S, \tau)$, where S $(0 < S \leq \infty)$ is the asset price and τ $(0 < \tau \leq T)$ is the time to maturity. We employ the usual conditions: markets are perfect and trading occurs continuously, the asset price follows lognormal diffusion process S_t satisfying

$$dS_t = \mu S_t dt + \sigma S_t dz_t,$$

for positive constants μ (the expected rate of return of the asset) and σ (the volatility of the asset). Here z_t is a one-dimensional standard Brownian motion defined on a suitable probability space. We also assume that the risk-free interest rate is constant r.

As seen in enormous literature such as Mckean (1965), Merton (1973), and Myneni (1992), the value of an American put is considered as the solution to a free boundary problem containing a parabolic partial differential equation (PDE). Throughout this paper, we assume that the optimal exercise boundary B_τ of the American put is uniquely determined and all its sample paths are continuous. Then, the free boundary problem is to find the function $P(S, \tau)$, which is $C^{2,1}$ on $(B_\tau, \infty) \times [0, T]$, satisfying the Black-Scholes PDE

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = \frac{\partial P}{\partial \tau},$$

subject to a terminal condition

$$P(S, 0) = 0 \quad \text{for all} \quad S \geq B_0,$$

and boundary conditions

$$\lim_{S \uparrow \infty} P(S, \tau) = 0, \quad \lim_{S \downarrow B_\tau} P(S, \tau) = K - B_\tau, \quad \lim_{S \downarrow B_\tau} \frac{\partial P(S, \tau)}{\partial S} = -1 \tag{1}$$

for all $\tau \in (0, T]$.

Kim (1990) derives a valuation formula for American options, which contains optimal exercise boundary as a function of time to expiration, and an implicit-form integral equation with respect to optimal exercise boundary.

The valuation formula for a *live* American put is, for $S > B_\tau$,

$$P(S,\tau) = p(S,\tau) + \int_0^\tau rKe^{-r(\tau-\xi)}\aleph(-d_2(S,\tau-\xi;B_\xi))d\xi, \tag{2}$$

where $\aleph(\cdot)$ is the *unit normal distribution function*,

$$\begin{aligned} d_1(S,\tau;B) &= \frac{\ln(S/B) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \\ d_2(S,\tau;B) &= d_1(S,\tau;B) - \sigma\sqrt{\tau}, \end{aligned}$$

and $p(S,\tau)$ represents the Black-Scholes pricing formula for European put option:

$$p(S,\tau) = Ke^{-r\tau}\aleph(-d_2(S,\tau;K)) - S\aleph(-d_1(S,\tau;K)).$$

The integral equation for optimal exercise boundary B_τ is

$$K - B_\tau = p(B_\tau,\tau) + \int_0^\tau rKe^{-r(\tau-\xi)}\aleph(-d_2(B_\tau,\tau-\xi;B_\xi))d\xi, \tag{3}$$

and we call it the early exercise premium representation for B_τ. Note that the right-hand side of (3) contains a double integral since $\aleph(\cdot)$ is substantially an integral.

Differentiating both sides of (2) with respect to S, we obtain a delta hedging formula[1]: if $S > B_\tau$,

$$\Delta_\tau \equiv \frac{\partial P(S,\tau)}{\partial S} = -\aleph(-d_1(S,\tau;K)) - \frac{rK}{\sqrt{2\pi}\sigma S}\int_0^\tau \frac{1}{\sqrt{\tau-\xi}}e^{-r(\tau-\xi)-\frac{(d_2(S,\tau-\xi;B_\xi))^2}{2}}d\xi, \tag{4}$$

and, if $0 < S \le B_\tau$ and $\Delta_\tau = -1$.

As a result, we can calculate the value of the live American put $P(S,\tau)$ in (2) and the hedge ratio Δ_τ in (4) by using the optimal boundary B_τ obtained from (3). The optimal exercise boundary at expiration should be given as

$$B_0 = K.$$

[1] Differentiating (2) with respect to other variables, we also obtain explicit formulas for other Greeks, and the argument for the delta can be applied to them in the same way.

