美式看跌期权的闭合公式计算方法

The Closed Form solution for Pricing American Put Options
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本文提出了基础资产为无红利分配股票的美式看跌期权的第一个闭合计算公式。美式看跌期权赋予其持有人在期权存续期的任一时刻、以约定价格出售股票的权利但非义务。在过去的几十年中，特别是在 Black-Scholes 模型给出欧式期权的定价公式后，人们在美式期权的定价方面做了大量探索，提出了不少方法，但尚无闭合公式求法。本文提出了一个美式看跌期权提前行权的最优策略，即当且仅当一个美式看跌期权被提前行权时的收益大于其对应的欧式看跌期权的价值时，该美式看跌期权才会被提前行权。基于这一策略，本文提出了一系列紧密关联的定理并最终推出了一个闭合计算公式。另外，基于该闭合公式得出的结论，本文还指出了 Merton（1973）有关永久美式看跌期权(perpetual American put option)的模型是不妥的，明确指出永久美式看跌期权（股票无红利）的价格等于该期权的执行价格。

This paper proposes a closed form solution for pricing an American put option on a non-dividend paying stock. An American put option grants its holder rights, but not obligation to sell a stock in a fixed price at any time up until maturity. In the past decades, there is no closed form solution for pricing American options although many people made great efforts. In this paper, an optimally early exercise strategy of an American put option on a non-dividend paying stock is set up. That is, an American put option should be early-exercised when the maximum option premium of early exercise is no less than the value of its European counterpart; otherwise, it should not be early-exercised. Based on this strategy, a series of lemmas is proposed and a closed form formula is drawn. Also, this paper shows that Merton (1973)'s formula does not do a good job for pricing perpetual American put options and shows the price of a perpetual American put option on a non-dividend paying stock is equal to the strike price.

Keywords: American put option, Closed-form formula, Assets Pricing

关键词：美式看跌期权，闭合公式，资产定价

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1. Introduction

In 1973, the publication of the celebrated work of Black and Scholes made great contribution to the world's financial markets. After that, the pricing of European options becomes easier and cheaper. Comparing to American options, European options are simpler financial derivatives that give their holder the rights, but not the obligation, to buy or sell a unit of asset at a fixed time, for a fixed price.

Black and Scholes gave the following famous closed-form formula for pricing European options on non-dividend paying stock:

$$C_E(S_0, K, r, T, \sigma) = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{1.1}$$

$$P_E(S_0, K, r, T, \sigma) = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \tag{1.2}$$

Where

$$d_1 = \frac{\ln\frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{1.3}$$

$$d_2 = \frac{\ln\frac{S_0}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \tag{1.4}$$

American options differ from European options by virtue of the fact that they can be exercised at any time during the lifetime of the options. This makes for a more complicated pricing problem. In the past decades, there is no closed form formula for pricing American options although many people made great efforts.

An American Call Option on a non-dividend-paying stock should never be exercised prior to expiration, so an American call option on a non-dividend paying stock has the same price as its European counterpart, which means

$$C_A(S_0, K, r, T, \sigma) = C_E(S_0, K, r, T, \sigma) \tag{1.5}$$

However, an American put option may be rationally exercised, no matter there is dividend paying or not.

The pricing of an American put option on a non-dividend paying stock could be described as an optimization problem. When the maximum option premium of early exercise is no less than the value of its European counterpart, the holder of an American put option would prefer early exercise; otherwise, the holder should not early exercise. Based on this early-exercise strategy, this paper proposes a closed form formula for pricing American put options on a non-dividend paying stock.

This paper proceeds as follows. Section 2 gives assumptions. Section 3 derives a closed form formula to price an American put option on a non-dividend paying stock by setting up a series of Lemmas. Section 4 shows the price of a perpetual American put option on a non-dividend paying stock is equal to the strike price. The final section makes concluding remarks.

All the mathematics notations are given in the Appendix.

2. Assumptions

In order to price an American put option on a non-dividend paying stock, we take the following assumptions.

(1) There are no transaction costs or taxes.

(2) We are in a risk-neutral economy, which means all the market expected return is equal to the risk –free interest rate. The risk-free interest rate is known and constant; borrowing and lending are possible at the risk-free interest rate.

(3) There are non-dividends for the underlying stock, and short-sell such underlying stock is allowed and possible.

A short sale of such an underlying stock entails market participants borrowing a share of underlying stock and then selling it, receiving cash. Later, buy back an underlying stock, paying cash for it, and return it to the lender (Since there are non-dividends for the underlying stock, only an underlying stock is needed to return to the lender). A short sale can be viewed, then, just as a way of borrowing money.

(4) The stock price is Log normal distributed.

$\ln(S_t / S_0)$ is normally distributed with mean $(r - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$:

$$\ln(S_t / S_0) \sim N[(r - \frac{1}{2}\sigma^2)t, \sigma^2 t] \tag{2.1}$$

This gives us two equivalent ways to write an expression for the stock prices:

$$\ln(S_t / S_0) = (r - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}z \tag{2.2}$$

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}z} \tag{2.3}$$

Where z is a standard normal random variable. $z \sim N(0,1)$

(5) The variation of the stock's continuous combine return is known and constant.

(6) The market participants take advantage of arbitrage opportunities as they occur. In other words, there

are no risk-free arbitrage opportunities in the market.

3. Pricing American put options on non-dividend paying stocks

First of all, the following two Lemmas are given without proof.

Lemma 3.1: An European put option with a higher strike price is at least as valuable as an otherwise identical one with a lower strike price. That is, if $K_1 \geq K_2$, then

$$P_E(S_0, K_1, r, T, \sigma) \geq P_E(S_0, K_2, r, T, \sigma) \tag{3.1}$$

Lemma 3.2: Upper and lower bounds for an American put option on a non-dividend paying stock is:

$$K - S_0 \leq P_A(S_0, K, r, T, \sigma) \leq C_A(S_0, K, r, T, \sigma) + (K - S_0) \leq K$$

From (1.5), we know above formula can be written as

$$K - S_0 \leq P_A(S_0, K, r, T, \sigma) \leq C_E(S_0, K, r, T, \sigma) + (K - S_0) \leq K \tag{3.2}$$

In addition, the following formula should also be satisfied.

$$P_E(S_0, K, r, T, \sigma) \leq P_A(S_0, K, r, T, \sigma) \tag{3.3}$$

Now, let's begin to use probabilistic approach to price an American put option on a non-dividend paying stock.

Lemma 3.3: The value of an American put option $P_A(S_0, K, r, T, \sigma)$ on a non-dividend paying stock is equal to the expected value of the maximum option premium.

$$P_A(S_0, K, r, T, \sigma) \\ = E^Q\{\max[P_E(S_0, K, r, T, \sigma), Max\Pr emium(earlyexercise)]\} \tag{3.4}$$

Where $Max\Pr emium(earlyexercise)$ is the maximum option premium when an American put option is optimally early-exercised.

