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Exact Results on Maximally Supersymmetric Conformal Field Theories in Three Dimensions

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Abstract

In this thesis, we present exact results on $\mathcal{N} = 8$ superconformal field theories (SCFTs) in three dimensions. In the first part, we extend previous work by beginning the setup for the $\mathcal{N} = 8$ superconformal bootstrap in three dimensions. In particular, we express the four-point functions of superconformal primaries in the $(8, B, +)_{[0020]}$ and $(8, B, +)_{[0040]}$ multiplets as a sum of $SO(8)$ harmonics multiplied by conformal blocks. Using the associativity of the operator algebra, as well as the Ward identities, we determine the maximal set of independent crossing equations relating the conformal blocks. In the second part of this work, we use the results of recent work on supersymmetric localization to obtain exact analytic expressions for the OPE coefficients of certain low-lying $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS operators in the $U(N)_1 \times U(N)_{-1}$ ABJM theories $(N = 2, 3)$ and for the $SU(2)_3 \times SU(2)_{-3}$ BLG theory. We also find that the partition function values, the normalized four-point functions, and the OPE coefficients all match for the interacting sector of the $U(3)_1 \times U(3)_{-1}$ ABJM theory and for the $SU(2)_3 \times SU(2)_{-3}$ BLG theory. We therefore conjecture a previously unknown duality between the two theories.

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To my family.

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1 Introduction

Quantum field theory (QFT) is a broad mathematical framework that spans practically all areas of theoretical physics, from elementary physics, such as of particles and strings, to more applied areas like condensed matter and many-body systems. Under this framework, physics has enjoyed enormous success with some of the most accurate predictions of nature to date. The quintessential example, the aptly-named Standard Model of Particle Physics, classifies all known elementary particles and their interactions – the strong, weak, and electromagnetic forces. On the thermodynamic side, a system near a second-order phase transition can be described by a conformal field theory [1]. Additionally, the subject has flourished under the discovery of a number of dualities, such as between quantum field theory and statistical mechanics, or with the AdS/CFT correspondence between gravity in the bulk of anti-de Sitter space and conformal field theory on the boundary [2, 3].

However, the perturbative techniques that led to many of quantum field theory's successes cannot adequately address certain modern problems. The following four examples illustrate this point. First, it is of interest to classify all potential quantum field theories. However, the "space" of potential theories is vast, and general shared symmetries such as Lorentz or rotational invariance are insufficient to uniquely determine their specific features. In addition, strongly coupled fields theories with distinctly non-perturbative features, such as the continuum limit of a statistical mechanical system undergoing a second order phrase transition (i.e. at the critical point), cannot be studied using perturbation theory.[1] Next, it is believed that some quantum field theories do not have a microscopic (Lagrangian) description, and standard perturbation theory can only be applied to theories with a well-defined Lagrangian [4]. Lastly, basic philosophical arguments indicate that gravity should be quantum-mechanical on length scales on the order of the Planck length. However, it is well-known that quantum gravity is a non-renormalizable field theory, implying that it does not have a good definition as a quantum field theory.

In hopes of understanding the non-perturbative features of quantum field theory, we must first address the basic but difficult question of "what is a QFT?". One potential answer lies in the ideas of the renormalization group, whose field theoretic

[1] In some cases perturbative techniques, such as the large-N or ϵ expansions, are still applicable.

origins are due to in large part to work done in the 1970s by Kenneth Wilson [5–7]. The Wilsonian renormalization group (RG) is a formalism that describes how the interactions of a QFT and their relative strengths vary as one "integrates out" the high energy degrees of freedom. This process of "zooming out" generates a flow in the space of quantum field theories from a region in the ultraviolet (UV) with large energies and momenta to the smoothed-out infrared (IR) region of large distances and small energies. In the RG language, a UV-complete quantum field theory is thus defined as an RG flow in theory space from the UV to the IR, where each point in the flow gives some effective field theory at the particular length scale under consideration.

On a more technical level, the RG flow is generated by a set of differential equations known as the renormalization group equations [8, 9]. This motivates us to consider their fixed points, which play an important role in determining the entirety of the resulting flow. It is often the case that these fixed points are none other than the endpoints of the flow, with one point in the UV and the other in the IR. Moreover, the RG fixed points can be interpreted physically as representing second order phase transitions of either a quantum or statistical mechanical system. In fact, they are also necessarily scale-invariant with no particle excitations. Rather surprisingly, it is often the case that the scale invariance of these fixed points is enhanced to conformal invariance, whose study falls under the domain of conformal field theory [10].

Conformal field theories are quantum field theories invariant under conformal transformations, including rotations and Lorentz transformations, as well as local and global rescalings known as dilatations. Upon first glance, it would seem as though these theories are physically irrelevant and only a mathematical curiosity. Indeed, many field theories possesses a characteristic length scale that breaks both scale and conformal invariance. For instance, QFTs such as the Standard Model are scale-dependent by virtue of the fact that they contain massive particles. Even massless theories which have classical conformal invariance, such as Maxwell's electromagnetism, often break this symmetry on the quantum level due to the presence of hidden length scales. In the case of electromagnetism, this characteristic scale manifests itself as the infamous Landau pole, signaling a breakdown of the theory at high energies.

On the contrary, conformal field theories are not only ubiquitous throughout nature, but they also play a crucial role in quantum field theory in general. First, as previously mentioned, all second-order phase transitions in nature are described by a CFT, regardless of the particular details of the microscopic theory. More generically, as the endpoints of RG flows, CFTs potentially contain information about the entire flow, and hence about the intermediate non-conformal QFTs. Therefore, one might hope to gain information on general quantum field theories by studying conformal field theory.

With a large amount of symmetry, conformal field theories represent one of the best avenues for tackling the daunting task of classifying all possible quantum field theories. Conformal invariance imposes highly restrictive constraints on the correlation functions of local operators (generalized fields) in the theory. For instance, the functional forms of the two- and three-point correlation functions are exactly determined to within a multiplicative constant. Another powerful distinguishing feature of conformal field theory is the idea of the operator product expansion (OPE), which endows CFTs with an associative operator algebra. Under this algebra, a product of local operators can be approximated by a sum over all operators in the theory. In quantum field theory, the OPE is not necessarily guaranteed to exist [11]. Even if it does, it may only exist as an asymptotic series, and in the limit where operators insertions are taken to be arbitrarily close together in spacetime [12]. For conformal field theories, the OPE is not only a convergent series, but one with a finite radius of convergence [13]. Most importantly, these features of conformal invariance, in addition to being non-perturbative, are completely general and do not depend on the particular microscopic definition of the theory.

One of the most promising non-perturbative techniques in conformal field theory is the conformal bootstrap. Originally developed by [14–16], the essential idea of the bootstrap is that the associativity of the OPE can be used to generate an infinite set of relations that impose restrictive constraints on the operator spectrum in any given theory. The bootstrap was first used successfully in two dimensions, where conformal symmetry is enhanced to an "infinite" local symmetry, to exactly compute the correlation functions of the so-called minimal models [17] as well as of Liouville theory [18].

In more than two dimensions, conformal invariance is less restrictive: after its inception and application to various two-dimensional theories, it was subsequently abandoned. However, the technique was recently revived when it was found that the bootstrap can be formulated as a numerical optimization problem [19]. Using optimization methods, one can use the bootstrap to place rigorous non-perturbative bounds on the operator spectrum as well as on various theory-specific constants known as OPE coefficients. When a theory saturates these bounds, it is possible to extract actual numerical values for this "CFT data." In extraordinary cases, the bootstrap is able to confine the theory space to an "island" in theory space that shrinks with increasing computational precision [20]. For these cases, the CFT data can be determined to arbitrary precision, essentially solving the corresponding theory.

At present, the numerical bootstrap has been successfully applied to a variety of CFTs in three and greater dimensions. One of the most notables successes is that of the Ising Model in three dimensions, for which the bootstrap uncovered a shrinking island. On a practical level, this means that the numerical bootstrap can calculate the theory's critical exponents to what is in principle arbitrary precision [20–22].

We would like to use the numerical bootstrap to find more instances of islands in the space of CFTs. Unfortunately, like the space of QFTs, this space is itself quite large. To make any progress, we must study more specific types of field theories. Since QFTs with more symmetry tend to also be more constrained, the corresponding parameter spaces are likely to be smaller in size. Thus, our best chances of finding new islands resides in applying the bootstrap to theories with the largest amount of symmetry.

In addition to conformal symmetry, it is possible to consider internal symmetries that act by "rotating" the fields themselves instead of the underlying spacetime. A natural question to ask is whether the two types can be combined nontrivally into a single, larger symmetry. For these types of symmetries, the generators induce simultaneous transformations on the spacetime itself and among the fields. Famously, Coleman and Mandula produced a no-go theorem stating that this is not possible. According to the Coleman-Mandula theorem, the unbroken symmetry group of a relativistic quantum field theory can only take the form of a direct product between the spacetime symmetry group (e.g., Poincaré group or the conformal group) and the internal symmetry group [23, 24]. Any other symmetry group has a vanishing S-matrix, rendering the theory trivial. Practically, this means that the spacetime generators commute with the generators of the internal symmetry.

In a sense, the Coleman-Mandula theorem restricts all possible quantum field theories from having exotic symmetries, with the most intricate allowed spacetime symmetry being conformal symmetry. However, there exists a loophole in that the theorem only applies to the commutation relations of the generators. In 1975, Haag et. al. showed that it was possible to nontrivially extend the Poincaré symmetry in four dimensions by including a new type of symmetry called supersymmetry (SUSY). In particular, the Haag-Lopuszański-Sohnius theorem extends the Poincaré algebra to the super-Poincaré algebra, which includes spin 1/2 fermionic generators with nontrivial *anti*-commutation relations [25].

Supersymmetric (SUSY) field theories are quantum field theories with the feature that every boson has an associated fermion partner, and vice-versa [26]. Supersymmetry acts by exchanging these partners. The corresponding generator, which transforms as a spinor under Lorentz transformations, is called the Poincaré supercharge. Acting with the supercharge on a bosonic field yields a fermionic field. Acting again with the supercharge gives back the original bosonic field, but now translated to some new position. There is where the nontrivial anti-commutation relations come into play: the supercharges anti-commute to form translations. We can thus think of supercharges in a rough sense as the square root of the momentum generator.

It is also possible to consider supersymmetric QFTs with more than a single supercharge. For instance, such theories might pair a scalar (spin 0) with a spinor (spin 1/2) through a single supercharge, and then the same spinor with a vector (spin 1) through another. Theories with more than one supercharge are said to have extended SUSY with $\mathcal{N}$ spinorial supercharges ($2\mathcal{N}$ real supercharges). These theories have an additional internal symmetry, known as R-symmetry, that rotates the supercharges into one another. In three dimensions, the R-symmetry group for $\mathcal{N}$ supercharges is given by the double cover of the rotation group $SO(\mathcal{N})$, i.e. Spin($\mathcal{N}$). It is believed that all relevant quantum field theories (without gravity) have particles with spin of at most 1 (the photon/gauge boson) — this places a bound on the maximum amount of supersymmetry. In both three and four dimensions, the maximum is given by 16 real supercharges, which is denoted by $\mathcal{N} = 8$ in three dimensions and $\mathcal{N} = 4$ in four dimensions [27].

Another extension one can consider is the supersymmetric version of conformal field theories, known as superconformal field theories (SCFTs) [28]. For these theories, the number of supercharges is doubled to include a new type of supercharge called the conformal supercharge. Just as the Poincaré super charge anti-commutes to the momentum, the conformal supercharge anti-commutes to the generator of special conformal transformations.

Due to their large amount of symmetry, superconformal field theories are some of the most constrained quantum field theories. It is also known that SCFTs in three dimensions with $\mathcal{N} \geq 4$ SUSY have no continuous parameters. Therefore, we are most likely to find isolated theories (shrinking islands) for these types of theories and, in particular, for $\mathcal{N} = 8$ maximally supersymmetric CFTs in three dimensions. Maximally supersymmetric CFTs in three dimensions were previously studied in [29, 30] through the single-correlator bootstrap. The results yielded both upper and lower bounds on the operator spectrum, but no nontrivial islands were found. In this paper, we will continue their search by beginning the setup for the mixed superconformal bootstrap, which is expected to produce stronger bounds on the CFT data.

Adding information on explicit theories to the bootstrap has the potential to impose more stringent bounds on the space of theories. This motivates us to study explicit examples of $\mathcal{N} = 8$ SCFTs. All known $\mathcal{N} = 8$ SCFTs in three dimensions are given by or are dual to Chern-Simons-matter theories [31] with a product gauge group coupled to two matter hypermultiplets transforming in the bifundamental representation. Broadly, all of these theories belong to one of three infinite families, given by the $SU(2)_k \times SU(2)_{-k}$ theories of Bagger-Lambert-Gustavsson (BLG) [32–35] with Chern-Simons level k; the $U(N)_k \times U(N)_{-k}$ theories of Ahrony-Bergman-Jafferis-Maldacena (ABJM) [36] with integer N and Chern-Simons level $k = 1, 2$; and the $U(N+1)_2 \times U(N)_{-2}$ theories of Ahrony-Bergman-Jafferis (ABJ) [37]. All theories have manifest $\mathcal{N} = 8$ SUSY except for the ABJM theories, whose $\mathcal{N} = 4$ SUSY is enhanced to $\mathcal{N} = 8$ by monopole operators [38]. For convenience, we will label the ABJ(M) theories by $\text{ABJ(M)}_{N,k}$ and the BLG theories by BLG_k.

There are various known dualities between these infinite families and super Yang-Mills theories in the IR. In particular, the $U(N)$ and $O(2N)$ super Yang-Mills (SYM) theories are believed to flow to the same IR fixed point as the $\text{ABJM}_{N,k}$ theories for

$k = 1, 2$. The $SO(2N+1)$ SYM theories are thought to flow to the ABJ$_{N,2}$ theories [38–42]. In the IR, it is known that the ABJM/SYM theories split into the direct product of two $\mathcal{N} = 8$ SCFTs, one free and one interacting. For the SYM theories, one can explicitly decompose all of the fields into trace and traceless parts that do not interact, where the trace part is free and the traceless part is a superconformal $SU(N)$ gauge theory. This product decomposition can be directly observed at the classical level for the SYM theories, whereas it is only visible at the quantum level for the ABJM theories [40].

There are also examples of dualities between the ABJ(M) and BLG theories. In general, the two families are likely distinct, since the number of degrees of freedom in ABJ(M) theories grows arbitrarily large in the limit of large N, whereas it remains finite for all values of k for the BLG theories. However, there are a few known coincidental dualities for low values of N, k, which are given by [41, 43]:

$$\boxed{U(2)_1 \times U(2)_{-1} \text{ ABJM and } (SU(2)_1 \times SU(2)_{-1})/\mathbb{Z}_2 \text{ BLG,}} \tag{1.1}$$

$$\boxed{U(2)_2 \times U(2)_{-2} \text{ ABJM and } SU(2)_2 \times SU(2)_{-2} \text{ BLG,}} \tag{1.2}$$

$$\boxed{U(3)_2 \times U(2)_{-2} \text{ ABJM and } (SU(2)_4 \times SU(2)_{-4})/\mathbb{Z}_2 \text{ BLG.}} \tag{1.3}$$

In this thesis, we focus on determining certain quantities exactly for a few instances of these infinite $\mathcal{N} = 8$ superconformal families. We apply the results of the localization technique developed [44] to the IR limit of SYM theories with gauge group $U(N)$ for $N = 2, 3$, as well as to the BLG$_3$ theory. The $U(N)$ gauge theories flow to the same $\mathcal{N} = 8$ SCFT as the ABJM$_{N,1}$ theories. We find evidence for an unknown duality between the interacting sector of the ABJM$_{3,1}$ theory and the BLG$_3$ theory. We therefore conjecture a new duality between the

$$\boxed{\text{Interacting sector of } U(3)_1 \times U(3)_{-1} \text{ ABJM and } (SU(2)_3 \times SU(2)_{-3})/\mathbb{Z}_2 \text{ BLG}} \tag{1.4}$$

Due to the large amount of symmetry in SCFTs, it is sometimes possible to compute quantities in such theories exactly using only general symmetry considerations. For instance, it is possible to determine exact properties of certain operators that reside in a one-dimensional topological sector of $\mathcal{N} = 4$ SCFTs [30, 45, 46]. In particular, various crossing relations for the OPE coefficients were derived in [30] using the topological crossing symmetry of the single correlator. In this case, there were sufficiently many equations to determine a few of the OPE coefficients of low-lying BPS operators. In this thesis, we will extend the work of [30] and derive crossing equations for the mixed correlator with the goal of computing more coefficients. We are also interested in computing the OPE coefficients for specific examples of $\mathcal{N} = 8$ SCFTs.

When focusing on a certain theory, determining local quantities often boils down to computing correlation functions directly or indirectly through the Feynman path integral [47]. The Feynman integral is an important theoretical tool in quantum field theory that, whenever available, contains all information on a given QFT. For Lorentzian field theories, the path integral is given by a functional integral over all possible field configurations, weighted by an exponential factor containing the imaginary unit multiplied by the action. As an infinite-dimensional oscillatory integral, this type of path integral is ill-defined mathematically. It is common to instead work with the Euclidean path integral, which is weighted by a decaying exponential that is often sufficient for convergence. All observables of interest in quantum field theory are given by correlation functions, which can be determined by inserting the relevant operators into the Euclidean path integral. The results for the Lorentzian field theory can usually be recovered by performing a Wick rotation from Euclidean (imaginary) time to real time.

In practice, path integrals are all but impossible to calculate except for a few exceptional cases, such as for exactly solvable Gaussian models. Indeed, most of the time, these integrals only represent interesting formal constructs used in high-level field theoretic arguments. However, for certain supersymmetric gauge theories, a mathematical technique called supersymmetric localization is able to reduce the infinite dimensional path integral to a finite matrix integral [31, 48]. Recently, the localization technique was extended to localizing the full path integral of supersymmetric gauge theories to a matrix model coupled to a one-dimensional Gaussian theory; this model was then used to calculate the correlation functions of special types of operators called Higgs branch operators [44]. In this paper, we will use the results of [44] to compute the correlation functions of $\frac{1}{2}$-BPS operators for certain $\mathcal{N} = 8$ theories.

This thesis is organized as follows. In Section 2, we begin by introducing the necessary background on conformal field theory, and in particular the conformal bootstrap. In Section 3, we discuss the algebraic aspects of superconformal field theories in three dimensions and present the conformal content of two $\frac{1}{2}$-BPS operators, the superconformal primaries of the $(8, B, +)_{[0020]}$ and $(8, B, +)_{[0040]}$ multiplets. Additionally, we begin the setup for the mixed bootstrap for these two operators and find the maximal set of independent crossing equations. In Section 4, we review the construction of the one-dimensional topological sector in [30] and derive a general set of crossing relations among the OPE coefficients for certain low-lying $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS operators for $\mathcal{N} = 8$ theories. In Section 5, we apply the localization results of [44] to the $\mathrm{ABJM}_{N,1}$ theories for $N = 1, 2$ as well as to the BLG_3 theory, exactly computing the four-point functions in the corresponding localized theories. Using these functions, as well as the crossing constraints, we manage to extract analytic forms for all of the OPE coefficients. We also provide evidence for a new duality between $\mathrm{ABJM}_{3,1}$ and BLG_3.

2 Conformal Field Theory

2.1 Conformal Symmetry

A conformal transformation $f : M \to M$ is a transformation on the underlying spacetime M which preserves the metric tensor g up to an overall, possibly position-dependent scale factor $\Omega > 0$. Denoting the transformed metric by g', we have that [49]

$$f^* g' = \Omega g. \tag{2.1}$$

For the case where $\Omega = 0$, the metric is preserved, and f is simply an isometry of M. It is also convenient to consider the above expression in coordinate form. In particular, we introduce a set of local coordinates $x \in U \subset M$, with the transformed coordinates denoted by $y = f(x)$. The definition of a conformal transformation then takes the form

$$\frac{\partial y^\sigma}{\partial x^\mu}\frac{\partial y^\rho}{\partial x^\nu} g_{\sigma\rho}(y) = \Omega(x) g_{\mu\nu}(x). \tag{2.2}$$

where the Greek indices $\mu, \nu, \sigma, \rho = 0, 1, \ldots, d-1$ with $d = \dim(M) \geq 3$.[2] In this thesis, we will only concern ourselves with the d-dimensional Minkowski spacetime $(\mathbb{R}^{d-1,1}, \eta)$. Unless we specify otherwise, we will take η to be the usual Minkowski metric in the"mostly plus" convention, i.e. $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$. The set of all conformal transformations on flat spacetime forms a group $\mathrm{Conf}(\mathbb{R}^{d-1,1}) \simeq O(d,2)/\mathbb{Z}_2$ known as the conformal group[3].

As is often the case in physics, we are interested in analyzing the infinitesimal solutions of (2.2) to obtain a simplified description of the conformal group. In mathematics terms, these solutions are the generators of $\mathrm{Conf}(\mathbb{R}^{d-1,1})$ and are known as *conformal Killing vector fields*. We begin by considering an infinitesimal

[2] We will only consider the conformal group in $d > 2$ dimensions, as the case $d = 2$ is special and a subject worth studying in its own right.

[3] The connected component of the conformal group containing the identity element is isomorphic to $SO(d,2)$, implying that the two share the same Lie algebra $\mathfrak{so}(d,2)$. When there is no chance for confusion, we will sometimes refer to $SO(d,2)$ as the"conformal group."

displacement of the coordinates,

$$x^\mu \to x^\mu + \epsilon X^\mu + O(\epsilon^2). \tag{2.3}$$

When this expression is inserted into (2.2), we obtain a set of differential equations for the generators of the conformal group in flat Minkowski space given by

$$\partial_\mu X_\nu + \partial_\nu X_\mu = \frac{2}{d}(\partial \cdot X)\eta_{\mu\nu}, \tag{2.4}$$

where $\partial \cdot X = \partial^\mu X_\mu$. Although the above equations are explicit, their coordinate form is somewhat cryptic. We can elucidate them through a more natural definition from differential geometry. In particular, we can identify the left hand side with the Lie derivative $\mathcal{L}_X$ of the Minkowski metric in flat space, and the right hand side with the scale factor Ω. A tidier definition of the conformal Killing fields is thus given by

$$\mathcal{L}_X \eta = \frac{2}{d}(\partial \cdot X)\eta. \tag{2.5}$$

Solving the equations in (2.4) yields $d(d-1)/2$ independent conformal Killing fields, which are unique up to a change of basis. In field theory, they are the conserved charges which generate the conformal symmetries. One choice of solution is given by[4]

$$\begin{aligned}
P_\mu &= -i\partial_\mu, \\
D &= x^\mu \partial_\mu, \\
M_{\mu\nu} &= -i(x_\mu \partial_\nu - x_\nu \partial_\mu), \\
K_\mu &= i(2x_\mu(x \cdot \partial) - x^2 \partial_\mu).
\end{aligned} \tag{2.6}$$

We briefly discuss the generates given above. Familiar from ordinary relativistic quantum field theory are P_μ and $M_{\mu\nu}$, which generate spacetime translations and Lorentz transformations, respectively. These transformations form the group of isometries of Minkowski spacetime, ISO($\mathbb{R}^{d-1,1}$), commonly referred to as the

[4]We note that the conformal generators, as presented above, are vector fields on $\mathbb{R}^{d-1,1}$ and *not* quantum operators. It is important to distinguish to these, as they act on different spaces (the first on the underlying spacetime, and the second on various functional spaces).

Poincaré group. The new generators are D, the generator of dilatations or rescalings, and K_μ, the generator of special conformal transformations (SCTs). The conformal transformations associated with these generators are listed below:

$$\begin{aligned} P_\mu &: x^\mu \to x^\mu + a^\mu, \\ M_{\mu\nu} &: x^\mu \to \Lambda^\mu{}_\nu x^\nu, \\ D &: x^\mu \to \lambda x^\mu, \\ K_\mu &: x^\mu \to \mathrm{SCT}_b(x^\mu) \equiv \frac{x^\mu - b^\mu x^2}{1 - 2(b\cdot x) + b^2 x^2}, \end{aligned} \tag{2.7}$$

where a and b are constant vectors, $\Lambda^\mu{}_\nu \in SO(d-1,1)$ is a Lorentz transformation, and $\lambda > 0$ is a scale parameter. The most unfamiliar type of transformation, the special conformal transformation, is what differentiates the conformal group from the set of dilatations. To shed some light on its properties, we can rewrite it as

$$\mathrm{SCT}_b = I \circ T_b \circ I, \tag{2.8}$$

where $I : x^\mu \to x^\mu / x^2$ is an inversion across the unit $d-1$-sphere S^{d-1} and T_b is a translation by b. Hence, an SCT is an inversion followed by a translation followed by another inversion. Equation (2.8) gives us a more transparent geometric picture of SCT. In fact, we can also think of SCTs roughly as the counterpart to translations: a translation fixes the point at infinity and moves the origin, whereas an SCT fixes the origin and moves the point at infinity.

2.2 The Conformal Algebra

It is not difficult to check that the generators of the conformal Lie algebra, as defined in (2.6), satisfy the following commutation relations (the rest are zero):

$$\begin{aligned}
[M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \\
[M_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}), \\
[D, P_\mu] &= P_\mu, \\
[D, K_\mu] &= -K_\mu, \\
[K_\mu, P_\nu] &= -2iM_{\mu\nu} + 2\eta_{\mu\nu}D,
\end{aligned} \tag{2.9}$$

where $[\cdot,\cdot] : T\mathbb{R}^{d-1,1} \to T\mathbb{R}^{d-1,1}$ is the Lie bracket on $\mathbb{R}^{d-1,1}$. We will later show that this algebra is isomorphic to $\mathfrak{so}(d,2)$ and will henceforth refer to $\mathfrak{so}(d,2)$ and (2.9) as the "conformal algebra."[5]. In fact, these commutation relations are sufficiently general and will remain true regardless of the details of the specific representation or type of bracket. We may therefore think of (2.9) more abstractly as existing independent of the explicit forms of the generators.

Let us now dissect the commutation relations of the conformal algebra. The first two equations indicate that P_μ and K_μ transform individually as vectors under the action of the Lorentz group. The third equation is the usual set of commutation relations for the Lorentz algebra $\mathfrak{so}(d-1,1)$. The relations in the last three lines are special to the conformal algebra: they will allow us to treat P_μ and K_μ as raising/lowering operators for dilatation eigenstates of D in the representation theory. Aside from some differences due to non-compactness, the set of operators $\{D, P_\mu, K_\mu\}$ in the conformal algebra are analogous to the spin operators $\{S_z, S_+, S_-\}$ in $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{so}(3)_\mathbb{C}$.

[5] To be mathematically precise, we should really be talking about the complexified Lie algebra $\mathfrak{so}(d,2)_\mathbb{C} = \mathfrak{so}(d+2)_\mathbb{C}$. We will assume this is what we mean by $\mathfrak{so}(d,2)$ without mentioning so explicitly unless there is the possibility of confusion.

2.3 Representations of the Conformal Algebra

2.3.1 States

With the conformal algebra in hand, we can now discuss how the conformal charges act on quantum operators (e.g. fields) in a CFT. That is, we are interested in determining the infinite-dimensional unitary irreducible representations of $\mathfrak{so}(d,2)$. Under radial quantization, the eigenstates of D form a basis for the Hilbert space living on a radial slice of the spacetime (when analytically continued to Euclidean signature). Such a state is partially specified by two quantum numbers – its Lorentz spin ℓ and its scaling dimension Δ. The details of the Lorentz spin can be found in any standard QFT textbook and involves the representation theory of the Lorentz subgroup (i.e. the little group) of the Poincaré group. The new quantum number, Δ, appears due to the scale invariance of CFTs. We can thus label states in part as $|\Delta, \ell\rangle$. The action of the conformal generators on these states is given by

$$\begin{aligned} D\,|\Delta, \ell\rangle &\propto |\Delta, \ell\rangle\,, \\ P\,|\Delta, \ell\rangle &\propto |\Delta + \ell, \ell + 1\rangle\,, \\ K\,|\Delta, \ell\rangle &\propto |\Delta - \ell, \ell + 1\rangle\,, \end{aligned} \tag{2.10}$$

where we have suppressed the spacetime indices on the charges for brevity. Upon inspection, we see that P acts as a raising operator for dimension and K as a lowering operator. We can thus define a highest weight state $|\Delta, \ell\rangle^{\text{h.w.}}$ as an eigenstate of D which is annihilated by K, i.e.

$$K\,|\Delta, \ell\rangle^{\text{h.w.}} = 0. \tag{2.11}$$

Due to unitarity, the highest weight states satisfy the so-called conformal unitary bounds, which are given by [50]

$$\begin{cases} \Delta \geq \frac{d-2}{2}, & \text{for } \ell = 0, \\ \Delta \geq \frac{d-1}{2} & \text{for } \ell = \frac{1}{2}, \\ \Delta \geq \ell + d - 2, & \text{otherwise.} \end{cases} \tag{2.12}$$

A whole family of states, called descendants, can be generated through successive applications of the raising operator P. Thus, we find that eigenstates of D organize themselves into representations called *conformal multiplets*. A conformal multiplet (Δ, ℓ) is a representation of the conformal algebra, labeled by the dimension Δ and spin ℓ of its highest weight state. It is given by the infinite sequence

$$(\Delta, \ell): \quad |\Delta, \ell\rangle^{\text{h.w.}} \underset{P}{\rightarrow} |\Delta+1, \ell+1\rangle \underset{P}{\rightarrow} \cdots \tag{2.13}$$

2.3.2 Operators

So far, we have only discussed the states found in a CFT, and have made no mention of the operators. Typically, states and operators are very different mathematical objects. States live in Hilbert spaces (attached to radial slices of the spacetime) whereas local operators exist at a single point. In most QFTs, there are many more operators than states. Indeed, it is often the case that a single state can often be generated by through the action of many different local operators. However, for field theories with conformal symmetry, something special happens. In a CFT, states are in bijective correspondence with local operators inserted at the origin. This bijection is referred to as the state-operator correspondence. In particular, given a state $|\mathcal{O}\rangle$, there exists a unique local operator $\mathcal{O}(x)$ satisfying

$$\mathcal{O}(0)\,|0\rangle = |\mathcal{O}\rangle\,, \tag{2.14}$$

where $|0\rangle$ is the vacuum of the theory, i.e. the unique state that's eliminated by all the conformal charges.

The state-operator correspondence serves as a powerful device for analyzing CFTs, and we will use it throughout this work. We begin by analyzing the representation theory of operators through the results for states in the theory as presented above. The action of a charge Q on the state $|\mathcal{O}\rangle$ induces an action on the corresponding operator $\mathcal{O}(0)$. Let Π be the bijection that takes states to operators. The induced action is given by

$$Q \cdot \mathcal{O}(0) = [\hat{Q}, \mathcal{O}(0)] \equiv \Pi(\hat{Q}\,|\mathcal{O}\rangle), \tag{2.15}$$

where $\hat{Q}$ is the charge in the quantized theory corresponding to the generator Q, and $[\cdot,\cdot]$ is the commutator, defined by $[A,B] \equiv AB - BA$.[6] The operator corresponding to the highest weight state in a multiplet is known as the *conformal primary*, which we will denote by $\mathcal{O}_{(\Delta,\ell)}$ (we have suppressed spin indices as well as any global symmetry indices for brevity). The conformal primary thus satisfies the operator equation[7]

$$[K_\mu, \mathcal{O}_{(\Delta,\ell)}] = 0. \tag{2.16}$$

All of the decsendant operators in a multiplet can be generated through successive applications of P to $\mathcal{O}_{(\Delta,\ell)}$. Thus, a conformal multiplet also corresponds to the sequence

$$(\Delta,\ell): \quad \mathcal{O}_{(\Delta,\ell)} \underset{P}{\rightarrow} P_\mu \mathcal{O}_{(\Delta,\ell)} \underset{P}{\rightarrow} P_\nu P_\mu \mathcal{O}_{(\Delta,\ell)} \underset{P}{\rightarrow} \cdots \tag{2.17}$$

We can now work out the action of the charges on a general operator $\mathcal{O}^a(x)$ with dimension and spin $(\Delta_\mathcal{O}, \ell_\mathcal{O})$, where the Latin index $a = \mu_1 \ldots \mu_{\ell_\mathcal{O}}$ labels the spin indices. In the adjoint representation, the commutation relations take the form

$$\begin{aligned} [P_\mu, \mathcal{O}^a(0)] &= -i\partial_\mu \mathcal{O}^a(0), \\ [D, \mathcal{O}^a(0)] &= -\Delta_\mathcal{O} \mathcal{O}^a(0), \\ [M_{\mu\nu}, \mathcal{O}^a(0)] &= (\mathcal{S}_{\mu\nu})^a{}_b\, \mathcal{O}^b(0), \end{aligned} \tag{2.18}$$

where $\mathcal{S}_{\mu\nu}$ are the spin $\ell_\mathcal{O}$ matrices. Notice that the dimension $\Delta_\mathcal{O}$, the eigenvalue of D for the eigenstate $\mathcal{O}\,|0\rangle$, is also the scaling dimension of the operator, with

$$\mathcal{O}(\lambda x) = \lambda^{-\Delta_\mathcal{O}} \mathcal{O}(x). \tag{2.19}$$

For the classical theory, it is simply the canonical mass dimension $\Delta_\mathcal{O} = [\mathcal{O}]$. In the quantized theory, it can receive quantum corrections collectively referred to as the anomalous dimension. The action of the generators on $\mathcal{O}^a(0)$ fully specifies

[6]Operators and charges of a CFT live in the same space, so they most naturally act on one another through the adjoint representation.