III An Iteration Method

We introduce a new early exercise premium representation with only a *single* integral by exploiting Little *et al.* (2000)'s idea.[2] When the stock price is below the early exercise boundary it is optimal to exercise the option, thus, we can let $P_\tau = K - S_\tau$ in (2). Hence, we obtain

$$K - S_\tau = p(S_\tau, \tau) + \int_0^\tau rKe^{-r(\tau-\xi)}\aleph(-d_2(S_\tau, \tau-\xi; B_\xi))d\xi,$$

Now we can substitute S_τ as ϵB_τ with $\epsilon \in (0, 1]$. Then

$$K - \epsilon B_\tau = p(\epsilon B_\tau, \tau) + \int_0^\tau rKe^{-r(\tau-\xi)}\aleph(-d_2(\epsilon B_\tau, \tau-\xi; B_\xi))d\xi,$$

By differentiating with respect to ϵ, we have

$$\begin{aligned}&B_\tau \aleph(d_1(\epsilon B_\tau, \tau; K)) + \epsilon B_\tau \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2}d_1(\epsilon B_\tau,\tau;K)^2} \\ &= \frac{1}{\sigma\sqrt{2\pi\tau}} K e^{-\{r\tau + \frac{1}{2}d_2(\epsilon B_\tau,\tau;K)^2\}} + rK\int_0^\tau \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}} re^{-\{r(\tau-\xi)+\frac{1}{2}d_2(\epsilon B_\tau,\tau;B_\xi)^2\}}d\xi\end{aligned}$$

Taking the limit of $\epsilon \uparrow 1$ and rearranging it, we can get the following equation:

$$B_\tau = \frac{\frac{1}{\sigma\sqrt{2\pi\tau}} K e^{-\{r\tau + \frac{1}{2}d_2(B_\tau,\tau;K)^2\}} + rK\int_0^\tau \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}} re^{-\{r(\tau-\xi)+\frac{1}{2}d_2(B_\tau,\tau;B_\xi)^2\}}d\xi}{\aleph(d_1(B_\tau,\tau;K)) + \frac{1}{\sigma\sqrt{2\pi\tau}}e^{-\frac{1}{2}d_1(B_\tau,\tau;K)^2}} \tag{5}$$

The equation (5) gives us an implicit form defining the optimal exercise boundary B_τ. Viewed differently, it can be interpreted as an equation in which the left-hand side B_τ is determined by the right hand side. Therefore, we can think of the left-hand side of (5) as a functional form of the iteration method for calculating B_τ.

More specifically, we begin with the function

$$B^0_\tau = K,$$

and it can be used on the right-hand side to obtain the left-hand side as the first round

[2] It does not have the same form with that in Little *et al.* (2000). In fact, our further analysis doesn't go well if one uses the representation in Little *et al.* (2000).

approximation denoted by B^1_τ. The first round approximation can be derived explicitly:

$$B^1_\tau = K \cdot \frac{\frac{1}{\sigma\sqrt{2\pi\tau}} e^{-(r\tau + \frac{1}{2} d_2(K,\tau;K)^2)} + \frac{2\sigma r}{2r+\sigma^2} (2\aleph(\frac{(2r+\sigma^2)\sqrt{\tau}}{2\sigma}) - 1)}{\aleph(d_1(K,\tau;K)) + \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{1}{2} d_1(K,\tau;K)^2}} \tag{6}$$

and $B^1_0 = K$.

The first round approximation B^1_τ is substituted on the right-hand side to get the second round approximation B^2_τ. This procedure is repeated until convergence is obtained. On each round, we set the approximate optimal boundary at $\tau = 0$ to be K. We can use any method of numerical integration, e.g. *trapezoidal rule* or *Gaussian quadrature rule*, in order to approximate the integral in (5) after first round.

In this paper we use *Gauss-Kronrod rule*, which is one of the most prevalent method for calculating numerical integrations, but there is no matter if one use more advanced numerical integration schemes in order to enhance accuracy. The Gauss-Kronrod rule is an adaptive Gaussian quadrature rule for numerical integration with error estimation based on evaluation at special points known as *Kronrod points*. The details for this numerical scheme are appeared in the original paper of Kronrod (1964) and the survey paper of Gautschi (1999).

Our iteration method is conceptually very simple, easy to implement and accurate as long as one takes the number of nodes for time to expiration sufficiently large. This implies that it may take fairly enormous computing time for long-dated put options, therefore it would be useful to develop an accelerated method to compute the optimal exercise boundary fast. We use the *polynomial interpolation* for the optimal exercise boundary in order to accelerate our method.[3]

Precisely speaking, we can approximate the optimal exercise boundary B_τ by a polynomial of degree n that interpolates all points in the set of $\{B^k_{\tau_i}\}_{0\leq i\leq n}$ at each k-th round iteration. We affirm that this polynomial interpolation provides us with a dramatic reduction of computing time without sacrifice any accuracy.

To sum up, we implement the iteration method according to the following procedure:

Step 0. Set $(n+1)$ to be the number of nodes for time to expiration and k the number

[3]One may use other interpolation methods such as interpolation using piecewise polynomials (e.g. cubic splines) in order to get rid of unwanted oscillations. However, we could not find any evidence that such methods can provide us with superior results.

of iteration. Denote the maturity of the option by T.

