Asymmetric butterfly velocities in local time-independent Hamiltonians

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We have asymmetric butterfly velocities.

We'll want to cite a bunch of people [18, 19, 21–24, 26–46].

ACKNOWLEDGEMENTS

We thank many people.

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[64] Here $\tilde{v}_B(\hat{\mathbf{n}})$, with a tilde, denotes the normal propagation speed of a straight front whose normal is parallel to $\hat{\mathbf{n}}$. In Ref. [43] this was denoted $v_B(\hat{\mathbf{n}})$, but here we use $v_B(\hat{\mathbf{n}})$ to denote the speed at which an initially local operator spreads away from the origin in the direction $\hat{\mathbf{n}}$. These differ because in the absence of rotational symmetry the operator's front is not in general perpendicular to the radial vector, but they are related by a geometrical construction known from classical droplet growth [43, 87, 88].

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[69] The expression on the left-hand side of (**??**) becomes a "partition function" for two paths. The local weights $\partial u(\mathbf{y}_{i+1}, i+1)/\partial u(\mathbf{y}_i, i)$ depend not only on $\mathbf{y}_{i+1}$ and $\mathbf{y}_i$ but also on the configuration $u(\mathbf{y_i}, i)$. The chaotic time-dependence of $u(\mathbf{y}_i, i)$ means that the configurational average has a similar effect to averaging over weakly correlated randomness in the weights. Since we are averaging the "partition function", rather than its logarithm, this is an annealed average, and $-\lambda(\mathbf{v})t$ is an annealed "free energy" for the pair of paths. The quenched free energy, in which we take the logarithm before averaging, would give the more conventional definition of the Lyapunov exponent [66–68].

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[71] Inside the light cone there is a large deviation form governing convergence to the saturation value: $C^{\mathrm{rc}}_{1d}(x,t) \sim 1 - \exp\left(-\frac{(v-v_B)^2}{2D} t\right)$. The exponent here is the continuation of $\lambda(v)$ outside the front. However, in the higher dimensional examples, the large deviation form inside the front scales with a distinct power of t, t^d in d spatial dimensions [74]. In the presence of additional conserved densities (like energy or charge), the late time saturation of the OTOC is a power-law in time instead of exponential [45, 46].

[72] In random circuits related random walk pictures underlie the calculation of both the OTOC and the second Renyi entropy [43, 44]. In these random systems this yields a relation between $\lambda(v)$ and the "entanglement line tension" defined in [59], specifically the line tension $\mathcal{E}_2(v)$ for the second Renyi entropy. This motivates the conjecture, for non-random systems, that $\lambda(v)|_{\mathrm{cont}} = -s_{\mathrm{eq}}(\mathcal{E}_2(v) - v)$, where s_{eq} is the thermal entropy density. The left hand side denotes the analytic continuation of $\lambda(v)$ from $v > v_B$ to values $v < v_B$. In random circuits we must distinguish different kinds of averages. The line tension extracted from a calculation of $\overline{e^{-S_2}}$ determines $\lambda(v)$ for the average OTOC $\overline{C(x,t)}$ by the above formula. It is natural to expect that the line tension determined by the more natural direct average $\overline{S_2}$ determines $\lambda(v)$ for the typical value of the OTOC, $\exp \overline{\ln C(x,t)}$. The average and typical values of the OTOC are parametrically close in the region close to the front, but they may differ significantly in the far-front regime where both are exponentially small.

[73] In some circuit models in $d > 1$ (which do not have continuous spatial rotation symmetry) some sections of the operator's front can be "glued" to the strict lightcone defined by the discrete time circuit [43]. This is a peculiar case where $v_B(\hat{\mathbf{n}}) = v_{\mathrm{LC}}(\hat{\mathbf{n}})$ for some directions $\hat{\mathbf{n}}$ in space, so that no nontrivial $\lambda(\mathbf{v})$ can be defined for these directions of $\mathbf{v}$.

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[85] Let the probability distribution for weak-link "waiting times" be $P(\tau) \sim \tau^{-a-2}$. At weak disorder ($1 < a$) the broadening of the operator's front [50] is diffusive, as in the clean system. At intermediate disorder ($0 < a < 1$) the front broadens more strongly, giving $\lambda(v) \sim -(v - v_B)^{(a+1)/a}$. For strong disorder ($-1 < a < 0$) the butterfly speed vanishes: in this regime $\lambda(v) \sim -|v|^{1-|a|}$. In the disordered system the definition

of $\lambda(v)$ depends on whether we consider e.g. the mean or the typical value of the OTOC, but this should not change these exponents.

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