Proof: According to the definition of an American put option, the holder has the rights, but not obligation to exercise it at any time before the expiration date. As we know, when an American put option is not early-exercised, the premium will be equal to its European counterpart.

$$P_A(S_0, K, r, T, \sigma) = P_E(S_0, K, r, T, \sigma)$$

The holder of an American put option should take an optimal exercise strategy that gets the maximum option premium. So the pricing of an American put option is such an optimization problem:

(1) When the maximum option premium of optimally early exercise is no less than $P_E(S_0, K, r, T, \sigma)$, the American put option should be optimally early-exercised and get the max premium:

$$P_A(S_0, K, r, T, \sigma) = Max\Pr emium(earlyexercise) \tag{3.5}$$

(2) Otherwise, the American put option should not be early-exercised and get the same premium as its European counterpart:

$$P_A(S_0, K, r, T, \sigma) = P_E(S_0, K, r, T, \sigma) \tag{3.6}$$

$$\therefore P_A(S_0, K, r, T, \sigma) \\ = E^Q\{\max[P_E(S_0, K, r, T, \sigma), Max\Pr emium(earlyexercise)]\}$$

□

What is the economic meaning of an American put option contract? When the buyer goes into the contract with the writer of an American put option on a non-dividend paying stock, the writer gives the buyer rights, but not obligation to borrow an underlying stock from the writer and short-sell it to the writer in the strike price K at any time before the expiration date. We can also say the writer gives the buyer rights, but not obligation to borrow the strike price K from the writer at any time before the expiration date. Of course, if the buyer uses the rights, he/she will have obligation to buy back an underlying stock in the market price from the writer and return it to the writer before the expiration date, or we say the buyer will have obligation to return money (the market price of an underlying stock) to the writer before the expiration date.

Since the buyer has rights whether exercise the option or not, and when exercise the option, he/she will take an optimal exercise strategy to gain maximum profits, this will bring loss to the writer of option. In fact, the price of the option actually is the compensation for the option writer.

Lemma 3.4: As long as an American put option $P_A(S_0, K, r, T, \sigma)$ on a non-dividend paying stock is optimally early-exercised, an underlying stock should be shorted in the strike price K at time 0.

Proof: There are two steps in exercising an American put option on a non-dividend paying stock: short-sell one underlying stock in the strike price K, and buy back one in the market price.

Suppose $P_A(S_0, K, r, T, \sigma)$ is optimally early-exercised, and an underlying stock is shorted in the strike price K at time t $(0 < t < T)$. The premium will be equal to

(1) If buy back an underlying stock at time t immediately

$$\Pr emium(earlyexercise) = Payoff\,(S_t < K)$$

$$= [K - \mathrm{E}(S_t \,/\, S_t < K)]e^{-rt} \Pr o(S_t < K)^{1} = P_E\,(S_0, K, r, t, \sigma)$$

(2) If buy back an underlying stock at time ψ $(t < \psi \le T)$

At time ψ $(t < \psi \le T)$, the holder will have $Ke^{r(\psi - t)}$,

$$\Pr emium(earlyexercise) = Payoff\,(S_\psi < Ke^{r(\psi - t)})$$

$$= [Ke^{r(\psi - t)} - \mathrm{E}(S_\psi \,/\, S_\psi < Ke^{r(\psi - t)})]e^{-r\psi} \Pr o(S_\psi < Ke^{r(\psi - t)})$$

$$= P_E\,(S_0, Ke^{r(\psi - t)}, r, \psi, \sigma)$$

However, if an underlying stock is shorted in the strike price K at time 0, the premium will be equal to

(1) If buy back an underlying stock at time t $(0 < t < T)$

At time t, the holder will have Ke^{rt},

$$\Pr emium(earlyexercise) = Payoff\,(S_t < Ke^{rt})$$

$$= [Ke^{rt} - \mathrm{E}(S_t \,/\, S_t < Ke^{rt})]e^{-rt} \Pr o(S_t < Ke^{rt})$$

$$= P_E\,(S_0, Ke^{rt}, r, t, \sigma)$$

(2) If buy back an underlying stock at time ψ $(t < \psi \le T)$

At time ψ, the holder will have $Ke^{r\psi}$,

$$\Pr emium(earlyexercise) = Payoff\,(S_\psi < Ke^{r\psi})$$

$$= [Ke^{r\psi} - \mathrm{E}(S_\psi \,/\, S_\psi < Ke^{r\psi})]e^{-r\psi} \Pr o(S_\psi < Ke^{r\psi})$$

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[1] $\Pr o(S_t < K)$: Means probability of $S_t < K$

$= P_E(S_0, Ke^{r\psi}, r, \psi, \sigma)$

According to **Lemma 3.1**, they are greater than $P_E(S_0, K, r, t, \sigma)$ or $= P_E(S_0, Ke^{r(\psi - t)}, r, \psi, \sigma)$, respectively.

So, as long as an American put option on a non-dividend paying stock is optimally early-exercised, an underlying stock should be shorted in the strike price K at time 0.

□

Lemma 3.5: As long as an American put option $P_A(S_0, K, r, T, \sigma)$ on a non-dividend paying stock is optimally early-exercised, the maximum option premium of optimally early exercise will be:

$$E[Max\,\Pr emium(earlyexercise)] = P_E(S_0, Ke^{rT}, r, T, \sigma)\; N(-d_4) + (K - S_0)\; N(d_4) \qquad (3.7)$$

The maximum option premium under the optimally early-exercised strategy is $P_E(S_0, Ke^{rT}, r, T, \sigma)$ or $(K - S_0)$, and the probability to take the value is $N(-d_4)$ or $N(d_4)$, respectively.

Where

$$P_E(S_0, Ke^{rT}, r, T, \sigma) = KN(-d_4) - S_0 N(-d_3) \qquad (3.8)$$

$$d_3 = \frac{\ln\frac{S_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \qquad (3.9)$$

$$d_4 = \frac{\ln\frac{S_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \qquad (3.10)$$

Proof: According to **Lemma 3.4**, as long as an American put option $P_A(S_0, K, r, T, \sigma)$ on a non-dividend paying stock is optimally early-exercised, an underlying stock should be shorted in the strike price K at time 0. Then at time ς $(0 < \varsigma \le T)$, the option holder will have $Ke^{r\varsigma}$, since the holder can keep the money K in a risk-free banking account during the time period $0 \sim \varsigma$.