Step 1. Calculate B^1_τ by using (6).

Step 2. Calculate B^k_τ.

Step 2-1. Calculate the value $\{B^2_{t_i}\}_{0\le i\le n}$ of the approximate optimal exercise boundary by replacing the function B_τ in the righthand side of (5) by the function B^1_τ in Step 1. Here, $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$ and $t_i - t_{i-1} = \dfrac{T}{n}$, and we use the Gauss-Kronrod rule for calculating the integration in (5).

Step 2-2. Construct the function B^2_τ by interpolating the values of $\{B^2_{t_i}\}_{0\le i\le n}$ with a polynomial of order n.

Step 2-3. Repeat Step 2-1 and Step 2-2 $(k-1)$ times.

Step 3. Calculate the value of the option(or/and the hedge ratio) in the formula (2)((4), respectively). The Gauss-Kronrod rule is used to calculate the integration in it.

IV Convergence, Accuracy, and Speed

Figure 1 shows the graph of a calculated optimal exercise boundary as a function of time to expiration τ until fifth round iteration. We use the iteration method with parameters $r = 0.05$, $\sigma = 0.2$, $K = 45\$$, $T = 0.5$ *year*, and the number of nodes $(n+1) = 17$. It tells us that the optimal exercise boundary does not change much even after four iterations. Since the first round approximation B^1_τ can be derived explicitly in (6), it seems that three iterations are enough to obtain a suitably accurate optimal exercise boundary for this case. This is true for various other parameter values. We will show that the root of the mean squared errors of the values of the American puts for various parameters, which are obtained by sixth round iteration, is relatively very small compared with the results obtained by other numerical methods.

Insert Figure 1 here.

In Table 1, we display the calculated values of optimal exercise boundaries B^{10}_τ at $\tau = T$ and those of an American put corresponding such boundaries, when the number of nodes n increases doubly. We can see that the values of the American put obtained by using more than 4 nodes have errors less than 10^{-2},[4] namely, we can obtain the approximate value of American put with error less than 10^{-2} by using only the polynomial interpolation of order 4.

Insert Table 1 here.

Table 2 reports the computing times and root mean squared errors (RMSEs) in calculating the values of 601 American put option contracts with 6-month maturity by various methods. In Table 2, the computing times by the iteration method and the randomization method of Carr (1998) are almost the same, but the iteration method is more accurate. If we compare the results by the iteration method with those by Ju (1998), we can find that the iteration method is more faster under the condition with the similar RMSE. On the other hand, both the computing time and RMSE by the iteration method are superior to those by the binomial tree method by Cox *et al.* (1979). Compared with two approximation methods of Brone-Adesi and Whaley (1987) and Bjerksund and Stensland (1993), the iteration method is more accurate but requires more computing time. However, as we have already mentioned, the two approximation methods have a critical weak point: there is no way of improving accuracy.

Insert Table 2 here.

Table 3 reports the computing times and RMSEs in calculating the values of 1601 American put option contracts with 5-year maturity. It seems that the results in the table are almost the same with those in Table 2 except the fact that the iteration method gives us strictly better performance than Ju (1998)'s approximation method.

Insert Table 3 here.

Table 4 and 5 report the computing times and root mean squared errors (RMSEs) when calculating the values of 520 American put option contracts with various short maturities (from 3 months to 6 months) and 4020 American put option contracts with long maturities (from 3 Years to 5 Years) by various computational methods, respectively.[5] The results in the tables also show better performances of the iteration method.

[4] Although results from the numerical simulation indicate that they are convergent to the exact value, the proof of the convergence still remains open.

[5] We remove the results by Ju's (1998) approximation method, since the method takes considerably much computing time under the condition with the similar accuracy compared with our results.

Insert Table 4 here.

Insert Table 5 here.

Table 6 and 7 report the computing times and root mean squared errors (RMSEs) when calculating the hedge ratios (deltas) of 401 American put option contracts with 6-month maturity and 701 American put option contracts with 5-year maturity, respectively. The results in the tables show that the iteration method gives us strictly better performances compared with all four methods except Ju (1998). Also, compared with Ju (1998), it seems that the iteration method is much faster with little sacrifice of accuracy.

Insert Table 6 here.

Insert Table 7 here.

In Table 8 we display the computing times and RMSEs in calculating the hedge ratios (deltas) for 520 American put option contracts with various short maturities (from 3 months to 6 months). We can see that all of the five methods provide us with the similar computational speed but our iteration method gives us the most precise results.

Insert Table 8 here.