[7]In the rest of this paper, we will drop the hat on the quantized charges and assume whether we mean Q or $\hat{Q}$ will be clear from context.

their action on the translated operator $\mathcal{O}^a(x) \equiv e^{-iP\cdot x}\mathcal{O}^a e^{iP\cdot x}$. Using the conformal algebra, we find

$$\begin{aligned}
[P_\mu, \mathcal{O}^a(x)] &= -i\partial_\mu \mathcal{O}^a(x), \\
[D, \mathcal{O}^a(x)] &= -(x^\mu\partial_\mu + \Delta)\mathcal{O}^a(x), \\
[K_\mu, \mathcal{O}^a(x)] &= i(2x_\mu(x\cdot\partial) - x^2\partial_\mu + 2\Delta x_\mu)\mathcal{O}^a(x) + 2x^\nu(\mathcal{S}_{\mu\nu})_b{}^a\mathcal{O}^a(x), \\
[M_{\mu\nu}, \mathcal{O}^a(x)] &= -i(x_\mu\partial_\nu - x_\nu\partial_\mu)\mathcal{O}^a(x) + (\mathcal{S}_{\mu\nu})_b{}^a\mathcal{O}^b(x).
\end{aligned} \tag{2.20}$$

2.4 Correlation Functions

We now turn our attention to computing observables in CFTs. Conformal symmetry imposes restrictive constraints on the forms of the correlation functions. Indeed, the forms of the two- and three-point correlation functions in a CFT are completely determined up to a multiplicative factor only by kinematic arguments. While four-point functions can vary in form, they are still constrained to be functions of conformal invariants up to a fixed covariant prefactor.

Although the constraints from conformal symmetry are enough to determine certain correlation functions [1], it is difficult to do so due to the inherently nonlinear action of the conformal group on the spacetime. It is easier to derive their forms through a technique known as the projective null cone embedding [51]. Originally formulated by Dirac, this formalism involves embedding the spacetime as a null hypersurface in some higher-dimensional indefinite space [52]. The crux of the idea lies in the fact that the connected component of $\mathrm{Conf}(\mathbb{R}^{d-1,1})$ is isomorphic to $SO(d,2)$. This isomorphism induces a map from the nonlinear action of $\mathrm{Conf}(\mathbb{R}^{d-1,1})$ on $\mathbb{R}^{d-1,1}$ to one that is *linear* on the null surface Λ. This map is projective in that $\mathbb{R}^{d-1,1}$ is identified with the quotient space $\Lambda/\sim$, where $X \sim \lambda X$ for $X \in \Lambda$ and $\lambda > 0$.[8]

We will now work out the conformal kinematics of the null surface. The inner product between two vectors $X, Y \in \mathbb{R}^{d,2}$ is given by

$$X\cdot Y \equiv \eta_{AB}X^AY^B = -X^{-1}Y^{-1} - X^0Y^0 + \sum_{i=1}^{d} X^iY^i, \tag{2.21}$$

[8] Our approach differs from the conventional one, as in [50], since we are working in Minkowski (not Euclidean) spacetime.

where $A, B = -1, 0, \dots d$ and $\eta_{AB} = \mathrm{diag}(-1, -1, 1, \dots, 1)$. The null hypersurface $\Lambda \subset \mathbb{R}^{d,2}$ is the submanifold specified by $\Lambda = \{X \in \mathbb{R}^{d,2} | X \cdot X = 0\}$. One choice of embedding is the map $X : \mathbb{R}^{d-1,1} \to \Lambda$ with

$$x^\mu \mapsto X^A = \left(\frac{1 - x^2}{2}, x^\mu, \frac{1 + x^2}{2} \right). \tag{2.22}$$

Under this embedding, it follows that $X \cdot X = 0$. The map between the inner product on Λ and the one on $\mathbb{R}^{d-1,1}$ is given by

$$-2X \cdot Y = (x - y)^2. \tag{2.23}$$

Now consider a scalar conformal primary $\phi(x)$ with dimension Δ.[9] The operator can be lifted to Λ by requiring $\phi(X)$ be a homogeneous polynomial, i.e.

$$\phi(\lambda X) = \lambda^{-\Delta} \phi(X). \tag{2.24}$$

Crucially, (2.24) remains true for conformal primaries even when λ is a local function of x. This is not true in general for arbitrary operators in a CFT. Given a set of scalar primaries $\{\phi_1, \dots, \phi_4\}$ with respective dimensions $\{\Delta_1, \dots, \Delta_4\}$, we can easily determine the forms of the correlation functions on the null hypersurface. Besides being homogeneous polynomials of the coordinates, these functions must be invariant under $SO(d, 2)$ transformations. The only invariant associated with this group is given by the inner product between two vectors. Using these two conditions, it is easy to show that the two-point function must be equal to

$$\langle \phi_1(X_1) \phi_2(X_2) \rangle = \frac{Z}{(X_1 \cdot X_2)^\Delta}, \quad \Delta_1 = \Delta_2 = \Delta, \tag{2.25}$$

where Z is a constant related to the normalization of the operators; we are always free to set it to unity. The three-point function involves three points in Λ, which have three associated invariants. It is given by

$$\langle \phi_1(X_1) \phi_2(X_2) \phi_3(X_3) \rangle = \frac{\lambda_{123}}{(X_1 \cdot X_2)^{\alpha_{123}} (X_2 \cdot X_3)^{\alpha_{231}} (X_3 \cdot X_1)^{\alpha_{312}}}, \tag{2.26}$$

[9] In this paper, we are only interested in correlation functions of scalar operators. However, it is not difficult to extend our discussion to show that correlation functions of operators with spin have the same overall form modulo tensor structures.

with $\alpha_{ijk} = \Delta_i + \Delta_j - \Delta_k$. The coefficient λ_{123} is a theory-specific constant commonly called a structure constant. Unlike the normalization factor in the two-point function, which can be set for each operator, the entire set of structure constants for a CFT are fixed modulo an overall factor.

Finally, we consider the four-point functions of a CFT. With more than three points, it is now possible to construct functions with scale invariance. In particular, for four points there are two independent invariants known as the *conformal cross-ratios*. They are given by

$$u = \frac{(X_1 \cdot X_2)(X_3 \cdot X_4)}{(X_1 \cdot X_3)(X_2 \cdot X_4)}, \quad v = \frac{(X_3 \cdot X_2)(X_1 \cdot X_4)}{(X_1 \cdot X_3)(X_2 \cdot X_4)}. \tag{2.27}$$

The four-point function is only determined up to an arbitrary function of these invariants. Carrying out the same same analysis as for the two- and three-point functions yields

$$\begin{aligned} &\langle \phi_1(X_1)\phi_2(X_2)\phi_3(X_3)\phi_4(X_4)\rangle = \\ &\frac{1}{(X_1 \cdot X_2)^{\frac{\Delta_1+\Delta_2}{2}}(X_3 \cdot X_4)^{\frac{\Delta_2+\Delta_3}{2}}} \left(\frac{X_2 \cdot X_4}{X_1 \cdot X_4}\right)^{\frac{\Delta_{12}}{2}} \left(\frac{X_1 \cdot X_4}{X_1 \cdot X_3}\right)^{\frac{\Delta_{34}}{2}} g(u,v), \end{aligned} \tag{2.28}$$

where $g(u, v)$ can in principle be any function of the cross-ratios.

To determine these correlation functions on the original spacetime, we simply project $\Lambda/\sim \to \mathbb{R}^{d-1,1}$. Using (2.23), we recover the two-point function

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{Z}{|x_1 - x_2|^{2\Delta}}, \quad \Delta_1 = \Delta_2 = \Delta \tag{2.29}$$

and the three-point function

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{x_{12}^{2\alpha_{123}} x_{23}^{2\alpha_{231}} x_{31}^{2\alpha_{312}}}. \tag{2.30}$$

After projecting, we also obtain the conformal cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{32}^2 x_{14}^2}{x_{13}^2 x_{24}^2} \tag{2.31}$$

and the four-point function

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{1}{x_{12}^{\Delta_1+\Delta_2}x_{34}^{\Delta_3+\Delta_4}}\left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{12}}\left(\frac{x_{14}}{x_{13}}\right)^{\Delta_{34}} g(u,v). \quad (2.32)$$

The existence of two (conformally invariant) degrees of freedom u, v among the four spacetime coordinates x_i is not so transparent in the original spacetime. To make this invariance manifest, we can exploit the underlying conformal symmetry. First, we take the point $x_4 \to \infty$ using a special conformal transformation. We then translate x_1 to the origin. Taking x_1 and x_3 to lie along the same line, we next apply a dilatation such that $x_3 = (1, 0, \ldots 0)$. Finally, using Lorentz transformations (rotations in Euclidean space), we can take the point $x_2 = (x, y, 0, \ldots 0)$ to lie in the $x^0 - x^1$ plane. Introducing complex coordinates $z = x + iy$ and $\overline{z} = x - iy$, we find that the cross-ratios reduce to functions of two real degrees of freedom:

$$u = z\overline{z}, \quad v = (1-z)(1-\overline{z}). \quad (2.33)$$

We can interpret these formulas geometrically. In Euclidean space, u represents the distance squared of z from the origin and v represents the distance of z from the point $(1, 0, \ldots, 0)$.

2.5 The Operator Product Expansion

In ordinary quantum field theories, there is the notion of the asymptotic *operator product expansion* (OPE). Given two operators $\mathcal{O}_i(x)$ and $\mathcal{O}_j(y)$, the product $\mathcal{O}_i(x)\mathcal{O}_j(y)$ roughly tends towards a sum over all the operators in the theory as $x \to y$. Due to the state-operator correspondence, this expansion is both well-defined and convergent in conformal field theory. In particular, a CFT consists of an operator algebra $\mathcal{A}$ with the product map given by the operator product expansion OPE : $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. This map is given explicitly by

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_{\mathcal{O}_k \in \mathcal{A}} c_{ijk}\mathcal{O}_k(x_3) \quad (2.34)$$

for some constants c_{ijk}. The series above converges for all points x_3 in a ball of radius $|x_{12}|$ centered at x_1.

Conformal symmetry imposes certain relations among the c_{ijk} in the sum in (2.34). Since all of the descendants in a conformal multiplet are given by derivatives of the corresponding primary, it follows that all operators in a given multiplet (Δ, ℓ) can be packaged together into a single term given by a differential operator $C_{ij\mathcal{O}}$ acting on the primary $\mathcal{O}_{(\Delta,\ell)}$. It follows that the OPE can be written as a sum over primaries [50]:

$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_{\mathcal{O}\in\mathcal{O}_i\times\mathcal{O}_j} \lambda_{ij\mathcal{O}} C_{ij\mathcal{O}}(x_{12}, \partial_2)\mathcal{O}_{(\Delta,\ell)}(x_2), \tag{2.35}$$

where $\lambda_{ij\mathcal{O}}$ is the structure constant in the three-point function $\langle\mathcal{O}_i\mathcal{O}_j\mathcal{O}\rangle$. From now on, we will refer to the structure constant $\lambda_{ij\mathcal{O}}$ as the *OPE coefficient* for the operator $\mathcal{O}$ in the OPE $\mathcal{O}_i \times \mathcal{O}_j$. The above sum is taken to be over all conformal multiplets (Δ, ℓ) appearing in the decomposition of the product $\mathcal{O}_i \times \mathcal{O}_j$.[10] With this understanding, from now on we will drop some of the cumbersome notation for convenience.

The OPE is a powerful tool in conformal theory; indeed, by applying it consecutively we can reduce any product of operators to a string of single operators. From this it follows that any n-point correlation function can be replaced by a sum over one-point functions. The only nonzero one point function in a standard CFT is that of the unit operator. Thus, in principle, the OPE can be used to calculate any correlation function.

For the purposes of the conformal bootstrap, we will now consider the OPE as applied to the correlation function of four possibly different scalars. In the s-channel expansion, we pair $(1, 2)$ and $(3, 4)$, which yields an expression of the form

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = I_{1234}(x_i)\sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\, g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(u, v). \tag{2.36}$$

Let us take the time to break down this expression into manageable parts. The function

$$I_{1234}(x_i) = \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{12}} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_{34}} \tag{2.37}$$

[10]The expansion in (2.35) is somewhat schematic, as we have left out possible spin indices of all the operators; for operators that have spin, the $C_{ij\mathcal{O}}$ may have spin indices as well.

is the same conformally-covariant prefactor found in (2.32). The sum in (2.36) is taken to be over conformal multiplets that appear in both $\phi_1 \times \phi_2$ and $\phi_3 \times \phi_4$. The $g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(u,v)$ are functions of the cross-ratios which depend only on the quantum numbers (Δ, ℓ) of the conformal multiplet, as well as the differences Δ_{12} and Δ_{34}. These functions are called *conformal blocks* since their functional form is determined entirely by the underlying representation theory of $\mathfrak{so}(d,2)$.

For convenience, we will refer to a set of correlation functions involving multiple operators as a *mixed correlator*. Additionally, to be precise, the $g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(u,v)$ are called *mixed blocks* since they appear in the mixed correlator. For the case of only a single scalar ϕ with dimension Δ_ϕ, equation (2.36) simplifies to

$$\langle \overbrace{\phi(x_1)\phi}(x_2)\overbrace{\phi(x_3)\phi}(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 \, g_{\Delta,\ell}(u,v), \tag{2.38}$$

where $g_{\Delta,\ell}(u,v) \equiv g_{\Delta,\ell}^{0,0}(u,v)$ are just the regular conformal blocks.

2.6 Conformal Blocks

2.6.1 The Conformal Casimir

To determine the conformal blocks, we require an operator which distinguishes different conformal multiplets. The quintessential example of such an operator is the Casimir invariant, i.e. an operator that commutes with all of the generators. For the conformal algebra, we only need consider the quadratic Casimir $L^2_{\mathfrak{so}(d,2)}$, which can be expressed as a linear combination of the squares of the generators. Due to conformal invariance, the terms are restricted to be Lorentz scalars. Given that $[K,P]$ is the only commutator containing D, we can conclude that terms can be at most linear in P and K. Therefore, the Casimir must have the form

$$L^2_{\mathfrak{so}(d,2)} = D^2 + c_1 M_{\mu\nu}M^{\mu\nu} + c_2 P_\mu K^\mu + c_3 K_\mu P^\mu, \tag{2.39}$$

with three yet-determined constants c_i, with $i = 1, 2, 3$. Our choice of the coefficient for D^2 corresponds to specifying an overall normalization. Consider the following commutators of the Casimir with P:

$$\begin{aligned}
[D^2, P_\mu] &= DP_\mu + P_\mu D, \\
[M_{\mu\nu}M^{\mu\nu}, P_\rho] &= -2iP^\mu M_{\mu\rho} - 2iM_{\mu\rho}P^\mu, \\
[P_\mu K^\mu, P_\rho] &= 2P_\rho D - 2iP^\mu M_{\mu\rho}, \\
[K_\mu P^\mu, P_\rho] &= 2DP_\rho - 2iM_{\mu\rho}P^\mu.
\end{aligned} \tag{2.40}$$

Requiring $[C_{\mathfrak{so}(d,2)}, P_\mu] = 0$ fixes all of the coefficients to be $c_2 = c_3 = -c_1 = -1/2$. It follows that the Casimir is given by

$$L^2_{\mathfrak{so}(3,2)} = D^2 + \frac{1}{2}M_{\mu\nu}M^{\mu\nu} - \frac{1}{2}(P_\mu K^\mu + K_\mu P^\mu) \tag{2.41}$$

up to an overall constant.

The conformal Casimir is useful because it distinguishes among different representations of the conformal algebra. Indeed, we can show that the value of $L^2_{\mathfrak{so}(d,2)}$ is constant across states in a given multiplet. Consider the multiplet (Δ, ℓ) with heighest weight state $|\Delta, \ell\rangle^{\text{h.w.}}$. Using the conformal algebra, we find

$$\begin{aligned}
L^2_{\mathfrak{so}(3,2)} |\Delta, \ell\rangle^{\text{h.w.}} &= \left[D^2 + \frac{1}{2}M_{\mu\nu}M^{\mu\nu} - \frac{1}{2}([K_\mu, P^\mu] + 2P_\mu K^\mu)\right] |\Delta, \ell\rangle^{\text{h.w.}} \\
&= [\Delta(\Delta - d) + \ell(\ell + d - 2)] |\Delta, \ell\rangle^{\text{h.w.}}.
\end{aligned} \tag{2.42}$$

Since the Casimir commutes with all of the generators, and in particular P_μ, it follows that

$$L^2_{\mathfrak{so}(d,2)} |\Delta', m; \Delta, \ell\rangle = [\Delta(\Delta - d) + \ell(\ell + d - 2)] |\Delta', m; \Delta, \ell\rangle, \tag{2.43}$$

for any state $|\Delta', m; \Delta, \ell\rangle \in (\Delta, \ell)$.

2.6.2 The Casimir Action

The action of the conformal charges on operators induces an action of $L^2_{\mathfrak{so}(d,2)}$ on the operator algebra $\mathcal{A}$. In the context of the operator product expansion, we wish to extend this action to a product of two operators. Roughly, the operator product lives in the space $\mathcal{A} \times \mathcal{A}$, so we can extend the action of the Casimir by simply extending that of the generators. We define this extension of a generator Q as

$$Q \to Q \otimes 1 + 1 \otimes Q, \tag{2.44}$$

It follows that the Casimir acts on the four-point function G_{1234} as

$$L^2_{\mathfrak{so}(d,2)} \cdot G_{1234} = I(x_i)\, \mathcal{D} g(u,v), \tag{2.45}$$

where $\mathcal{D}$ is some differential operator and $I(x_i)$ is the prefactor from (2.37). We observe that the Casimir action passes through the prefactor and only acts on invariant function $g(u,v)$. To determine $\mathcal{D}$, we work in planar coordinates akin to those used in (2.33). Using conformal transformations, we can work in a coordinate system where the four spacetime points can be written in Euclidean signature as

$$\begin{aligned} &x_1 = 0, \quad x_3 = (1, 0, \ldots, 0), \\ &x_4 = (q, 0, \ldots, 0), \quad x_2 = \left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}, 0, \ldots, 0 \right), \end{aligned} \tag{2.46}$$

where q is the position of x_4 on the x^0-axis. To reproduce the coordinates of (2.33), we take $q \to \infty$:

$$L^2_{\mathfrak{so}(d,2)} \cdot G_{1234} = I(x_i) \lim_{q \to \infty} \mathcal{D} g_q(z, \overline{z}). \tag{2.47}$$

where we have defined $g_q(z,\overline{z}) = g(u(z,\overline{z}), v(z,\overline{z}))$. A tedious computation reveals that the differential operator in the above equation is given by

$$\begin{aligned} \mathcal{D}^{(\Delta_{12},\Delta_{34})}_{\mathfrak{so}(d,2)} &= z^2(1-z)\partial_z^2 - \left(\frac{1}{2}\Delta_{21} + \frac{1}{2}\Delta_{34} + 1 \right) z^2 \partial_z - \frac{1}{4}\Delta_{21}\Delta_{34} z \\ &+ \overline{z}^2(1-\overline{z})\partial_{\overline{z}}^2 - \left(\frac{1}{2}\Delta_{21} + \frac{1}{2}\Delta_{34} + 1 \right) \overline{z}^2 \partial_{\overline{z}} - \frac{1}{4}\Delta_{21}\Delta_{34} \overline{z} \\ &+ (d-2)\frac{z\overline{z}}{z - \overline{z}} \left((1-z)\partial_z - (1-\overline{z})\partial_{\overline{z}} \right). \end{aligned} \tag{2.48}$$

Substituting back the cross-ratios u and v yields

$$\begin{aligned}\mathcal{D}_{\mathfrak{so}(d,2)}^{(\Delta_{12},\Delta_{34})} = (1-u-v)\partial_v\left(v\partial_v + \frac{1}{2}\Delta_{21} + \frac{1}{2}\Delta_{34}\right) + u\partial_u\left(2u\partial_u - d\right) \\ -(1+u-v)\left(u\partial_u + v\partial_v + \frac{1}{2}\Delta_{21}\right)\left(u\partial_u + v\partial_v + \frac{1}{2}\Delta_{34}\right).\end{aligned} \tag{2.49}$$

Since the conformal Casimir is constant across representations, it follows that the mixed blocks can be determined by solving the eigenvalue equation

$$\mathcal{D}_{\mathfrak{so}(d,2)}^{(\Delta_{12},\Delta_{34})} g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(u,v) = [\Delta(\Delta-d) + \ell(\ell+d-2)] g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(u,v). \tag{2.50}$$

In general, both the mixed and regular conformal blocks are complicated functions of the cross-ratios. The regular blocks all have the general form [50]:

$$g_{\Delta,0}^{0,0}(u,v) = u^{\Delta/2}(1+\cdots) \tag{2.51}$$

Additionally, in even dimensions, they can be solved analytically. For instance, in four dimensions the mixed blocks are given by [53]

$$g_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(z,\overline{z}) \overset{4d}{=} \frac{z\overline{z}}{z-\overline{z}}\left(k_{\Delta-\ell-2}(z)k_{\Delta+\ell}(\overline{z}) - k_{\Delta+\ell}(z)k_{\Delta-\ell-2}(\overline{z})\right) \tag{2.52}$$

where

$$k_\alpha(x) \equiv x^{\alpha/2}\,{}_2F_1\left(\frac{\alpha-\Delta 12}{2}, \frac{\alpha+\Delta_{34}}{2}, \alpha, x\right), \tag{2.53}$$

and ${}_2F_1$ is a hypergeometric function. In odd dimensions, the conformal blocks do not have closed form solutions. However, they can still be determined order-by-order in an infinite polynomial series.

2.7 Conformal Bootstrap

2.7.1 Crossing Symmetry

In Section 2.5, we used the operator algebra to express the four-point function as a sum over all the conformal multiplets in the overlap of two OPEs. In particular, we carried out an s-channel expansion, where we carried out the OPE through the pairings (1,2) and (3,4). However, since the objects inside the brackets of a correlation function are really functions, and not operators, we should be able to rearrange them as we please; this implies that the OPE can be used on *any* two operators inside a correlation function, and not just on neighboring ones. This freedom of choice is known as *crossing symmetry*. For instance, we could choose the t-channel expansion with the pairings (3,2) and (1,4). This is equivalent to interchanging the indices $1 \leftrightarrow 3$. Since $u \leftrightarrow v$ under this interchange, it follows that the four-point function in (2.38) is equal to

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{x_{32}^{2\Delta_\phi} x_{14}^{2\Delta_\phi}} \sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 \, g_{\Delta,\ell}(v,u). \tag{2.54}$$

Combining the t-channel expansion (2.54) with the s-channel expansion (2.38) yields a set of equations known as *crossing equations*, given by

$$\sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 F_{\Delta,\ell} = 0, \tag{2.55}$$

where we have introduced the crossed blocks

$$F_{\Delta,\ell}(u,v) = v^{\Delta_\phi} g_{\Delta,\ell}(u,v) - u^{\Delta_\phi} g_{\Delta,\ell}(v,u). \tag{2.56}$$

2.7.2 Mixed Correlator

We can extend the ideas from the previous section to the mixed correlator, where correlation functions can contain different operators. Each unique correlation function yields an equation with the form of (2.55). Following the theme of this work, we focus on the mixed correlator of two scalars ϕ_1 and ϕ_2. Applying crossing symmetry to the correlation functions $\langle \phi_1\phi_1\phi_1\phi_1 \rangle$, $\langle \phi_2\phi_2\phi_2\phi_2 \rangle$, $\langle \phi_1\phi_2\phi_1\phi_2 \rangle$, and

$\langle\phi_1\phi_1\phi_2\phi_2\rangle$, we find

$$\begin{aligned}
&\sum_{\mathcal{O}} \lambda_{11\mathcal{O}}^2 [v^{\Delta_1} g_{\Delta,\ell}^{(0,0)}(u,v) - u^{\Delta_1} g_{\Delta,\ell}^{(0,0)}(v,u)] = 0, \\
&\sum_{\mathcal{O}} \lambda_{22\mathcal{O}}^2 [v^{\Delta_2} g_{\Delta,\ell}^{(0,0)}(u,v) - u^{\Delta_2} g_{\Delta,\ell}^{(0,0)}(v,u)] = 0, \\
&\sum_{\mathcal{O}} \lambda_{12\mathcal{O}}^2 \left[v^{\frac{\Delta_1+\Delta_2}{2}} g_{\Delta,\ell}^{(\Delta_{12},\Delta_{12})}(u,v) - u^{\frac{\Delta_1+\Delta_2}{2}} g_{\Delta,\ell}^{(\Delta_{12},\Delta_{21})}(v,u) \right] = 0, \\
&\sum_{\mathcal{O}} \left[\lambda_{11\mathcal{O}}\lambda_{22\mathcal{O}} v^{\frac{\Delta_1+\Delta_2}{2}} g_{\Delta,\ell}^{(0,0)}(u,v) - \lambda_{12\mathcal{O}}^2 u^{\Delta_1} g_{\Delta,\ell}^{(\Delta_{12},\Delta_{21})}(v,u) \right] = 0.
\end{aligned} \tag{2.57}$$

By adding these four equations, we arrive at a single crossing equation of the form

$$\sum_{\mathcal{O}} \begin{pmatrix} \lambda_{11\mathcal{O}} & \lambda_{22\mathcal{O}} \end{pmatrix} \begin{pmatrix} F_{\Delta,\ell}^{(1,1)} & F_{\Delta,\ell}^{(1,2)} \\ F_{\Delta,\ell}^{(2,1)} & F_{\Delta,\ell}^{(2,2)} \end{pmatrix} \begin{pmatrix} \lambda_{11\mathcal{O}} \\ \lambda_{22\mathcal{O}} \end{pmatrix} + F_{\Delta,\ell}\lambda_{12\mathcal{O}}^2 \;=\; 0, \tag{2.58}$$

where the matrix and $F_{\Delta,\ell}$ are both functions of u and v, analogous to (2.56). In fact, these functions can all be written in terms of the mixed conformal blocks $g_{\Delta,\ell}^{\Delta_{ij},\Delta_{kl}}(u,v)$, which only depend on the underlying representation theory of $\mathfrak{so}(d,2)$. The only quantity that varies in this sum is the local CFT data $(\{\Delta_i\},\{\lambda_{ijk}\})$ that specifies the particular theory under consideration.

A natural question to ask is whether an arbitrary set of scaling dimensions and OPE coefficients necessarily corresponds to a CFT. The answer, no, comes from the restrictiveness of the consistency conditions in (2.58). If the LHS of this equation does not equal zero for a given set of data, then it cannot coincide with a valid CFT. This is the crux of the idea of the conformal bootstrap, which can determine whether a CFT is disallowed using conformal kinematics alone. In the context of the mixed correlator, it is often referred to as the mixed bootstrap. It is formulated as follows.

Mixed Bootstrap

Consider a set of scaling dimensions and OPE coefficients $(\{\Delta_i\},\{\lambda_{ijk}\})$. Suppose there exists a linear functional α such that $\alpha(F_{\Delta,\ell}^{(1,1)}) = 1$ for the unit operator and

$$\begin{pmatrix} \alpha(F_{\Delta,\ell}^{(1,1)}) & \alpha(F_{\Delta,\ell}^{(1,2)}) \\ \alpha(F_{\Delta,\ell}^{(2,1)}) & \alpha(F_{\Delta,\ell}^{(2,2)}) \end{pmatrix} \succeq 0, \quad \alpha(F_{\Delta,\ell}) \geq 0, \tag{2.59}$$

for all crossed blocks. It follows that the sum on the LHS in (2.58) is strictly positive, implying $0 > 0$. It follows that the data $(\{\Delta_i\}, \{\lambda_{ijk}\})$ cannot constitute a valid CFT.

3 Superconformal Field Theory

Having discussed some basic facts about conformal field theories in $d > 2$ dimensions, we now focus our attention on superconformal field theories in $d = 3$ dimensions. After a discussion on the superconformal algebra and its representations, we develop the necessary tools to implement the mixed superconformal bootstrap.

3.1 Superconformal Algebra

Superconformal field theories in three dimensions with $2\mathcal{N}$ Poincaré supercharges are invariant under the superconformal group $OSp(\mathcal{N}|4)$. The corresponding superconformal algebra $\mathfrak{osp}(\mathcal{N}|4) = \mathrm{Lie}(OSp(\mathcal{N}|4))$ is a Lie superalgebra [28], i.e. a $\mathbb{Z}_2$-graded algebra including both bosonic and fermionic generators. Its maximal bosonic subalgebra is given by $\mathfrak{sp}(4) \oplus \mathfrak{so}(\mathcal{N})_R$, where we can identity $\mathfrak{sp}(4) \simeq \mathfrak{so}(3,2)$ as the conformal algebra in $2+1$ dimensions, and $\mathfrak{so}(\mathcal{N})_R$ as the R-symmetry algebra. The conformal generators are the standard ones given by $\{P_\mu, M_{\mu\nu}, D, K_\mu\}$. The R-symmetry generators can be written as antisymmetric matrices R_{rs}, where $r, s = 1, \ldots, \mathcal{N}$ are fundamental $\mathfrak{so}(\mathcal{N})_R$ indices. The fermionic generators consist of $2\mathcal{N}$ Poincaré supercharges (corresponding to P_μ) and $2\mathcal{N}$ superconformal charges (corresponding to K_μ), both of which transform as Majorana spinors under the Lorentz subalgebra $\mathfrak{so}(2,1)$, and as fundamental vectors under the R-symmetry algebra. We denote the Poincaré supercharges by $Q_{\alpha r}$ and the superconformal charges by $S^\alpha_{\ r}$, where the spinor index $\alpha = 1, 2$. We will take a pedagogical approach and build the superconformal algebra from the bottom up, beginning with the conformal algebra, and then extending it to the full superconformal algebra by introducing the fermionic and R-symmetry generators.