(1) When $S_{\varsigma} < Ke^{r\varsigma} \quad (0 \le \varsigma \le T)$

$\Pr emium(earlyexercise) =$

$[Ke^{r\varsigma} - \mathrm{E}(S_{\varsigma} / S_{\varsigma} < Ke^{r\varsigma})]e^{-r\varsigma} \Pr o(S_{\varsigma} < Ke^{r\varsigma}) = P_E(S_0, Ke^{r\varsigma}, r, \varsigma, \sigma)$

(2) When $S_{\varsigma} > Ke^{r\varsigma} \quad (0 \le \varsigma \le T)$

$\Pr emium(earlyexercise) =$

$[Ke^{r\varsigma} - \mathrm{E}(S_{\varsigma} / S_{\varsigma} > Ke^{r\varsigma})]e^{-r\varsigma} \Pr o(S_{\varsigma} > Ke^{r\varsigma}) = -C_E(S_0, Ke^{r\varsigma}, r, \varsigma, \sigma)$

It can be prove that $\dfrac{\partial P_E(S_0, Ke^{r\varsigma}, r, \varsigma, \sigma)}{\partial \varsigma}$>0 (see Appendix 1), which means $P_E(S_0, Ke^{r\varsigma}, r, \varsigma, \sigma)$ increases while $\varsigma \quad (0 \le \varsigma \le T)$ increases.

Since $C_E(S_0, Ke^{r\varsigma}, r, \varsigma, \sigma) = P_E(S_0, Ke^{r\varsigma}, r, \varsigma, \sigma) - (K - S_0)$

So $\dfrac{\partial[-C_E(S_0, Ke^{r\varsigma}, r, \varsigma, \sigma)]}{\partial \varsigma} < 0$, which means $-C_E(S_0, Ke^{r\varsigma}, r, \varsigma, \sigma)$ decreases while ς $(0 \le \varsigma \le T)$ increases.

Now we suppose T' is a point at $[0,T]$, $0 < T' < T$, let $T = n\Delta$,$T' = m\Delta$, Δ is a very short time period， m and n are nature numbers, $m < n$.

As long as $P_A(S_0, K, r, T, \sigma)$ is early-exercised under the optimally early-exercise strategy, the option holder will take the following optimal strategy to get the maximum gains:

When $S_T < Ke^{rT}$, $Max\Pr emium(earlyexercise) = P_E(S_0, Ke^{rT}, r, T, \sigma)$

When $S_T > Ke^{rT}$ and $S_{T-\Delta} < Ke^{r(T-\Delta)}$,

$Max\Pr emium(earlyexercise) = P_E(S_0, Ke^{r(T-\Delta)}, r, T-\Delta, \sigma) - C_E(S_0, Ke^{rT}, r, T, \sigma) \approx (K - S_0)$

When $S_T > Ke^{rT}$、 $S_{T-\Delta} > Ke^{r(T-\Delta)}$ and $S_{T-2\Delta} < Ke^{r(T-2\Delta)}$,

$Max\Pr emium(earlyexercise) = P_E(S_0, Ke^{r(T-2\Delta)}, r, T-2\Delta, \sigma) - C_E(S_0, Ke^{r(T-\Delta)}, r, T-\Delta, \sigma)$

$\approx (K - S_0)$

……

When $S_T > Ke^{rT}$、 $S_{T-\Delta} > Ke^{r(T-\Delta)}$、$S_{T-2\Delta} > Ke^{r(T-2\Delta)}$, ……and $S_{T-m\Delta} < Ke^{r(T-m\Delta)}$,

$Max\Pr emium(earlyexercise)$

$= P_E(S_0, Ke^{r(T-m\Delta)}, r, T-m\Delta, \sigma) - C_E(S_0, Ke^{r[T-(m-1)\Delta]}, r, T-(m-1)\Delta, \sigma) \approx (K - S_0)$

……

When $S_T > Ke^{rT}$、 $S_{T-\Delta} > Ke^{r(T-\Delta)}$、$S_{T-2\Delta} > Ke^{r(T-2\Delta)}$、…… $S_{T-m\Delta} > Ke^{r(T-m\Delta)}$、……and $S_\Delta > Ke^{r\Delta}$,

$Max\Pr emium(earlyexercise) = (K - S_0)$

As long as $P_A(S_0, K, r, T, \sigma)$ is early-exercised under the optimally early-exercise strategy, the maximum gains of the option holder will be:

$E[Max\Pr emium(earlyexercise)] =$

$P_E(S_0, Ke^{rT}, r, T, \sigma)\ \Pr o(S_T < Ke^{rT})$

$+ (K - S_0)\ \Pr o(S_T > Ke^{rT})\ \Pr o(S_{T-\Delta} < Ke^{r(T-\Delta)})$

$$+(K-S_0)\ \Pr o(S_T > Ke^{rT})\ \Pr o(S_{T-\Delta} > Ke^{r(T-\Delta)})\ \Pr o(S_{T-2\Delta} < Ke^{r(T-2\Delta)})$$

……

$$+(K-S_0)\ \Pr o(S_T > Ke^{rT})\ \Pr o(S_{T-\Delta} > Ke^{r(T-\Delta)}) \ldots\ldots \Pr o(S_{T-m\Delta} < Ke^{r(T-m\Delta)})$$

……

$$+(K-S_0)\ \Pr o(S_T > Ke^{rT})\ \Pr o(S_{T-\Delta} > Ke^{r(T-\Delta)}) \ldots\ldots \Pr o(S_{\Delta} > Ke^{r\Delta})$$

Now let Δ tends to be zero, and T' tends to T, we will have:

$$\Pr o(S_T < Ke^{rT}) \approx \Pr o(S_{T-\Delta} < Ke^{r(T-\Delta)}) \approx \Pr o(S_{T-2\Delta} < Ke^{r(T-2\Delta)}) \ldots\ldots$$

$$\approx \Pr o(S_{T-m\Delta} < Ke^{r(T-m\Delta)}) = N(-d_4) \tag{3.11}$$

$$\Pr o(S_T > Ke^{rT}) \approx \Pr o(S_{T-\Delta} > Ke^{r(T-\Delta)}) \approx \Pr o(S_{T-2\Delta} > Ke^{r(T-2\Delta)}) \ldots\ldots$$

$$\approx \Pr o(S_{T-m\Delta} > Ke^{r(T-m\Delta)}) = N(d_4) \tag{3.12}$$

Since $N(d_4) < 1$, it is easy to have:

$$E[Max \Pr emium(earlyexercise)] =$$

$$P_E(S_0, Ke^{rT}, r, T, \sigma)\ N(-d_4) + (K-S_0)\ N(-d_4)\ [N(d_4) + N(d_4)^2 + \ldots + N(d_4)^m]$$

$$= P_E(S_0, Ke^{rT}, r, T, \sigma)\ N(-d_4) + (K-S_0)\ N(-d_4)\ \frac{N(d_4)[1-N(d_4)^{m-1}]}{1-N(d_4)}$$

$$= P_E(S_0, Ke^{rT}, r, T, \sigma)\ N(-d_4) + (K-S_0)\ N(d_4)$$

This tells us that the maximum option premium under the optimally early-exercised strategy is $P_E(S_0, Ke^{rT}, r, T, \sigma)$ or $(K-S_0)$, and the probability to take the value is $N(-d_4)$ or $N(d_4)$, respectively.