The results in Table 9 take on a little different aspect for longer maturity. We can see that the IFDM, the randomization method by Carr (1998) and the iteration method has similar accuracy, but the iteration method gives us the least computing time. For instance, the computing times by the iteration method is about 4 or 5 times less than those by Carr's (1998) method. This is caused by the fact that the iteration method gives us the way of calculating the hedge ratios by using the exact formula (see (4)), while the others cannot do that. If we utilize the implicit FDM by Brennan and Schwartz (1976) and the randomization method by Carr (1998), we should calculate hedge ratios by using a difference form (e.g., central difference). In this paper, a central difference form is used in calculating those hedge ratios.

Insert Table 9 here.

V An Extension to American Options with Dividends

We apply our iteration method to the valuation of American puts with proportional dividends. We are convinced that the method proposed in this paper can be used to price any derivative securities with American feature if they have an early exercise premium representation.

Consider an American put written on an asset that pays continuous proportional dividends at a rate of $\alpha > 0$ and all other circumstances are the same as those in the second section. For this case, the price dynamics for the underlying asset can be represented by

$$dS_t = (\mu - \alpha)S_t dt + \sigma S_t dz_t.$$

The Black-Scholes PDE for the live American put takes the form of

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - \alpha)S\frac{\partial P}{\partial S} - rP = \frac{\partial P}{\partial \tau},$$

subject to a terminal condition

$$P(S, 0) = \max\{0, K - S\} \text{ for all } S \geq B_0,$$

and boundary conditions in (1).

Then, as in Kim (1990), the valuation formula for the live American put is, for $S > B_\tau$,

$$P(S,\tau) = \tilde{p}(S,\tau) + \int_0^\tau \left[rKe^{-r(\tau-\xi)}\aleph(-\tilde{d}_2(S,\tau-\xi;B_\xi)) - \alpha S e^{-\alpha(\tau-\xi)}\aleph(-\tilde{d}_1(S,\tau-\xi;B_\xi))\right] d\xi,$$

and the early exercise premium representation is

$$K - B_\tau = \tilde{p}(B_\tau,\tau) + \int_0^\tau \left[rKe^{-r(\tau-\xi)}\aleph(-\tilde{d}_2(B_\tau,\tau-\xi;B_\xi)) - \alpha B_\tau e^{-\alpha(\tau-\xi)}\aleph(-\tilde{d}_1(B_\tau,\tau-\xi;B_\xi))\right] d\xi, \tag{7}$$

where

$$\tilde{d}_1(S,\tau;B) = \frac{\ln(S/B) + (r - \alpha + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},$$
$$\tilde{d}_2(S,\tau;B) = \tilde{d}_1(S,\tau;B) - \sigma\sqrt{\tau},$$

and European put value function $\tilde{p}$ in (7) is defined as

$$\tilde{p}(S,\tau) = Ke^{-r\tau}\aleph(-\tilde{d}_2(S,\tau;K)) - Se^{-\alpha\tau}\aleph(-\tilde{d}_1(S,\tau;K)).$$

As already mentioned by Kim (1990), the early exercise boundary at expiration should be given by $B_0 = \min\left\{K, \frac{r}{\alpha}K\right\}$.

An iteration method stems from the following equation equivalent to (7):

$$\begin{aligned} B_\tau = \quad & \Big(\frac{1}{\sigma\sqrt{2\pi\tau}}Ke^{-\{r\tau+\frac{1}{2}d_2(B_\tau,\tau;K)^2\}} + rK\int_0^\tau \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}}re^{-\{r(\tau-\xi)+\frac{1}{2}d_2(B_\tau,\tau-\xi;B_\xi)^2\}}d\xi\Big) \\ & \Big/\Big(e^{-\alpha\tau}\aleph(d_1(B_\tau,\tau;K)) + \frac{1}{\sigma\sqrt{2\pi\tau}}e^{-\{\alpha\tau+\frac{1}{2}d_1(B_\tau,\tau;K)^2\}} \\ & \quad +\alpha\int_0^\tau e^{-r(\tau-\xi)}\aleph(d_1(B_\tau,\tau-\xi;B_\xi)) + \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}}re^{-\{r(\tau-\xi)+\frac{1}{2}d_1(B_\tau,\tau-\xi;B_\xi)^2\}}d\xi\Big) \end{aligned}$$

We should begin the iteration with the function

$$B_\tau^0 = \min\left\{K, \frac{r}{\alpha}K\right\}.$$

The first round approximation B_τ^1 and the valuation formula for the hedge ratio Δ_τ can be easily derived, so the specific derivations are left to the reader.