The conformal algebra $\mathfrak{so}(3,2)$ in three dimensions has the nonzero commutation relations:

$$\begin{gathered}
[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu), \quad [M_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu), \\
[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma}), \\
[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu, \quad [K_\mu, P_\nu] = -2i M_{\mu\nu} + 2\eta_{\mu\nu} D.
\end{gathered} \tag{3.1}$$

The supercharges have nontrivial commutation relations with both the conformal and the R-symmetry generators. However, the supercharges carry spinor indices, whereas the conformal charges have Lorentz indices. The commutation relations acquire a much simpler form if both sets of spacetime generators are written with the same type of index. From a group theoretic viewpoint, we wish to make explicit the fact that the spin 1 (vector) representation of $\mathfrak{so}(2,1) \simeq \mathfrak{su}(2)$ appears in the tensor product between two spin-1/2 representations. On a practical level, every Lorentz index can be thought of as two spinor indices. The conversion between the two is given by the γ matrices, defined as

$$(\gamma^\mu)_{\alpha\beta} \equiv (1, \sigma^1, \sigma^3), \quad (\overline{\gamma}^\nu)^{\alpha\beta} \equiv (-1, \sigma^1, \sigma^3), \tag{3.2}$$

where the σ^i are the usual Pauli matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3}$$

It follows that the conformal charges can be written in the spinor basis as

$$P_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta} P_\mu, \quad K^{\alpha\beta} = (\overline{\gamma}^\mu)^{\alpha\beta} K_\mu, \quad M_\alpha{}^\beta = \frac{i}{2} (\gamma^\mu \overline{\gamma}^\nu)_\alpha{}^\beta M_{\mu\nu}. \tag{3.4}$$

Written as matrices, they are given by

$$\begin{gathered} P_{\alpha\beta} = \begin{pmatrix} P_0 + P_1 & P_1 \\ P_1 & P_0 - P_2 \end{pmatrix}, \quad K_{\alpha\beta} = \begin{pmatrix} K_0 + K_1 & K_1 \\ K_1 & K_0 - K_2 \end{pmatrix}, \\ M_\alpha{}^\beta = i \begin{pmatrix} M_{02} & M_{01} - M_{12} \\ M_{01} + M_{12} & -M_{02} \end{pmatrix}. \end{gathered} \tag{3.5}$$

In particular, the generators $M_\alpha{}^\beta$ can be written in terms of the $\mathfrak{su}(2)$ generators as

$$M_\alpha{}^\beta = \begin{pmatrix} J_0 & J_+, \\ J_- & -J_0 \end{pmatrix}, \quad [J_+, J_-] = 2J_0, \quad [J_0, J_\pm] = \pm J_\pm. \tag{3.6}$$

The conformal algebra in (3.1) can then be rewritten in the spinor basis as

$$\begin{aligned}
[M_\alpha{}^\beta, P_{\gamma\delta}] &= \delta_\gamma{}^\beta P_{\alpha\delta} + \delta_\delta{}^\beta P_{\alpha\gamma} - \delta_\alpha{}^\beta P_{\gamma\delta},\\
[M_\alpha{}^\beta, K^{\gamma\delta}] &= -\delta_\alpha{}^\delta K^{\beta\delta} - \delta_\alpha{}^\delta K^{\beta\gamma} + \delta_\alpha{}^\beta P_{\gamma\delta},\\
[M_\alpha{}^\beta, M_\gamma{}^\delta] &= -\delta_\alpha{}^\delta M_\gamma{}^\beta + \delta_\gamma{}^\beta M_\alpha{}^\delta,\\
[D, P_{\alpha\beta}] &= P_{\alpha\beta},\\
[D, K^{\alpha\beta}] &= -K^{\alpha\beta},\\
[K^{\alpha\beta}, P_{\gamma\delta}] &= 4\delta_{(\gamma}^{(\alpha} M_{\delta)}^{\beta)} + 4\delta_{(\gamma}^{\alpha}\delta_{\delta)}^{\beta} D,
\end{aligned} \tag{3.7}$$

We are now ready to present the full superconformal algebra. Introducing the anti-commutator $\{\cdot,\cdot\}$, with $\{A, B\} \equiv AB - BA$, the conformal algebra in (3.7) is extended to $\mathfrak{osp}(\mathcal{N}|4)$ with the following nonzero (anti)-commutation relations for the supercharges [54]:

$$\begin{aligned}
\{Q_{\alpha r}, Q_{\beta s}\} &= 2\delta_{rs}P_{\alpha\beta}, & \{S^\alpha{}_r, S^\beta{}_s\} &= -2\delta_{rs}K^{\alpha\beta},\\
[K^{\alpha\beta}, Q_{\gamma r}] &= -i(\delta_\gamma{}^\alpha S^\beta{}_r + \delta_\gamma{}^\beta S^\alpha{}_r), & [P_{\alpha\beta}, S^\gamma{}_r] &= -i(\delta_\alpha{}^\gamma Q_{\beta r} + \delta_\beta{}^\gamma Q_{\alpha r})\\
[M_\alpha{}^\beta, Q_{\gamma r}] &= \delta_\gamma{}^\beta Q_{\alpha r} - \frac{1}{2}\delta_\alpha{}^\beta Q_{\gamma r}, & [M_\alpha{}^\beta, S^\gamma{}_r] &= -\delta_\alpha{}^\gamma S^\beta{}_r + \frac{1}{2}\delta_\alpha{}^\beta S^\gamma{}_r,\\
[D, Q_{\alpha r}] &= \frac{1}{2}Q_{\alpha r}, & [D, S^\alpha{}_r] &= -\frac{1}{2}S^\alpha{}_r,\\
[R_{rs}, Q_{\alpha t}] &= i(\delta_{rt}Q_{\alpha s} - \delta_{st}Q_{\alpha r}), & [R_{rs}, S^\alpha{}_t] &= i(\delta_{rt}S^\alpha{}_s - \delta_{st}S^\alpha{}_r),\\
\{Q_{r\alpha}, S^\beta{}_s\} &= 2i(\delta_{rs}(M_\alpha{}^\beta + \delta_\alpha{}^\beta D) - i\delta_\alpha{}^\beta R_{rs}),
\end{aligned} \tag{3.8}$$

as well as by the $\mathfrak{so}(\mathcal{N})$ commutation relations:

$$[R_{rs}, R_{tu}] = i(\delta_{rt}R_{su} - \delta_{st}R_{ru} - \delta_{ru}R_{st} + \delta_{su}R_{rt}). \tag{3.9}$$

Under radial quantization, the conjugation properties of the $\mathfrak{osp}(\mathcal{N}|4)$ generators are

$$\begin{aligned}
&(P_{\alpha\beta})^\dagger = K^{\alpha\beta}, \quad (K_{\alpha\beta})^\dagger = P^{\alpha\beta}, \quad (M_\alpha{}^\beta)^\dagger = M_\beta{}^\alpha, \quad D^\dagger = D,\\
&(Q_{\alpha r})^\dagger = -iS^\alpha{}_r, \quad (S^\alpha{}_r)^\dagger = -iQ_{\alpha r}, \quad (R_{rs})^\dagger = R_{rs}.
\end{aligned} \tag{3.10}$$

3.2 Superconformal Casimir

We can determine the superconformal Casimir $L^2_{\mathfrak{osp}(\mathcal{N}|4)}$ through the same method used for the conformal Casimir. In particular, we restrict our consideration to an operator that is quadratic in the generators and whose terms are all Lorentz scalars. It follows that terms quadratic in $Q_{\alpha r}$ and $S^\alpha_{\ r}$ can be excluded using identical arguments as were made for P_μ and K_μ. It follows that the Casimir has the general form

$$L^2_{\mathfrak{osp}(\mathcal{N}|4)} = L^2_{\mathfrak{so}(3,2)} + c_4 R_{rs}R_{rs} + c_5 Q_{\alpha r}S_r^{\ \alpha} + c_6 S_r^{\ \alpha}Q_{\alpha r}, \tag{3.11}$$

with three undetermined constants c_4, c_5, c_6. Since the Casimir commutes with all the generators, we use the fact that $[L^2_{\mathfrak{osp}(\mathcal{N}|4)}, Q_{r\alpha}] = 0$ to fix the values of the coefficients. It is sufficient to consider the commutators:

$$\begin{aligned}
[Q_{\alpha r}S^\alpha_{\ r}, Q_{\beta t}] &= 2P_{\alpha\beta}S^\alpha_{\ t} + 2Q_{\beta r}R_{rt} - 2iQ_{\alpha t}M_\beta^{\ \alpha} - 2iQ_{\beta t}D,\\
[S^\alpha_{\ r}Q_{\alpha r}, Q_{\beta t}] &= -2S^\alpha_{\ t}P_{\alpha\beta} - 2R_{rt}Q_{\beta r} + 2iM_\beta^{\ \alpha}Q_{\alpha t} + 2iDQ_{\beta t},\\
[R_{rs}R_{rs}, Q_{\alpha t}] &= -2iR_{rt}Q_{\alpha r} - 2iQ_{\alpha r}R_{rt},\\
[D^2, Q_{\alpha r}] &= \frac{1}{2}DQ_{\alpha r} + \frac{1}{2}Q_{\alpha r}D,\\
[P_{\alpha\beta}K^{\alpha\beta}, Q_{\gamma r}] &= 2iP_{\alpha\gamma}S^\alpha_{\ r},\\
[K^{\alpha\beta}P_{\alpha\beta}, Q_{\gamma r}] &= 2iS^\alpha_{\ r}P_{\alpha\gamma},\\
[M_\alpha^{\ \beta}M_\beta^{\ \alpha}, Q_{\gamma r}] &= M_\gamma^{\ \beta}Q_{\beta r} + Q_{\alpha r}M_\gamma^{\ \alpha}.
\end{aligned} \tag{3.12}$$

Since $P_{\alpha\beta}K^{\alpha\beta} = 2P_\mu K^\mu$ and $M_\alpha^{\ \beta}M_\beta^{\ \alpha} = 2M_{\mu\nu}M^{\mu\nu}$, it follows that $c_5 = -c_6 = ic_4 = -i/4$. The superconformal Casimir is thus given by

$$L^2_{\mathfrak{osp}(\mathcal{N}|4)} = L^2_{\mathfrak{so}(3,2)} - \frac{1}{4}R_{rs}R_{rs} - \frac{i}{4}(Q_{\alpha r}S_r^{\ \alpha} - S_r^{\ \alpha}Q_{\alpha r}). \tag{3.13}$$

Similar to the conformal Casimir, the superconformal Casimir is constant for individual representations of the $\mathfrak{osp}(\mathcal{N}|4)$ superconformal algebra. In principle, its differential representation in superspace could be used to determine representation-dependent functions called superconformal blocks. While these functions are important for the superconformal bootstrap, they are usually derived from another set of symmetry equations known as the superconformal Ward identities.

3.3 Representations of the $\mathcal{N} = 8$ Superconformal Algebra

In the previous two sections, we derived the superconformal algebra for an arbitrary number of real supercharges $4\mathcal{N}$, with $1 \leq \mathcal{N} \leq 8$. For the purposes of this thesis, we are mainly concerned with theories with $\mathcal{N} = 8$ supersymmetry, so we will specialize to the $\mathfrak{osp}(8|4)$ superconformal algebra. From a conceptual standpoint, we seek the unitary irreducible representations of $\mathfrak{osp}(8|4)$ with physically relevant quantum numbers. Its representations can be specified by the quantum numbers for representations of its maximal bosonic subalgebra $\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)_R$, as well as by additional restrictions known as *shortening conditions* [27].

3.3.1 Representations of $\mathfrak{so}(3,2)$

The unitary irreducible representations of the conformal algebra, the conformal multiplets, were worked out in Section 2.3; we briefly review the essential details. A conformal multiplet is specified by the conformal dimension $\Delta \geq 0$ and the Lorentz spin $\ell \in \frac{1}{2}\mathbb{N}$ of its highest weight state, and hence also of its conformal primary, which is annihilated by the conformal charge K_μ. Its two quantum numbers (Δ, ℓ) are the eigenvalues of (D, L_0), where D is the generator of dilatations, and L_0 is the axial spin generator for the little subgroup of the Lorentz group.[11] The generator of translations, P_μ, acts as a raising operator for Δ. Acting with P_μ on the highest weight state or conformal primary generates an infinite family of descendant states/operators. The conformal primaries satisfy the unitarity bounds in three dimensions, given by [50]

$$\begin{cases} \Delta \geq \frac{1}{2}, & \text{for } \ell = 0, \\ \Delta \geq 1 & \text{for } \ell = \frac{1}{2}, \\ \Delta \geq \ell + 1, & \text{otherwise.} \end{cases} \tag{3.14}$$

These bounds are saturated by free multiplets and conserved current multiplets.

[11] As mentioned in [54], all of the generators of $\mathfrak{osp}(8|4)$ are Hermitian except for ${M_\alpha}^\beta$ and D, which are anti-Hermitian. For the conformal unitarity bounds to be satisfied, we can employ a similarity transformation to make the two operators Hermitian.

3.3.2 Representations of $\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)_R$

Because $\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)_R$ takes the form of a direct sum algebra, it follows that its unitary irreducible representations are given as a direct product of representations of the $\mathfrak{so}(3,2)$ conformal algebra and of representations of the $\mathfrak{so}(8)_R$ R-symmetry algebra. The physically relevant irreducible representations of $\mathfrak{so}(8)_R$ can be determined through standard group theoretic arguments involving $\mathfrak{su}(2)^4 \subset \mathfrak{so}(8)_R$. Briefly, one can construct four generators H_m with $m = 1, \ldots, 4$ in the Cartan of $\mathfrak{so}(8)_R$ using linear combinations of the antisymmetric generators R_{rs}. It follows that the relevant representations of $\mathfrak{so}(8)_R$ are specified by four half-integers $\mathbf{r} = (r_1, r_2, r_3, r_4) \in (\frac{1}{2}\mathbb{Z})^4$ which satisfy $r_1 \geq r_2 \geq r_3 \geq |r_4|$. In particular, r_m is the eigenvalue of H_m for the highest weight state in the representation [55].[12] It is convenient to label representations of $\mathfrak{so}(8)_R$ by their Dynkin indices $\mathbf{a} \equiv [a_1, a_2, a_3, a_4] \in \mathbb{N}^4$. The two sets of quantum numbers are related by

$$[a_1, a_2, a_3, a_4] = [r_1 - r_2, r_2 - r_3, r_3 + r_4, r_3 - r_4] \tag{3.15}$$

Combining representations of the conformal and R-symmetry algebras yields those of $\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)_R$. Its representations are conformal multiplets $(\Delta, \ell)_{\mathbf{a}}$ parametrized by $\mathfrak{so}(8)_R$representations $\mathbf{a} \equiv [a_1, a_2, a_3, a_4]$. All the states in $(\Delta, \ell)_{\mathbf{a}}$ transform in the same representation $\mathbf{a}$ of $\mathfrak{so}(8)_R$. In particular, the representations $(\Delta, \ell)_{[0000]}$ in the single [0000] are the standard representations of the conformal algebra.

As an aside, we note that the representation theory for the $\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)_R$ bosonic subalgebra does not rely on representations of the $\mathfrak{osp}(8|4)$ superconformal algebra. Indeed, the irreducible representations of $\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)_R$ can be found in any CFT with a global $SO(8)$ symmetry. For these theories, the operators carry $\mathfrak{so}(8)$ indices. Additionally, the expansion of four-point correlation functions can be written as a sum over conformal blocks multiplied by analogous representation-dependent functions for $SO(8)$. We discuss these modifications to operators and to the OPE in Section 3.5.

[12] This can be thought of as a higher-dimensional analogue for the representations of $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$, which can be labeled by their spin eigenvalue under J_0.

3.3.3 Representations of $\mathfrak{osp}(8|4)$

We are now ready to introduce the representations of the full $\mathfrak{osp}(8|4)$ superconformal algebra [54]. The unitary irreducible representations of $\mathfrak{osp}(8|4)$, called *superconformal multiplets*, each contain a highest weight state that corresponds to an operator known as a *superconformal primary.* The highest weight states are specified by their quantum numbers $(\Delta, \ell, \mathbf{r})$ under the maximal bosonic subalgebra. We denote such a state by $|\Delta, \ell, \mathbf{r}\rangle^{\text{h.w}}$, which satisfies

$$(D, L_0, H_m)\,|\Delta, \ell, \mathbf{r}\rangle^{\text{h.w}} = (\Delta, \ell, r_m)\,|\Delta, \ell, \mathbf{r}\rangle^{\text{h.w}}\,, \tag{3.16}$$

and is annihilated by the bosonic charges $K^{\alpha\beta}$ and J_+, as well as by certain $\mathfrak{so}(8)_R$ generators outside of the Cartan. Additionally, the highest weight state of superconformal multiplet is annihilated by all of the conformal supercharges:

$$S^{\alpha}_{\ r}\,|\Delta, \ell, \mathbf{r}\rangle^{\text{h.w}} = 0. \tag{3.17}$$

From the anti-commutation relations in (3.8), we observe that $S^{\alpha}_{\ r}$ can roughly be identified as the square root of $K^{\alpha\beta}$. Indeed, the conformal supercharges also act as lowering operators for the conformal dimension, but with $\Delta \to \Delta - 1/2$. Similarly, the Poincaré supercharge $Q_{\alpha r}$ is roughly the square root of $P_{\alpha\beta}$, acting as a raising operator for the dimension with $\Delta \to \Delta + 1/2$. Acting twice with the same conformal supercharge is equivalent to acting with K; similarly, acting twice with the same Poincaré supercharge is equivalent to acting with $P_{\alpha\beta}$. Starting with a highest weight state, we can thus generate a whole family of superconformal descendants by acting with the Poincaré supercharges. However, acting with different supercharges does *not* correspond to acting with $P_{\alpha\beta}$. Since the superconformal primaries are also conformal primaries, these "middle" states must be conformal primaries as well. We therefore observe that a superconformal multiplet consists of a finite sequence

Type	BPS	Δ	Spin	$\mathfrak{so}(8)_R$
$(8, A, 0)$ (long)	0	$\geq \Delta_0 + \ell + 1$	ℓ	$[a_1, a_2, a_3, a_4]$
$(8, A, 1)$	1/16	$\Delta_0 + \ell + 1$	ℓ	$[a_1, a_2, a_3, a_4]$
$(8, A, 2)$	1/8	$\Delta_0 + \ell + 1$	ℓ	$[0, a_2, a_3, a_4]$
$(8, A, 3)$	3/8	$\Delta_0 + \ell + 1$	ℓ	$[0, 0, a_3, a_4]$
$(8, A, +)$	1/4	$\Delta_0 + \ell + 1$	ℓ	$[0, 0, a_3, 0]$
$(8, A, -)$	1/4	$\Delta_0 + \ell + 1$	ℓ	$[0, 0, 0, a_4]$
$(8, B, 1)$	1/8	Δ_0	0	$[a_1, a_2, a_3, a_4]$
$(8, B, 2)$	1/4	Δ_0	0	$[0, a_2, a_3, a_4]$
$(8, B, 3)$	3/8	Δ_0	0	$[0, 0, a_3, a_4]$
$(8, B, +)$	1/2	Δ_0	0	$[0, 0, a_3, 0]$
$(8, B, -)$	1/2	Δ_0	0	$[0, 0, 0, a_4]$
$(8, \text{cons.})$	5/16	$\ell + 1$	ℓ	$[0, 0, 0, 0]$

Table 1: Unitary irreducible representations (multiplets) of $\mathfrak{osp}(8|4)$ and the quantum numbers of their corresponding superconformal primary operator. As usual, the Lorentz spin can be any non-negative half-integer. The dimension Δ is given in terms of $\Delta_0 \equiv a_1 + a_2 + (a_3 + a_4)/2$. The irreducible representations of the $\mathfrak{so}(8)_R$ R-symmetry algebra are labeled by their Dykin labels $[a_1, a_2, a_3, a_4]$, where each a_i is a non-negative integer. The BPS of a given multiplet is the amount of preserved supersymmetry, i.e. the fraction of supercharges conserved for all states.

of conformal multiplets. Schematically, it is given by the sequence:

$$\begin{aligned}
(\Delta, \ell, \mathbf{a}) : \quad & \{|\Delta\rangle^{\text{h.w}}_{\text{scf}} \underset{Q^2}{\longrightarrow} |\Delta + 1\rangle \underset{Q^2}{\longrightarrow} \cdots\} \\
& \underset{Q}{\longrightarrow} \{|\Delta + 1/2\rangle^{\text{h.w.}}_{\text{cf}} \underset{Q^2}{\longrightarrow} |\Delta + 3/2\rangle \underset{Q^2}{\longrightarrow} \cdots\} \\
& \underset{Q}{\longrightarrow} \cdots \underset{Q}{\longrightarrow} \{|\Delta + 4\rangle^{\text{h.w}}_{\text{scf}} \underset{Q^2}{\longrightarrow} |\Delta + 5\rangle \underset{Q^2}{\longrightarrow} \cdots\}
\end{aligned} \tag{3.18}$$

where "cf" indicates a conformal primary and "scf" a superconformal primary, and where we have dropped the spin and $\mathfrak{so}(8)_R$ quantum numbers for clarity. The subsequences in (3.18) correspond to individual conformal multiplets.

There are two families of superconformal multiplets, an A series and a B series, which are distinguished by the unitarity bounds of their superconformal primaries [56]:

$$\begin{aligned}
A : \qquad & \Delta_A \geq \Delta_0 + \ell + 1 \,, \\
B : \qquad & \ell = 0 \,, \quad \Delta_B = \Delta_0 \,,
\end{aligned} \tag{3.19}$$

where $\Delta_0 = a_1 + a_2 + (a_3 + a_4)/2$. The A series is further decomposed into $(8, A, 0)$, the *long multiplets*, and into $(8, A, \cdot)$, the *semi-short* multiplets. The long multiplets have no shortening conditions, so their superconformal primaries can have arbitrary spin and dimension. The semi-short multiplets preserve some of the supersymmetry, implying that their states are annihilated by a fraction of the Poincaré supercharges. This fixes the scaling dimensions of their superconformal primaries to be $\Delta_{\text{A,semi-short}} = \Delta_0 + \ell + 1$. There is a special type of multiplet that saturates the A series unitarity bounds in (3.19): the conserved current multiplet. Since this multiplet contains higher-spin conserved currents, it can only appear in the free theory.

The B series consists of supersymmetry-preserving multiplets called *BPS multiplets*. Superconformal primaries in these multiplets are constrained to have zero spin and fixed scaling dimensions $\Delta_B = \Delta_0$. These multiplets are special because the scaling dimensions of their operators are protected and thus cannot receive quantum corrections. Operators in a BPS multiplet that preserve a fraction k of the total supersymmetry are usually called k-BPS operators. For instance, operators in the $(8, B, +)$ multiplets preserve half of the total supersymmetry, and are known as $\frac{1}{2}$-BPS operators. We list all of the $\mathfrak{osp}(8|4)$ multiplets in Table 1.

3.4 The $(8, B, +)_{[0020]}$ and $(8, B, +)_{[0040]}$ Multiplets

In this thesis, we will focus on the mixed correlator of two low-lying $(8, B, +)$ multiplets, $(8, B, +)_{[0020}$ and $(8, B, +)_{[0040]}$. The first multiplet contains the stress tensor as a conformal primary, i.e. a spin 2 operator with dimension 3 that transforms trivially under $\mathfrak{so}(8)_R$. This multiplet, which we refer to as the *stress tensor multiplet*, was studied with single correlator bootstrap techniques in [29, 30]. Its superconformal primary, denoted by $\mathcal{O}_{\mathbf{35}_c}$, consists of 35 scalars of dimension 1 that transform as the $\mathbf{35}_c$ under $\mathfrak{so}(8)_R$. The second multiplet, $(8, B, +)_{[0040]}$, is the next lowest nontrivial multiplet of type $(8, B, +)$ with $\mathfrak{so}(8)_R$ Dynkin label $[0, 0, 2m, 0]$. Its superconformal primary consists of 294 scalars of dimension 2 which transforms in the $\mathbf{294}_c$ under the R-symmetry algebra. By decomposing the $\mathfrak{osp}(8|4)$ characters into those of $\mathfrak{so}(3, 2)$ and $\mathfrak{so}(8)_R$, we can determine the conformal primary content of these multiplets [54, 57]. The results of this decomposition are listed in Table 2.

$(B,+)_{[0020]}$

Δ	ℓ	$\mathfrak{so}(8)_R$ irrep
1	0	$\mathbf{35}_c = [0020]$
3/2	1/2	$\mathbf{56}_v = [0011]$
2	0	$\mathbf{35}_s = [0002]$
2	1	$\mathbf{28} = [0100]$
5/2	3/2	$\mathbf{8}_v = [1000]$
3	2	$\mathbf{1} = [0000]$

$(B,+)_{[0040]}$

Δ	ℓ	$\mathfrak{so}(8)_R$ irrep
2	0	$\mathbf{294}_c = [0040]$
5/2	1/2	$\mathbf{672}_{cs} = [0031]$
3	0	$\mathbf{840}'_v = [0022]$
3	1	$\mathbf{576}_c = [0120]$
7/2	1/2	$\mathbf{840}_c = [0111]$
7/2	3/2	$\mathbf{224}_{cv} = [1020]$
4	0	$\mathbf{300} = [0200]$
4	1	$\mathbf{350} = [1011]$
4	2	$\mathbf{35}_c = [0020]$
9/2	1/2	$\mathbf{160}_v = [1100]$
9/2	3/2	$\mathbf{56}_v = [0011]$
5	0	$\mathbf{35}_v = [2000]$
5	1	$\mathbf{28} = [0100]$
11/2	1/2	$\mathbf{8}_v = [1000]$
6	0	$\mathbf{1} = [0000]$

Table 2: Decomposition of $\mathfrak{osp}(8|4)$ primaries into $\mathfrak{so}(3,2)$ primaries, which are labeled by their dimension Δ, Lorentz spin ℓ, and $\mathfrak{so}(8)_R$ irrep $[a_1, a_2, a_3, a_4]$.

We are especially interested in the mixed four-point functions for $\mathcal{O}_{\mathbf{35}_c}$ and $\mathcal{O}_{\mathbf{294}_c}$. We will call this set of correlation functions the *mixed correlator*. The unique four-point functions are given by

$$\text{mixed correlator}\begin{cases}\langle\mathcal{O}_{\mathbf{35}_c}\mathcal{O}_{\mathbf{35}_c}\mathcal{O}_{\mathbf{35}_c}\mathcal{O}_{\mathbf{35}_c}\rangle\\ \langle\mathcal{O}_{\mathbf{294}_c}\mathcal{O}_{\mathbf{294}_c}\mathcal{O}_{\mathbf{294}_c}\mathcal{O}_{\mathbf{294}_c}\rangle\\ \langle\mathcal{O}_{\mathbf{35}_c}\mathcal{O}_{\mathbf{35}_c}\mathcal{O}_{\mathbf{294}_c}\mathcal{O}_{\mathbf{294}_c}\rangle\\ \langle\mathcal{O}_{\mathbf{35}_c}\mathcal{O}_{\mathbf{294}_c}\mathcal{O}_{\mathbf{35}_c}\mathcal{O}_{\mathbf{294}_c}\rangle\end{cases} \tag{3.20}$$

3.5 $SO(8)$ Global Symmetry

3.5.1 Correlation Functions

In preparation for the mixed bootstrap, we now probe CFTs with a global $SO(8)$ symmetry.[13] Consider a set of scalars that transform in the rank k traceless

[13] Although our aim is to eventually study the mixed correlator for $\mathcal{O}_{\mathbf{35}_c}$ and $\mathcal{O}_{\mathbf{294}_c}$, the analysis of this section is general in nature, applying to any CFT with an $SO(8)$ global symmetry.

symmetric representation of $\mathfrak{so}(8)_R$, i.e. $[00k0]$. The corresponding operator is denoted by $\mathcal{O}_{i_1,\ldots,i_k}$, where the global indices run from $i_1,\ldots,i_k = 1,\ldots,8$.[14] It is quite cumbersome to work with the tensor structures found in correlation functions of these operators. To simplify notation, we introduce a set of auxiliary variables Y that commute:

$$Y_1^i Y_2^j = Y_2^j Y_1^i, \tag{3.21}$$

and which have null norm, i.e.

$$Y \cdot Y \equiv \sum_{i=1}^{8} Y_i Y^i = 0. \tag{3.22}$$

We can then contract the $SO(8)$ indices of the Ys with those of the operators to form a composite operator $\mathcal{O}_k(x,Y)$, which depends on both the spacetime position x and on the $\mathbb{R}^8$ "position" Y. It is defined as

$$\mathcal{O}_k(x,Y) \equiv \mathcal{O}_{i_1,\ldots,i_k} Y^{i_1} \cdots Y^{i_k}. \tag{3.23}$$

Based on the above definition, these operators must be homogeneous polynomials in the Ys, with

$$\mathcal{O}_k(x,\rho Y) = \rho^k \mathcal{O}_k(x,Y). \tag{3.24}$$

From this, it is apparent that the composite operators contain exactly the same information as their covariant counterparts. Extracting the component $\mathcal{O}_{i_1,\ldots,i_k}$ simply amounts to determining the coefficient of $Y^{i_1}\cdots Y^{i_k}$ in the polynomial $\mathcal{O}_k(x,Y)$. This nice result is due to the fact that the properties of the Ys directly correspond to those of the covariant operator. Their commutativity is related to the fact that $\mathcal{O}^{i_1,\ldots,i_k}$ is a symmetric tensor, and their null norm corresponds with its tracelessness.

The utility of the composite operators in (3.23) comes into play when we consider the correlation functions of the theory. Instead of having to work with tensor structures, we now simply consider $SO(8)$ invariants built from the Ys. In general,

[14]We have used a new index label i_j to distinguish the $\mathfrak{so}(8)$ representation $[00k0] \in \otimes^k[0010]$ from the representation $[1000]$ of the superconformal charges.

the n-point function for a set of n operators $\{\mathcal{O}_{k_i}\}$ with scaling dimensions $\{\Delta_i\}$ must contain k_i copies of Y_i. These two requirements are sufficient to fix the forms of the correlation functions for $n = 2, 3$. For example, for $n = 2$ there are only two variables Y_1 and Y_2, of which there is only a single invariant: $Y_1 \cdot Y_2$. It follows that the two-point function is given by

$$\langle \mathcal{O}_k(x_1, Y_1)\mathcal{O}_k(x_2, Y_2)\rangle = \frac{(Y_1 \cdot Y_2)^k}{|x_{12}|^{2\Delta_k}} \tag{3.25}$$

up to an overall c-number related to the normalization of $\mathcal{O}_k$. Proceeding with the same reasoning for $n = 3$, we find that the three-point function has the form

$$\langle \mathcal{O}_{k_1}(x_1, Y_1)\mathcal{O}_{k_2}(x_2, Y_2)\mathcal{O}_{k_3}(x_3, Y_3)\rangle = \lambda \frac{(Y_1 \cdot Y_2)^{k_{123}}(Y_3 \cdot Y_3)^{k_{231}}(Y_3 \cdot Y_1)^{k_{312}}}{x_{12}^{2\alpha_{123}} x_{23}^{2\alpha_{231}} x_{31}^{2\alpha_{312}}}, \tag{3.26}$$

where λ is none other than the OPE coefficient, and where we have defined the constants $k_{ijk} = (k_i + k_j - k_k)/2$ and $\alpha_{ijk} = (\Delta_i + \Delta_j - \Delta_k)/2$, with $i, j, k = 1, 2, 3$. In particular, for operators with dimension $\Delta_i = k_i/2$ (e.g. $(8, B, +)_{[00k0]}$ primaries), the correlation functions in (3.25) and (3.26) reduce to

$$\begin{aligned}
\langle \mathcal{O}_k(x_1, Y_1)\mathcal{O}_k(x_2, Y_2)\rangle &= \left(\frac{Y_1 \cdot Y_2}{|x_{12}|}\right)^k, \\
\langle \mathcal{O}_{k_1}(x_1, Y_1)\mathcal{O}_{k_2}(x_2, Y_2)\mathcal{O}_{k_3}(x_3, Y_3)\rangle &= \lambda \left(\frac{Y_1 \cdot Y_2}{|x_{12}|}\right)^{k_{123}} \left(\frac{Y_2 \cdot Y_3}{|x_{23}|}\right)^{k_{231}} \left(\frac{Y_3 \cdot Y_1}{|x_{31}|}\right)^{k_{312}}.
\end{aligned} \tag{3.27}$$

3.5.2 Four-point Functions

Just as before, we give special consideration to the four-point function, which plays an important role in the mixed correlator. For convenience, it is denoted by

$$G(x_i, Y_i) = \langle \mathcal{O}_{k_1}(x_1, Y_1)\mathcal{O}_{k_2}(x_2, Y_2)\mathcal{O}_{k_3}(x_3, Y_3)\mathcal{O}_{k_4}(x_4, Y_4)\rangle\,. \tag{3.28}$$

To determine its general form, we can simply apply the analysis of the previous section and seek out $SO(8)$-invariant quantities constructed from the Y_i. Although $SO(8)$ is certainly different from the conformal group, it is useful to introduce $SO(8)$

invariants analogous to the conformal cross-ratios; they are given by

$$U = \frac{(Y_1 \cdot Y_2)(Y_3 \cdot Y_4)}{(Y_1 \cdot Y_2)(Y_3 \cdot Y_4)}, \quad V = \frac{(Y_3 \cdot Y_2)(Y_1 \cdot Y_4)}{(Y_1 \cdot Y_2)(Y_3 \cdot Y_4)}. \tag{3.29}$$

Besides being invariant under $SO(8)$ rotations, the ratios U and V are independent under rescalings of the form $Y \to \rho Y$. Using (3.29) and the fact that G is a homogeneous polynomial in Ys, we can write down the s-channel expansion of G as

$$G_{1234}(x_i, Y_i) = I_{1234}(x_i, Y_i)\, \mathcal{G}(u, v; \sigma, \tau), \tag{3.30}$$

where I_{1234} is the four-point prefactor from (2.37) with an added $SO(8)$ term:

$$\begin{aligned} I_{1234} = (Y_1 \cdot Y_2)^{\frac{k_1+k_2}{2}} (Y_3 \cdot Y_4)^{\frac{k_4+k_3}{2}} \left(\frac{Y_2 \cdot Y_4}{Y_1 \cdot Y_4}\right)^{\frac{k_{12}}{2}} \left(\frac{(Y_1 \cdot Y_2)(Y_3 \cdot Y_4)}{(Y_1 \cdot Y_4)(Y_2 \cdot Y_4)}\right)^{\frac{k_{43}}{2}} \\ \times \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_2+\Delta_3}} \left(\frac{x_{24}}{x_{14}}\right)^{\Delta_{12}} \left(\frac{x_{14}}{x_{13}}\right)^{\Delta_{34}}, \end{aligned} \tag{3.31}$$

where $k_{ij} \equiv k_i - k_j$ and $\mathcal{G}(u, v; \sigma, \tau)$ is an undetermined function of the conformal cross-ratios, as well as a polynomial in σ and τ, where

$$\sigma = \frac{1}{U}, \quad \tau = \frac{V}{U}. \tag{3.32}$$

3.5.3 Differential Representations of $SO(8)$

Thus far, we have found that the four-point function of operators transforming as the $[00k0]$ carry additional structure in terms of the polarization vectors Y_i. Its form in (3.30) is completely determined by the value of k, as well as by the operator scaling dimensions, up to an overall multiplicative function $\mathcal{G}(u, v; \sigma, \tau)$. This is completely analogous to the case of a CFT with no global symmetry, where the four-point function is determined up to an arbitrary function $g(u, v)$. In fact, just as g can be expanded in terms of conformal blocks and written as a sum over representations of the conformal algebra $\mathfrak{so}(3, 2)$, so too can $\mathcal{G}$ be expanded in terms of representations of $\mathfrak{so}(3, 2) \oplus \mathfrak{so}(8)$. Because this is a direct sum, we should be able to decompose $\mathcal{G}$ separately into representations of $\mathfrak{so}(3, 2)$ and $\mathfrak{so}(8)$. Schematically, this looks like

$$\mathcal{G}(u,v;\sigma,\tau) \sim \sum_{\rho_{\mathfrak{so}(8)}} \sum_{\rho_{\mathfrak{so}(3,2)}} T_{\rho_{\mathfrak{so}(8)}}(\sigma,\tau) g_{\rho_{\mathfrak{so}(8)},\rho_{\mathfrak{so}(3,2)}}(u,v), \tag{3.33}$$

where $\rho_{\mathfrak{g}}$ is an irreducible representation of the Lie algebra $\mathfrak{g}$ and $g_{\rho_1,\rho_2}(u,v)$ is a conformal block parametrized by the $\mathfrak{so}(8)$ representations. We can eliminate the inner sum by defining a new function Γ as a sum over the conformal blocks. This yields

$$\mathcal{G}(u,v;\sigma,\tau) \sim \sum_{\rho_{\mathfrak{so}(8)}} T_{\rho_{\mathfrak{so}(8)}}(\sigma,\tau) \Gamma_{\rho_{\mathfrak{so}(8)}}(u,v). \tag{3.34}$$

The sum in (3.34) lends itself to various interpretations. For instance, we can interpret the polynomials $T_\rho(\sigma,\tau)$ superficially as a choice of basis for the four-point function $\mathcal{G}$. On a deeper level, they represent the infinite-dimensional representations of the $SO(8)$ algebra.