□

Now, it is the time to propose the closed form formula for pricing an American put option on a non-dividend paying stock.

Lemma 3.6: The price of an American put option $P_A(S_0, K, r, T, \sigma)$ on a non-dividend paying stock is:

$$P_A(S_0, K, r, T, \sigma) =$$

$$P_E(S_0, Ke^{rT}, r, T, \sigma)N(-d_4) + \max[(K - S_0), P_E(S_0, K, r, T, \sigma)]N(d_4) \quad (3.13)$$

Where,

$$P_E(S_0, K, r, T, \sigma) = Ke^{-rT}N(-d_2) - S_0N(-d_1) \quad (3.14)$$

$$P_E(S_0, Ke^{rT}, r, T, \sigma) = KN(-d_4) - S_0N(-d_3) \quad (3.15)$$

$$d_1 = \frac{\ln\frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln\frac{S_0}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \quad (3.16)$$

$$d_3 = \frac{\ln\frac{S_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_4 = \frac{\ln\frac{S_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \quad (3.17)$$

Proof: According to **Lemma 3.3** and **Lemma 3.5**, we have:

$$P_A(S_0, K, r, T, \sigma)$$
$$= E^Q\{\max[P_E(S_0, K, r, T, \sigma), Max\Pr emium(earlyexercise)]\}$$

Where $Max\Pr emium(earlyexercise) = P_E(S_0, Ke^{rT}, r, T, \sigma)$ or $(K - S_0)^+$, with the probability $N(-d_4)$ or $N(d_4)$, respectively.

Since $P_E(S_0, Ke^{rT}, r, T, \sigma) > P_E(S_0, K, r, T, \sigma)$

$$\therefore P_A(S_0, K, r, T, \sigma)$$
$$= E^Q\{P_E(S_0, Ke^{rT}, r, T, \sigma), \max[(K - S_0)^+, P_E(S_0, K, r, T, \sigma)]\}$$
$$= E^Q\{P_E(S_0, Ke^{rT}, r, T, \sigma), \max[(K - S_0), P_E(S_0, K, r, T, \sigma)]\}$$
$$= P_E(S_0, Ke^{rT}, r, T, \sigma)N(-d_4) + \max[(K - S_0), P_E(S_0, K, r, T, \sigma)]N(d_4)$$

□

Next, we will use this formula to present the price of a perpetual American put option on a non-dividend paying stock.

4. Perpetual American put option on a non-dividend paying stock

A perpetual American put option is a special kind of American put option. It grants its holder rights, but not obligation to sell an underlying stock in a fixed price at any time up until infinity future. Since the maturity time of such an option is infinity future, it is also called an expiration-less option.

Obviously, a perpetual American put option on a non-dividend paying stock should at least satisfy:

$$K - S_0 \le P_{PA}(S_0, K, r, \infty, \sigma) \le K \tag{4.1}$$

And

$$P_E(S_0, K, r, t, \sigma) \le P_{PA}(S_0, K, r, \infty, \sigma) \tag{4.2}$$

Where $0 \le t \le \infty$

4.1 Merton's formula

Merton (1973) proposed a formula for pricing a perpetual American put option. McDonald and Siegel (1986) discussed the link between the perpetual American put and perpetual American call.

For one dividend-paying stock, with current stock price S_0, strike price K, risk-free interest rate r, expiration time ∞, volatility σ and continuous dividend-paying rate δ, the formula[2] for pricing a perpetual American put option is:

$$P_{PA}(S_0, K, r, \infty, \sigma, \delta) = \frac{K}{1-h_2}\left(\frac{h_2-1}{h_2}\frac{S_0}{K}\right)^{h_2} \tag{4.3}$$

Where,

$$h_2 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}\partial} \tag{4.4}$$

For non-dividend paying stock, $\delta = 0$, the formulas become:

$$P_{PA}(S_0, K, r, \infty, \sigma) = \frac{K}{1-h_2}\left(\frac{h_2-1}{h_2}\frac{S_0}{K}\right)^{h_2} \tag{4.5}$$

Where,

$$h_2 = -\frac{2r}{\sigma^2} \tag{4.6}$$

Formula (4.5) tells us that the price of a perpetual American put option on a non-dividend paying stock is related to S_0, K, r and σ.

Now, we give some examples to show formula (4.5) does not do a good job. **Figure 4.1.1-4.1.3** [3]shows the results based on formula (4.5) for the price of perpetual American put options in different initial stock price, different risk-free interest rate and different volatility, respectively. It is very clear that in some cases, the price of the perpetual American put option is greater than the strike price, which is contrary to the upper

[2] Please see Robert L. McDONALD (2003), "Derivatives Markets", *Pearson Education, Inc.* 392-393
[3] All the figures in this paper are drawn with MATLAB

limit of perpetual American put option (4.1). This is apparently not reasonable.

4.2 Formula for pricing PAPO

In fact, the value of a perpetual American put option on a non-dividend paying stock could be drawn from **Lemma 3.6**. For a perpetual American put option on a non-dividend paying stock, the following lemma is proposed.

Lemma 4.2.1: The price of a perpetual American put option $P_{PA}(S_0, K, r, \infty, \sigma)$ on a non-dividend paying stock is equal to the strike price:

$$P_{PA}(S_0, K, r, \infty, \sigma) = K \qquad (4.7)$$

Proof: A perpetual American put option on a non-dividend paying stock is a special kind of American put option on a non-dividend paying stock:

$$P_{PA}(S_0, K, r, \infty, \sigma) = P_A(S_0, K, r, T, \sigma) \ (when T \to \infty)$$

From **Lemma 3.6**, we know that

$$P_A(S_0, K, r, T, \sigma) =$$

$$P_E(S_0, Ke^{rT}, r, T, \sigma)N(-d_4) + \max[(K - S_0), P_E(S_0, K, r, T, \sigma)]N(d_4)$$

$$d_4 = \frac{\ln\frac{S_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \xrightarrow{T\to\infty} -\infty$$

$$d_3 = \frac{\ln\frac{S_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = \xrightarrow{T\to\infty} \infty$$

$$\therefore N(d_4) \xrightarrow{T\to\infty} 0,\ \ N(-d_4) \xrightarrow{T\to\infty} 1,\ \ N(-d_3) \xrightarrow{T\to\infty} 0$$

$$P_E(S_0, Ke^{rT}, r, T, \sigma) = KN(-d_4) - S_0 N(-d_3) = \xrightarrow{T\to\infty} K$$

$$P_A(S_0, K, r, T, \sigma) = P_E(S_0, Ke^{rT}, r, T, \sigma) N(-d_4)$$

$$+\max[(K - S_0), P_E(S_0, K, r, T, \sigma)] N(d_4) = \xrightarrow{T\to\infty} K$$

So $P_{PA}(S_0, K, r, \infty, \sigma) = K$

□

This result can be seen clearly from **Figure 4.2.1-4.2.3**. Based on formula (3.13), **Lemma 3.6**, the relationship between the price of an American put option (at the money, in the money, out of money, respectively) and the maturity time T is show in **Figure 4.2.1-4.2.3**, from which we can notice that the price of an American put option rises while maturity time T increases and tends to K while the maturity time T is big enough.