The iteration method also can be used for valuing American call options with proportional dividends. It can be done by the early exercise premium representation for American call options by Kim (1990). On the other hand, we can use the parity property suggested by McDonald and Schröder (1998). They verify the fact that the value of an American call option with underlying asset price S, exercise price K, risk-free interest rate r, and dividend rate α is equal to the value of an American put option with underlying asset price K, exercise price S, risk-free interest rate α, and dividend rate r. Using it, we can obtain the value of American call option with proportional dividends by calculating the value of the American put option satisfying the parity property.

VI Conclusion

In this paper, we provide a simple but innovative iteration method to obtain optimal exercise boundary of American options, and we show that it allows us to compute the values of

American options and their hedge ratios rapidly as well as accurately.

We derive a new early exercise premium representation with only a single integral by following Little *et al.* (2000)'s idea. By using the representation we suggest a simple but efficient iterative numerical algorithm. According to numerical results, our iteration method is computationally superior to the methods currently available without mentioning the binomial method of Cox *et al.* (1979), the implicit FDM by Brennan and Schwartz (1976), and the randomization method by Carr (1998).

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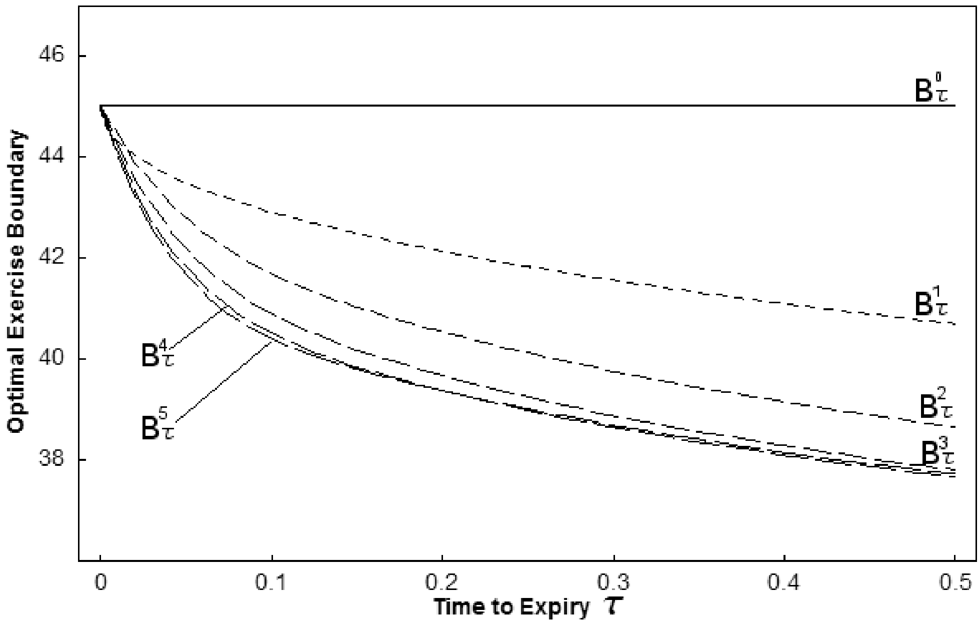
Figures

Figure 1: The convergency of optimal exercise boundary. The parameters are $r = 0.05,\ \sigma = 0.2,\ K = 45\$,\ T = 0.5\ year$, and $n = 16$.



Tables

Table 1: The convergency of the calculated values of optimal exercise boundary B_τ^{10} at $\tau = T$ and the corresponding values of the American put when n increases doubly.

n	B_T^{10}	American put
2	37.6824	2.1202
4	37.7490	2.1011
8	37.7602	2.0963
16	37.7629	2.0950
32	37.7624	2.0947
benchmark		2.0950

The parameters are $r = 0.05,\ \sigma = 0.2,\ S = 45\$,\ K = 45\$,\ T = 0.5\ year$, and the result of the 'benchmark' case is obtained by using the binomial tree method by Cox *et al.* (1979) with 10000 time steps.

Table 2: The computing times and RMSEs in calculating the values of 601 American puts with a short maturity (6M).