To determine such representations, we begin by analyzing the the $\mathfrak{so}(8)$ Casimir $L^2_{\mathfrak{so}(8)}$, which can be expressed in terms of the generators R_{ij} as[15]

$$L^2_{\mathfrak{so}(8)} = \frac{1}{4} R_{ij} R^{ij}, \tag{3.35}$$

where we have included a factor of $1/4$, without loss of generality, to match the superconformal Casimir in (3.13) up to an overall sign. In the differential representation of $\mathfrak{so}(8)$, the action of the generators on $\mathcal{O}_k$ is given by

$$[R_{ij}, \mathcal{O}_k(x,Y)] = \left(Y_i \frac{\partial}{\partial Y^j} - Y_j \frac{\partial}{\partial Y^i} \right) \mathcal{O}_k(x,Y). \tag{3.36}$$

This action can be extended to the product of two operators in the same fashion as the conformal case in Section 2.6.2. It follows that the action of $L^2_{\mathfrak{so}(8)}$ on $\mathcal{G}$ is

$$L^2_{\mathfrak{so}(8)} G_{1234} = I_{1234} \mathcal{D}^{(a,b)}_{\mathfrak{so}(8)} \mathcal{G}, \tag{3.37}$$

[15]The $\mathfrak{so}(8)$ generators transform in the adjoint [0100]. They can be packaged together either as an antisymmetric matrix R_{rs} with $\mathbf{8}_v$ indices, or as R_{ij} with $\mathbf{8}_c$ indices.

where the differential representation of $L^2_{\mathfrak{so}(8)}$ is given by

$$L^2_{\mathfrak{so}(8)} \to \frac{1}{4}\left(Y_{1,i}\frac{\partial}{\partial Y_1^j} - Y_{1,j}\frac{\partial}{\partial Y_1^i} + Y_{2,i}\frac{\partial}{\partial Y_j^2} - Y_{2,j}\frac{\partial}{\partial Y_i^2}\right)^2, \tag{3.38}$$

and where $\mathcal{D}^{(a,b)}_{\mathfrak{so}(8)}$ is some differential operator depending on the the quantities

$$a = \frac{1}{2}(k_{12} - k_{34}), \quad b = -\frac{1}{2}(k_{12} + k_{34}). \tag{3.39}$$

Similar to the conformal case, we observe that the action of the Casimir passes through the prefactor in (3.37).

3.5.4 Null Space Projection

To determine the form of $\mathcal{D}^{(a,b)}_{\mathfrak{so}(8)}$, we use a method analogous to the null surface formalism developed in Section 2.4. The key insight in this case is that $SO(8)$, being locally isomorphic to $SO(6,2)$, acts linearly on the null vector Y. Instead of embedding the position $x \in \mathbb{R}^{2,1}$ into a null vector $X \in \mathbb{R}^{3,2}$, we employ the reverse method: we project $Y \in \mathbb{R}^{6,2}$ to a Minkowski vector $y \in \mathbb{R}^{5,1}$. Using (2.22) as a guide, we can write the projection map explicitly as

$$Y^i = \left(\frac{1-y^2}{2}, y^{\widetilde{\mu}}, \frac{1+y^2}{2}\right) \mapsto y^{\widetilde{\mu}}, \tag{3.40}$$

where the pseudo-spacetime indices are given by $\widetilde{\mu}, \widetilde{\nu} = 0, \ldots, 6$, and the metric on $\mathbb{R}^{5,1}$ is $\eta_{\widetilde{\mu}\widetilde{\nu}} = \mathrm{diag}(-1, 1, \ldots, 1)$. Under the projection (3.40), the $SO(6,2)$ product is given by the familiar expression

$$-2Y_1 \cdot Y_2 = (y_1 - y_2)^2. \tag{3.41}$$

It follows that the $SO(8)$ invariants U and V can be viewed as bona fide $SO(6,2)$ conformal cross-ratios, with

$$U = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}, \quad V = \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}, \quad y_{ij} \equiv |y_i - y_j|. \tag{3.42}$$

Continuing the analogy, we introduce complex variables α and $\overline{\alpha}$ similar to $z, \overline{z}$ in (2.33). However, the difference in this case is that T is a polynomial in σ and

τ, which gives these variables more importance than U and V. We therefore write $\sigma(\alpha, \overline{\alpha})$ and $\tau(\alpha, \overline{\alpha})$ as functions of the complex variables. Temporarily continuing to Euclidean space, the pseudo-spacetime points can be written as

$$\begin{aligned} y_1 &= (1, 0, \ldots, 0), \quad y_2 = 0, \\ y_3 &= (q, 0, \ldots, 0), \quad y_4 = \left(\frac{\alpha + \overline{\alpha}}{2}, \frac{\alpha - \overline{\alpha}}{2i}, 0, \ldots, 0 \right). \end{aligned} \tag{3.43}$$

In the limit where $q \to \infty$, we find

$$\sigma = \alpha\overline{\alpha}, \quad \tau = (1 - \alpha)(1 - \overline{\alpha}). \tag{3.44}$$

Defining $\mathcal{G}_q(\alpha, \overline{\alpha}) \equiv \mathcal{G}(\sigma(\alpha, \overline{\alpha}), \tau(\alpha, \overline{\alpha}))$ for arbitrary values of q, the differential equation in (3.37) becomes

$$L^2_{\mathfrak{so}(8)} G_{1234} = I_{1234} \lim_{q \to \infty} \mathcal{D}^{(a,b)}_{\mathfrak{so}(8)_R} \mathcal{G}_q(\alpha, \overline{\alpha}). \tag{3.45}$$

Evaluating the limit above gives $\mathcal{D}^{(a,b)}_{\mathfrak{so}(8)_R}$ in terms of α and $\overline{\alpha}$. We can convert back to the variables σ and τ by using (3.44). This yields

$$\begin{aligned} \mathcal{D}^{(a,b)}_{\mathfrak{so}(8)_R} = \mathcal{D}_{\mathfrak{so}(8)_R} &+ (1 - \sigma - \tau) \left(a \frac{\partial}{\partial \tau} + b \frac{\partial}{\partial \sigma} \right) \\ &- 2a\sigma \frac{\partial}{\partial \sigma} - 2b\tau \frac{\partial}{\partial \tau} - \frac{1}{2}(a + b)(a + b + 6)\,, \end{aligned} \tag{3.46}$$

where

$$\mathcal{D}_{\mathfrak{so}(8)_R} = (1 - \sigma - \tau) \left(\frac{\partial}{\partial \sigma} \sigma \frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \tau} \tau \frac{\partial}{\partial \tau} \right) - 4\sigma\tau \frac{\partial^2}{\partial \sigma \partial \tau} - 6 \left(\sigma \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} \right). \tag{3.47}$$

For correlation functions involving a single operator with $k_i = k$ ($a = b = 0$), the differential operator $\mathcal{D}^{(a,b)}_{\mathfrak{so}(8)}$ reduces to $\mathcal{D}_{\mathfrak{so}(8)}$.

3.5.5 $SO(8)$ Harmonics

With the differential form of the $\mathfrak{so}(8)$ Casimir, we can now determine the differential representations of $SO(8)$. We consider the eigenvalue equation [58]:

$$\mathcal{D}^{(a,b)}_{\mathfrak{so}(8)_R} T^{(a,b)}_{nm} = -[n(n+5)+m(m+1)]T^{(a,b)}_{nm}. \tag{3.48}$$

The solutions to (3.48) are polynomials of degree $n+(a+b)/2$, where the function $T^{(a,b)}_{nm}(\sigma,\tau)$ corresponds with the $\mathfrak{so}(8)$ representation $[0,n-m,2m,0]$. They are analogous to the spherical harmonics, the eigenfunctions of the Laplacian on $\mathbb{R}^3$ restricted to the sphere S^2. In our case, the $\{T^{(a,b)}_{nm}\}$ are the eigenfunctions of the $\mathbb{R}^{6,2}$ Laplacian restricted to the submanifold $Y^2=0$. To determine their coefficients, we simply insert an *ansatz* given by

$$T^{(a,b)}_{nm}(\sigma,\tau) = \sum_{p+q\leq n+(a+b)/2} c_{p,q}\, \sigma^p \tau^q \tag{3.49}$$

into (3.48) and solve for $c_{p,q}$ in the resulting linear equations. In particular, the solutions can be chosen to be orthogonal with respect to the inner product [58]:

$$\langle T^{(a,b)}_{nm}, T^{(c,d)}_{pq}\rangle = \int_0^1 \int_0^{(1-\tau^{1/2})^2} \mathrm{d}\tau \mathrm{d}\sigma\; w(\sigma,\tau) T^{(a,b)}_{nm}(\sigma,\tau) T^{(c,d)}_{pq}(\sigma,\tau), \tag{3.50}$$

where $w=\Lambda^{3/2}$ is a weight function, and

$$\Lambda(\sigma,\tau) = (\sigma^{1/2}+\tau^{1/2}+1)(\sigma^{1/2}+\tau^{1/2}-1)(\sigma^{1/2}-\tau^{1/2}+1)(\sigma^{1/2}-\tau^{1/2}-1). \tag{3.51}$$

Under our conventions, the eigenfunctions are normalized to $T^{(a,b)}_{(a+b)/2,(a+b)/2}=1$, such that under crossing ($\sigma \leftrightarrow \tau$) they satisfy $T^{(0,2)}_{nm}(\sigma,\tau) = T^{(2,0)}_{nm}(\tau,\sigma)$.

3.6 Towards the Mixed Bootstrap

The $\mathcal{N}=8$ single-correlator bootstrap was implemented by [29, 30]. Using a semidefinite problem solver, the authors studied the crossing equations for the single correlator of $\mathcal{O}_{\mathbf{35}_c}$ and extracted various non-perturbative bounds for the OPE coefficients and scaling dimensions of low-lying operators in the spectrum. In the interest of producing stricter bounds, we begin the setup for the mixed bootstrap between $\mathcal{O}_{\mathbf{35}_c}$ and $\mathcal{O}_{\mathbf{294}_c}$. Specifically, we derive the crossing equations for the mixed correlator and construct a method for determining their dependency relations.

3.6.1 Four-point Functions

The operators $\mathcal{O}_{\mathbf{35}_c}$ and $\mathcal{O}_{\mathbf{294}_c}$ have $\mathfrak{so}(8)_R$ weights [0020] and [0040], respectively. Using (3.30), the s-channel expansion of the mixed correlator can be written as

$$\begin{aligned}
&\langle \mathcal{O}_{\mathbf{35}_c}(x_1,Y_1)\mathcal{O}_{\mathbf{35}_c}(x_2,Y_2)\mathcal{O}_{\mathbf{35}_c}(x_3,Y_3)\mathcal{O}_{\mathbf{35}_c}(x_4,Y_4)\rangle \\
&\qquad = \frac{(Y_1\cdot Y_2)^2(Y_3\cdot Y_4)^2}{x_{12}^2x_{34}^2}\mathcal{G}_{2222}(u,v;\sigma,\tau), \\
&\langle \mathcal{O}_{\mathbf{294}_c}(x_1,Y_1)\mathcal{O}_{\mathbf{294}_c}(x_2,Y_2)\mathcal{O}_{\mathbf{294}_c}(x_3,Y_3)\mathcal{O}_{\mathbf{294}_c}(x_4,Y_4)\rangle \\
&\qquad = \frac{(Y_1\cdot Y_2)^4(Y_3\cdot Y_4)^4}{x_{12}^4x_{34}^4}\mathcal{G}_{4444}(u,v;\sigma,\tau), \\
&\langle \mathcal{O}_{\mathbf{35}_c}(x_1,Y_1)\mathcal{O}_{\mathbf{35}_c}(x_2,Y_2)\mathcal{O}_{\mathbf{294}_c}(x_3,Y_3)\mathcal{O}_{\mathbf{294}_c}(x_4,Y_4)\rangle \\
&\qquad = \frac{(Y_1\cdot Y_2)^2(Y_3\cdot Y_4)^4}{x_{12}^2x_{34}^4}\mathcal{G}_{2244}(u,v;\sigma,\tau), \\
&\langle \mathcal{O}_{\mathbf{35}_c}(x_1,Y_1)\mathcal{O}_{\mathbf{294}_c}(x_2,Y_2)\mathcal{O}_{\mathbf{35}_c}(x_3,Y_3)\mathcal{O}_{\mathbf{294}_c}(x_4,Y_4)\rangle \\
&\qquad = \frac{(Y_1\cdot Y_2)^2(Y_3\cdot Y_4)^2(Y_2\cdot Y_4)^2}{x_{12}^2x_{34}^2x_{12}^2}\mathcal{G}_{2424}(u,v;\sigma,\tau).
\end{aligned} \tag{3.52}$$

From now on, we will drop the prefactor and refer to $\mathcal{G}_{k_1,k_2,k_3,k_4}$ as the four-point function. Following the previous section, we can write these functions in terms of the $SO(8)$ harmonics, with

$$\mathcal{G}_{k_1,k_2,k_3,k_4}(u,v;\sigma,\tau) = \sum_{n,m} T_{nm}^{(a,b)}(\sigma,\tau)\Gamma_{nm}(u,v). \tag{3.53}$$

The $\Gamma_{nm}(u,v)$ in the above expression are functions of the conformal cross-ratios. We have purposely left the summation indices in (3.53) vague; to determine the harmonics that enter the sum, we need to know which $\mathfrak{so}(8)_R$ representations are relevant. The OPEs found in the mixed correlator are $\mathcal{O}_{\mathbf{35}_c}\times\mathcal{O}_{\mathbf{35}_c}$, $\mathcal{O}_{\mathbf{294}_c}\times\mathcal{O}_{\mathbf{294}_c}$, and $\mathcal{O}_{\mathbf{35}_c}\times\mathcal{O}_{\mathbf{294}_c}$. These correspond to the $\mathfrak{so}(8)_R$ products $[0020]\otimes[0020]$, $[0040]\otimes[0040]$, and $[0020]\otimes[0040]$, respectively. The tensor product between two $\mathfrak{so}(8)_R$ representations with Dynkin indices $[00k0]$ can be decomposed into a direct sum of irreducible representations of the form $[0, 2m, n, 0]$. It is given by

$$[00k_i0] \otimes [00k_j0] = \bigoplus_{p=0}^{\max(k_i,k_j)} \bigoplus_{q=0}^{p} [0, p-q, 2q+|k_{ij}|, 0]. \tag{3.54}$$

The tensor products of interest admit the decompositions:

$$\begin{aligned}
[0020] \otimes [0020] &= [0000] \oplus [0100] \oplus [0020] \oplus [0040] \oplus [0200] \oplus [0120], \\
[0020] \otimes [0040] &= [0020] \oplus [0040] \oplus [0120] \oplus [0060] \oplus [0140] \oplus [0220], \\
[0040] \otimes [0040] &= [0000] \oplus [0100] \oplus [0020] \oplus [0040] \oplus [0200] \oplus [0120] \\
&\quad \oplus [0060] \oplus [0300] \oplus [0140] \oplus [0220] \oplus [0080] \oplus [0400] \\
&\quad \oplus [0160] \oplus [0320] \oplus [0240].
\end{aligned} \tag{3.55}$$

The harmonic $T_{nm}^{(a,b)}$ corresponds to the $\mathfrak{so}(8)_R$ representation $[0, n-m, 2m, 0]$ with $a = (k_{12} - k_{34})/2$ and $b = -(k_{12} + k_{34})/2$. Therefore, the harmonic corresponding to $[0, n-m, 2m, 0]$ enters into (3.53) only if

$$[0, n-m, 2m, 0] \in ([00k_10] \otimes [00k_20]) \cap ([00k_30] \otimes [00k_40]). \tag{3.56}$$

The relevant $SO(8)$ harmonics are listed in Appendix A.

3.6.2 Crossing Equations

We are now ready to determine the crossing equations for the mixed correlator. In the previous section, we used the s-channel decomposition to express the four-point functions $\mathcal{G}_{k_1,k_2,k_3,k_4}$ as finite sums over the $SO(8)$ harmonics. In this basis, the components $\Gamma_{nm}(u, v)$ can be further expanded in terms of the $\mathfrak{so}(3, 2)$ mixed blocks $g_{\Delta,\ell}^{\Delta_{ij},\Delta_{kl}}(u, v)$. For notational convenience, we first introduce crossed blocks defined by

$$F_{\mp,\Delta,\ell,n,m}^{ij,kl}(u, v) = v^{\frac{\Delta_k+\Delta_j}{2}} g_{\Delta,\ell,n,m}^{\Delta_{ij},\Delta_{kl}}(u, v) \mp u^{\frac{\Delta_k+\Delta_j}{2}} g_{\Delta,\ell,n,m}^{\Delta_{ij},\Delta_{kl}}(v, u), \tag{3.57}$$

where the $g^{\Delta_{ij},\Delta_{kl}}_{\Delta,\ell,n,m}(u,v)$ can be written in terms of conformal blocks. The crossed blocks are analogous to those in (2.56). We also define

$$\widetilde{T}^{(a,b)}_{nm}(\sigma,\tau) = \tau^{\frac{k_1+k_2+k_3-k_4}{2}} T^{(a,b)}_{nm}(\sigma\tau^{-1},\tau^{-1}), \tag{3.58}$$

which correspond with the $SO(8)$ harmonics under crossing.

Next, we define $\lambda_{ij\mathcal{O}}$ as the OPE coefficient for the operator $\mathcal{O}$ in the OPE corresponding with $[00k_i0]\otimes[00k_j0]$. For instance, $\lambda_{24\mathcal{O}}$ is the OPE coefficient for $\mathcal{O}\in\mathcal{O}_{\mathbf{35}_c}\times\mathcal{O}_{\mathbf{294}_c}$. The operators under consideration are real, and consequently so are the OPE coefficients. It follows that we have that identity

$$\lambda_{\phi\phi\mathcal{O}} = (-1)^{\ell}\lambda_{\mathcal{O}\phi\phi}, \tag{3.59}$$

where ℓ is the spin of $\mathcal{O}$.

To find the t-channel expansion of (3.52), we simply interchange indices $1\leftrightarrow 3$ in (3.53). This yields a set of mixed crossing equations:

$$\sum_{n=0}^{2}\sum_{m=0}^{n} s^{2222}_{n,m}(-1)^{\ell}\lambda^2_{22\mathcal{O}}F^{22,22}_{\mp,\Delta,\ell,n,m}(u,v)\left[T^{(0,0)}_{nm}(\sigma,\tau)\pm\widetilde{T}^{(0,0)}_{nm}(\sigma,\tau)\right]=0 \tag{3.60}$$

$$\sum_{n=0}^{4}\sum_{m=0}^{n} s^{4444}_{n,m}(-1)^{\ell}\lambda^2_{44\mathcal{O}}F^{44,44}_{\mp,\Delta,\ell,n,m}(u,v)\left[T^{(0,0)}_{nm}(\sigma,\tau)\pm\tilde{T}^{(0,0)}_{nm}(\sigma,\tau)\right]=0 \tag{3.61}$$

$$\sum_{n=1}^{3}\sum_{m=1}^{n} s^{2424}_{n,m}(-1)^{\ell}\lambda^2_{24\mathcal{O}}F^{24,24}_{\mp,\Delta,\ell,n,m}(u,v)\left[T^{(0,2)}_{nm}(\sigma,\tau)\pm\tilde{T}^{(0,2)}_{nm}(\sigma,\tau)\right]=0 \tag{3.62}$$

$$\begin{aligned}&\sum_{n=1}^{3}\sum_{m=1}^{n} s^{4224}_{n,m}\lambda^2_{24\mathcal{O}}F^{42,24}_{\mp,\Delta,\ell,n,m}(u,v)T^{(2,0)}_{nm}(\sigma,\tau)\\ \pm&\sum_{n=0}^{2}\sum_{m=0}^{n} s^{2244}_{n,m}(-1)^{\ell}\lambda_{22\mathcal{O}}\lambda_{44\mathcal{O}}F^{22,44}_{\mp,\Delta,\ell,n,m}(u,v)\tilde{T}^{(0,0)}_{nm}(\sigma,\tau)=0\end{aligned} \tag{3.63}$$

Based on how the $T^{(a,b)}_{nm}$ are defined, it follows that $s^{\text{4-pt}}_{n,m}=1$. The above crossing equations can be combined into a single vector equation. Schematically, it is given by

$$\sum_{\mathcal{O}}\lambda^T_{\mathcal{O}}F_{\mathcal{O}}\lambda_{\mathcal{O}}+\lambda^2_{24\mathcal{O}}\ \text{terms} \ = \ 0, \tag{3.64}$$

where $\lambda_{\mathcal{O}} = (\lambda_{22\mathcal{O}}, \lambda_{44\mathcal{O}})^T$ and $F_{\mathcal{O}}$ is a 2×2 matrix.

3.6.3 Dependency Relations

In superconformal field theories, the conformal blocks are no longer independent, and are now grouped together into sets called *superconformal blocks*. Just as conformal blocks individually correspond with conformal multiplets, so too do superconformal blocks correspond with superconformal multiplets. The crossing equations in (3.64) only depend on the bosonic subalgebra $\mathfrak{so}(3,2) \oplus \mathfrak{so}(8)_R$. The $\mathfrak{osp}(8|4)$ superconformal algebra imposes additional relations among them. These relations can be derived explicitly through the superconformal Ward identities.

It is convenient to first rewrite the four-point function $\mathcal{G}(z, \overline{z}; \alpha, \overline{\alpha})$ in terms of both the conformal complex variables $z, \overline{z}$, as well as the $SO(8)$ complex variables $\alpha, \overline{\alpha}$. These are related to the conventional cross-ratios by

$$\begin{aligned} u &= z\overline{z}, \quad v = (1-z)(1-\overline{z}), \\ \sigma &= \alpha\overline{\alpha}, \quad \tau = (1-\alpha)(1-\overline{\alpha}). \end{aligned} \tag{3.65}$$

In terms of these variables, the four-point function is given by

$$\widehat{\mathcal{G}}(x, \overline{x}; \alpha, \overline{\alpha}) = u^{\frac{\Delta_{34}}{2}} \sum_{n,m} T_{nm}(\sigma(\alpha, \overline{\alpha}), \tau(\alpha, \overline{\alpha})) g_{nm}(u(z, \overline{z}), v(z, \overline{z})), \tag{3.66}$$

where we have introduced a factor of $u^{\frac{\Delta_{34}}{2}}$ to match [59]. The superconformal Ward identities for $\mathfrak{osp}(8|4)$ are given by [59]:

$$\begin{aligned} (z\partial_z - (\alpha/2)\partial_\alpha)\widehat{\mathcal{G}}(z, \overline{z}; \alpha, \overline{\alpha})|_{\alpha=1/z} = 0, \\ (\overline{z}\partial_{\overline{z}} - (\overline{\alpha}/2)\partial_{\overline{\alpha}})\widehat{\mathcal{G}}(z, \overline{z}; \alpha, \overline{\alpha})|_{\overline{\alpha}=1/\overline{z}} = 0. \end{aligned} \tag{3.67}$$

Summing over the conformal multiplets, we can rewrite the crossed blocks in (3.57) in terms of functions dependent on the $\mathfrak{so}(8)_R$ representations alone:

$$F^{ij,kl}_{\mp,n,m}(u, v) = \sum_{\mathcal{O}} F^{ij,kl}_{\mp,\Delta,\ell,n,m}(u, v). \tag{3.68}$$

To remove their u, v dependence, we consider an expansion of the function

$$f_{nm}(z,\overline{z}) \equiv v^{\frac{\Delta_k+\Delta_j}{2}} g_{nm}(u(z,\overline{z}), v(z,\overline{z})) \tag{3.69}$$

around the crossing-symmetric point:

$$\begin{aligned} f_{nm}(z,\overline{z}) &= \sum_{p,q=0}^{\infty} \frac{1}{p!q!}\left(z-\frac{1}{2}\right)^p \left(\overline{z}-\frac{1}{2}\right)^q \tilde{f}_{nm}^{(p,q)}, \\ \tilde{f}_{nm}^{(p,q)} &\equiv \partial^p \overline{\partial}^q f_{nm}(z,\overline{z})\big|_{z=\overline{z}=\frac{1}{2}}. \end{aligned} \tag{3.70}$$

Under crossing, $z\overline{z} \leftrightarrow (1-z)(1-\overline{z})$. It follows that

$$f_{nm}(z,\overline{z})\Big|_{u\leftrightarrow v} = \sum_{p,q=0}^{\infty} \frac{(-1)^{p+q}}{p!q!}\left(z-\frac{1}{2}\right)^p \left(\overline{z}-\frac{1}{2}\right)^q \tilde{f}_{nm}^{(p,q)}. \tag{3.71}$$

We can then expand (3.68) in terms of the $\tilde{f}_{n,m}^{(p,q)}$:

$$\begin{aligned} F_{+,n,m}^{ij,kl} &= \sum_{p+q=\text{even}}^{\infty} \frac{2}{p!q!}\left(z-\frac{1}{2}\right)^p \left(\overline{z}-\frac{1}{2}\right)^q \tilde{f}_{n,m}^{(p,q)}, \\ F_{-,n,m}^{ij,kl} &= \sum_{p+q=\text{odd}}^{\infty} \frac{2}{p!q!}\left(z-\frac{1}{2}\right)^p \left(\overline{z}-\frac{1}{2}\right)^q \tilde{f}_{n,m}^{(p,q)}. \end{aligned} \tag{3.72}$$

After summing over the conformal multiplets, further information from the crossing equations (3.60)-(3.63) can be extracted by requiring them to be satisfied for each independent tensor structure $\sigma^r\tau^s$ in the $T_{n,m}^{(a,b)}(\sigma,\tau)$. The resulting set of equations takes the schematic form

$$\vec{h}_{\mp}(z,\overline{z}) \equiv \sum_{n,m} \vec{c}_{\mp,n,m} F_{\mp,n,m}^{ij,kl}(z,\overline{z}) = 0 \tag{3.73}$$

for some set of numbers $\{\vec{c}_{\pm,n,m}\}$. The LHS can thus be expanded as

$$\begin{aligned} \vec{h}_{+} &= \sum_{n,m}\sum_{p+q=\text{even}} \frac{2\vec{c}_{+,n,m}\tilde{f}_{n,m}^{(p,q)}}{p!q!}\left(z-\frac{1}{2}\right)^p \left(\overline{z}-\frac{1}{2}\right)^q, \\ \vec{h}_{-} &= \sum_{n,m}\sum_{p+q=\text{odd}} \frac{2\vec{c}_{-,n,m}\tilde{f}_{n,m}^{(p,q)}}{p!q!}\left(z-\frac{1}{2}\right)^p \left(\overline{z}-\frac{1}{2}\right)^q. \end{aligned} \tag{3.74}$$

Using this expansion, as well as the Ward identities in (3.67), we can determine, order by order, the maximal set of independent $h^i_{\mp}$ (i.e. crossing equations). The only remaining pieces of information needed to implement the $\mathcal{N} = 8$ mixed bootstrap are the superconformal blocks, which can also be computed using the Ward identities, and the conformal primaries appearing in the OPE, which can be determined using the character decomposition of $\mathfrak{osp}(8|4)$. These tasks are outside the scope of this work.

4 Twisted Cohomology

There exists a one-dimensional topological sector within certain $\mathcal{N} = 4$ SCFTs in three dimensions. The symmetries of this sector impose powerful constraints that restrict the $\mathcal{N} = 4$ OPE to a finite series. Using this series, it is possible to find a finite set of crossing constraints for the OPE coefficients of operators in this sector. In this section, we will use a slight variant of the twisting procedure outlined in [30] to produce crossing constraints for low-lying $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS operators in the $\mathcal{N} = 8$ mixed correlator[16].

4.1 Twisted Cohomology for $\mathcal{N} = 4$ SCFTs

To construct the topological sector, we begin with a discussion on the representation theory of the $\mathfrak{osp}(4|4)$ superconformal algebra. The unitary irreducible representations of $\mathfrak{osp}(4|4)$ (superconformal multiplets) are labeled by their scaling dimension Δ, Lorentz spin ℓ, and $\mathfrak{so}_4 \cong \mathfrak{su}(2)_l \oplus \mathfrak{su}(2)_r$ R-symmetry spins $(j_l, j_r) \in \frac{1}{2}\mathbb{Z}_+ \times \frac{1}{2}\mathbb{Z}_+$, in addition to various shortening conditions which truncate the multiplet, as detailed in [54]. There are a total of 8 distinct superconformal multiplets; we list them in Table 3.

For the $\mathcal{N} = 4$ representation theory, there are three classes of multiplets, just as for the $\mathcal{N} = 8$ theory. Those in the A series have a superconformal primary with arbitrary spin, satisfying the first bound in (3.19). The long multiplet $(4, A, 0)$ preserves no supersymmetry, so its superconformal primary can have arbitrarily large scaling dimension. The semi-short multiplets in the A series, denoted $(4, A, \cdot)$, preserve some amount of supersymmetry. The dimensions of their superconformal primaries are restricted to $\Delta_A = j_l + j_r + \ell + 1$. The multiplets in the B series, denoted $(4, B, \cdot)$, have superconformal primaries with zero spin; these are the BPS multiplets. Their superconformal primaries also satisfy the second unitarity bound in (3.19) and have fixed scaling dimension $\Delta_B = j_l + j_r$. In particular, the multiplets $(4, B, +)$ and $(4, B, -)$ are $\frac{1}{2}$-BPS, conserving the maximal amount (half) of supersymmetry. The conserved current multiplet, $(4, \text{cons.})$, saturates the unitarity bounds in (3.19) with $\Delta = \ell + 1$. It contains higher spin currents, so it can only appear in free SCFTs.

[16] Any $\mathcal{N} = 8$ SCFT in three dimensions can be interpreted as an $\mathcal{N} = 4$ SCFT in three dimensions with a global flavor symmetry.

Type	BPS	Δ	Spin	$\mathfrak{su}(2)_l \oplus \mathfrak{su}(2)_r$
$(4, A, 0)$ (long)	0	$\geq j_l + j_r + \ell + 1$	ℓ	(j_l, j_r)
$(4, A, 1)$	1/8	$j_l + j_r + \ell + 1$	ℓ	(j_l, j_r)
$(4, A, +)$	1/4	$j_l + j_r + \ell + 1$	ℓ	$(j_l, 0)$
$(4, A, -)$	1/4	$j_l + j_r + \ell + 1$	ℓ	$(0, j_r)$
$(4, B, 1)$	1/4	$j_l + j_r$	0	(j_l, j_r)
$(4, B, +)$	1/2	$j_l + j_r$	0	$(j_l, 0)$
$(4, B, -)$	1/2	$j_l + j_r$	0	$(0, j_r)$
$(4, \text{cons.})$	3/8	$\ell + 1$	ℓ	$(0, 0)$

Table 3: Unitary irreps (multiplets) of $\mathfrak{osp}(4|4)$ and the quantum numbers of their corresponding superconformal primary operator. As usual, the Lorentz spin can be any non-negative half-integer. The irreducible representations of the $\mathfrak{so}(4)_R \cong \mathfrak{su}(2)_l \oplus \mathfrak{su}(2)_r$ R-symmetry are labeled by their spins (j_l, j_r), each of which also takes values in the non-negative half-integers. The BPS of a given multiplet is the amount of preserved supersymmetry, i.e. the fraction of supercharges that are conserved for all states.