It will be easier to understand this result from the economic meaning of an American put option in **Section 3**:

When $T \to \infty$, since the present value of $\mathrm{E}(S_T / S_T < Ke^{rT})$ is

$$\mathrm{E}(S_T / S_T < Ke^{rT}) e^{-rT} = S_0 e^{rT} \frac{N(-d_3)}{N(-d_4)} e^{-rT} = S_0 \frac{N(-d_3)}{N(-d_4)} \xrightarrow{T\to\infty} 0$$

And $\mathrm{Pr}o(S_T / S_T < Ke^{rT}) = N(-d_4) \xrightarrow{T\to\infty} 1$

If the writer of an American put option gives the buyer rights, but not obligation to borrow money K

from the writer at any time before expiration date, and return money $\mathrm{E}(S_T / S_T < Ke^{rT})$ in infinite future. How much of compensation should the writer ask for? Of course, the compensation should be K !

4.3 Arbitrage Opportunity

Now, let's prove there will be arbitrage opportunity if the price of a perpetual American put option on a non-dividend paying stock is not equal to the strike price.

Lemma 4.3.1: The price of a perpetual American put option $P_{PA}(S_0, K, r, \infty, \sigma)$ on a non-dividend paying stock is equal to the strike price K . Otherwise, there will be arbitrage opportunity.

Proof: (1) If the price of a perpetual American put option

$$P_{PA}(S_0, K, r, \infty, \sigma) > K$$

Writing a cash-secured put would earn arbitrage profits.[4]

(2) If the price of a perpetual American put option

$$P_{PA}(S_0, K, r, \infty, \sigma) < K$$

Let $P_{PA}(S_0, K, r, \infty, \sigma) = K' < K$

An arbitrager can make risk-free profits as follows: buy one perpetual American put option $P_{PA}(S_0, K, r, \infty, \sigma)$ in K' (His cost is K'), exercise this option immediately (short-sell a stock in K), and keep the money K in a banking account. The only thing the arbitrager needs to do next is just

[4] Please see YUH-DAUH LYUU (2002), " Financial Engineering and Computation, Principles, Mathematics, Algorithms", P86

waiting to time t_x until the stock price S_{t_x} at time t_x is less than $(K-K')e^{rt_x}$, which means

$$S_{t_x} < (K-K')e^{rt_x} \tag{4.8}$$

Then buy back a stock in S_{t_x} at time t_x. Since before infinite future, the arbitrager surely has such an opportunity, so the present value of risk-free profit of the arbitrager will be

$$\mathrm{Pr}ofit = K - K' - S_{t_x}e^{-rt_x} > 0 \tag{4.9}$$

In a word, the price of a perpetual American put option $P_{PA}(S_0, K, r, \infty, \sigma)$ on a non-dividend paying stock could not be greater or less than strike price K, so

$$P_{PA}(S_0, K, r, \infty, \sigma) = K$$

□

This conclusion tells us that time could 'erase' all differences. Since the maturity time of a perpetual American put option is the infinite future, as long as the strike prices are same, the value of these perpetual American put options will be same. They will be equal to the strike price.

5. Concluding remarks

An American put option grants its holder the rights, but not obligation to sell an underlying stock in the strike price at any time up until maturity, so the holder of an American put option has more rights than that of an otherwise equivalent European put option. The pricing of an American put option on a non-dividend paying stock could be described as an optimization problem: when the maximum option premium of early exercise is no less than the value of its European counterpart, the option should be early-exercised; otherwise, it should not be early-exercised. The price of an American put option on a non-dividend paying stock is equal to the expected value of maximum option premium.

The value of a perpetual American put option on a non-dividend paying stock is equal to the strike price, no matter the current stock price, the risk-free interest rate, and the volatility are. This can be shown by the no-arbitrage theory and by the formula of pricing American put option on a non-dividend paying stock. On the other hand, this also proves that Merton (1973)'s formula is not reasonable.

Based on formula (3.13), **Lemma 3.6**, we could also find that:

(1) Keep others constant, as the risk-free interest rate r increases, the value of $P_A(S_0, K, r, T, \sigma)$ decreases (See **Figure 5.1**); as the time to expiration T or volatility σ increases, the value of $P_A(S_0, K, r, T, \sigma)$ also increases (See **Figure 5.2**)

(2) The upper and lower bounds for an American put option on a non-dividend paying stock in **Lemma 3.2** are satisfied.

Appendix 1

Now, let's prove

$$\frac{\partial P_E(S_0, Ke^{r\varsigma}, r, \varsigma, \sigma)}{\partial \varsigma} > 0$$

Proof: $\dfrac{\partial P_E(S_0, Ke^{r\varsigma}, r, \varsigma, \sigma)}{\partial \varsigma}$

$$= \frac{\partial[KN(-\frac{\ln\frac{S_0}{K} - \frac{1}{2}\sigma^2\varsigma}{\sigma\sqrt{\varsigma}}) - S_0 N(-\frac{\ln\frac{S_0}{K} + \frac{1}{2}\sigma^2\varsigma}{\sigma\sqrt{\varsigma}})]}{\partial \varsigma}$$

$$= K\frac{\partial N(-\frac{\ln\frac{S_0}{K} - \frac{1}{2}\sigma^2\varsigma}{\sigma\sqrt{\varsigma}})}{\partial \varsigma} - S_0\frac{\partial N(-\frac{\ln\frac{S_0}{K} + \frac{1}{2}\sigma^2\varsigma}{\sigma\sqrt{\varsigma}})}{\partial \varsigma}$$

$$= S_0 N'(d_3)\frac{\partial\frac{\ln\frac{S_0}{K} + \frac{1}{2}\sigma^2\varsigma}{\sigma\sqrt{\varsigma}}}{\partial \varsigma} - KN'(d_4)\frac{\partial\frac{\ln\frac{S_0}{K} - \frac{1}{2}\sigma^2\varsigma}{\sigma\sqrt{\varsigma}}}{\partial \varsigma}$$

Where

$$N'(d_3) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d_3)^2}, \quad N'(d_4) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(d_4)^2}$$