Method		Numerical		Approximation				
		Binomial	IFDM	BW	BS	Carr	Ju	Iteration
$\sigma = 0.2$	Computing time(sec)	53.882	4.008	0.982	1.202	2.043	4.431	1.903
	RMSE	6.98e-3	7.65e-3	2.72e-2	6.28e-2	4.48e-3	8.39e-4	1.29e-3
$\sigma = 0.4$	Computing time(sec)	54.74	4.041	0.968	1.155	2.09	4.46	1.872
	RMSE	0.019e-2	3.56e-2	2.06e-2	6.96e-2	1.11e-2	1.49e-3	1.92e-3

The results are the computing times and RMSEs in calculating the values of the 601 American put option contracts with 601 underlying stock prices (from 90\$ to 120\$, 0.05\$ steps). The default parameters are $r = 0.05$ and $K = 100\$$. We obtained the results for the benchmark case (the exact values) by the binomial tree method with 10000 time steps, which is introduced by Cox *et al.* (1979). The results on the third column were obtained by the binomial tree method with 100 time steps by Cox *et al.* (1979), on the fourth column by the implicit FDM in Brennan and Schwartz (1976) with 100 time steps and 2000 underlying state steps, on the fifth column by the approximation method in Barone-Adesi and Whaley (1987), on the sixth column by the approximation method in Bjerksund and Stensland (1993), on the seventh column by the randomization method in Carr (1998) with 5-point Richardson extrapolation and on the eighth column by the multi-piece exponential function method in Ju (1998) with 3-point Richardson extrapolation. The ninth column represents the results obtained by the iteration method in this paper with 6 iterations and 10 node points (i.e., polynomial interpolation of order 9). Computing time is the time required to compute the values for all 601 contracts and all routines were programmed using `Mathematica` language and ran on a 3.0-GHz Pentium computer.

Table 3: The computing times and RMSEs in calculating the values of 1601 American puts with a long maturity (5Y).

Method		Numerical		Approximation				
		Binomial	IFDM	BW	BS	Carr	Ju	Iteration
$\sigma = 0.2$	Computing time(sec)	145.268	10.608	2.388	3.386	5.428	10.186	6.302
	RMSE	3.26e-2	1.48e-2	1.75e-1	5.55e-2	4.23e-3	3.31e-3	2.45e-3
$\sigma = 0.4$	Computing time(sec)	144.861	10.64	2.246	3.135	5.32	10.187	6.192
	RMSE	1.06e-1	2.66e-1	5.35e-1	1.60e-1	1.96e-2	7.27e-3	3.32e-3

The results are the computing times and RMSEs in calculating the values of the 1601 American put option contracts with 1601 underlying stock prices (from 80$ to 160$, 0.05$ steps). The default parameters are $r = 0.05$ and $K = 100$$. We obtained the results for the benchmark case (the exact values) by the binomial tree method with 10000 time steps, which is introduced by Cox *et al.* (1979). The results on the third column were obtained by the binomial tree method with 100 time steps by Cox *et al.* (1979), on the fourth column by the implicit FDM in Brennan and Schwartz (1976) with 100 time steps and 3000 underlying state steps, on the fifth column by the approximation method in Barone-Adesi and Whaley (1987), on the sixth column by the approximation method in Bjerksund and Stensland (1993), on the seventh column by the randomization method in Carr (1998) with 5-point Richardson extrapolation and on the eighth column by the multi-piece exponential function method in Ju (1998) with 3-point Richardson extrapolation. The ninth column represents the results obtained by the iteration method in this paper with 6 iterations and 25 node points (i.e., polynomial interpolation of order 24). Computing time is the time required to compute the values for all 1601 contracts and all routines were programmed using `Mathematica` language and ran on a 3.0-GHz Pentium computer.

Table 4: The computing times and RMSEs in calculating the values of 520 American puts with various short maturities (3M-6M).

Method		Numerical		Approximation			
		Binomial	IFDM	BW	BS	Carr	Iteration
$\sigma = 0.2$	Computing time(sec)	48.36	1.842	0.919	1.123	1.934	1.669
	RMSE	1.88e-3	2.27e-3	9.18e-3	1.93e-2	1.39e-3	9.95e-4
$\sigma = 0.4$	Computing time(sec)	47.783	1.84	0.905	1.046	1.841	1.654
	RMSE	6.52e-3	2.91e-3	8.03e-3	2.11e-2	3.63e-3	1.51e-3

The results are the computing times and RMSEs in calculating the values of the 520 American put option contracts with 20 underlying stock prices (from 41$ to 60$, 1$ steps) and 26 maturities (from 0.25 year to 0.50 year, 0.01 year steps). The default parameters are $r = 0.05$ and $K = 45$$. We obtained the results for the benchmark case (the exact values) by the binomial tree method with 10000 time steps, which is introduced by Cox *et al.* (1979). The results on the third column were obtained by the binomial tree method with 100 time steps by Cox *et al.* (1979), on the fourth column by the implicit FDM in Brennan and Schwartz (1976) with 500 time steps and 200 underlying state steps, on the fifth column by the approximation method in Barone-Adesi and Whaley (1987), on the sixth column by the approximation method in Bjerksund and Stensland (1993), and on the seventh column by the randomization method in Carr (1998) with 5-point Richardson extrapolation. The eighth column represents the results obtained by the iteration method in this paper with 6 iterations and 10 node points (i.e., polynomial interpolation of order 9). Computing time is the time required to compute the values for all 520 contracts and all routines were programmed using `Mathematica` language and ran on a 3.0-GHz Pentium computer.