4.1.1 Twisted Algebra

One of the key ideas in constructing the topological sector is the fact that there is a one-dimensional superconformal subalgebra $\mathfrak{su}(2|2) \subset \mathfrak{osp}(4|4)$. We can take $\mathfrak{su}(2|2)$ to be generated by the set

$$\mathfrak{su}(2|2): \quad \{\underbrace{P, K, D}_{\mathfrak{sl}(2)}, \underbrace{R_a{}^b}_{\mathfrak{su}(2)_R}, \underbrace{Q_{1a}, Q_{2a}, S^1{}_a, S^2{}_a}_{\text{fermionic generators}}\}, \tag{4.1}$$

where P is the generator of translations along x^2; K is the generator of special conformal transformations along x^2; D is the dilatation generator; $R_a{}^b$ are the R-symmetry generators with $a, b = 1, 2$; Q_{1a} and Q_{2a} are the Poincaré supercharges; and S^1_a and S^2_a are the conformal supercharges.

Under our conventions, the $\mathfrak{su}(2|2)$ generators in (4.1) relate to the $\mathfrak{osp}(4|4)$ algebra as follows. The spacetime generators, $P = P_2$ and $K = K_2$, originate from the one-dimensional conformal subalgebra $\mathfrak{sl}(2) \subset \mathfrak{so}(3,2)$. The R-symmetry generators $R_a{}^b$ come from the left copy of $\mathfrak{su}(2)_l$ in the $\mathfrak{osp}(4|4)$ R-symmetry algebra given by $\mathfrak{so}(4)_R = \mathfrak{su}(2)_l \oplus \mathfrak{su}(2)_r$. Finally, the supercharges in the fermionic part of the two superalgebras are identified through $Q_{1a} = Q_{1a\dot{2}}$, $Q_{1a} = Q_{2a\dot{1}}$, $S^1{}_a = S^1{}_{a\dot{2}}$, and $S^2{}_a = S^2{}_{a\dot{1}}$.

In any quantized theory, we can introduce an inner product among states in the Hilbert space. This induces a map from the Hilbert space to its dual. Through this map, we can define the notion of an adjoint (dual) operator. Given an operator $\mathcal{O}$, its adjoint $\mathcal{O}^\dagger$ is defined by

$$(\mathcal{O}\,|\psi\rangle)^\dagger = \langle\psi|\,\mathcal{O}^\dagger, \tag{4.2}$$

where $|\psi\rangle$ is some state and $\langle\psi|$ is its dual. Under radial quantization, the $\mathfrak{su}(2|2)$ generators have the conjugates

$$\begin{gathered} P^\dagger = K, \quad D^\dagger = D, \quad (R_a{}^b)^\dagger = R_b{}^a, \\ (Q_{1a})^\dagger = i\epsilon^{ab} S^1{}_b, \quad (Q_{2a})^\dagger = -i\epsilon^{ab} S^2{}_b, \end{gathered} \tag{4.3}$$

where $\epsilon^{12} = -\epsilon^{21} = 1$ is the antisymmetric Levi-Civita symbol.

We now consider two new supercharges, which are linear combinations of the Poincaré and conformal supercharges. With respect to the full $\mathfrak{osp}(4|4)$ algebra, there are multiple choices for these combination supercharges. We will work with the same choice as [44], defining

$$\begin{aligned} \mathcal{Q}_1 &= Q_{11} + \frac{1}{2r} S^2{}_2, \quad \mathcal{Q}_1^\dagger = -i(2rQ_{21} - S^1_2), \\ \mathcal{Q}_2 &= Q_{22} + \frac{1}{2r} S^1{}_2, \quad \mathcal{Q}_2^\dagger = i(2rQ_{12} - S^2_1), \end{aligned} \tag{4.4}$$

where r is some parameter with units of length, which will eventually be identified as the radius of a three-sphere S^3. Both of the supercharges are nilpotent with $\mathcal{Q}_1^2 = \mathcal{Q}_2^2 = 0$.

We now seek a set of $\mathcal{Q}$-exact generators.[17] This set necessarily generates transformations which preserve the cohomological classes of $\mathcal{Q}$. Inspecting (4.4), we observe that there are no conformal generators or R-symmetry generators that are individually exact; however, certain "twisted" linear combinations of them are

[17] We will often drop the subscript on the $\mathcal{Q}$s when we are referring to both supercharges simultaneously.

exact. Consider the following set of generators:

$$\begin{aligned}
\widehat{L}_0 &\equiv -D + R_1{}^1 = -\frac{1}{8}\{\mathcal{Q}_1, \mathcal{Q}_1^\dagger\} = -\frac{1}{8}\{\mathcal{Q}_2, \mathcal{Q}_2^\dagger\}, \\
\widehat{L}_- &\equiv P + \frac{i}{2r} R_2{}^1 = -\frac{1}{4}\{\mathcal{Q}_1, Q_{22}\} = \frac{1}{4}\{\mathcal{Q}_2, Q_{12}\}, \\
\widehat{L}_+ &\equiv K - 2irR_1{}^2 = -\frac{1}{4}\{\mathcal{Q}_1, S^1{}_1\} = \frac{1}{4}\{\mathcal{Q}_2, S^2{}_1\}.
\end{aligned} \tag{4.5}$$

The above generators $\{\widehat{L}_0, \widehat{L}_\pm\}$ form an $\mathfrak{sl}(2)$ triplet, with $[L_0, \widehat{L}_\pm] = \pm\widehat{L}_\pm$ and $[\widehat{L}_+, \widehat{L}_-] = -2\widehat{L}_0$. Because the generators of this algebra are a combination of the spacetime generators and the external R-symmetry generators, we will refer to it as the *twisted algebra* and denote it by $\widehat{\mathfrak{sl}}(2)$. We call $\widehat{L}_0$ the generator of twisted dilations and $\widehat{L}_-$ the generator of twisted translations.[18]

4.1.2 Twisted $\mathcal{Q}$-Cohomology

Now that we have a collection of nilpotent supercharges, we can construct a notion of cohomology on the states of the theory. Using the state-operator correspondence, we can implement this procedure through the operator algebra $\mathcal{A}$. An operator $\mathcal{O} \in \mathcal{A}$ in the theory is considered $\mathcal{Q}$-closed (co-closed) if $\mathcal{Q}\mathcal{O} = 0$ ($\mathcal{Q}^\dagger\mathcal{O} = 0$). It is called $\mathcal{Q}$-exact (co-exact) if there exists an operator $\mathcal{O}' \in \mathcal{A}$ such that $\mathcal{O} = \mathcal{Q}\mathcal{O}'$ ($\mathcal{O} = \mathcal{Q}^\dagger\mathcal{O}'$). Since the twisted supercharges are nilpotent, any operator that is $\mathcal{Q}$-exact (co-exact) is also $\mathcal{Q}$-closed (co-closed). We define the $\mathcal{Q}$-cohomology $\mathcal{H}_\mathcal{Q}(\mathcal{A})$ as the set of operators which are $\mathcal{Q}$-closed modulo those that are $\mathcal{Q}$-exact, i.e.

$$\mathcal{H}_\mathcal{Q}(\mathcal{A}) \equiv \frac{\ker \mathcal{Q}}{\mathrm{im}\mathcal{Q}}. \tag{4.6}$$

The cohomology classes of $\mathcal{H}_\mathcal{Q}(\mathcal{A})$ are given by representatives, denoted by $[\mathcal{O}]$ for some $\mathcal{Q}$-closed operator $\mathcal{O} \in \mathcal{A}$. It is easy to see a representative is only unique up to an exact operator, with $[\mathcal{O}] = [\mathcal{O} + \mathcal{Q}\mathcal{O}']$. Taking advantage of the inner product on the space of states, we can use standard arguments from Hodge theory to show that the twisted-dilatation generator $\widehat{L}_0 \sim \{\mathcal{Q}, \mathcal{Q}^\dagger\}$ induces a vector space

[18] $\widehat{L}_+$ generates twisted special conformal transformations, but it is not relevant for this paper outside of generating the $\widehat{\mathfrak{sl}}(2)$ twisted algebra.

isomorphism $\varphi : \ker \widehat{L}_0 \to \mathcal{H}_{\mathcal{Q}}(\mathcal{A})$ such that

$$\ker \widehat{L}_0 \cong \mathcal{H}_{\mathcal{Q}}(\mathcal{A}). \tag{4.7}$$

This implies that there is a unique choice of representative given by operators that are annihilated by $\widehat{L}_0$.[19]

As demonstrated in [30], it is possible to determine the operators in $\ker \widehat{L}_0$ for representations of $\mathfrak{osp}(4|4)$. A rough sketch of the argument proceeds as follows. Any operator annihilated by $\widehat{L}_0 = -D + {R_1}^1$ satisfies $\Delta = m_l$. The unitarity bound $-\widehat{L}_0 \geq 0$ places an additional constraint on the possible representatives. After a systematic consideration of each representation of $\mathfrak{osp}(4|4)$, we find that the only operators which satisfy these requirements are the (4,B,+) primaries. Therefore, the two vector spaces are isomorphic, with[20]

$$\ker \widehat{L}_0 \cong \text{superconformal primaries of type } (4, B, +). \tag{4.8}$$

Given the above isomorphisms in (4.8) and (4.7), we can study the nontrivial cohomology classes of $\mathcal{H}_{\mathcal{Q}}(\mathcal{A})$ through the $(4, B, +)$ superconformal primaries, and vice-versa. Using the twisted-translation generator $\widehat{L}_-$, we can place these three-dimensional (3d) operators on the line x^2 and study the resulting one-dimensional (1d) topological theory. In general, a state in $(4, B, +)$ which transforms as the $(j_l, 0)$ irrep of $\mathfrak{su}(2)_l \oplus \mathfrak{su}(2)_r$ corresponds to an operator $\mathcal{O}_{a_1,\ldots,a_k}(0)$ inserted at the origin, where $k = 2j_l$. Under our conventions, the highest weight state corresponds to the primary with $a_i = 1$. Since $\widehat{L}_-$ is $\mathcal{Q}$-exact, we can define a *twisted operator* in the corresponding 1d theory:

$$\begin{aligned} \widehat{\mathcal{O}}_k(s) &\equiv e^{-is\widehat{L}_-}\mathcal{O}_{1,\ldots,1}(0)e^{is\widehat{L}_-} \\ &= y^{a_1}(s)\cdots y^{a_k}(s)\mathcal{O}_{a_1\cdots a_k}(x)\big|_{x=(0,0,s)}, \end{aligned} \tag{4.9}$$

where $y^a(s) = (1, s/2r)$, is in the same class as $\mathcal{O}_{1,\ldots,1}(0)$. The label k refers to the fact that the 1d operator $\widehat{\mathcal{O}}_k$ corresponds with the 3d operator $\mathcal{O}_{a_1\cdots a_k}$, which

[19]The cohomology $\ker \widehat{L}_0$ is identical for both of the nilpotent supercharges. Since $\widehat{L}_0$ has the same form for both $\mathcal{Q}_1$ and $\mathcal{Q}_2$, the two must share the same cohomology.

[20]The $(4, B, +)$ superconformal primaries, as the representatives of the nontrivial cohomology classes of $\mathcal{H}_{\mathcal{Q}}(\mathcal{A})$, also appear in [44] under the name of Higgs branch operators.

transforms in the $(k/2,0)$ of the $\mathfrak{so}(4)_R$ R-symmetry algebra.

This 1d theory of twisted operators is special: its correlation functions do not depend on the positions of non-coincident operator insertions. This is due to the fact that twisted operators commute with the twisted-translation generator. Given a set of twisted operators $\{\widehat{\mathcal{O}}_i\}$ for $i = 1, \ldots, n$, we find that derivatives of the n-point correlation function vanish:

$$\begin{aligned} \frac{\partial}{\partial s_i} \langle \widehat{\mathcal{O}}_1(s_1) \cdots \widehat{\mathcal{O}}_n(s_n) \rangle &= \langle \widehat{\mathcal{O}}_1(s_1) \cdots [\widehat{L}_0, \widehat{\mathcal{O}}_i(s_i)] \cdots \widehat{\mathcal{O}}_n(s_n) \rangle \\ &= 0, \end{aligned} \tag{4.10}$$

where we have used the supersymmetric Ward identity and have assumed $s_i \neq s_j$ for $i \neq j$. The above statement implies that all correlation functions of the 1d theory can only depend on the ordering of the operators up to coincident points. Hence, the 1d twisted theory can be interpreted as a sort of topological quantum mechanics.

The twisted $\mathcal{Q}$-cohomology is analogous to the de Rham cohomology from Hodge theory. In particular, the supercharge $\mathcal{Q}$ and its adjoint $\mathcal{Q}^\dagger$ can be associated with the exterior derivative d and its adjoint δ, respectively. Operators that are $\mathcal{Q}$-closed (exact) correspond to differential forms which are closed (exact). The de Rham cohomology classes, defined as closed forms modulo exact forms, each admit a unique representative, the harmonic form, for which the Laplacian $\Delta = \{\mathrm{d}, \delta\}$ vanishes. Likewise, each $\mathcal{Q}$-cohomology class has a unique representative, a $(4, B, +)$ primary with $\Delta = m_l = j_l$, which is automatically annihilated by $\widehat{L}_0 \sim \{\mathcal{Q}, \mathcal{Q}^\dagger\}$. Finally, in Hodge theory, a representative $[\omega]$ of a certain de Rham cohomology class remains in the same class even when shifted by a total derivative, i.e. $[\omega] \to [\omega + \mathrm{d}\eta]$. Analogously, a $(4, B, +)$ primary $\mathcal{O}(0)$ and its twisted counterpart $\widehat{\mathcal{O}}(s) = \exp(is\widehat{L}_-)\mathcal{O}(0)\exp(-is\widehat{L}_-)$ are in the same $\mathcal{Q}$-cohomology class.

We conclude this section by briefly summarizing the twisting procedure. Within the $\mathfrak{su}(2|2) \subset \mathfrak{osp}(4|4)$ subalgebra, it is possible to construct a set of nilpotent supercharges $\mathcal{Q}$ from the fermionic generators. One can then form new generators by combining those in the one-dimensional conformal algebra $\mathfrak{sl}(2)$ and the R-symmetry algebra $\mathfrak{su}(2)_l$, forming a $\mathcal{Q}$-exact twisted algebra $\widehat{\mathfrak{sl}}(2)$ generated by the triplet $\{\widehat{L}_0, \widehat{L}_\pm\}$. The $\mathfrak{osp}(4|4)$ operators which are both exact and co-exact (operators annihilated by $\widehat{L}_0$), the $(4, B, +)$ primaries, are unique representatives of the nontrivial $\mathcal{Q}$-cohomology classes of the $\mathcal{Q}$-cohomology $\mathcal{H}_{\mathcal{Q}}(\mathcal{A})$. They remain in the same class even when twist-translated along the line spanned by x^2. This implies that correlation functions along x^2 only depend on the overall ordering of such operators, rendering the corresponding one-dimensional theory topological.

4.2 Twisting $\mathcal{N} = 8$ Operators

4.2.1 From $\mathcal{N} = 4$ to $\mathcal{N} = 8$

An $\mathcal{N} = 8$ SCFT in three dimensions can be viewed as an $\mathcal{N} = 4$ SCFT with an additional flavor symmetry. In a rough sense, this is achieved by "ignoring" a fraction of the total supersymmetry. We can make this identification explicit by expressing the $\mathcal{N} = 8$ superconformal algebra $\mathfrak{osp}(8|4)$ in terms of the $\mathcal{N} = 4$ superconformal algebra $\mathfrak{osp}(4|4)$. Both share the same conformal subalgebra, namely $\mathfrak{so}(3,2)$, so decomposing the bosonic part of $\mathfrak{osp}(8|4)$ amounts to determining the maximal subalgebra of the $\mathcal{N} = 8$ R-symmetry group. It is given by [60]:

$$\mathfrak{so}(8)_R \supset \underbrace{\mathfrak{su}(2)_l \oplus \mathfrak{su}(2)_r}_{\mathfrak{so}(4)_R} \oplus \underbrace{\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2}_{\mathfrak{so}(4)_F} . \tag{4.11}$$

We can identify the first two copies of $\mathfrak{su}(2)$ with the $\mathfrak{so}(4)_R \cong \mathfrak{su}(2)_l \oplus \mathfrak{su}(2)_r$ R-symmetry algebra and the second two as an additional flavor symmetry algebra given by $\mathfrak{so}(4)_F \cong \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$.

Consider a general state with $\mathfrak{so}(8)_R$ weights $[a_1, a_2, a_3, a_4]$. The corresponding state in the $\mathcal{N} = 4$ picture has $\mathfrak{su}(2)^4$ weights (m_l, m_r, m_1, m_2), where [29]:

$$[a_1 a_2 a_3 a_4] \to (m_l, m_r, m_1, m_2) = \left(\frac{a_1 + 2a_2 + a_3 + a_4}{2}, \frac{a_1}{2}, \frac{a_3}{2}, \frac{a_4}{2} \right) . \tag{4.12}$$

With regard to the twisting procedure, we are only interested in states in irreducible representations of $\mathfrak{osp}(8|4)$ which map to $(4, B, +)$ primaries of $\mathfrak{osp}(4|4)$. Using (4.12), we resolve that the $\mathfrak{osp}(8|4)$ states must satisfy:

$$(4, B, +) \text{ primary conditions} \begin{cases} \Delta = a_2 + (a_3 + a_4)/2, \\ \ell = 0, \\ a_1 = 0. \end{cases} \tag{4.13}$$

The only states that satisfy (4.13) are the superconformal primaries of $(8, B, 2)$, $(8, B, 3)$, $(8, B, +)$, and $(8, B, -)$. The corresponding projection from their $\mathfrak{so}(8)_R$ weights to the $\mathfrak{su}(2)^4$ weights is given by

$$\begin{aligned}
(8, B, 2) : [0a_2a_3a_4] &\to \left(\frac{2a_2 + a_3 + a_4}{2}, 0, \frac{a_3}{2}, \frac{a_4}{2}\right), \\
(8, B, 3) : [00a_3a_4] &\to \left(\frac{a_3 + a_4}{2}, 0, \frac{a_3}{2}, \frac{a_4}{2}\right), \\
(8, B, +) : [00a_30] &\to \left(\frac{a_3}{2}, 0, \frac{a_3}{2}, 0\right), \\
(8, B, -) : [000a_4] &\to \left(\frac{a_4}{2}, 0, 0, \frac{a_4}{2}\right).
\end{aligned} \tag{4.14}$$

In particular, the $\mathfrak{osp}(8|4)$ superconformal primaries in $(8, B, +)_{[0020]}$ and $(8, B, +)_{[0040]}$ can be decomposed into $\mathfrak{osp}(4|4)$ operators, with

$$\begin{aligned}
(8, B, +)_{[0020]} &\to (4, B, +)_{(1,0)} \otimes (1, 0) \oplus \cdots , \\
(8, B, +)_{[0040]} &\to (4, B, +)_{(2,0)} \otimes (2, 0) \oplus \cdots ,
\end{aligned} \tag{4.15}$$

where we have only listed the multiplets of type $(4, B, +)$, and where the quantum numbers $(j_1, 0)$ are those of the $\mathfrak{so}(4)_F$ flavor symmetry.

We can construct this projection explicitly using the $\mathfrak{so}(8)_R$ null vectors. As before, we denote the composite operator corresponding to the superconformal primary of $(8, B, +)_{[00k0]}$ by $\mathcal{O}_k(x, Y)$. The null vector Y can be written as [29]:

$$Y^i = \frac{1}{\sqrt{2}} y^a \overline{y}^{\dot{a}} \sigma^i_{a\dot{a}}, \quad Y^5 = Y^6 = Y^7 = Y^8 = 0. \tag{4.16}$$

where $i = 1, \ldots, 4$ and $\sigma^i_{a\dot{a}} = (1, i\sigma^1, \sigma^2, \sigma^3)$. The variables y^a and $\overline{y}^{\dot{a}}$ can be thought of as the respective polarization vectors of $\mathfrak{su}(2)_l$ and $\mathfrak{su}(2)_1$. It follows that $\mathcal{O}_k$ can be written as

$$\mathcal{O}_k(x, y, \overline{y}) = \frac{1}{2^{k/2}} \mathcal{O}_{i_1 \cdots i_k}(x)(y\sigma^{i_1}\overline{y}) \cdots (y\sigma^{i_k}\overline{y}). \tag{4.17}$$

Applying the twisting procedure in (4.9) to the above expression yields the corresponding operator in the twisted theory:

$$\widehat{\mathcal{O}}_k(s, \overline{y}) \equiv \mathcal{O}_k(x, y, \overline{y})\big|_{\substack{x=(0,0,s) \\ y=(1,s/2r)}} . \tag{4.18}$$

Therefore, a $(8, B, +)_{[00k0]}$ superconformal primary corresponds to a twisted operator with isospin $j_1 = k/2$ in the 1d topological theory.

4.2.2 Twisted Correlation Functions

To compute correlation functions in the topological theory, we simply project the correlation functions in Section 3.5 using the map (4.18). Up to an overall normalization factor, the two- and three-point functions are given by

$$\langle \widehat{\mathcal{O}}_k(s_1, \overline{y}_1)\widehat{\mathcal{O}}_k(s_2, \overline{y}_2)\rangle = \langle \overline{y}_1, \overline{y}_2\rangle^k\,(\text{sgn}\ s_{12})^k, \tag{4.19}$$

$$\begin{aligned}\langle \widehat{\mathcal{O}}_{k_1}(s_1, \overline{y}_1)\widehat{\mathcal{O}}_{k_2}(s_2, \overline{y}_2)\widehat{\mathcal{O}}_{k_3}(s_3, \overline{y}_3)\rangle = \lambda\,&\langle \overline{y}_1, \overline{y}_2\rangle^{\frac{k_1+k_2-k_3}{2}}\,\langle \overline{y}_2, \overline{y}_3\rangle^{\frac{k_2+k_3-k_1}{2}}\,\langle \overline{y}_3, \overline{y}_1\rangle^{\frac{k_3+k_1-k_2}{2}}\\ &\times(\text{sgn}\ s_{12})^{\frac{k_1+k_2-k_3}{2}}(\text{sgn}\ s_{23})^{\frac{k_2+k_3-k_1}{2}}(\text{sgn}\ s_{31})^{\frac{k_3+k_1-k_2}{2}}\end{aligned} \tag{4.20}$$

where $s_{ij} \equiv s_i - s_j$, λ is the OPE coefficient, and

$$\langle \overline{y}_i, \overline{y}_j\rangle \equiv \epsilon_{\dot{a}\dot{b}}\overline{y}_i^{\dot{a}}\overline{y}_j^{\dot{b}}, \quad (\epsilon^{12} = -\epsilon_{12} = 1) \tag{4.21}$$

is a product between $\mathfrak{su}(2)_1$ polarizations, commonly referred to as the $\mathfrak{sl}(2)$ product. As expected, the correlation functions in (4.19) and (4.20) only depend on the overall ordering of the operators.

4.2.3 Twisted Four-point Functions

The four-point function of twisted operators has the general form ($s_1 < s_2 < s_3 < s_4$):

$$\begin{aligned}&\langle \widehat{\mathcal{O}}_{k_1}(s_1, \overline{y}_1)\widehat{\mathcal{O}}_{k_2}(s_2, \overline{y}_2)\widehat{\mathcal{O}}_{k_3}(s_3, \overline{y}_3)\widehat{\mathcal{O}}_{k_4}(s_4, \overline{y}_4)\rangle =\\ &\langle \overline{y}_1, \overline{y}_2\rangle^{\frac{k_1+k_2}{2}}\,\langle \overline{y}_3, \overline{y}_4\rangle^{\frac{k_3+k_4}{2}}\left(\frac{\langle \overline{y}_1, \overline{y}_4\rangle}{\langle \overline{y}_2, \overline{y}_4\rangle}\right)^{\frac{k_{12}}{2}}\left(\frac{\langle \overline{y}_1, \overline{y}_3\rangle}{\langle \overline{y}_1, \overline{y}_4\rangle}\right)^{\frac{k_{34}}{2}} t(\overline{w}),\end{aligned} \tag{4.22}$$

where $k_{ij} \equiv k_i - k_j$ and $t(\overline{w})$ is an arbitrary function of the $\mathfrak{su}(2)$ "cross-ratio," defined as

$$\overline{w} = \frac{\langle \overline{y}_1, \overline{y}_2\rangle\,\langle \overline{y}_3, \overline{y}_4\rangle}{\langle \overline{y}_1, \overline{y}_3\rangle\,\langle \overline{y}_2, \overline{y}_4\rangle}. \tag{4.23}$$

To determine the composition of $t(\overline{w})$, we must project $(8, B, +)$ primaries from the 3d theory, written as $(4, B, +)$ operators, onto the topological theory. Using arguments developed in [29], it is possible to show that only certain operators in $\mathcal{O}_{k_i} \times \mathcal{O}_{k_j}$ survive the twisting procedure, i.e. those in $\mathcal{H}_{\mathcal{Q}}(\mathcal{A})$. In particular, an operator $\mathcal{O} \in \mathcal{O}_{k_i} \times \mathcal{O}_{k_j}$ is also in $\mathcal{H}_{\mathcal{Q}}(\mathcal{A})$ only if it is a superconformal primary from one of the $\mathcal{N} = 8$ multiplets which reduce to those of type $(4, B, +)$. These are the multiplets $(8, B, 2)$, $(8, B, 3)$, $(8, B, +)$, and $(8, B, -)$. Of these, only $(8, B, 2)$ and $(8, B, +)$ appear in $\mathcal{O}_{k_i} \times \mathcal{O}_{k_j}$. Explicitly, the nonvanishing representations in the OPE, given in terms of their $\mathfrak{so}(8)_R$ Dynkin labels, are

$$[00k_i0] \otimes [00k_j0] = \bigoplus_{n=0}^{k_j} \underbrace{[00(k_i + k_j - n)0]}_{(8,B,+)} \oplus \bigoplus_{n=0}^{k_j} \bigoplus_{m=1}^{n} [0(n)(k_i + k_j - 2n - 2m)0] \tag{4.24}$$

where we assume $k_j \geq k_i$ WLOG. When $k_i = k_j$ there is an additional kinematic restriction such that only the multiplets in the symmetric tensor product remain.

Using the same OPE coefficient normalizations as in [29], along with various projection arguments made in [30], we find that (4.22) has the s-channel decomposition:

$$\begin{aligned}&\langle \widehat{\mathcal{O}}_{k_1}(s_1, \overline{y}_1) \widehat{\mathcal{O}}_{k_2}(s_2, \overline{y}_2) \widehat{\mathcal{O}}_{k_3}(s_3, \overline{y}_3) \widehat{\mathcal{O}}_{k_4}(s_4, \overline{y}_4) \rangle = \\ &\langle \overline{y}_1, \overline{y}_2 \rangle^{\frac{k_1+k_2}{2}} \langle \overline{y}_3, \overline{y}_4 \rangle^{\frac{k_3+k_4}{2}} \left(\frac{\langle \overline{y}_1, \overline{y}_4 \rangle}{\langle \overline{y}_2, \overline{y}_4 \rangle} \right)^{\frac{k_{12}}{2}} \left(\frac{\langle \overline{y}_1, \overline{y}_3 \rangle}{\langle \overline{y}_1, \overline{y}_4 \rangle} \right)^{\frac{k_{34}}{2}} \sum_{n,m} t_m(\overline{w}) 4^{-n} \lambda_{k_i k_j n m} \lambda_{k_i k_j n m}.\end{aligned} \tag{4.25}$$

The function $t_m(\overline{w})$, the $\mathfrak{su}(2)$ analogue of the $SO(8)$ harmonics $T_{nm}^{(a,b)}$, obeys the eigenvalue equation:

$$(1 - \overline{w}) \overline{w}^2 \frac{d^2 t_m}{d\overline{w}^2} + \frac{1}{2}(k_{34} - k_{12} - 2) \overline{w}^2 \frac{dt_m}{d\overline{w}} + \frac{1}{4} k_{12} k_{34} \overline{w} t_m = m(m+1) t_m \,. \tag{4.26}$$

Up to normalization, the regular solution can be written in terms of the Jacobi polynomials $P_n^{(a,b)}(x)$ as

$$t_m(\overline{w}) = \overline{w}^{\frac{k_{34}}{2}} P_{m+\frac{k_{34}}{2}}^{\left(\frac{k_{12}-k_{34}}{2}, -\frac{k_{12}+k_{34}}{2}\right)} \left(\frac{2}{\overline{w}} - 1 \right) . \tag{4.27}$$

When the above expansion is inserted into (4.22), the resulting expression is a polynomial in the $\overline{y}$'s. Upon performing crossing ($1 \leftrightarrow 3$), we find the following general relations between the various OPE coefficients (where $\langle k_1 k_2 k_3 k_4\rangle$ corresponds with the crossing equations for $\langle \widehat{\mathcal{O}}_{k_1}\widehat{\mathcal{O}}_{k_2}\widehat{\mathcal{O}}_{k_3}\widehat{\mathcal{O}}_{k_4}\rangle$):

$$
\begin{aligned}
&\langle 2222\rangle : 4\lambda^2_{2,2,1,1} - 5\lambda^2_{2,2,2,2} + \lambda^2_{2,2,2,0} + 16 = 0,\\
&\langle 4444\rangle : \begin{cases} 64\lambda^2_{4,4,1,1} + 48\lambda^2_{4,4,2,2} + 4\lambda^2_{4,4,3,1} - 16\lambda^2_{4,4,3,3} + 3\lambda^2_{4,4,4,2} - 60\lambda^2_{4,4,4,4} = 0,\\ 32\lambda^2_{4,4,2,2} + 2\lambda^2_{4,4,4,2} + 9\lambda^2_{4,4,4,4} - 16\lambda^2_{4,4,2,0} - 20\lambda^2_{4,4,3,3} - \lambda^2_{4,4,4,0} - 256 = 0, \end{cases}\\
&\langle 2424\rangle : 16\lambda^2_{2,4,1,1} + 4\lambda^2_{2,4,2,1} + 4\lambda^2_{2,4,2,2} + \lambda^2_{2,4,3,1} + \lambda^2_{2,4,3,2} - 14\lambda^2_{2,4,3,3} = 0,\\
&\langle 4224\rangle : \begin{cases} 5\lambda^2_{2,4,3,1} + 80\lambda^2_{2,4,1,1} + 20\lambda^2_{2,4,2,1} - 12\lambda_{2,2,2,0}\lambda_{4,4,2,0} + 72\lambda_{2,2,1,1}\lambda_{4,4,1,1}\\ -42\lambda_{2,2,2,2}\lambda_{4,4,2,2} - 192 = 0,\\ 3\lambda^2_{2,4,3,2} + 12\lambda^2_{2,4,2,2} - 14\lambda_{2,2,2,2}\lambda_{4,4,2,2}\\ -8\lambda_{2,2,1,1}\lambda_{4,4,1,1} + 4\lambda_{2,2,2,0}\lambda_{4,4,2,0} + 64 = 0,\\ 15\lambda^2_{2,4,3,3} - 4\lambda_{2,2,2,2}\lambda_{4,4,2,2} - 16\lambda_{2,2,1,1}\lambda_{4,4,1,1} - 4\lambda_{2,2,2,0}\lambda_{4,4,2,0} - 64 = 0. \end{cases}
\end{aligned}
\tag{4.28}
$$

The above crossing relations are completely general in nature, and the OPE coefficients for the mixed correlator in any $\mathcal{N} = 8$ SCFT in 3d must satisfy them.

5 Supersymmetric Localization

In the previous section, we found general results for the topological sector within certain $\mathcal{N} = 8$ SCFTs in three dimensions. In particular, we derived crossing constraints for the OPE coefficients in certain low-lying $\frac{1}{2}$-BPS and $\frac{1}{4}$-BPS operators in the $\mathcal{N} = 8$ mixed correlator. In this section, we analyze a few examples of explicit $\mathcal{N} = 8$ theories. Using recent results on supersymmetric localization in [44], we make contact with the topological sector of these theories and compute various correlation functions.

