Asymmetric butterfly velocities in local time-independent Hamiltonians

The butterfly velocity v_B is the velocity at which initially local operators spread. In many 1-D systems this velocity is independent of the direction of spreading. This need not be the case. In fact, with arbitrarily nonlocal Hamiltonians, or arbitrarily deep circuit models, the ratio of the two butterfly velocities may be made arbitrarily large. We provide a class of circuits whose limiting behavior shows this arbitrarily large ratio. We also describe a local Hamiltonian with an asymmetric v_B, presenting various methods to measure the asymmetry.

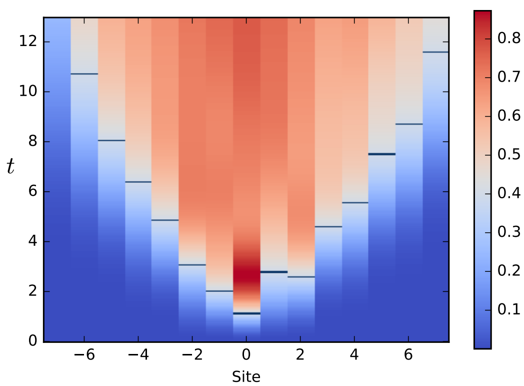

FIG. 1. Illustration of the initially local operator. The bars indicate the time at which the OTOC passes 0.4, to emphasize the asymmetry.

INTRODUCTION

Thermalization is important because...

Asymmetric transport is seen in "staircase" and "glider" circuits, but we wanted to show it is also possible in time-independent Hamiltonians.

In a 1-D circuit, how asymmetric can the spreading be?

On the edge of a 2-D system, spreading can be chiral even with a finite circuit depth [**?**].

To be completely chiral with only 1 dimension, the circuit will have to be of infinite depth.

Given a constraint on the depth, how asymmetric can the spreading be?

In this paper we will start by discussing a local Hamiltonian with asymmetric spreading. We show that it is a general Hamiltonian, and provide multiple methods for measuring v_B for left and right spreading. We then discuss staircase circuits in the small- and large-staircase limit and show that in the latter limit the circuit is completely chiral.

CONTENTS

LOCAL HAMILTONIANS

In order to define a local Hamiltonian with asymmetric spreading, we have to move away from 2-site interactions because these will have to be symmetric. The space of 3-site Hamiltonians is large ($q^6 = 64$) so we restrict to SU(2)-symmetric terms.

This space is still large Does it matter how large?, but we know we want Hamiltonians that are different in opposite directions. If we restrict further to Hamiltonians antisymmetric under inversion of the spin chain, we are left with only one option, the triple product of spins. The Hamiltonian on the full chain is then

$$H = \sum_{i=1}^{L-2} \mathbf{S}_i \cdot (\mathbf{S}_{i+1} \times \mathbf{S}_{i+2})$$

Degeneracy and Generality

As is, the model is not general, presenting a large degeneracy at $E = 0$. This can be traced to the inversion antisymmetry, along with other related antisymmetries in the model [**?**]. Should we go in-depth to show that various parts of the antisymmetry can be broken, etc? We can fully break this degeneracy within each U(1) block by introducing a random field in the Z direction, so the total Hamiltonians is

$$H = \sum_{i=1}^{L-2} \mathbf{S}_i \cdot (\mathbf{S}_{i+1} \times \mathbf{S}_{i+2}) + \sum_{i=1}^{L} h_i S_i^z, \qquad (1)$$

where each h_i has a uniform probability distribution on $[-h, h]$. This field breaks the SU(2) symmetry but leaves the U(1) subgroup intact.

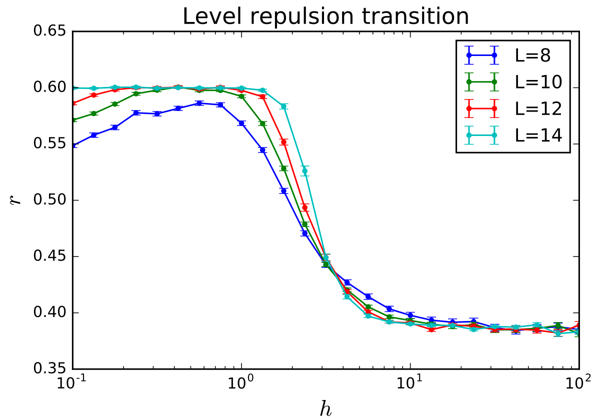

FIG. 2. Phase transition for the model, with level repulsion parameter plotted against field strength. Note that in the thermalizing phase the ratio is 0.6 instead of 0.53 because the statistics are GUE instead of GOE. I think I remember Vedika saying this but I can't find where.