Since

$$N'(d_4) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_4)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_3 - \sigma\sqrt{\varsigma})^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_3)^2 + d_3\sigma\sqrt{\varsigma} - \frac{1}{2}\sigma^2\varsigma}$$

$$= N'(d_3)\frac{S_0}{K}$$

So, we have

$$\frac{\partial P_E(S_0, Ke^{r\varsigma}, r, \varsigma, \sigma)}{\partial\varsigma} = S_0 N'(d_3)[\frac{\partial \frac{\ln\frac{S_0}{K} + \frac{1}{2}\sigma^2\varsigma}{\sigma\sqrt{\varsigma}}}{\partial\varsigma} - \frac{\partial \frac{\ln\frac{S_0}{K} - \frac{1}{2}\sigma^2\varsigma}{\sigma\sqrt{\varsigma}}}{\partial\varsigma}]$$

$$= S_0 N'(d_3)[\frac{\frac{1}{2}\sigma^2\varsigma - \ln\frac{S_0}{K}}{2\sigma\varsigma^{\frac{3}{2}}} - \frac{-\frac{1}{2}\sigma^2\varsigma - \ln\frac{S_0}{K}}{2\sigma\varsigma^{\frac{3}{2}}}]$$

$$= S_0 N'(d_3)\frac{\sigma}{2\sqrt{\varsigma}} > 0$$

Appendix 2: Mathematics Notations

In this paper, we use the following notations:

APO: American put option

$PAPO$: Perpetual American put option

S_0: Current stock price (Stock price at time 0)

K: Strike price of option, or exercise price of option

T: Time to expiration of option

t: A future point in time

S_T: Stock price at maturity (Stock price at time T)

r:[5] Continuously compounded risk-free interest rate for an investment maturing in time T

σ: Volatility of the stock price

$N(x)$: Cumulative probability that a variable with a standardized normal distribution is less than x. A standardized normal distribution is a normal distribution has a mean of zero and standard deviation of 1.0.

$P_E(S_0, K, r, T, \sigma)$: Value of an European put option to sell one non-dividend paying stock, with current stock price S_0, strike price K, risk-free interest rate r, expiration time T, volatility σ

$C_E(S_0, K, r, T, \sigma)$: Value of an European call option to buy one non-dividend paying stock, with current stock price S_0, strike price K, risk-free interest rate r, expiration time T, volatility σ

[5] r is the normal rate of interest, not the real rate of interest, $r > 0$

$P_A(S_0, K, r, T, \sigma)$: Value of an American put option to sell one non-dividend paying stock, with current stock price S_0, strike price K, risk-free interest rate r, expiration time T, volatility σ

$C_A(S_0, K, r, T, \sigma)$:[6] Value of an American call option to buy one non-dividend paying stock, with current stock price S_0, strike price K, risk-free interest rate r, expiration time T, volatility σ

$P_{PA}(S_0, K, r, \infty, \sigma)$:[7] Value of a perpetual American put option to sell one non-dividend paying stock, with current stock price S_0, strike price K, risk-free interest rate r, expiration time ∞, volatility σ

[6] For non-dividend paying stock, $C_A(S_0, K, r, T, \sigma) = C_E(S_0, K, r, T, \sigma)$

[7] It is known as expiration-less option

References

[1] Xiaodong Wang (2004): “ A closed form formula for pricing American put options”, *The Chinese Journal of Economic Theory, Volume 1, No.2, 2004*, 360-374

[2] Jamil Baz, and George Chacko (2004): “ Financial Derivatives, *Pricing, Applications, and Mathematics*”, *Cambridge University Press*, 48-65

[3] John G. Hull (2003): “ Options, Futures, and Other Derivatives” (Fifth edition), *Pearson Education International*, 254-256

[4] Robert L. McDONALD (2003): “ Derivatives Markets”, *Pearson Education, Inc.* 12,286,300,577-584

[5] YUH-DAUH LYUU (2002): “Financial Engineering and Computation, Principles, Mathematics, Algorithms”, *Cambridge University Press*, 84-91

[6] Black, and Scholes (1973): “ The Pricing of Options and Corporate Liabilities”, *Journal of Political Economy*

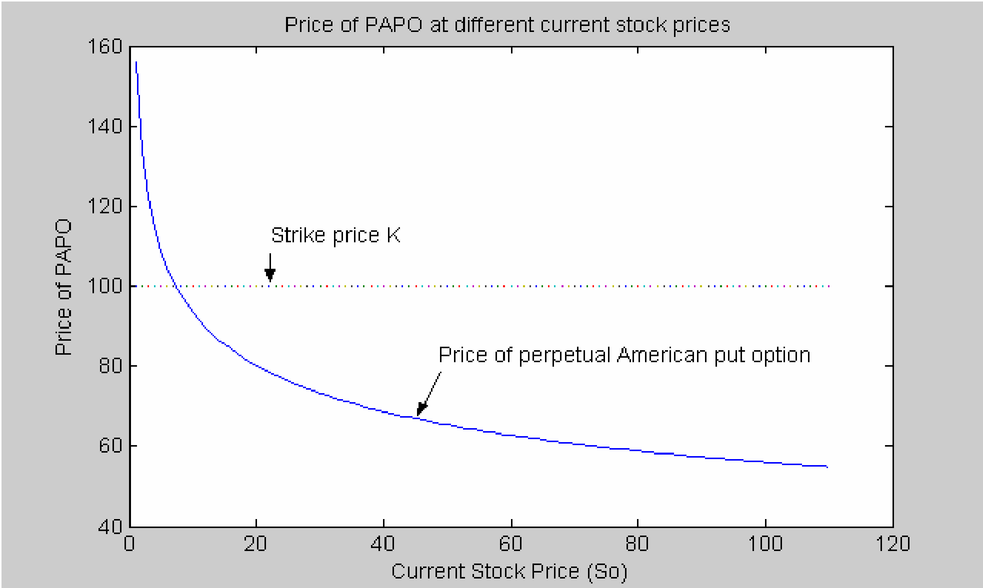

Figure 4.1.1. Price of PAPO at different current stock prices based on Formula (4.5)

$(K = 100, r = 0.04, \sigma = 0.6)$

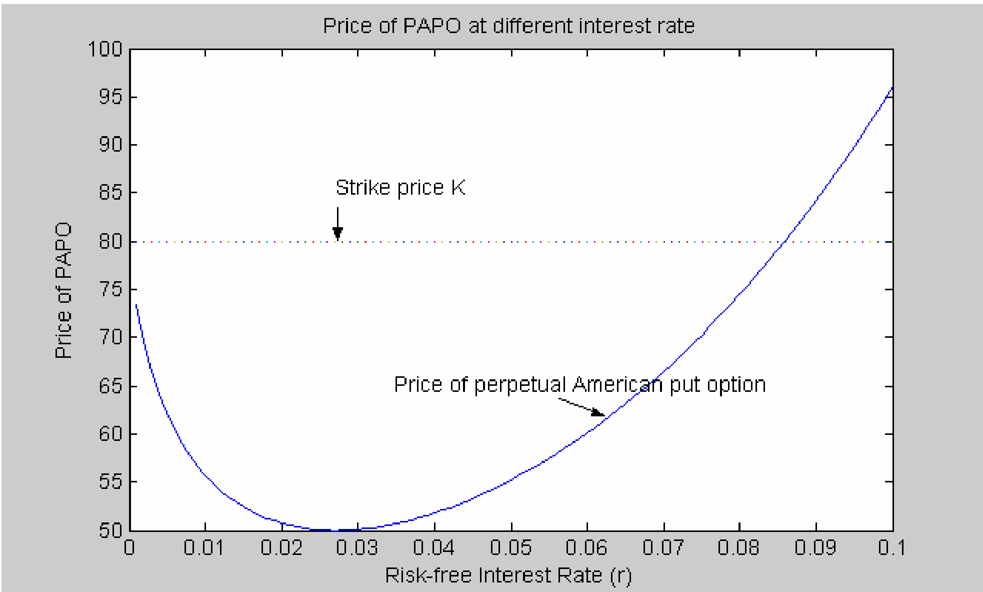

Figure 4.1.2. Price of PAPO at different risk-free interest rate based on Formula (4.5)