5.1 Supersymmetric Gauge Theory

5.1.1 Yang-Mills Theory and the Chern-Simons Action

All known examples of $\mathcal{N} = 8$ SCFTs are supersymmetric gauge theories. We first focus on gauge theories without supersymmetry. These theories have a gauge field A, which is properly viewed as a $\mathfrak{g}$-valued 1-form on the spacetime M.[21] The field strength is given by the curvature 2-form

$$F = \mathrm{d}A + A \wedge A, \tag{5.1}$$

where $\wedge$ is the standard wedge product on differential forms. The Yang-Mills action can be written in coordinate-free language as

$$S_{\mathrm{YM}} = \int_M \mathrm{tr}(F \wedge *F), \tag{5.2}$$

where $*$ is the Hodge dual. If M has odd dimension, there is another gauge-invariant form, known as the Chern-Simons form, which can be included in the action. In three dimensions, it is given by

$$S_{\mathrm{CS}} = -\frac{k}{4\pi} \int_M \mathrm{tr}(A \wedge \mathrm{d}A + \frac{2i}{3} A \wedge A \wedge A). \tag{5.3}$$

[21] In more general terms, the gauge field A is the pullback of a connection 1-form on a principal bundle over M with typical fiber G.

The Chern-Simons action serves as a field theory in its own right. Classically it has no dynamics, since the equations of motion for $\delta S_{\mathrm{CS}} = 0$ are

$$\frac{k}{2\pi} F = 0, \tag{5.4}$$

i.e. A is pure-gauge. As a quantum field theory, the Chern-Simons theory is an example of a topological field theory whose correlation functions are metric-independent.

We can combine the Chern-Simons action with the Yang-Mills action. The resulting theory is identical to classical Yang-Mills, but different on the quantum level. We can also add matter fields to the Chern-Simons theory to form a Chern-Simons-matter theory. These theories are no longer topological field theories, but they often have interesting dynamics for spacetimes with a nontrivial global topology.

5.1.2 Supersymmetric Yang-Mills Theory

We now consider supersymmetric Yang-Mills (SYM) theory, which consists of a Yang-Mills theory with gauge group G and with $\mathcal{N} = 4$ supersymmetry. For these theories, the gauge field has an associated vectormultiplet V given by

$$V = (A, \lambda_{a\dot{a}}, \Phi_{\dot{a}\dot{b}}, D_{ab}), \tag{5.5}$$

where A is the gauge field, $\lambda_{a\dot{a}}$ is a spin-1/2 field called the gaugino, and $\Phi_{\dot{a}\dot{b}}$ and D_{ab} are scalars. The fields transform respectively in the $(\mathbf{1},\mathbf{1}), (\mathbf{2},\mathbf{2}), (\mathbf{3},\mathbf{1}), (\mathbf{1},\mathbf{3})$ under the $\mathfrak{su}(2)_l \oplus \mathfrak{su}(2)_r$ R-symmetry. Additionally, all of the fields in V transform in the adjoint representation of the gauge group G.

We can add matter content to the theory in the form of a hypermultiplet

$$H = (q_a, \widetilde{q}^a, \psi_{\dot{a}}, \widetilde{\psi}_{\dot{a}}), \tag{5.6}$$

where $q_a, \widetilde{q}^a$ are complex scalars transforming in $(\mathbf{1},\mathbf{2})$ and $(\mathbf{1},\overline{\mathbf{2}})$ under the R-symmetry algebra and in the unitary representations ρ and $\overline{\rho}$ of G, respectively. The fields $\psi_{\dot{a}}$, $\widetilde{\psi}_{\dot{a}}$ are spinors transforming in $(\mathbf{2},\mathbf{1})$ and $(\overline{\mathbf{2}},\mathbf{1})$ representations of the R-symmetry algebra, and also in some unitary representation ρ and $\overline{\rho}$ of G.

The $\mathfrak{osp}(4|4)$-invariant action on a conformally-flat manifold M for the hypermultiplet is given by

$$S_{\text{hyper}}[H,V] = \int_M \mathrm{d}^3x\sqrt{g}\left(D^\mu \widetilde{q}^a \mathcal{D}_\mu q_a - i\widetilde{\psi}^{\dot{a}} \not{\mathcal{D}} \psi_{\dot{a}} + \cdots\right), \tag{5.7}$$

where we have left out the interaction terms for brevity. In the above expression, $g = \det g_{\mu\nu}$ and $\mathcal{D}_\mu = \nabla_\mu - iA_\mu$ is the double covariant derivative. The curved-space analogue of the YM action that preserves half the supersymmetry is given by

$$S_{\text{YM}}[V] = \frac{1}{g^2_{\text{YM}}} \int_M \mathrm{d}^3x\sqrt{g}\,\mathrm{tr}\left(F^{\mu\nu}F_{\mu\nu} - \mathcal{D}^\mu \Phi^{\dot{c}\dot{d}} \mathcal{D}_\mu \Phi_{\dot{c}\dot{d}} + i\lambda^{a\dot{a}} \not{\mathcal{D}} \lambda_{a\dot{a}} + \cdots\right), \tag{5.8}$$

where g_{YM} is the Yang-Mills coupling constant, and where we have again left out the interaction terms. The path integral for the SYM theory is thus given by

$$\mathcal{I} = \int DHDV e^{-S_{\text{hyper}}[H,V]-S_{\text{YM}}[V]}. \tag{5.9}$$

To ensure its convergence, the fields in H and V all satisfy certain realty conditions.[22]

5.2 Supersymmetric Localization

We now turn to a method for actually computing the path integral of supersymmetric gauge theories, including (5.9). The idea is roughly as follows. Suppose there is a nilpotent Grassmannian symmetry δ with $\delta^2 = 0$. Now consider a theory described by an action S invariant under δ. The basic idea is to deform the action by

$$S(t) = S + t\delta V, \tag{5.10}$$

where V is a Grassmann-odd operator. Assuming δ is a symmetry on the quantum level (i.e. of the path integral), it follows that the integral

$$Z(t) = \int D\phi\, e^{-S(t)} \tag{5.11}$$

[22]These conditions, as well as the interaction terms for the actions, can be found in [44]. There are additional subtleties in the SYM construction that are outside the scope of this paper.

is independent of t. Taking the limit $t \to \infty$ localizes the integral to a submanifold that consists of the zero modes of δV, which are also the saddle points of V. Each zero is associated with contributions from the classical action, as well as from the one-loop determinant that arises from integrating out the fluctuations around the saddle. Interestingly, for these theories the one-loop results contain the quantum information of the theory to arbitrary loop order. Amazingly, after integrating out the Gaussian modes, one often finds that the path integral has been reduced to a finite-dimensional matrix integral. Moreover, the localization procedure extends past the partition function to include any operator insertions of the form $\delta\mathcal{O}$, which are evaluated at the saddle points.

5.2.1 The Poincaré-Hopf Index Theorem

Following [61], we will first demonstrate the principle of localization for a finite-dimensional manifold.[23] Our motivation is proving the Poincaré-Hopf index theorem, which relates the local properties of smooth vector fields on a manifold to its global topology. On the topology side, we have the Euler-characteristic $\chi(M)$ for a manifold M – a topological invariant that measures the global properties of M.[24] On the analytic side, we have the index of a vector field X. Consider a closed ball $B \subset M$ around an isolated zero x of X. The index $\mathrm{ind}_X(x)$ of X at the zero x is defined to be the degree of the map $\widehat{X} : \partial B \to S^{\dim M - 1}$ given by $\widehat{X} = X(y)/|X(y)|$ for any regular point $y \in B$. Equivalently, the index is given by

$$\mathrm{ind}_X(x) = \frac{\det[\partial_\mu X^\nu(x)]}{|\det[\partial_\mu X^\nu(x)]|}. \tag{5.12}$$

Intuitively, the index measures the local curvature of the vector field. For instance, in the plane, a source or a sink is assigned a value of $+1$ and a saddle-point is assigned the value -1.

The Poincaré-Hopf index theorem can be formulated as follows. Consider a manifold M with Euler-characteristic $\chi(M)$. Let X be a smooth vector field on M

[23] The concepts from this section can be generalized to the Feynman path integral, albeit in a slightly complicated fashion.

[24] For compact surfaces, $\chi(M)$ is related to the genus g of the manifold by $\chi(M) = 2 - 2g$.

with isolated zeros $\{x_k\}$. Then [62]

$$\sum_{x_k} \mathrm{ind}_X(x_k) = \chi(M). \tag{5.13}$$

The Poincaré-Hopf theorem can be considered a generalized version of the hairy ball theorem, which states that there are no nonvanishing continuous vector fields on S^{2n}. For example, let us consider the standard two-sphere S^2 with Euler-characteristic $\chi(S^2) = 2$. Since the index is always nonzero, it follows that every vector field must have at least two zeros on S^2.

We will now prove this theorem for manifolds without boundary.[25] Let TM be the tangent bundle of some Riemannian manifold M of real dimension $2n$, equipped with a metric $g_{\mu\nu}$ and a vielbein e^μ_a. Let V^μ be a smooth section of TM. From a supersymmetry perspective, we can introduce local "supercoordinates" given by the coordinate doublets (x^μ, θ^μ) for M and $(\overline{\theta}_\mu, p_\mu)$ for the typical fiber $\mathbb{R}^{2n}$, where x^μ, p_μ are bosonic variables and $\theta^\mu, \overline{\theta}_\mu$ are fermionic. The coordinates are related by a nilpotent Grassmannian symmetry δ, with

$$\begin{aligned} \delta x^\mu &= \theta^\mu, \quad \delta\overline{\theta}_\mu = p_\mu, \\ \delta\theta^\mu &= 0, \quad \delta p_\mu = 0. \end{aligned} \tag{5.14}$$

To further the analogy, we consider the "partition function"

$$Z(t) = \frac{1}{(2\pi)^{2n}} \int_M \mathrm{d}x\mathrm{d}\theta\mathrm{d}\overline{\theta}\mathrm{d}p\, e^{-S(t)} \tag{5.15}$$

for an "action" S defined by

$$S(t) = \delta\Psi, \quad \Psi = \frac{1}{2}\overline{\theta}_\mu(p^\mu + 2itV^\mu + \Gamma^\sigma_{\tau\nu}\overline{\theta}_\sigma\theta^\nu g^{\mu\tau}), \tag{5.16}$$

where $\Gamma^\sigma_{\tau\nu} = \frac{1}{2}g^{\sigma\mu}(\partial_\nu g_{\mu\tau} + \partial_\tau g_{\mu\nu} - \partial_\mu g_{\tau\nu})$ are the usual Christoffel symbols with respect to the local coordinates x^μ. After some algebra, one finds that the "field" p_μ is Gaussian with mean value

$$p^\mu = -itV^\mu - g^{\mu\nu}\Gamma^\sigma_{\tau\nu}\overline{\theta}_\sigma\theta^\nu. \tag{5.17}$$

[25] The theorem holds for manifolds with a boundary as well, but the proof is more difficult.

Integrating it out yields

$$Z(t) = \frac{1}{(2\pi)^n} \int_M \mathrm{d}x\mathrm{d}\theta\mathrm{d}\psi \, e^{-S_{\text{eff}}(t)} \tag{5.18}$$

for the "effective action"

$$S_{\text{eff}}(t) = \frac{t^2}{2} g_{\mu\nu} V^\mu V^\nu - \frac{1}{4} R^{\rho\sigma}{}_{\mu\nu} \bar{\theta}_\rho \bar{\theta}_\sigma \theta^\mu \theta^\nu - it \nabla_\mu V^\nu \bar{\theta}_\nu \theta^\mu. \tag{5.19}$$

We can introduce orthonormal coordinates defined through the inverse vielbein,

$$\psi_a = e^\mu_a \bar{\theta}_\mu, \tag{5.20}$$

such that (5.19) becomes

$$S_{\text{eff}}(t) = \frac{t^2}{2} g_{\mu\nu} V^\mu V^\nu - \frac{1}{4} R^{ab}{}_{\mu\nu} \psi_a \psi_b \theta^\mu \theta^\nu - it \nabla_\mu V^\nu e^a_\nu \psi_a \theta^\mu. \tag{5.21}$$

In this form, it is easy to see that the effective action can be decomposed as

$$S_{\text{eff}}(t) = S_{\text{eff}}(0) + t\delta V, \quad V = i\bar{\theta}_\mu V^\mu. \tag{5.22}$$

Because $R^{ab}{}_{\mu\nu}$ is anti-symmetric in the first two indices, it follows that $\delta S(0) = 0$. The integral is therefore independent of t, since

$$\frac{dZ}{dt} = -\int_M \mathrm{d}x\mathrm{d}\theta\mathrm{d}\psi \, \delta V e^{-S_{\text{eff}}(0) - t\delta V} = \int_M \mathrm{d}x\mathrm{d}\theta\mathrm{d}\psi \, \delta \left(e^{-S_{\text{eff}}(0) - t\delta V} \right) = 0. \tag{5.23}$$

We first consider the case where $t = 0$. It follows that

$$Z(0) = \frac{1}{(2\pi)^n} \int_M \mathrm{d}x\mathrm{d}\theta\mathrm{d}\psi \, e^{\frac{1}{4} R^{ab}{}_{\mu\nu} \psi_a \psi_b \theta^\mu \theta^\nu} = \frac{1}{(2\pi)^n} \int_M \mathrm{d}x \, \mathrm{Pf}(R) = \chi(M), \tag{5.24}$$

where $\mathrm{Pf}(R)$ is the Pfaffian of the curvature two-form, and in the last equality we made use of Gauss-Bonnet-Chern theorem for Riemannian manifolds [63].

Next, we consider the limit $t \to \infty$, which localizes the integral to a submanifold $\{x_k\} \subset M$ consisting of the zeros of δV. We expand V_μ in a Taylor series around the zero x_k,

$$V^\mu(x) = \sum_{p\geq 1} \frac{1}{p!}\partial_{\mu_1}\cdots\partial_{\mu_p}V^\mu(x_k)\xi^{\mu_1}\cdots\xi^{\mu_p}, \tag{5.25}$$

where $\xi^\mu = x^\mu - x_k^\mu$. Rescaling the variables by

$$\xi \to t^{-1}\xi, \quad \theta \to t^{-1/2}\theta, \quad \psi \to t^{-1/2}\psi, \tag{5.26}$$

and then inserting (5.25) into (5.21) eliminates the interaction terms. This yields

$$S_{\text{eff}}(t) = \frac{1}{2}g_{\mu\nu}W_\alpha^{(k)\mu}W_\beta^{(k)\mu}\xi^\alpha\xi^\beta - iW_\mu^{(k)\nu}e_\nu^a\psi_a\theta^\mu + O(t^{-1/2}), \tag{5.27}$$

where for convenience we have defined $W_\alpha^{(k)\mu} \equiv \partial_\alpha V^\mu(x_k)$. We therefore find that the partition function reduces to a Gaussian integral given by

$$\lim_{t\to\infty} Z(t) = \sum_{x_k}\int_M \mathrm{d}\xi\mathrm{d}\theta\mathrm{d}\psi\, \exp\left(-\frac{1}{2}g_{\mu\nu}W_\alpha^{(k)\mu}W_\beta^{(k)\mu}\xi^\alpha\xi^\beta + iW_\mu^{(k)\nu}e_\nu^a\psi_a\theta^\mu\right). \tag{5.28}$$

Proceeding with the computation, we find

$$\begin{aligned}
\lim_{t\to\infty} Z(t) &= \sum_{x_k}\int_M e^{-\frac{1}{2}g_{\mu\nu}W_\alpha^{(k)\mu}W_\beta^{(k)\mu}\xi^\alpha\xi^\beta}\int_M \mathrm{d}\theta\mathrm{d}\psi\, e^{iW_\mu^{(k)\nu}e_\nu^a\psi_a\theta^\mu} \\
&= \sum_{x_k}\frac{1}{\sqrt{g}\,|\det W^{(k)}|}\det e_\mu^\alpha \det W^{(k)} \\
&= \sum_{x_k}\frac{\det W^{(k)}}{|\det W^{(k)}|}.
\end{aligned} \tag{5.29}$$

Thus, the integral over M localized to a sum over the submanifold $\{x_k\}$. The equality of (5.29) with (5.24) completes the proof.

5.2.2 Localization of $\mathcal{N}=4$ SYM Theories

We are now ready to discuss the localization procedure for a SYM theory with gauge group G. We are free to place the theory on any conformally-flat surface, and in particular S^3. The exact procedure carried out in [44] is outside the scope of this thesis, and we will simply state its results. The path integral for the SYM theory in (5.9) localizes to a matrix model coupled to a 1d Gaussian theory given by

$$\mathcal{I} = \frac{1}{|\mathcal{W}|}\int_{\text{Cartan}} \mathrm{d}\sigma\, \det{}'_{\text{adj}}[2\sinh(\pi\sigma)] \int DQD\widetilde{Q} \exp\left[-\ell\int_{-\pi}^{\pi} d\varphi \left(\widetilde{Q}\partial_\varphi Q + \widetilde{Q}\sigma Q\right)\right]. \tag{5.30}$$

Let us illuminate the theory given by the above path integral. The quantity $|\mathcal{W}|$ is the order of the Weyl group of G. The matrix model consists of a field σ, the matrix degree of freedom, that takes values in the Cartan of $\mathfrak{g}$. The 1d theory consists of anti-periodic scalar fields $(Q, \widetilde{Q})$ which transform in some representation ρ of G. These fields live on a circle $S^1 \subset S^3$ parametrized by the angle $\varphi \in [-\pi, \pi)$. The constant $\ell = -4\pi r$ is related to the radius r of S^3. From the point of view of the 1d model, σ acts like an auxiliary field. The localized path integral in (5.30) also allows for operator insertions. However, all operator insertions in the original path integral (5.9) vanish except for those in $\mathcal{H}_{\mathcal{Q}}(\mathcal{A})$. Thus, this theory corresponds with the same 1d topological sector from Section 4.

Integrating out the Gaussian degrees of freedom, we find [44, 64]:

$$Z = \frac{1}{|\mathcal{W}|}\int_{\text{Cartan}} \mathrm{d}\sigma\, \frac{\det{}'_{\text{adj}}[2\sinh(\pi\sigma)]}{\det_\rho[2\cosh(\pi\sigma)]}. \tag{5.31}$$

We now consider the observables in the theory. Correlation functions in the full theory for (5.30) are given by

$$\langle \mathcal{O}_1(\varphi_1)\cdots\mathcal{O}_n(\varphi_n)\rangle = \frac{1}{Z|\mathcal{W}|}\int_{\text{Cartan}} \mathrm{d}\sigma\, \det{}'_{\text{adj}}[2\sinh(\pi\sigma)]\, \langle \mathcal{O}_1(\varphi_1)\cdots\mathcal{O}_n(\varphi_n)\rangle_\sigma\,, \tag{5.32}$$

where $\langle(\cdots)\rangle_\sigma$ is the correlation function at fixed σ, i.e. for the Gaussian theory:

$$\langle \mathcal{O}_1(\varphi_1)\cdots\mathcal{O}_n(\varphi_n)\rangle_\sigma = \int DQD\widetilde{Q} e^{-S_\sigma[Q,\widetilde{Q}]}\mathcal{O}_1(\varphi_1)\cdots\mathcal{O}_n(\varphi_n), \tag{5.33}$$

where

$$S_\sigma[Q,\widetilde{Q}] = \int_{-\pi}^{\pi} \mathrm{d}\varphi \left(\widetilde{Q}\partial_\varphi Q + \widetilde{Q}\sigma Q\right). \tag{5.34}$$

The propagator for the Gaussian theory is given by [44]:

$$\langle Q(\varphi_1)\widetilde{Q}(\varphi_2)\rangle_\sigma = \frac{\mathrm{sgn}(\varphi_1-\varphi_2)+\tanh(\pi\sigma)}{2\ell}e^{-\sigma(\varphi_1-\varphi_2)}. \tag{5.35}$$

Using the localization results, we can compute all quantities of interest in a relatively straightforward manner. We outline the procedure below:

1. Given a specific $\mathcal{N}=8$ SCFT, write down the localized path integral using (5.30). The partition function can be evaluated using the corresponding matrix model in (5.31).
2. Find operators in the 1d theory, written as functions of $Q, \widetilde{Q}$, which correspond to the 3d operators of interest in $\mathcal{H}_{\mathcal{Q}}(\mathcal{A})$.
3. All correlation functions for the Gaussian model can be computed using Wick's theorem and the propagator given in (5.35).
4. Correlation functions for the full theory in (5.30) can be determined by integrating over σ.

5.3 Applications to ABJM Theory

5.3.1 Localized Partition Function

Consider the IR limit of a $U(N)$ gauge theory with an adjoint hypermultiplet and a fundamental hypermultiplet. This theory is believed to flow to the same IR fixed point as $\mathrm{ABJM}_{N,1}$. The theory splits into two parts: a trace sector described by a $U(1)$ gauge theory that flows to a free $\mathcal{N}=8$ SCFT in the IR, and a traceless sector described by an $SU(N)$ gauge theory which flows to a separate interacting $\mathcal{N}=8$ SCFT. We denote the twisted fields in the adjoint hypermultiplet by $(X^i{}_j, \widetilde{X}_i{}^j)$ and those in the fundamental hypermultiplet by $(Q^i, \widetilde{Q}_i)$, where $i,j=1,\ldots,N$. The matrix degree of freedom σ lives in the Cartan of $\mathfrak{u}(N)$. After localization, the S^3 partition function is given by

$$Z_{\mathrm{ABJM}_{N,1}} = \frac{1}{N!}\int \mathrm{d}^N\sigma \prod_{i<j} 4\sinh^2(\pi(\sigma_i-\sigma_j))\int DQ^i D\widetilde{Q}_i \int DX^i{}_j D\widetilde{X}_i{}^j e^{-S_{\mathrm{ABJM}_{N,1}}}, \tag{5.36}$$

with the 1d action

$$S_{\mathrm{ABJM}_{N,1}} = \ell \int \mathrm{d}\varphi \left[\widetilde{Q}_i \dot{Q}^i + \widetilde{X}_i{}^j \dot{X}^i{}_j + \sum_i \sigma_i \widetilde{Q}_i Q^i + \sum_{i<j} (\sigma_i - \sigma_j)(\widetilde{X}_i{}^j X^i{}_j - \widetilde{X}_j{}^i X^j{}_i) \right] \tag{5.37}$$

We observe that $(\widetilde{X}, X^T)$ transforms as a doublet under an $\mathfrak{su}(2)_F$ flavor symmetry. With respect to the $\mathcal{N} = 8$ theory in 3d, we can interpret $\mathfrak{su}(2)_F$ as $\mathfrak{su}(2)_1$ in $\mathfrak{su}(2)^4 \subset \mathfrak{so}(8)_R$. Integrating out the Xs and Qs, we obtain the matrix model

$$Z_{\mathrm{ABJM}_{N,1}} = \frac{1}{N!} \int \mathrm{d}^N \sigma \frac{\prod_{i<j} 4 \sinh^2(\pi \sigma_{ij})}{\prod_{i,j} 2 \cosh(\pi \sigma_{ij}) \prod_i 2 \cosh(\pi \sigma_i)}, \tag{5.38}$$

where $\sigma_{ij} \equiv \sigma_i - \sigma_j$. The partition function in (5.38) is directly related to the $\mathrm{ABJM}_{N,1}$ matrix model through mirror symmetry [65].

5.3.2 Operator Content

Correlation functions in the full theory (5.36) are given by

$$\begin{aligned} \langle (\cdots) \rangle &= \frac{1}{Z_{\mathrm{ABJM}_{N,1}}} \int \mathrm{d}^N \sigma \, Z_{N,\sigma} \, \langle (\cdots) \rangle_\sigma \,, \\ Z_{N,\sigma} &\equiv \frac{1}{N!} \frac{\prod_{i<j} 4 \sinh^2(\pi \sigma_{ij})}{\prod_{i,j} 2 \cosh(\pi \sigma_{ij}) \prod_i 2 \cosh(\pi \sigma_i)}, \end{aligned} \tag{5.39}$$

where $\langle (\cdots) \rangle_\sigma$ is the correlation function for an arbitrary insertion $(\cdots)$ in the path integral of the Gaussian theory, i.e. at fixed σ. It is given by

$$\langle (\cdots) \rangle_\sigma = \int DQ^i D\widetilde{Q}_i \int DX^i{}_j D\widetilde{X}_i{}^j e^{-S_{\mathrm{ABJM}_{N,1}}} (\cdots) \tag{5.40}$$

For the Gaussian theory, the field σ acts a Lagrange multiplier. The equations of motion for σ are

$$\widetilde{Q}_i Q^i + \widetilde{X}_i{}^j X^i{}_j - \widetilde{X}_j{}^i X^j{}_i = 0. \tag{5.41}$$

All correlation functions in the theory are gauge invariant, meaning that they cannot involve individual Qs, and at a minimum contain the product $\widetilde{Q}_i Q^i$. Using (5.41), we can therefore systematically replace $\widetilde{Q}_i Q^i$ by $\widetilde{X}_j{}^i X^j{}_i - \widetilde{X}_i{}^j X^i{}_j$ and write all correlation functions solely in terms of the Xs. In fact, given that the Gaussian theory is free, we really only need to consider the propagator [44]

$$\langle X^i{}_j(\varphi_1)\widetilde{X}_k{}^l(\varphi_2)\rangle_\sigma = \delta^i{}_k\delta_j{}^l\frac{\operatorname{sgn}(\varphi_1-\varphi_2)+\tanh(\pi\sigma_{ij})}{2\ell}e^{-\sigma_{ij}(\varphi_1-\varphi_2)}. \tag{5.42}$$

Using (5.42), computing correlation functions amounts to repeated application of Wick's theorem, as is the case for any Gaussian field theory.

We now describe the operator content of the ABJM$_{N,1}$ theory. In particular, we focus on a subset of operators in the 3d theory that are in $\mathcal{H}_\mathcal{Q}(\mathcal{A})$. These are given by the multiplets $(8,B,+)_{[0010]}$, $(8,B,+)_{[0020]}$, and $(8,B,+)_{[0040]}$.

$(8,B,+)_{[0010]}$ Free Multiplet

The $\mathcal{N}=8$ free multiplet contains 8 scalar operators of scaling dimension $1/2$ and and 8 spin-$1/2$ operators of scaling dimension 1. Of these, 4 scalars are relevant in the $\mathcal{N}=4$ picture. After performing the twisting procedure, the remaining 2 scalars transform with $j_F=1/2$ under $\mathfrak{su}(2)_F$. We denote the corresponding operator in the topological theory by $\mathcal{O}_{1,\text{free}}$, which can be written as

$$\mathcal{O}_{1,\text{free}}(\varphi,\overline{y}) = \operatorname{tr}\mathcal{X}(\varphi,\overline{y}), \tag{5.43}$$

where, for convenience, we have introduced the operator

$$\mathcal{X}(\varphi,\overline{y}) = \overline{y}^1\widetilde{X}(\varphi)+\overline{y}^2X^T(\varphi), \tag{5.44}$$

which naturally incorporates the $U(N)$ gauge symmetry as well as the $\mathfrak{su}(2)_F$ flavor symmetry. Since the single-traces $\operatorname{tr}X$ and $\operatorname{tr}\widetilde{X}$ only appear in the kinetic terms of (5.37), we can calculate correlation functions of $\mathcal{O}_{1,\text{free}}$ without having to perform any integrals over σ. Using the propagator (5.42) and (5.40), we find

$$\langle\mathcal{O}_{1,\text{free}}(\varphi_1,\overline{y}_1)\mathcal{O}_{1,\text{free}}(\varphi_2,\overline{y}_2)\rangle = N\,\langle\overline{y}_1,\overline{y}_2\rangle\,\frac{\operatorname{sgn}(\varphi_1-\varphi_2)}{2\ell}. \tag{5.45}$$

In fact, the twisted operator $\mathcal{O}_{k,\text{free}}$ corresponding to the $(8,B,+)_{[00k0]}$ free multiplet is given by

$$\mathcal{O}_{k,\text{free}}(\varphi,\overline{y}) = [\mathcal{O}_{1,\text{free}}(\varphi,\overline{y})]^k. \tag{5.46}$$

$(8, B, +)_{[0020]}$ Stress Tensor Multiplets

An $\mathcal{N} = 8$ stress tensor multiplet $(8, B, +,)_{[0020]}$ contains 35 scalar operators with scaling dimension 1, of which 9 are relevant in the $\mathcal{N} = 4$ language. After twisting, only 3 scalars remain; they transform with $j_F = 1$ under $\mathfrak{su}(2)_F$. In the $U(2)$ and $U(3)$ theories, there are two $\mathcal{N} = 8$ stress tensor multiplets – one in the free sector and another in the interacting sector. We denote the corresponding operator in the free sector by $\mathcal{O}_{2,\text{free}}$. It can be written as

$$\mathcal{O}_{2,\text{free}}(\varphi,\overline{y}) = (\operatorname{tr}\mathcal{X})^2(\varphi,\overline{y}). \tag{5.47}$$

For the interacting sector, we denote the corresponding operator by $\mathcal{O}_{2,\text{int}}$. Any twisted operator $\mathcal{O}_2$ corresponding to $(8, B, +)_{[0020]}$ can be written in terms of single and double traces, with

$$\mathcal{O}_2 = a(\operatorname{tr}\mathcal{X})^2 + b\operatorname{tr}\mathcal{X}^2, \tag{5.48}$$

for $a, b \in \mathbb{R}$. Given that

$$\langle(\operatorname{tr}\mathcal{X}^2)(\varphi_1)(\operatorname{tr}\mathcal{X})^2(\varphi_2)\rangle = N\,\langle\overline{y}_1,\overline{y}_2\rangle^2\,\frac{1}{2\ell^2}, \tag{5.49}$$

requiring that $\mathcal{O}_{2,\text{int}}$ be orthogonal to $\mathcal{O}_{2,\text{free}}$ implies that

$$\mathcal{O}_{2,\text{int}}(\varphi,\overline{y}) = (\operatorname{tr}\mathcal{X}^2)(\varphi,\overline{y}) - \frac{1}{N}(\operatorname{tr}\mathcal{X})^2(\varphi,\overline{y}) \tag{5.50}$$

up to an overall c-number.

$(B, +)_{[0040]}$ Multiplets

The $\mathcal{N} = 8$ multiplet $(8, B, +,)_{[0040]}$ contains 284 scalar operators with scaling dimension 2. From these, there are 25 that are important from the $\mathcal{N} = 4$ perspective. After twisting, we are left with 5 scalars that transform as $j_F = 2$ under $\mathfrak{su}(2)_F$. For the $U(2)$ and $U(3)$ theories, any of the corresponding twisted operators $\mathcal{O}_4$ can be written as the linear combination

$$\mathcal{O}_4 = a(\text{tr}\,\mathcal{X})^4 + b(\text{tr}\,\mathcal{X}^2)(\text{tr}\,\mathcal{X})^2 + c(\text{tr}\,\mathcal{X}^2)^2. \tag{5.51}$$

Upon inspecting (5.51), we find that there are a total of three linearly independent operators corresponding to a $(8, B, +,)_{[0040]}$ multiplet. As a matter of convention, we define three operators $\mathcal{O}_{4,\text{free}}$, $\mathcal{O}_{4,\text{int}}$, and $\mathcal{O}_{4,\text{mixed}}$, as the unique $\mathcal{O}_4$ appearing in certain $\mathcal{O}_2 \times \mathcal{O}_2$ OPEs. Schematically, we require

$$\begin{aligned} \mathcal{O}_{4,\text{free}} &\in \mathcal{O}_{2,\text{free}} \times \mathcal{O}_{2,\text{free}}, \\ \mathcal{O}_{4,\text{int}} &\in \mathcal{O}_{2,\text{int}} \times \mathcal{O}_{2,\text{int}}, \\ \mathcal{O}_{4,\text{mixed}} &\in \mathcal{O}_{2,\text{free}} \times \mathcal{O}_{2,\text{int}}. \end{aligned} \tag{5.52}$$

The operators $\mathcal{O}_{4,\text{free}}$ and $\mathcal{O}_{4,\text{int}}$ lie in the free and interacting sectors of the $U(2)$, $U(3)$ theories, respectively. The so-called mixed operator $\mathcal{O}_{4,\text{mixed}}$ is an artifact of the nonvanishing of the $\mathcal{O}_{2,\text{free}} \times \mathcal{O}_{2,\text{int}}$ OPE. Since the two sectors are decoupled in the IR, we are free to ignore the mixed operator in our calculations. Requiring that three operators be orthogonal within correlation functions (5.40) for the 1d theory fixes them such that

$$\begin{aligned} \mathcal{O}_{4,\text{free}}(\varphi, \overline{y}) &= [\mathcal{O}_{2,\text{free}}(\varphi, \overline{y})]^2, \\ \mathcal{O}_{4,\text{int}}(\varphi, \overline{y}) &= [\mathcal{O}_{2,\text{int}}(\varphi, \overline{y})]^2, \\ \mathcal{O}_{4,\text{mixed}}(\varphi, \overline{y}) &= \mathcal{O}_{2,\text{free}}(\varphi, \overline{y})\mathcal{O}_{2,\text{int}}(\varphi, \overline{y}). \end{aligned} \tag{5.53}$$

5.4 $U(2)$ Gauge Theory

We now consider specific $U(N)$ theories. For the $U(2)$ theory, the localized partition function on S^3 is given by

$$
\begin{aligned}
Z_{\mathrm{ABJM}_{2,1}} &= \int \mathrm{d}^2\sigma Z_{2,\sigma} \\
&= \frac{1}{32}\int \mathrm{d}^2\sigma \tanh^2(\pi\sigma_{12})\mathrm{sech}(\pi\sigma_1)\mathrm{sech}(\pi\sigma_2) \\
&= \frac{1}{16\pi}.
\end{aligned}
\tag{5.54}
$$

Our results match those for the $\mathrm{ABJM}_{2,1}$ partition function in [66].