Table 5: The computing times and RMSEs in calculating the values of 4020 American puts with various long maturities (3Y-5Y).

Method		Numerical		Approximation			
		Binomial	IFDM	BW	BS	Carr	Iteration
$\sigma = 0.2$	Computing time(sec)	371.469	23.483	6.038	7.566	14.008	11.31
	RMSE	1.36e-2	1.11e-3	6.85e-2	3.29e-2	2.28e-3	2.06e-3
$\sigma = 0.4$	Computing time(sec)	370.486	18.158	5.709	7.379	13.838	12.901
	RMSE	4.05e-2	2.21e-1	1.79e-1	7.31e-2	9.36e-3	1.81e-3

The results are the computing times and RMSEs in calculating the values of the 4020 American put option contracts with 20 underlying stock prices (from 41$ to 60$, 1$ steps) and 201 maturities (from 3 years to 5 years, 0.01 year steps). The default parameters are $r = 0.05$ and $K = 45$$. We obtained the results for the benchmark case (the exact values) by the binomial tree method with 10000 time steps, which is introduced by Cox *et al.* (1979). The results on the third column were obtained by the binomial tree method with 100 time steps by Cox *et al.* (1979), on the fourth column by the implicit FDM in Brennan and Schwartz (1976) with 500 time steps and 200 underlying state steps, on the fifth column by the approximation method in Barone-Adesi and Whaley (1987), on the sixth column by the approximation method in Bjerksund and Stensland (1993), and on the seventh column by the randomization method in Carr (1998) with 5-point Richardson extrapolation. The eighth column represents the results obtained by the iteration method in this paper with 6 iterations and 25 node points (i.e., polynomial interpolation of order 24). Computing time is the time required to compute the values for all 4020 contracts and all routines were programmed using `Mathematica` language and ran on a 3.0-GHz Pentium computer.

Table 6: The computing times and RMSEs in calculating the hedge ratios (deltas) of 401 American puts with a short maturity (6M).

Method		Numerical	Approximation				
		IFDM	BW	BS	Carr	Ju	Iteration
$\sigma = 0.2$	Computing time(sec)	3.931	1.309	1.45	1.637	5.351	1.279
	RMSE	4.69e-4	3.16e-3	3.31e-3	3.96e-4	5.48e-5	1.11e-4
$\sigma = 0.4$	Computing time(sec)	3.9	1.325	1.528	1.653	5.366	1.295
	RMSE	4.81e-4	2.24e-3	1.60e-3	3.48e-4	3.61e-5	6.74e-5

The results are the computing times and RMSEs in calculating the values of the 401 American put option contracts with 401 underlying stock prices (from 40\$ to 60\$, 0.05\$ steps). The default parameters are $r = 0.05$, $T = 0.5$ and $K = 45\$$. We obtained the results for the benchmark case (the exact values) by the implicit FDM with 5000 time steps and 10000 underlying state steps, which is introduced by Brennan and Schwartz (1976). The results on the third column were obtained by the implicit FDM in Brennan and Schwartz (1976) with 100 time steps and 2000 underlying state steps, on the fourth column by the approximation method in Barone-Adesi and Whaley (1987), on the fifth column by the approximation method in Bjerksund and Stensland (1993), and on the sixth column by the randomization method in Carr (1998) with 4-point Richardson extrapolation and on the seventh column by the multipiece exponential function method in Ju (1998) with 3-point Richardson extrapolation. The eighth column represents the results obtained by the iteration method in this paper with 6 iterations and 10 node points (i.e., polynomial interpolation of order 9). Computing time is the time required to compute the values for all 401 contracts and all routines were programmed using `Mathematica` language and ran on a 3.0-GHz Pentium computer.

Table 7: The computing times and RMSEs in calculating the hedge ratios (deltas) of 701 American puts with a long maturity (5Y).