$(S_0 = 30, K = 80, \sigma = 0.3)$

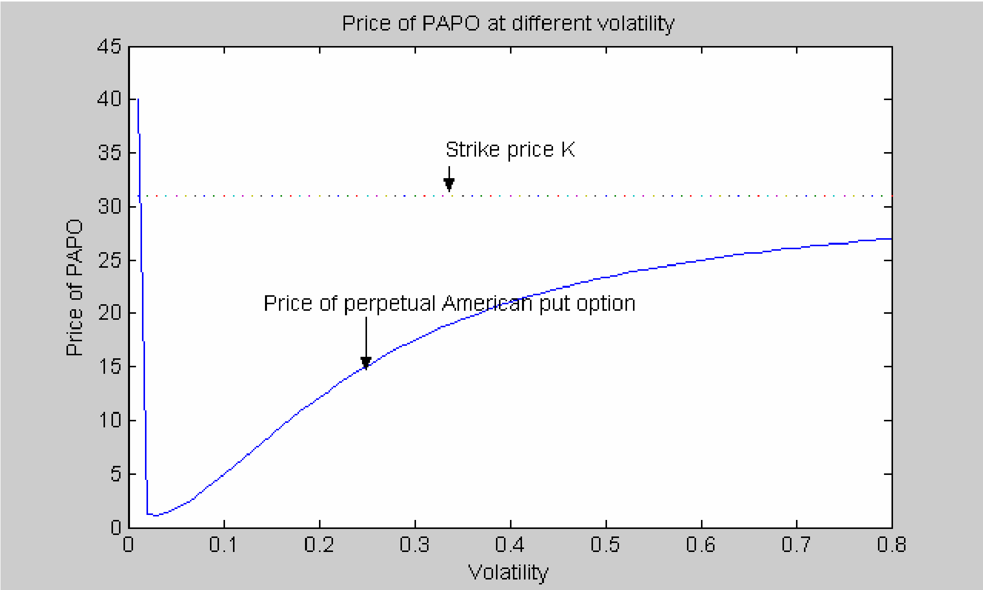

Figure 4.1.3. Price of PAPO at different volatility based on Formula (4.5)

$(S_0 = 30, K = 31, r = 0.01)$

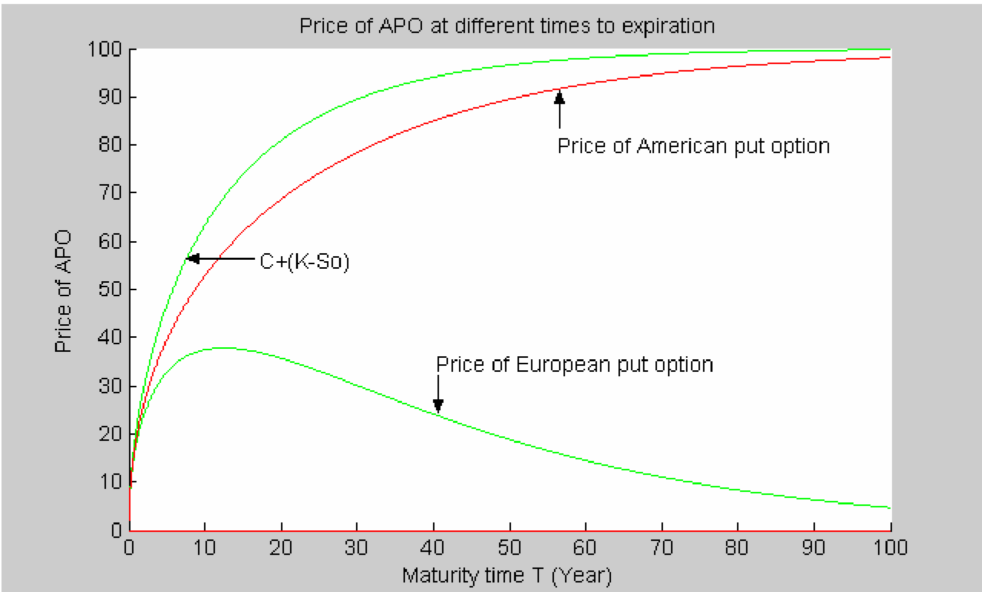

Figure 4.2.1. Price of APO at different T based on formula (3.13)

$(S_0 = 100, K = 100, r = 0.03, \sigma = 0.5)$

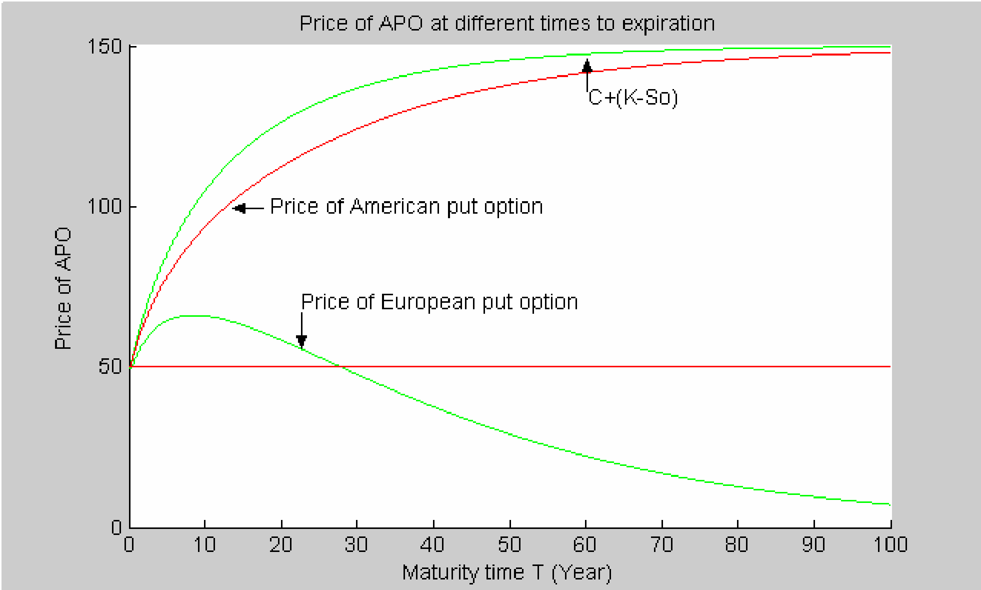

Figure 4.2.2. Price of APO at different T based on formula (3.13)