5.4.1 Two-point Functions

We use (5.45) and Wick contractions to compute the two-point functions for the free sector. We find that

$$
\langle \mathcal{O}_{2,\text{free}}(\varphi_1,\overline{y}_1)\mathcal{O}_{2,\text{free}}(\varphi_2,\overline{y}_2)\rangle = \langle \overline{y}_1,\overline{y}_2\rangle^2 \frac{2}{\ell^2}, \tag{5.55}
$$

$$
\langle \mathcal{O}_{4,\text{free}}(\varphi_1,\overline{y}_1)\mathcal{O}_{4,\text{free}}(\varphi_2,\overline{y}_2)\rangle = \langle \overline{y}_1,\overline{y}_2\rangle^4 \frac{24}{\ell^4}. \tag{5.56}
$$

The interacting sector contains double-trace operators with off-diagonal components of X and $\widetilde{X}$, so correlation functions of the topological theory will generally depend nontrivially on σ. Using (5.40) and (5.42), we compute

$$
\begin{aligned}
\langle \mathcal{O}_{2,\text{int}}(\varphi_1,\overline{y}_1)\mathcal{O}_{2,\text{int}}(\varphi_2,\overline{y}_2)\rangle &= \frac{1}{\ell^2}\frac{\langle \overline{y}_1,\overline{y}_2\rangle^4}{Z_{\mathrm{ABJM}_{2,1}}}\int \mathrm{d}^2\sigma\, Z_{2,\sigma}\left[\frac{3}{2}-\tanh^2(\pi\sigma_{12})\right] \\
&= \frac{2}{3\ell^2}\langle \overline{y}_1,\overline{y}_2\rangle^2,
\end{aligned}
\tag{5.57}
$$

and

$$
\begin{aligned}
&\langle \mathcal{O}_{4,\text{int}}(\varphi_1,\overline{y}_1)\mathcal{O}_{4,\text{int}}(\varphi_2,\overline{y}_2)\rangle \\
&= \frac{1}{\ell^4}\frac{\langle \overline{y}_1,\overline{y}_2\rangle^4}{Z_{\mathrm{ABJM}_{2,1}}}\int \mathrm{d}^2\sigma\, Z_{2,\sigma}\left[\frac{15}{2}-10\tanh^2(\pi\sigma_{12})+4\tanh^2(\pi\sigma_{12})\right] \\
&= \frac{32}{15\ell^2}\langle \overline{y}_1,\overline{y}_2\rangle^2.
\end{aligned}
\tag{5.58}
$$

5.4.2 Four-point Functions

We now compute the four-point correlation functions of operators in the free sector. There are four distinct four-point functions in total. Again using (5.45) along with the relevant Wick contractions, we find ($\varphi_1 < \varphi_2 < \varphi_3 < \varphi_4$):

$$\begin{aligned}\langle \mathcal{O}_{2,\text{free}}(\varphi_1, \overline{y}_1)\mathcal{O}_{2,\text{free}}(\varphi_2, \overline{y}_2)\mathcal{O}_{2,\text{free}}(\varphi_3, \overline{y}_3)\mathcal{O}_{2,\text{free}}(\varphi_4, \overline{y}_4)\rangle = \\ \left(\frac{2}{\ell^2}\right)^2 \langle \overline{y}_1, \overline{y}_2\rangle^2 \langle \overline{y}_3, \overline{y}_4\rangle^2 \frac{6 + 2\overline{w} - 2\overline{w}^2}{\overline{w}^2},\end{aligned} \tag{5.59}$$

$$\begin{aligned}\langle \mathcal{O}_{4,\text{free}}(\varphi_1, \overline{y}_1)\mathcal{O}_{4,\text{free}}(\varphi_2, \overline{y}_2)\mathcal{O}_{4,\text{free}}(\varphi_3, \overline{y}_3)\mathcal{O}_{4,\text{free}}(\varphi_4, \overline{y}_4)\rangle = \\ \left(\frac{24}{\ell^4}\right)^2 \langle \overline{y}_1, \overline{y}_2\rangle^4 \langle \overline{y}_3, \overline{y}_4\rangle^4 \frac{70 + 180\overline{w} - 174\overline{w}^2 - 12\overline{w}^3 + 6\overline{w}^4}{\overline{w}^4},\end{aligned} \tag{5.60}$$

$$\begin{aligned}\langle \mathcal{O}_{2,\text{free}}(\varphi_1, \overline{y}_1)\mathcal{O}_{2,\text{free}}(\varphi_2, \overline{y}_2)\mathcal{O}_{4,\text{free}}(\varphi_3, \overline{y}_3)\mathcal{O}_{4,\text{free}}(\varphi_4, \overline{y}_4)\rangle = \\ \left(\frac{2}{\ell^2}\right)\left(\frac{24}{\ell^4}\right) \langle \overline{y}_1, \overline{y}_2\rangle^2 \langle \overline{y}_3, \overline{y}_4\rangle^4 \frac{36 - 20\overline{w} - \overline{w}^2}{\overline{w}^2},\end{aligned} \tag{5.61}$$

$$\begin{aligned}\langle \mathcal{O}_{2,\text{free}}(\varphi_1, \overline{y}_1)\mathcal{O}_{4,\text{free}}(\varphi_2, \overline{y}_2)\mathcal{O}_{2,\text{free}}(\varphi_3, \overline{y}_3)\mathcal{O}_{4,\text{free}}(\varphi_4, \overline{y}_4)\rangle = \\ \left(\frac{2}{\ell^2}\right)\left(\frac{24}{\ell^4}\right) \langle \overline{y}_1, \overline{y}_2\rangle^2 \langle \overline{y}_3, \overline{y}_4\rangle^2 \langle \overline{y}_2, \overline{y}_4\rangle^2 \frac{15 + 12\overline{w} - 12\overline{w}^2}{\overline{w}^2}.\end{aligned} \tag{5.62}$$

Computing the four-point functions for interacting sector is more challenging and involves multiple integrals over σ. The resulting computations yield

$$\begin{aligned}\langle \mathcal{O}_{2,\text{int}}(\varphi_1, \overline{y}_1)\mathcal{O}_{2,\text{int}}(\varphi_2, \overline{y}_2)\mathcal{O}_{2,\text{int}}(\varphi_3, \overline{y}_3)\mathcal{O}_{2,\text{int}}(\varphi_4, \overline{y}_4)\rangle = \\ \left(\frac{2}{3\ell^2}\right)^2 \langle \overline{y}_1, \overline{y}_2\rangle^2 \langle \overline{y}_3, \overline{y}_4\rangle^2 \frac{6(4 + \overline{w} - \overline{w}^2)}{5\overline{w}^2}\end{aligned} \tag{5.63}$$

$$\begin{aligned}\langle \mathcal{O}_{4,\text{int}}(\varphi_1, \overline{y}_1)\mathcal{O}_{4,\text{int}}(\varphi_2, \overline{y}_2)\mathcal{O}_{4,\text{int}}(\varphi_3, \overline{y}_3)\mathcal{O}_{4,\text{int}}(\varphi_4, \overline{y}_4)\rangle = \\ \left(\frac{32}{15\ell^4}\right)^2 \langle \overline{y}_1, \overline{y}_2\rangle^4 \langle \overline{y}_3, \overline{y}_4\rangle^4 \frac{30(12 + 30\overline{w} - 29\overline{w}^2 - 2\overline{w}^2 + \overline{w}^4)}{7\overline{w}^4}\end{aligned} \tag{5.64}$$

$$\langle \mathcal{O}_{2,\text{int}}(\varphi_1, \overline{y}_1)\mathcal{O}_{2,\text{int}}(\varphi_2, \overline{y}_2)\mathcal{O}_{4,\text{int}}(\varphi_3, \overline{y}_3)\mathcal{O}_{4,\text{int}}(\varphi_4, \overline{y}_4)\rangle = \left(\frac{2}{3\ell^2}\right)\left(\frac{32}{15\ell^4}\right)\langle \overline{y}_1, \overline{y}_2\rangle^2 \langle \overline{y}_3, \overline{y}_4\rangle^4 \frac{3(64 - 36\overline{w} - \overline{w}^2)}{7\overline{w}^2} \tag{5.65}$$

$$\langle \mathcal{O}_{2,\text{int}}(\varphi_1, \overline{y}_1)\mathcal{O}_{4,\text{int}}(\varphi_2, \overline{y}_2)\mathcal{O}_{2,\text{int}}(\varphi_3, \overline{y}_3)\mathcal{O}_{4,\text{int}}(\varphi_4, \overline{y}_4)\rangle = \left(\frac{2}{3\ell^2}\right)\left(\frac{32}{15\ell^4}\right)\langle \overline{y}_1, \overline{y}_2\rangle^2 \langle \overline{y}_3, \overline{y}_4\rangle^2 \langle \overline{y}_2, \overline{y}_4\rangle^2 \frac{81 + 60\overline{w} - 60\overline{w}^2}{7\overline{w}^2} \tag{5.66}$$

Comparing the results of the mixed correlator to (4.25) yields a set of constraints among the OPE coefficients. Additionally, we can use the fact that the three-point function

$$\langle \mathcal{O}_{2,\text{int}}(\varphi_1, \overline{y}_1)\mathcal{O}_{4,\text{int}}(\varphi_2, \overline{y}_2)\mathcal{O}_{2,\text{int}}(\varphi_3, \overline{y}_3)\rangle \tag{5.67}$$

does not depend on the ordering of operators to product the additional constraints:

$$\lambda^2_{2,4,1,1} = \frac{3}{2}\lambda^2_{2,2,2,2}\,, \qquad \lambda^2_{2,4,2,2} = 2\lambda^2_{4,4,1,1}\,. \tag{5.68}$$

Miraculously, the combination of these equations with (4.28) and the restriction that $\lambda \in \mathbb{R}$ is sufficient to solve for the entire set of OPE coefficients in the mixed correlator. We list the OPE coefficients in Table 4. As expected, all coefficients for operators of type $(B, 2)$ are zero [30]. Up to an overall normalization factor that can be determined from the free theory, our results are in exact agreement with both [30] and [44].

5.5 $U(3)$ Gauge Theory

We will now determine the correlation functions for the topological theory of ABJM$_{3,1}$. Since the free sector of the $U(3)$ theory is isomorphic to that of the $U(2)$ theory (there is only a single free SCFT), we will focus on computing the correlation functions and OPE coefficients in the interacting sector. The computation is essentially the same, except that we now have many more Wick contractions in addition

λ^2	Free theory	$U(2)$ interacting sector
$\lambda^2_{2,2,1,1}$	16	12
$\lambda^2_{2,2,2,2}$	16	64/5
$\lambda^2_{2,2,2,0}$	0	0
$\lambda^2_{4,4,1,1}$	64	48
$\lambda^2_{4,4,2,2}$	576	20480/49
$\lambda^2_{4,4,3,3}$	1024	5184/7
$\lambda^2_{4,4,4,4}$	256	9216/49
$\lambda^2_{4,4,4,2}$	0	0
$\lambda^2_{4,4,4,0}$	0	0
$\lambda^2_{4,4,3,1}$	0	0
$\lambda^2_{4,4,2,0}$	0	0
$\lambda^2_{2,4,1,1}$	24	96/5
$\lambda^2_{2,4,2,2}$	128	96
$\lambda^2_{2,4,3,3}$	64	1728/35
$\lambda^2_{2,4,3,1}$	0	0
$\lambda^2_{2,4,3,2}$	0	0
$\lambda^2_{2,4,2,1}$	0	0

Table 4: Values of $(8, B, 2)$ and $(8, B, +)$ OPE coefficients, as defined in (4.25), for the free and interacting sectors of $\text{ABJM}_{2,1}$. There is only a single free $\mathcal{N} = 8$ SCFT in three dimensions, which is isomorphic to the free sector of the $U(2)$ theory.

to various triple integrals over the matrix eigenvalues. The S^3 partition function for the $U(3)$ theory is given by

$$\begin{aligned} Z_{\text{ABJM}_{3,1}} &= \int \mathrm{d}^3\sigma Z_{3,\sigma} \\ &= \frac{\pi - 3}{64\pi}, \end{aligned} \tag{5.69}$$

where

$$Z_{3,\sigma} = \frac{1}{384}\sigma \tanh^2(\pi\sigma_{12}) \tanh^2(\pi\sigma_{23}) \tanh^2(\pi\sigma_{31}) \text{sech}(\pi\sigma_1)\text{sech}(\pi\sigma_2)\text{sech}(\pi\sigma_3). \tag{5.70}$$

Our results match those for the $\text{ABJM}_{3,1}$ partition function in [66].

5.5.1 Two-point Functions

Using the same method from Section 5.4, we find that the two-point functions for the interacting sector of the $U(3)$ theory are

$$\begin{aligned} \langle \mathcal{O}_{2,\mathrm{int}}(\varphi_1, \overline{y}_1)\mathcal{O}_{2,\mathrm{int}} \rangle &= \frac{1}{\ell^2} \frac{\langle \overline{y}_1, \overline{y}_2 \rangle^2}{Z_{\mathrm{ABJM}_{3,1}}} \int \mathrm{d}^3\sigma \; Z_{3,\sigma} \left[4 - \sum_{i<j} \tanh^2(\pi\sigma_{ij}) \right] \\ &= \frac{10\pi - 31}{2(\pi - 3)\ell^2} \langle \overline{y}_1, \overline{y}_2 \rangle^2 , \end{aligned} \tag{5.71}$$

and

$$\begin{aligned} &\langle \mathcal{O}_{4,\mathrm{int}}(\varphi_1, \overline{y}_1)\mathcal{O}_{4,\mathrm{int}}(\varphi_2, \overline{y}_2) \rangle \\ &= \frac{1}{\ell^4} \frac{\langle \overline{y}_1, \overline{y}_2 \rangle^4}{Z_{\mathrm{ABJM}_{3,1}}} \int \mathrm{d}^3\sigma \; Z_{3,\sigma} \Bigg[40 - 20 \sum_{i<j} \tanh^2(\pi\sigma_{ij}) \\ &+ 4 \sum_{i<j} \sum_{k<l} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \Bigg] \\ &= \frac{840\pi - 2629}{10(\pi - 3)\ell^4} \langle \overline{y}_1, \overline{y}_2 \rangle^4 . \end{aligned} \tag{5.72}$$

5.5.2 Four-point Functions

Likewise, after an extraordinarily tedious computation, we find that the four-point functions are given by

$$\begin{aligned} &\langle \mathcal{O}_{2,\mathrm{int}}(\varphi_1, \overline{y}_1)\mathcal{O}_{2,\mathrm{int}}(\varphi_2, \overline{y}_2)\mathcal{O}_{2,\mathrm{int}}(\varphi_3, \overline{y}_3)\mathcal{O}_{2,\mathrm{int}}(\varphi_4, \overline{y}_4) \rangle = \\ &\qquad \frac{\langle \overline{y}_1, \overline{y}_2 \rangle^2 \langle \overline{y}_3, \overline{y}_4 \rangle^2}{10(\pi - 3)\ell^4} \left[(640\pi - 2009) + \frac{1}{\overline{w}}(2009 - 640\pi) + \frac{1}{\overline{w}^2}(840\pi - 2629) \right] \end{aligned} \tag{5.73}$$

$$
\begin{aligned}
&\langle \mathcal{O}_{4,\text{int}}(\varphi_1,\overline{y}_1)\mathcal{O}_{4,\text{int}}(\varphi_2,\overline{y}_2)\mathcal{O}_{4,\text{int}}(\varphi_3,\overline{y}_3)\mathcal{O}_{4,\text{int}}(\varphi_4,\overline{y}_4)\rangle = \\
&\qquad \frac{\langle \overline{y}_1,\overline{y}_2\rangle^4 \langle \overline{y}_3,\overline{y}_4\rangle^4}{188230(\pi-3)\pi\ell^8}\left[\frac{1}{\overline{w}^0}(3129033600+\pi(25243234800\pi-80295720779))\right. \\
&\qquad\qquad -\frac{2}{\overline{w}}(3129033600+\pi(25243234800\pi-80295720779)) \\
&\qquad\qquad +\frac{5}{\overline{w}^2}(1730171520+\pi(12304930512\pi-39218045869)) \\
&\qquad\qquad -\frac{6}{\overline{w}^3}(920304000+\pi(6046902960\pi-19299084761)) \\
&\qquad\qquad \left.+\frac{9}{\overline{w}^4}(429475200+\pi(4045834800\pi-12843489523))\right],
\end{aligned}
\tag{5.74}
$$

$$
\begin{aligned}
&\langle \mathcal{O}_{2,\text{int}}(\varphi_1,\overline{y}_1)\mathcal{O}_{2,\text{int}}(\varphi_2,\overline{y}_2)\mathcal{O}_{4,\text{int}}(\varphi_3,\overline{y}_3)\mathcal{O}_{4,\text{int}}(\varphi_4,\overline{y}_4)\rangle = \\
&\qquad \frac{\langle \overline{y}_1,\overline{y}_2\rangle^2 \langle \overline{y}_3,\overline{y}_4\rangle^4}{26890(\pi-3)\pi\ell^6}\left[\frac{1}{\overline{w}^0}(4930200+\pi(63500475\pi-201008128))\right. \\
&\qquad\qquad -\frac{4}{\overline{w}}(1643400+\pi(27190185\pi-85854392)) \\
&\qquad\qquad \left.+\frac{4}{\overline{w}^2}(1643400+\pi(31707705\pi-99993154))\right],
\end{aligned}
\tag{5.75}
$$

λ^2	$U(3)$ interacting sector
$\lambda^2_{2,2,1,1}$	≈ 5.447
$\lambda^2_{2,2,2,2}$	≈ 8.676
$\lambda^2_{2,2,2,0}$	≈ 5.593
$\lambda^2_{4,4,1,1}$	≈ 21.787
$\lambda^2_{4,4,2,2}$	≈ 173.226
$\lambda^2_{4,4,3,3}$	≈ 292.215
$\lambda^2_{4,4,4,4}$	≈ 89.790
$\lambda^2_{4,4,4,2}$	≈ 62.599
$\lambda^2_{4,4,4,0}$	≈ 45.660
$\lambda^2_{4,4,3,1}$	≈ 41.441
$\lambda^2_{4,4,2,0}$	≈ 20.662
$\lambda^2_{2,4,1,1}$	≈ 13.014
$\lambda^2_{2,4,2,2}$	≈ 43.575
$\lambda^2_{2,4,3,3}$	≈ 29.092
$\lambda^2_{2,4,3,1}$	≈ 24.754
$\lambda^2_{2,4,3,2}$	0
$\lambda^2_{2,4,2,1}$	0

Table 5: Values of $(8,B,2)$ and $(8,B,+)$ OPE coefficients, as defined in (4.25), for the interacting sector of ABJM$_{3,1}$.

$$
\begin{aligned}
&\langle \mathcal{O}_{2,\text{int}}(\varphi_1,\overline{y}_1)\mathcal{O}_{4,\text{int}}(\varphi_2,\overline{y}_2)\mathcal{O}_{2,\text{int}}(\varphi_3,\overline{y}_3)\mathcal{O}_{4,\text{int}}(\varphi_4,\overline{y}_4)\rangle = \\
&\frac{\langle \overline{y}_1,\overline{y}_2\rangle^2 \langle \overline{y}_3,\overline{y}_4\rangle^2 \langle \overline{y}_2,\overline{y}_4\rangle^2}{26890(\pi-3)\pi\ell^6} \Bigg[\frac{4}{\overline{w}^0}(1643400+\pi(18155145\pi-57576868)) \\
&\qquad -\frac{4}{\overline{w}}(1643400+\pi(18155145\pi-57576868)) \\
&\qquad +\frac{3}{\overline{w}^2}(1643400+\pi(27190185\pi-85854392))\Bigg].
\end{aligned} \tag{5.76}
$$

Using the mixed correlator results above, as well as (4.28), (5.68), and the restriction that $\lambda \in \mathbb{R}$, we again find all the desired OPE coefficients, which are listed below. In addition, their approximate values are provided for convenience in Table 5.

$$
\begin{aligned}
\lambda^2_{2,2,1,1} &= \frac{16(\pi-3)}{10\pi-31},\\
\lambda^2_{2,2,2,2} &= \frac{16\left(7887-5149\pi+840\pi^2\right)}{15(31-10\pi)^2},\\
\lambda^2_{2,2,2,0} &= \frac{16\left(3888-2557\pi+420\pi^2\right)}{3(31-10\pi)^2},\\
\lambda^2_{4,4,1,1} &= \frac{64(\pi-3)}{10\pi-31},\\
\lambda^2_{4,4,2,2} &= \frac{1280(\pi-3)\left(1643400-99993154\pi+31707705\pi^2\right)^2}{21692163\pi^2(840\pi-2629)^3},\\
\lambda^2_{4,4,3,3} &= \frac{6144\left(-4930200+259206576\pi-167424947\pi^2+27190185\pi^3\right)}{2689(2629-840\pi)^2\pi},\\
\lambda^2_{4,4,4,4} &= \frac{2304\left(-1288425600+38959943769\pi-24980993923\pi^2+4045834800\pi^3\right)}{131761(2629-840\pi)^2\pi},\\
\lambda^2_{4,4,4,2} &= (1062915987\pi^2(840\pi-2629)^3)^{-1}(5120(1588048973280000+65808627696504000\pi\\
&\quad -2350319629000295919\pi^2)+2243279625553884853\pi^3-722357008158321060\pi^4\\
&\quad +77762265431312700\pi^5),\\
\lambda^2_{4,4,4,0} &= (21692163(2629-840\pi)^2\pi^2(3888-2557\pi+420\pi^2))^{-1}(256(-4764146919840000\\
&\quad +10540426370997600\pi+2261712776837118687\pi^2)-2976914365216300800\pi^3\\
&\quad +1465981190907907177\pi^4-320622004004558340\pi^5+26284068925487100\pi^6)\\
\lambda^2_{4,4,3,1} &= (2689(2629-840\pi)^2\pi(10\pi-31))^{-1}(1024(2139706800-44280625377\pi\\
&\quad +41910326329\pi^2-13501834190\pi^3+1457515500\pi^4),\\
\lambda^2_{4,4,2,0} &= \frac{16\left(69022800-2292779055\pi+1478741996\pi^2-240619470\pi^3\right)^2}{21692163(2629-840\pi)^2\pi^2\left(3888-2557\pi+420\pi^2\right)},\\
\lambda^2_{2,4,1,1} &= \frac{8\left(7887-5149\pi+840\pi^2\right)}{5(31-10\pi)^2},\\
\lambda^2_{2,4,2,2} &= \frac{128(\pi-3)}{10\pi-31},\\
\lambda^2_{2,4,3,3} &= \frac{128\left(-4930200+259206576\pi-167424947\pi^2+27190185\pi^3\right)}{13445\pi\left(81499-52330\pi+8400\pi^2\right)},\\
\lambda^2_{2,4,3,1} &= (13445(31-10\pi)^2\pi(840\pi-2629))^{-1}(128(2139706800-44280625377\pi\\
&\quad +41910326329\pi^2-13501834190\pi^3+1457515500\pi^4),\\
\lambda^2_{2,4,3,2} &= 0,\\
\lambda^2_{2,4,2,1} &= 0.
\end{aligned}
\tag{5.77}
$$

5.6 Applications to BLG Theory

5.6.1 The S^3 Partition Function

We now apply the localization technique to the BLG_3 theory. We begin by considering the infrared limit of an $SU(2)_3 \times SU(2)_{-3}$ Chern-Simons matter theory with two hypermultiplets transforming in the bifundamental representation. The twisted fields in the fundamental hypermultiplet $Q, \widetilde{Q}$ transform separately under the gauge group as $\mathbf{2} \oplus \overline{\mathbf{2}}$ and $\overline{\mathbf{2}} \oplus \mathbf{2}$, respectively. It is convenient to express the fields in terms of their gauge indices, which we will denote by $a = 1, 2$ for the first $SU(2)$ and $\dot{a} = 1, 2$ for the second. The localization procedure for the BLG theory is analogous to the ABJM theory, except we must multiply by a factor of 2 in the partition function to account for the $\mathbb{Z}_2$ factor in the quotient. After localization, the S^3 partition function is given by

$$\begin{aligned} Z_{\mathrm{BLG}_3} = \frac{2}{2! \cdot 2^2} \int \mathrm{d}\lambda \mathrm{d}\mu \, e^{6i\pi(\lambda^2 - \mu^2)} \left(\frac{2\sinh(2\pi\lambda) 2\sinh(2\pi\mu)}{2\cosh(\pi(\lambda-\mu)) 2\cosh(\pi(\lambda+\mu))} \right)^2 \\ \times \int DQ_a{}^{\dot{a}} D\widetilde{Q}_{\dot{a}}{}^{a} e^{-S_{\mathrm{BLG}_3}}, \end{aligned} \tag{5.78}$$

where the action for the 1d theory is

$$\begin{aligned} S_{\mathrm{BLG}_3} = \ell \int_{-\pi}^{\pi} \mathrm{d}\varphi \, \Big[\widetilde{Q}_{\dot{a}}{}^{a} \partial_\varphi Q_a{}^{\dot{a}} + (\lambda - \mu) \widetilde{Q}_1{}^1 Q_1{}^1 - (\lambda + \mu) \widetilde{Q}_1{}^2 Q_2{}^1 \\ + (\lambda + \mu) \widetilde{Q}_2{}^1 Q_1{}^2 + (\mu - \lambda) \widetilde{Q}_2{}^2 Q_2{}^2 \Big], \end{aligned} \tag{5.79}$$

and where λ and μ are the matrix degrees of freedom in the Cartan of separate copies of $\mathfrak{su}(2)$. The theory has an additional $\mathfrak{su}_F(2)$ flavor symmetry, under which $(Q, \widetilde{Q}^T)$ forms a doublet. After integrating out the Qs, we are left with the matrix model

$$\begin{aligned} Z_{\mathrm{BLG}_3} &= \int \mathrm{d}\lambda_+ \mathrm{d}\lambda_- Z_\lambda \\ &= \frac{\pi - 3}{64\pi}, \end{aligned} \tag{5.80}$$

where

$$Z_\lambda = e^{\frac{6i}{\pi}\lambda_+\lambda_-}\left(\frac{\sinh(\lambda_+ + \lambda_-)\sinh(\lambda_+ - \lambda_-)}{8\pi\cosh^2(\lambda_-)\cosh^2(\lambda_+)}\right)^2 \tag{5.81}$$

and

$$\lambda_+ = \pi(\lambda+\mu), \quad \lambda_- = \pi(\lambda-\mu). \tag{5.82}$$

This value matches the partition function for both BLG_3 and $\mathrm{ABJM}_{3,1}$.

5.6.2 Propagators

We are interested in calculating correlation functions for the topological theory. Consider a set of operators, denoted by $(\cdots)$, which are functions of the twisted fields Q and $\widetilde{Q}$. Correlation functions of the full theory can be written as

$$\langle(\cdots)\rangle = \frac{1}{Z_{\mathrm{BLG}_3}}\int \mathrm{d}\lambda_+\mathrm{d}\lambda_-\ Z_\lambda \left\langle(\cdots)\right\rangle_\lambda , \tag{5.83}$$

where

$$\langle(\cdots)\rangle_\lambda = \int DQ_a{}^{\dot{a}} D\widetilde{Q}_{\dot{a}}{}^{a} e^{-S_{\mathrm{BLG}_3}}(\cdots), \tag{5.84}$$

is the corresponding correlation function of the 1d theory at fixed values of $\lambda_\pm$. For the 1d theory, the coupling terms involving the fields λ and μ look like mass terms. Introducing a quantity analogous to a mass matrix,

$$M_{\dot{a}}{}^{a} = \begin{pmatrix} \lambda-\mu & \lambda+\mu \\ \lambda+\mu & \mu-\lambda \end{pmatrix}, \tag{5.85}$$

we can rewrite the 1d action in (5.79) as

$$S_{1d} = \ell\int_{-\pi}^{\pi} \mathrm{d}\varphi\ \widetilde{Q}_{\dot{a}}{}^{a}(\delta_a{}^{\dot{a}}\partial_\varphi + M_a{}^{\dot{a}})Q_a{}^{\dot{a}}. \tag{5.86}$$

With the action in a more compact form, we can easily read off the propagators as

$$\langle Q_a{}^{\dot{a}}(\varphi_1)\widetilde{Q}_{\dot{b}}{}^{b}(\varphi_2)\rangle_\lambda = \delta_a{}^{b}\delta_{\dot{b}}{}^{\dot{a}}\ \frac{\mathrm{sgn}(\varphi_1-\varphi_2)+\tanh(\pi M_a{}^{\dot{a}})}{2\ell} e^{-M_a{}^{\dot{a}}(\varphi_1-\varphi_2)}. \tag{5.87}$$

5.6.3 Operator Content

We now determine the operator content of the topological theory for BLG$_3$.

$(8, B, +)_{[0020]}$ **Stress Tensor Multiplet**

It is possible to construct gauge invariant operators out of two of the twisted fields, namely:

$$\begin{aligned} \det Q &= Q_1{}^1 Q_2{}^2 - Q_1{}^2 Q_2{}^1, \quad \det \widetilde{Q} = \widetilde{Q}_1{}^1 \widetilde{Q}_2{}^2 - \widetilde{Q}_1{}^2 \widetilde{Q}_2{}^1, \\ \frac{1}{2}\mathrm{tr}(Q\widetilde{Q}) &= \frac{1}{2}\left(Q_1{}^1 \widetilde{Q}_1{}^1 + Q_1{}^2 \widetilde{Q}_2{}^1 + Q_2{}^1 \widetilde{Q}_1{}^2 + Q_2{}^2 \widetilde{Q}_2{}^2\right). \end{aligned} \tag{5.88}$$

Upon inspecting the action, one finds that the flavor doublets are $(Q_1{}^1, -\widetilde{Q}_2{}^2)$, $(Q_1{}^2, \widetilde{Q}_2{}^1)$, $(Q_2{}^1, \widetilde{Q}_1{}^2)$, and $(Q_2{}^2, -\widetilde{Q}_1{}^1)$. Let $\mathcal{O}_A$ denote such a doublet, where $A = 1, 2$ is an $\mathfrak{su}(2)_F$ flavor index. It is convenient to introduce auxiliary variables $\overline{y}^A = (\overline{y}^1, \overline{y}^2)$, such that a field transforming with $j_F = k/2$ under $\mathfrak{su}(2)_F$ can be expressed in index-free notation by contracting its flavor indices with those of the $\overline{y}$'s. In particular, we define the flavor-invariant operator

$$\mathcal{O}_k(\varphi, \overline{y}) \equiv \mathcal{O}_{A_1,\ldots,A_k}(\varphi)\overline{y}^{A_1}\cdots\overline{y}^{A_k}. \tag{5.89}$$

Introducing a matrix of flavor doublets,

$$\mathbf{Q}(\varphi, \overline{y}) = \begin{pmatrix} Q_1{}^1\overline{y}^1 - \widetilde{Q}_2{}^2\overline{y}^2 & Q_1{}^2\overline{y}^1 + \widetilde{Q}_2{}^1\overline{y}^2 \\ Q_2{}^1\overline{y}^1 + \widetilde{Q}_1{}^2\overline{y}^2 & Q_2{}^2\overline{y}^1 - \widetilde{Q}_1{}^1\overline{y}^1 \end{pmatrix}, \tag{5.90}$$

we find that the corresponding flavor-invariant operator transforming with $j_F = 1$ is conveniently given by

$$\begin{aligned} \mathcal{O}_2(\varphi, \overline{y}) &= \det \mathbf{Q}(\varphi, \overline{y}) \\ &= (\overline{y}^1)^2(\det Q)(\varphi) + (2\overline{y}^1\overline{y}^2)\left(\frac{1}{2}\mathrm{tr}(Q\widetilde{Q})\right)(\varphi) + (\overline{y}^2)^2(\det \widetilde{Q})(\varphi). \end{aligned} \tag{5.91}$$

This 1d operator corresponds to the superconformal primary in $(8, B, +)_{[0020]}$.

$(8, B, +)_{[0040]}$ **Multiplet**

The twisted operator corresponding to $(8, B, +)_{[0040]}$ is given by $\mathcal{O}_4$ in (5.89).