Method		Numerical	Approximation				
		IFDM	BW	BS	Carr	Ju	Iteration
$\sigma = 0.2$	Computing time(sec)	7.644	2.136	2.48	4.679	9.032	3.509
	RMSE	4.83e-4	5.61e-3	9.12e-4	1.90e-4	1.79e-4	1.55e-4
$\sigma = 0.4$	Computing time(sec)	7.489	2.105	2.559	4.71	9.173	3.338
	RMSE	1.57e-2	6.62e-3	8.93e-3	9.51e-3	9.43e-3	9.39e-3

The results are the computing times and RMSEs in calculating the values of the 701 American put option contracts with 701 underlying stock prices (from 35\$ to 70\$, 0.05\$ steps). The default parameters are $r = 0.05$, $T = 5$ and $K = 45$\$. We obtained the results for the benchmark case (the exact values) by the implicit FDM with 5000 time steps and 10000 underlying state steps, which is introduced by Brennan and Schwartz (1976). The results on the third column were obtained by the implicit FDM in Brennan and Schwartz (1976) with 200 time steps and 2000 underlying state steps, on the fourth column by the approximation method in Barone-Adesi and Whaley (1987), on the fifth column by the approximation method in Bjerksund and Stensland (1993), and on the sixth column by the randomization method in Carr (1998) with 5-point Richardson extrapolation and on the seventh column by the multipiece exponential function method in Ju (1998) with 3-point Richardson extrapolation. The eighth column represents the results obtained by the iteration method in this paper with 6 iterations and 25 node points (i.e., polynomial interpolation of order 9). Computing time is the time required to compute the values for all 701 contracts and all routines were programmed using `Mathematica` language and ran on a 3.0-GHz Pentium computer.

Table 8: The computing times and RMSEs in calculating the hedge ratios (deltas) of 520 American puts with various short maturities(3M-6M).

Method		Numerical	Approximation			
		IFDM	BW	BS	Carr	Iteration
$\sigma = 0.2$	Computing time(sec)	1.871	1.544	1.997	2.06	1.358
	RMSE	2.07e-4	2.77e-3	2.94e-3	3.75e-4	1.66e-4
$\sigma = 0.4$	Computing time(sec)	1.918	1.653	1.982	2.137	1.356
	RMSE	1.35e-4	1.92e-3	1.49e-3	3.68e-4	1.05e-4

The results are the computing times and RMSEs in calculating the values of the 520 American put option contracts with 20 underlying stock prices (from 41\$ to 60\$, 1\$ steps) and 26 maturities (from 0.25 year to 0.50 year, 0.01 year steps). The default parameters are $r = 0.05$ and $K = 45\$$. We obtained the results for the benchmark case (the exact values) by the implicit FDM with 5000 time steps and 5000 underlying state steps, which is introduced by Brennan and Schwartz (1976). The results on the third column were obtained by the implicit FDM in Brennan and Schwartz (1976) with 500 time steps and 200 underlying state steps, on the fourth column by the approximation method in Barone-Adesi and Whaley (1987), on the fifth column by the approximation method in Bjerksund and Stensland (1993), and on the sixth column by the randomization method in Carr (1998) with 4-point Richardson extrapolation. The seventh column represents the results obtained by the iteration method in this paper with 6 iterations and 10 node points (i.e., polynomial interpolation of order 9). Computing time is the time required to compute the values for all 520 contracts and all routines were programmed using `Mathematica` language and ran on a 3.0-GHz Pentium computer.

Table 9: The computing times and RMSEs in calculating the hedge ratios (deltas) of 4020 American puts with various long maturities (3Y-5Y).

Method		Numerical	Approximation			
		IFDM	BW	BS	Carr	Iteration
$\sigma = 0.2$	Computing time(sec)	18.422	12.043	14.29	26.956	6.052
	RMSE	8.36e-5	6.00e-3	1.12e-3	8.96e-5	2.58e-4
$\sigma = 0.4$	Computing time(sec)	18.268	11.544	14.742	27.113	5.663
	RMSE	1.44e-2	3.92e-3	1.25e-3	1.59e-3	1.46e-3

The results are the computing times and RMSEs in calculating the values of the 4020 American put option contracts with 20 underlying stock prices (from 41\$ to 60\$, 1\$ steps) and 201 maturities (from 3 years to 5 years, 0.01 year steps). The default parameters are $r = 0.05$ and $K = 45\$$. We obtained the results for the benchmark case (the exact values) by the implicit FDM with 5000 time steps and 5000 underlying state steps, which is introduced by Brennan and Schwartz (1976). The results on the third column were obtained by the implicit FDM in Brennan and Schwartz (1976) with 2000 time steps and 500 underlying state steps, on the fourth column by the approximation method in Barone-Adesi and Whaley (1987), on the fifth column by the approximation method in Bjerksund and Stensland (1993), and on the sixth column by the randomization method in Carr (1998) with 5-point Richardson extrapolation. The seventh column represents the results obtained by the iteration method in this paper with 6 iterations and 25 node points (i.e., polynomial interpolation of order 24). Computing time is the time required to compute the values for all 4020 contracts and all routines were programmed using `Mathematica` language and ran on a 3.0-GHz Pentium computer.