$(S_0 = 100, K = 150, r = 0.03, \sigma = 0.5)$

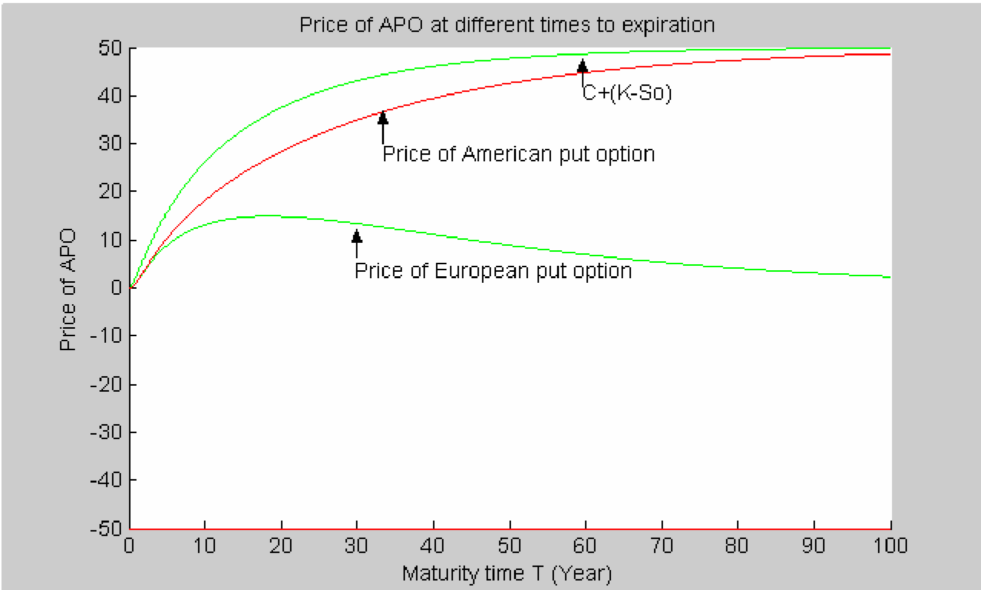

Figure 4.2.3. Price of APO at different T based on formula (3.13)

$(S_0 = 100, K = 50, r = 0.03, \sigma = 0.5)$

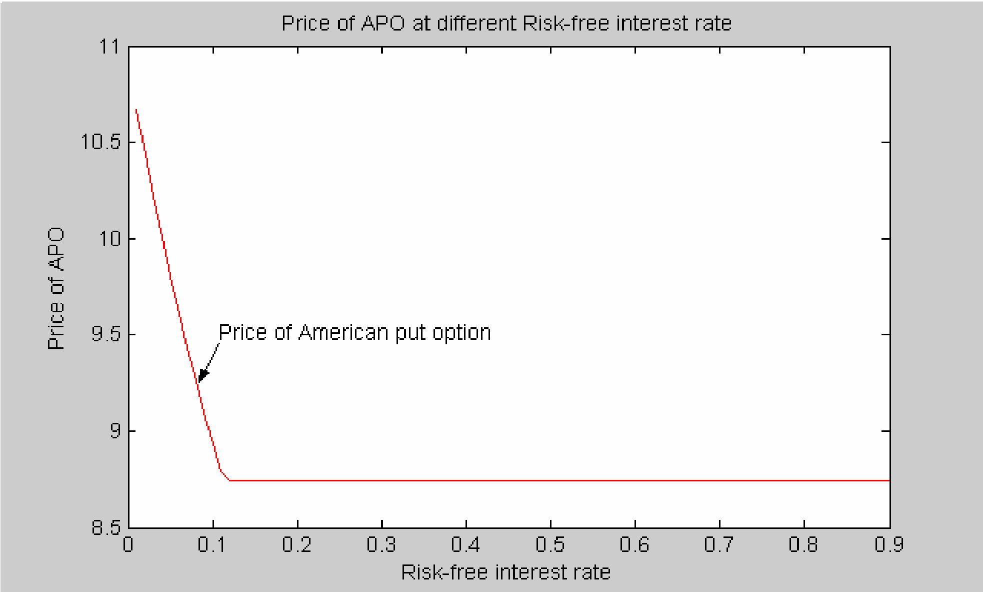

Figure 5.1. Price of APO at different risk-free interest rate r based on formula (3.13)

$(S_0 = 100, K = 105, T = 1, \sigma = 0.2)$

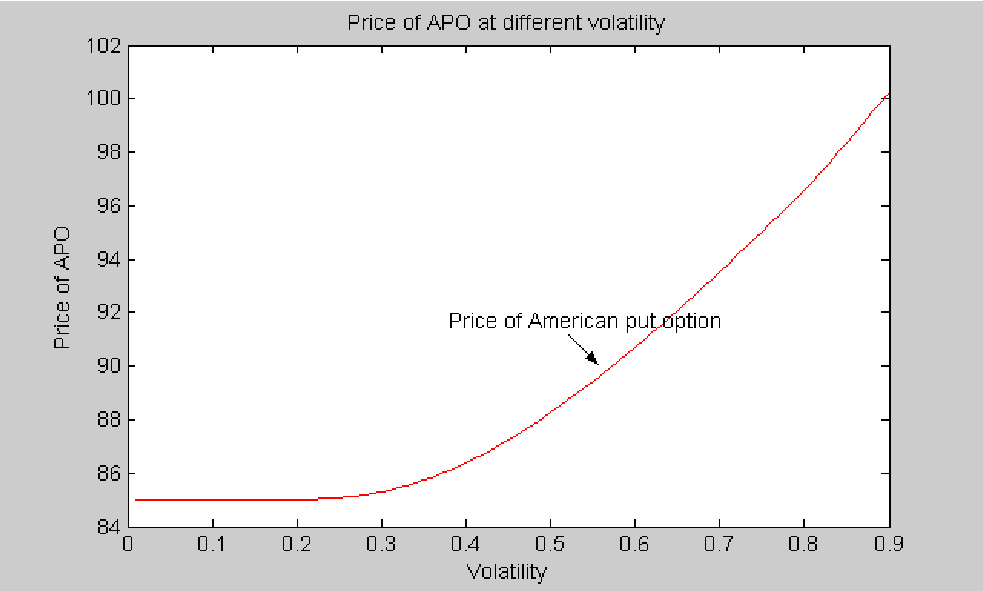

Figure 5.2. Price of APO at different volatility σ based on formula (3.13)

$(S_0 = 100, K = 185, T = 1, r = 0.09)$