Since there is only a single $(8, B, +)_{[0020]}$ multiplet, we can construct the 1d operator for $(8, B, +)_{[0040]}$ simply by squaring (5.91), i.e.

$$\mathcal{O}_4(\varphi, \overline{y}) = \mathcal{O}_2(\varphi, \overline{y})^2. \tag{5.92}$$

5.6.4 Two-point Functions

We are now ready to compute the two-point functions of the theory. Using the propagators in (5.87), we find that

$$\begin{aligned}
&\langle \mathcal{O}_2(\varphi_1, \overline{y}_1)\mathcal{O}_2(\varphi_2, \overline{y}_2)\rangle \\
&= \frac{\langle \overline{y_1}, \overline{y_2}\rangle^2}{8\ell^2} \int \mathrm{sech}(\lambda_-)^2 \mathrm{sech}(\lambda_+)^2 (2 + \cosh(2\lambda_-) + \cosh(2\lambda_+)) \\
&= \frac{10\pi - 31}{8(\pi - 3)\ell^2} \langle \overline{y}_1, \overline{y}_2\rangle^2
\end{aligned} \tag{5.93}$$

and

$$\langle \mathcal{O}_4(\varphi_1, \overline{y}_1)\mathcal{O}_4(\varphi_2, \overline{y}_2)\rangle = \frac{840\pi - 32629}{160(\pi - 3)\ell^4} \langle \overline{y}_1, \overline{y}_2\rangle^4 . \tag{5.94}$$

5.6.5 Four-point Functions

Computing the four-point functions for the BLG theory is somewhat complicated and involves multiple Fourier integrals. Instead of listing them explicitly, we can write the results in terms of the four-point functions for the interacting sector of ABJM$_{3,1}$ from Section 5.5. After a serious of computations, we find that

$$\begin{aligned}
\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle_{\mathrm{BLG}_3} &= \frac{1}{16} \langle \mathcal{O}_{2,\mathrm{int}}\mathcal{O}_{2,\mathrm{int}}\mathcal{O}_{2,\mathrm{int}}\mathcal{O}_{2,\mathrm{int}}\rangle_{\mathrm{ABJM}_{3,1}\ \mathrm{int}} \\
\langle \mathcal{O}_4\mathcal{O}_4\mathcal{O}_4\mathcal{O}_4\rangle_{\mathrm{BLG}_3} &= \frac{1}{256} \langle \mathcal{O}_{4,\mathrm{int}}\mathcal{O}_{4,\mathrm{int}}\mathcal{O}_{4,\mathrm{int}}\mathcal{O}_{4,\mathrm{int}}\rangle_{\mathrm{ABJM}_{3,1}\ \mathrm{int}} \\
\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_4\mathcal{O}_4\rangle_{\mathrm{BLG}_3} &= \frac{1}{64} \langle \mathcal{O}_{2,\mathrm{int}}\mathcal{O}_{2,\mathrm{int}}\mathcal{O}_{4,\mathrm{int}}\mathcal{O}_{4,\mathrm{int}}\rangle_{\mathrm{ABJM}_{3,1}\ \mathrm{int}} \\
\langle \mathcal{O}_2\mathcal{O}_4\mathcal{O}_2\mathcal{O}_4\rangle_{\mathrm{BLG}_3} &= \frac{1}{64} \langle \mathcal{O}_{2,\mathrm{int}}\mathcal{O}_{4,\mathrm{int}}\mathcal{O}_{2,\mathrm{int}}\mathcal{O}_{4,\mathrm{int}}\rangle_{\mathrm{ABJM}_{3,1}\ \mathrm{int}}
\end{aligned} \tag{5.95}$$

We can normalize the above four-point functions using (5.93) and (5.94). We find that the normalized four-point functions for BLG$_3$ match those from the interacting sector of $\mathrm{ABJM}_{3,1}$. It follows that all of the OPE coefficients in the mixed correlator must match as well.

5.7 Duality Conjecture

In the previous sections, we observed that the three-sphere partition function for $\mathrm{ABJM}_{3,1}$ matched that of $\mathbf{BLG}_3$. We also found that all of the normalized four-point functions for $\mathcal{O}_2$ and $\mathcal{O}_4$ in the BLG_3 topological theory exactly matched those of $\mathcal{O}_{2,\mathrm{int}}$ and $\mathcal{O}_{4,\mathrm{int}}$ for the $\mathrm{ABJM}_{3,1}$ topological theory. This implies that the spectrum of the OPE coefficients for the mixed correlator in the 3d theories is also identical. We therefore conjecture a new duality between the

$$\boxed{\text{Interacting sector of } U(3)_1 \times U(3)_{-1} \text{ ABJM and } (SU(2)_3 \times SU(2)_{-3})/\mathbb{Z}_2 \text{ BLG.}} \tag{5.96}$$

6 Conclusion

The beginning parts of this thesis consisted entirely of developing the necessary background to perform analytic work within the realm of superconformal field theory. In the first section, we introduced crucial machinery, such as the operator product expansion, which would later be used in determining the OPE coefficients of certain $\mathcal{N} = 8$ SCFTs.

In the second section, we discussed the superconformal algebra in three dimensions and its unitary irreducible representations. We then introduced the main superconformal multiplets under consideration, $(8, B, +)_{[0020]}$ and $(8, B, +)_{[0040]}$, and used character theory to derive their conformal contents. The latter parts of this section were dedicated to extending the work of [29, 30] through the $\mathcal{N} = 8$ three-dimensional mixed bootstrap. In particular, we began the set up for the mixed bootstrap by determining the relevant $SO(8)$ harmonics and working out the mixed crossing equations. From these, we determined the maximal set of independent equations. Future work is needed to implement the bootstrap to extract various non-perturbative bounds on the OPE coefficients and operator spectrum.

In the final section of this thesis, we focused the entirety of our efforts on analytically computing the OPE coefficients of the mixed correlator for the $(8, B, +)_{[0020]}$ and $(8, B, +)_{[0040]}$ superconformal primaries for certain $\mathcal{N} = 8$ theories. Applying the results from the supersymmetric localization technique developed in [44], we were able to calculate one-dimensional correlation functions for these theories analytically through repeated application of Wick's theorem.

As a warm-up exercise, we first computed the two- and four-point functions of certain Higgs branch operators in the topological sector of a $U(2)$ gauge theory containing a fundamental hypermultiplet and an adjoint hypermultiplet. From the correlation functions, we were able to extract information on the spectrum of OPE coefficients for both the free and interacting sectors of the theory. Our results for the single correlator of $(8, B, +)_{[0020]}$ matched those found using the same localization technique in [44]. Our results for the mixed correlator extended those of [44] and agreed with the OPE coefficients found through a different method in [30]. Although these quantities for the $U(2)$ theory were already known, our technique is easier to generalize to other theories, whereas the calculation in [30] relied on the absence of

$(8, B, 2)$ multiplets in the $U(2)$ theory (these multiplets are not expected to vanish for $N > 2$).

Motivated by the search for analytic results for other $\mathcal{N} = 8$ SCFTs, we then applied the localization technique to the more complicated case of a $U(3)$ gauge theory in the IR, again with a fundamental hypermultiplet and an adjoint hypermultiplet. After a series of tedious integrals, we were able to successfully compute the two- and four-point functions of the mixed correlator for the same Higgs branch operators. These functions yielded exact expressions for the OPE coefficients of the theory, which up to this point had not been known analytically nor numerically. The values of these OPE coefficients are well within the numerical bounds determined by the single-correlator bootstrap in [30].

Our method of computation for the $U(N)$ gauge theories encountered numerous difficulties that prevented us from generalizing our results to values of $N > 3$. First, calculating the four-point function $\langle \mathcal{O}_{k_1,\text{int}} \mathcal{O}_{k_2,\text{int}} \mathcal{O}_{k_3,\text{int}} \mathcal{O}_{k_4,\text{int}} \rangle_\sigma$ for the one-dimensional theory generally amounted to taking roughly N Wick contractions of around $2^{k_1 k_2 k_3 k_4}$ terms. Although this computation was optimized by taking various fortuitous cancellations into account, we were still left with an exponential number of operations. Since the combinatorics of Wick contractions make them fundamentally exponential, we do not see further optimization as an immediate solution to tackling larger values of N. Second, finding analytic expressions for the correlation functions of the full localized theory required the evaluation of complicated N-dimensional integrals. These calculations were performed using integration software with the standard symbolic multi-dimensional integration algorithms. An attempt was made to calculate them numerically, but it ultimately failed due to numerical instabilities arising from rapidly decaying integrands. Perhaps there are sophisticated pen-and-paper techniques which are more suited to doing these types of integrals.

Curiously, while calculating the various OPE coefficients, we noticed that the value of one of the OPE coefficients for the interacting sector of the $U(3)$ gauge theory was found to match that of $(SU(2)_3 \times SU(2)_{-3})/\mathbb{Z}_2$ BLG theory. The $U(3)$ gauge theory is believed to flow to the same CFT as the $U(3)_1 \times U(3)_{-1}$ ABJM theory, but there are no known results in the literature pertaining to the BLG theory. This led us to apply the the localization technique of [44] to a Chern-Simons-matter theory, namely the BLG theory. While the integrands of the BLG theory appeared to superficially differ from those of the $U(3)$ interacting sector, we proved that all of the normalized four-point functions and OPE coefficients for the two theories were an exact match. This led us to conjecture a previously unknown duality between the interacting sector of the ABJM theory and the BLG theory.

Recently, unpublished work in [67] has gathered much stronger evidence, proving that the moduli spaces and the superconformal indices are identical on both sides of the duality. Along with the matching of OPE coefficients done in this paper, there remains little reason to doubt the validity of the conjecture. In this realm, future work should attempt to address the question of whether there exists an explicit map between the localized versions of the two theories. If such a map exists, it would most likely be easiest to construct at the level of the matrix models, where the Gaussian degrees of freedom have been integrated out.

This work serves as a testament to the extraordinary analytic power offered by supersymmetric localization and, in particular, to the success of the specific technique developed in [44]. At present, it has only been applied to a few instances of $\mathcal{N} = 8$ SCFTs: $U(N)$ gauge theories for $N = 2, 3$, and now to a Chern-Simons-matter theory — the $(SU(2)_3 \times SU(2)_{-3})/\mathbb{Z}_2$ BLG theory. Since the main computational roadblocks are associated with Wick contractions and N-dimensional integrals, we foresee it being fairly straightforward to apply the methods of this thesis to theories with low-dimensional gauge groups, one example being the $(SU(2)_k \times SU(2)_{-k})/\mathbb{Z}_n$ BLG theory for general k and n. It would be especially interesting to investigate these theories along with $U(N_1)_1 \times U(N_2)_{-1}$ ABJ(M) theory for other values of N_1, N_2, with the potential of uncovering more dualities.[26]

Finally, the results of this thesis are promising with respect to the goal of classifying all conformal field theories. While there is no fundamental restriction against distinct theories sharing the same number of degrees of freedom, in the one case we investigated, it was determined that the two theories (ABJM, BLG) were indeed dual to one another. This gives hope that the mixed bootstrap will discover more unique theories and ultimately shed some light on the obscure nature of $\mathcal{N} = 8$ SCFTs.

[26]Dualities are unlikely for higher-dimensional gauge groups, since the number of degrees of freedom tends to infinity for the ABJM theories in the large-N limit, whereas they remain finite for the BLG theories.

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A $SO(8)$ Harmonics

We are primarily interested in the $T_{nm}^{(a,b)}$ that appear in the decomposition of the mixed correlator. The functions associated with $k_{12} = k_{34} = -2$ are given by

$$
\begin{aligned}
T_{11}^{(0,2)}(\sigma,\tau) &= 1, \\
T_{21}^{(0,2)}(\sigma,\tau) &= \sigma - \tau - \frac{1}{4}, \\
T_{22}^{(0,2)}(\sigma,\tau) &= \sigma + 3\tau - \frac{1}{2}, \\
T_{31}^{(0,2)}(\sigma,\tau) &= \sigma^2 - 2\sigma\tau - \frac{2\sigma}{3} + \tau^2 + \frac{1}{9}, \\
T_{32}^{(0,2)}(\sigma,\tau) &= \sigma^2 + 2\sigma\tau - \frac{26\sigma}{35} - 3\tau^2 + \frac{12\tau}{35} + \frac{3}{35}, \\
T_{33}^{(0,2)}(\sigma,\tau) &= \sigma^2 + 8\sigma\tau - \sigma + 6\tau^2 - 3\tau + \frac{3}{14}.
\end{aligned}
\tag{A.1}
$$

Those associated with $k_{12} = -k_{34} = 2$ are

$$
\begin{aligned}
T_{11}^{(2,0)}(\sigma,\tau) &= 1, \\
T_{21}^{(2,0)}(\sigma,\tau) &= -\sigma + \tau - \frac{1}{4}, \\
T_{22}^{(2,0)}(\sigma,\tau) &= 3\sigma + \tau - \frac{1}{2}, \\
T_{31}^{(2,0)}(\sigma,\tau) &= \sigma^2 - 2\sigma\tau - \frac{2\tau}{3} + \tau^2 + \frac{1}{9}, \\
T_{32}^{(2,0)}(\sigma,\tau) &= -3\sigma^2 + 2\sigma\tau + \frac{12\sigma}{35} + \tau^2 - \frac{26\tau}{35} + \frac{3}{35}, \\
T_{33}^{(2,0)}(\sigma,\tau) &= 6\sigma^2 + 8\sigma\tau - 3\sigma + \tau^2 - \tau + \frac{3}{14}\,.
\end{aligned}
\tag{A.2}
$$

Those associated with $k_i = 2$ or $k_i = 4$ are

$$
\begin{aligned}
T_{00}^{(0,0)}(\sigma,\tau) &= 1,\\
T_{10}^{(0,0)}(\sigma,\tau) &= \sigma - \tau,\\
T_{11}^{(0,0)}(\sigma,\tau) &= \sigma + \tau - \frac{1}{4},\\
T_{20}^{(0,0)}(\sigma,\tau) &= \sigma^2 + \tau^2 - 2\sigma\tau - \frac{1}{3}(\sigma+\tau) + \frac{1}{21},\\
T_{21}^{(0,0)}(\sigma,\tau) &= \sigma^2 - \tau^2 - \frac{2}{5}(\sigma-\tau),\\
T_{22}^{(0,0)}(\sigma,\tau) &= \sigma^2 + \tau^2 + 4\sigma\tau - \frac{2}{3}(\sigma+\tau) + \frac{1}{15},\\
T_{30}^{(0,0)}(\sigma,\tau) &= \sigma^3 - \tau^3 + 3(\sigma\tau^2 - \sigma^2\tau) + \frac{3}{4}(\tau^2-\sigma^2) + \frac{1}{6}(\sigma-\tau),\\
T_{31}^{(0,0)}(\sigma,\tau) &= \sigma^3 + \tau^3 - (\sigma^2\tau + \sigma\tau^2) - \frac{7}{9}(\tau^2+\sigma^2) - \frac{2}{9}\sigma\tau + \frac{13}{81}(\sigma+\tau) - \frac{1}{81},\\
T_{32}^{(0,0)}(\sigma,\tau) &= \sigma^3 - \tau^3 + 3(\sigma^2\tau - \sigma\tau^2) + \frac{6}{7}(\tau^2-\sigma^2) + \frac{1}{7}(\sigma-\tau),\\
T_{33}^{(0,0)}(\sigma,\tau) &= \sigma^3 + \tau^3 + 9(\sigma^2\tau + \sigma\tau^2) - \frac{9}{8}(\sigma^2+\tau^2) - \frac{9}{2}\sigma\tau + \frac{9}{28}(\sigma+\tau) - \frac{1}{56},\\
T_{40}^{(0,0)}(\sigma,\tau) &= \sigma^4 + \tau^4 - 4(\sigma^3\tau + \sigma\tau^3) + 6\sigma^2\tau^2 - \frac{6}{5}(\sigma^3+\tau^3) + \frac{6}{5}(\sigma^2\tau + \sigma\tau^2)\\
&\quad + \frac{51}{110}(\sigma^2+\tau^2) - \frac{18}{55}(\sigma\tau) - \frac{3}{55}(\sigma+\tau) + \frac{1}{330},\\
T_{41}^{(0,0)}(\sigma,\tau) &= \sigma^4 - \tau^4 + 2(\sigma\tau^3 - \sigma^3\tau) + \frac{17}{14}(\tau^3-\sigma^3)\\
&\quad + \frac{9}{14}(\sigma^2\tau - \sigma\tau^2) + \frac{36}{77}(\sigma^2-\tau^2) + \frac{9}{154}(\tau-\sigma),\\
T_{42}^{(0,0)}(\sigma,\tau) &= \sigma^4 + \tau^4 + 2(\sigma\tau^3 + \sigma^3\tau) - 6\sigma^2\tau^2 - \frac{5}{4}(\sigma^3+\tau^3) - \frac{3}{4}(\sigma^2\tau + \sigma\tau^2)\\
&\quad + \frac{21}{44}(\sigma^2+\tau^2) + \frac{9}{22}\sigma\tau - \frac{3}{44}(\sigma+\tau) + \frac{1}{308},\\
T_{43}^{(0,0)}(\sigma,\tau) &= \sigma^4 - \tau^4 + 8(\sigma^3\tau + \sigma\tau^3) + \frac{4}{3}(\tau^3-\sigma^3)\\
&\quad + 4(\sigma\tau^2 + \sigma^2\tau) + \frac{1}{2}(\sigma^2-\tau^2) + \frac{1}{21}(\tau-\sigma),\\
T_{44}^{(0,0)}(\sigma,\tau) &= \sigma^4 + \tau^4 + 16(\sigma^3\tau + \sigma\tau^3) + 36\sigma^2\tau^2 - \frac{8}{5}(\sigma^3+\tau^3) - \frac{72}{5}(\sigma\tau^2 + \sigma^2\tau)\\
&\quad + \frac{4}{5}(\sigma^2+\tau^2) + \frac{16}{5}\sigma\tau - \frac{2}{15}(\sigma+\tau) + \frac{1}{210}\,.
\end{aligned}
\tag{A.3}
$$

B Computations in the $U(N)$ Theory

B.1 Four-point Functions

We list some general four-point functions calculated for the $U(N)$ theories used in Sections 5.4 and 5.5. For convenience, we define

$$G_{N,\sigma}^{k_1,k_2,k_3,k_4} \equiv \ell^{\frac{1}{2}\sum_i k_i} \langle \overline{y}_1, \overline{y}_2 \rangle^{\frac{-k_1-k_2}{2}} \langle \overline{y}_3, \overline{y}_4 \rangle^{\frac{-k_3-k_4}{2}} \left(\frac{\langle \overline{y}_1, \overline{y}_4 \rangle}{\langle \overline{y}_2, \overline{y}_4 \rangle} \right)^{\frac{-k_{12}}{2}} \left(\frac{\langle \overline{y}_1, \overline{y}_3 \rangle}{\langle \overline{y}_1, \overline{y}_4 \rangle} \right)^{\frac{-k_{34}}{2}} \times \langle \mathcal{O}_{k_1,\mathrm{int}}(\varphi_1, \overline{y}_1) \mathcal{O}_{k_2,\mathrm{int}}(\varphi_2, \overline{y}_2) \mathcal{O}_{k_3,\mathrm{int}}(\varphi_3, \overline{y}_3) \mathcal{O}_{k_4,\mathrm{int}}(\varphi_4, \overline{y}_4) \rangle_\sigma . \tag{B.1}$$

The four-point functions for the interacting sector of the $U(2)$ theory at fixed σ are given by

$$\begin{aligned} G_{U(2),\sigma}^{2,2,2,2} &= \left(\frac{3}{2} - 6 \tanh^2(\pi\sigma_{12}) + 4 \tanh^4(\pi\sigma_{12} \right) \\ &+ \frac{1}{\overline{w}} \left(-\frac{3}{2} + 6 \tanh^2(\pi\sigma_{12}) - 4 \tanh^4(\pi\sigma_{12}) \right) \\ &+ \frac{1}{\overline{w}^2} \left(-\frac{15}{2} - 10 \tanh^2(\pi\sigma_{12}) + 4 \tanh^4(\pi\sigma_{12}) \right), \end{aligned} \tag{B.2}$$

$$\begin{aligned} G_{U(2),\sigma}^{4,4,4,4} = &\big(576 \tanh^8(\pi\sigma_{12}) - 1824 \tanh^6(\pi\sigma_{12}) + 2104 \tanh^4(\pi\sigma_{12}) \\ &- 1060 \tanh^2(\pi\sigma_{12}) + \frac{435}{2} \Big) + \frac{1}{\overline{w}} \big(-1152 \tanh^8(\pi\sigma_{12}) + 3648 \tanh^6(\pi\sigma_{12}) \\ &- 4208 \tanh^4(\pi\sigma_{12}) + 2120 \tanh^2(\pi\sigma_{12}) - 435\big) \\ + &\frac{1}{\overline{w}^2} \big(1728 \tanh^8(\pi\sigma_{12}) - 4704 \tanh^6(\pi\sigma_{12}) + 3112 \tanh^4(\pi\sigma_{12}) \\ &+ 1460 \tanh^2(\pi\sigma_{12}) - \frac{3975}{2} \Big) + \frac{1}{\overline{w}^3} \big(-1152 \tanh^8(\pi\sigma_{12}) + \\ &+ 2880 \tanh^6(\pi\sigma_{12}) - 1008 \tanh^4(\pi\sigma_{12}) - 2520 \tanh^2(\pi\sigma_{12}) + 2205\big) \\ + &\frac{1}{\overline{w}^4} \big(576 \tanh^8(\pi\sigma_{12}) - 2592 \tanh^6(\pi\sigma_{12}) \\ &+ 4536 \tanh^4(\pi\sigma_{12}) - 3780 \tanh^2(\pi\sigma_{12}) + \frac{2835}{2} \Big), \end{aligned} \tag{B.3}$$

$$
\begin{aligned}
G^{2,2,4,4}_{U(2),\sigma} = &-36\tanh^6(\pi\sigma_{12}) + 94\tanh^4(\pi\sigma_{12}) - \frac{155}{2}\tanh^2(\pi\sigma_{12}) + \frac{75}{4} \\
&+ \frac{1}{w}\big(48\tanh^6(\pi\sigma_{12}) - 168\tanh^4(\pi\sigma_{12}) + 210\tanh^2(\pi\sigma_{12}) - 105\big) \\
&+ \frac{1}{\overline{w}}\big(-48\tanh^6(\pi\sigma_{12}) + 200\tanh^4(\pi\sigma_{12}) - 290\tanh^2(\pi\sigma_{12}) + 165\big),
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
G^{2,4,2,4}_{U(2),\sigma} = &-48\tanh^6(\pi\sigma_{12}) + 104\tanh^4(\pi\sigma_{12}) - 50\tanh^2(\pi\sigma_{12}) - 15 \\
&+ \frac{1}{w}\big(48\tanh^6(\pi\sigma_{12}) - 104\tanh^4(\pi\sigma_{12}) + 50\tanh^2(\pi\sigma_{12}) + 15\big) \\
&+ \frac{1}{\overline{w}}\left(-36\tanh^6(\pi\sigma_{12}) + 126\tanh^4(\pi\sigma_{12}) - \frac{315}{2}\tanh^2(\pi\sigma_{12}) + \frac{315}{4}\right).
\end{aligned}
\tag{B.5}
$$

For the interacting sector of the $U(3)$ theory at fixed σ, the correlation functions of interest are

$$
\begin{aligned}
G^{2,2,2,2}_{3,\sigma} = &24 - 16\sum_{i<j}\tanh^2(\pi\sigma_{ij}) + 4\sum_{i<j}\sum_{k<l}\tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl}) \\
&+\frac{1}{w}\left(-24 + 16\sum_{i<j}\tanh^2(\pi\sigma_{ij}) - 4\sum_{i<j}\sum_{k<l}\tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl})\right) \\
&+\frac{1}{\overline{w}^2}\left(40 - 20\sum_{i<j}\tanh^2(\pi\sigma_{ij}) + 4\sum_{i<j}\sum_{k<l}\tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl})\right),
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
G_{3,\sigma}^{2,2,4,4} &= 400 - 380 \sum_{i<j} \tanh^2(\pi\sigma_{ij}) + 184 \sum_{i<j}\sum_{k<l} \tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl}) \\
&- 36 \sum_{i<j}\sum_{k<l}\sum_{m<n} \tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl})\tanh^2(\pi\sigma_{mn}) \\
&+\frac{1}{\overline{w}}\left(-960 + 720 \sum_{i<j} \tanh^2(\pi\sigma_{ij}) - 288 \sum_{i<j}\sum_{k<l} \tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl}) \right. \\
&\left. + 48 \sum_{i<j}\sum_{k<l}\sum_{m<n} \tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl})\tanh^2(\pi\sigma_{mn})\right) \\
&+\frac{1}{\overline{w}^2}\left(1280 - 880 \sum_{i<j} \tanh^2(\pi\sigma_{ij}) + 320 \sum_{i<j}\sum_{k<l} \tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl}) \right. \\
&\left. - 48 \sum_{i<j}\sum_{k<l}\sum_{m<n} \tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl})\tanh^2(\pi\sigma_{mn})\right),
\end{aligned}
\tag{B.7}
$$

$$
\begin{aligned}
G_{3,\sigma}^{2,4,2,4} &= 320 - 400 \sum_{i<j} \tanh^2(\pi\sigma_{ij}) + 224 \sum_{i<j}\sum_{k<l} \tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl}) \\
&- 48 \sum_{i<j}\sum_{k<l}\sum_{m<n} \tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl})\tanh^2(\pi\sigma_{mn}) \\
&+\frac{1}{\overline{w}}\left(-320 + 400 \sum_{i<j} \tanh^2(\pi\sigma_{ij}) - 224 \sum_{i<j}\sum_{k<l} \tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl}) \right. \\
&\left. + 48 \sum_{i<j}\sum_{k<l}\sum_{m<n} \tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl})\tanh^2(\pi\sigma_{mn})\right) \\
&+\frac{1}{\overline{w}^2}\left(720 - 540 \sum_{i<j} \tanh^2(\pi\sigma_{ij}) + 216 \sum_{i<j}\sum_{k<l} \tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl}) \right. \\
&\left. - 36 \sum_{i<j}\sum_{k<l}\sum_{m<n} \tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl})\tanh^2(\pi\sigma_{mn})\right),
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
G^{4,4,4,4}_{3,\sigma} = {} & 7360 - 9920 \sum_{i<j} \tanh^2(\pi\sigma_{ij}) + 7744 \sum_{i<j}\sum_{k<l} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \\
& - 3264 \sum_{i<j}\sum_{k<l}\sum_{m<n} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \tanh^2(\pi\sigma_{mn}) \\
& + 576 \sum_{i<j}\sum_{k<l}\sum_{m<n}\sum_{p<q} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \tanh^2(\pi\sigma_{mn}) \tanh^2(\pi\sigma_{pq}) \\
& + \frac{1}{\overline{w}} \Bigg(-14720 + 19840 \sum_{i<j} \tanh^2(\pi\sigma_{ij}) - 15488 \sum_{i<j}\sum_{k<l} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \\
& + 6528 \sum_{i<j}\sum_{k<l}\sum_{m<n} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \tanh^2(\pi\sigma_{mn}) \\
& - 1152 \sum_{i<j}\sum_{k<l}\sum_{m<n}\sum_{p<q} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \tanh^2(\pi\sigma_{mn}) \tanh^2(\pi\sigma_{pq}) \Bigg) \\
& + \frac{1}{\overline{w}^2} \Bigg(1600 - 15680 \sum_{i<j} \tanh^2(\pi\sigma_{ij}) + 18112 \sum_{i<j}\sum_{k<l} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \\
& - 9024 \sum_{i<j}\sum_{k<l}\sum_{m<n} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \tanh^2(\pi\sigma_{mn}) \\
& + 1728 \sum_{i<j}\sum_{k<l}\sum_{m<n}\sum_{p<q} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \tanh^2(\pi\sigma_{mn}) \tanh^2(\pi\sigma_{pq}) \Bigg) \\
& + \frac{1}{\overline{w}^3} \Bigg(5760 + 5760 \sum_{i<j} \tanh^2(\pi\sigma_{ij}) - 10368 \sum_{i<j}\sum_{k<l} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \\
& + 5760 \sum_{i<j}\sum_{k<l}\sum_{m<n} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \tanh^2(\pi\sigma_{mn}) \\
& - 1152 \sum_{i<j}\sum_{k<l}\sum_{m<n}\sum_{p<q} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \tanh^2(\pi\sigma_{mn}) \tanh^2(\pi\sigma_{pq}) \Bigg) \\
& + \frac{1}{\overline{w}^4} \Bigg(20160 - 20160 \sum_{i<j} \tanh^2(\pi\sigma_{ij}) + 12096 \sum_{i<j}\sum_{k<l} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \\
& - 4032 \sum_{i<j}\sum_{k<l}\sum_{m<n} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \tanh^2(\pi\sigma_{mn}) \\
& + 576 \sum_{i<j}\sum_{k<l}\sum_{m<n}\sum_{p<q} \tanh^2(\pi\sigma_{ij}) \tanh^2(\pi\sigma_{kl}) \tanh^2(\pi\sigma_{mn}) \tanh^2(\pi\sigma_{pq}) \Bigg) .
\end{aligned}
\tag{B.9}
$$

B.2 Useful Integrals

For convenience, we list some of the integrals computed in this paper. For the $U(2)$ theory in Section 5.4, we used:

$$\begin{aligned}
\frac{1}{Z_{\mathrm{ABJM}_{2,1}}}\int \mathrm{d}^2\sigma Z_{2,\sigma}\tanh^2(\pi\sigma_{12}) &= \frac{5}{6},\\
\frac{1}{Z_{\mathrm{ABJM}_{2,1}}}\int \mathrm{d}^2\sigma Z_{2,\sigma}\tanh^4(\pi\sigma_{12}) &= \frac{89}{120},\\
\frac{1}{Z_{\mathrm{ABJM}_{2,1}}}\int \mathrm{d}^2\sigma Z_{2,\sigma}\tanh^6(\pi\sigma_{12}) &= \frac{381}{560},\\
\frac{1}{Z_{\mathrm{ABJM}_{2,1}}}\int \mathrm{d}^2\sigma Z_{2,\sigma}\tanh^8(\pi\sigma_{12}) &= \frac{25609}{40320}.
\end{aligned} \tag{B.10}$$

For the $U(3)$ in Section 5.5, we used the integrals ($\sigma_{ij} \neq \sigma_{kl} \neq \sigma_{mn}$):

$$\begin{aligned}
\frac{1}{Z_{\mathrm{ABJM}_{3,1}}}\int \mathrm{d}^3\sigma Z_{3,\sigma}\tanh^2(\pi\sigma_{ij}) &= \frac{7-2\pi}{6(\pi-3)},\\
\frac{1}{Z_{\mathrm{ABJM}_{3,1}}}\int \mathrm{d}^3\sigma Z_{3,\sigma}\tanh^4(\pi\sigma_{ij}) &= \frac{133-30\pi}{360(\pi-3)},\\
\frac{1}{Z_{\mathrm{ABJM}_{3,1}}}\int \mathrm{d}^3\sigma Z_{3,\sigma}\tanh^6(\pi\sigma_{ij}) &= \frac{548\pi^2-4565}{2689\pi(\pi-3)},\\
\frac{1}{Z_{\mathrm{ABJM}_{3,1}}}\int \mathrm{d}^3\sigma Z_{3,\sigma}\tanh^8(\pi\sigma_{ij}) &= \frac{473-70\pi}{2688(\pi-3)},\\
\frac{1}{Z_{\mathrm{ABJM}_{3,1}}}\int \mathrm{d}^3\sigma Z_{3,\sigma}\tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl}) &= \frac{75\pi-232}{\pi-3},\\
\frac{1}{Z_{\mathrm{ABJM}_{3,1}}}\int \mathrm{d}^3\sigma Z_{3,\sigma}\tanh^4(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl}) &= \frac{1526-465\pi}{720(\pi-3)},\\
\frac{1}{Z_{\mathrm{ABJM}_{3,1}}}\int \mathrm{d}^3\sigma Z_{3,\sigma}\tanh^6(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl}) &= \frac{5794-1575\pi}{5040(\pi-3))},\\
\frac{1}{Z_{\mathrm{ABJM}_{3,1}}}\int \mathrm{d}^3\sigma Z_{3,\sigma}\tanh^4(\pi\sigma_{ij})\tanh^4(\pi\sigma_{kl}) &= \frac{-9488+3045\pi}{960(\pi-3))},\\
\frac{1}{Z_{\mathrm{ABJM}_{3,1}}}\int \mathrm{d}^3\sigma Z_{3,\sigma}\tanh^2(\pi\sigma_{ij})\tanh^2(\pi\sigma_{kl})\tanh^2(\pi\sigma_{mn}) &= \frac{63-20\pi}{2(\pi-3))},\\
\frac{1}{Z_{\mathrm{ABJM}_{3,1}}}\int \mathrm{d}^3\sigma Z_{3,\sigma}\tanh^4(\sigma_{ij})\tanh^2(\pi\sigma_{kl})\tanh^2(\pi\sigma_{mn}) &= \frac{-4829+1540\pi}{120(\pi-3))}.
\end{aligned} \tag{B.11}$$