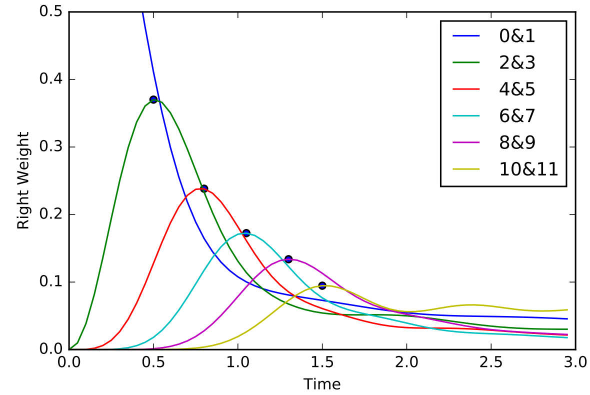

FIG. 3. Right weight at even sites for $L = 13$. The peak travels ballistically. Later peaks are smaller Is this due to broadening?

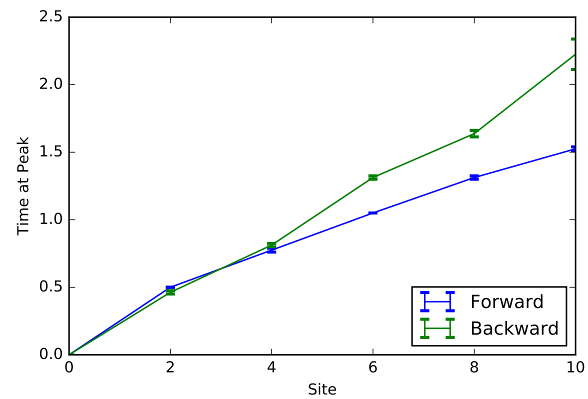

FIG. 4. Time of peak vs. site. Since this is plot of time as a function of distance, the larger slope in the left weight means that v_B is larger for propagation to the right. This has the wrong normalization but I'm working on it.

For sufficiently large h the model becomes localized. In the large-L limit the transition from ergodic to localized is a phase transition, described in [**?**]. The transition for the present model can be seen in Fig. 2, showing the ratio of adjacent energy gaps. Note that at smaller L the model also drifts away from GUE statistics at very small h, when the field is no longer large enough to fully lift the $E = 0$ degeneracy.

We will measure the asymmetry using two metrics. The first is the weight of all operators with right (left) endpoint on site i, which we will call the right (left) weight. The other is the OTOC. should we define these in this section? Make sure to point out use of initial operators as being on site 0 or $L - 1$.

In the thermalizing, generic phase, the right weights peak as the information front passes. Because of the three-site nature of each term in the Hamiltonian, the right weight and OTOC exhibit an "odd-even" effect. It is possible to account for these by averaging judiciously, or by only looking at even (or odd) sites. At $L = 13$, there are enough even sites that the asymmetry can be seen. For a picture of the rights weights with their successive peaks, see Fig. 3.

Fig. 4 shows the peaks traveling ballistically. The peaks reach equivalent sites at later times for the left-moving wave, implying $v_{B,l} < v_{B,r}$. We can extract $v_{B,l}$ and $v_{B,r}$ from these curves by fitting linear functions to the peak timings.

It is also possible to extract butterfly velocities from the the velocity-dependent Lyapunov exponents. The VDLEs quantify how fast signals decay along constant-velocity trajectories outside the lightcone. In particular, if the OTOC is measured at each site i at time $t_i = i/v$ for some v, then it should decay exponentially,

$$C(i, t) \sim e^{\lambda(v)t} \quad \text{for} \quad i = vt. \tag{2}$$

Ref. [**?**] gives a thorough explanation of VDLEs. The name comes from the fact that the Lyapunov exponent defines how fast a signal grows inside a lightcone in a classically chaotic system.

In the current system, the OTOCs are influenced by the previously-mentioned odd-even effects. We can once again look only at even sites for sufficiently large L to calculate $\lambda(v)$. Then v_B is the point at which $\lambda(v)$ smoothly goes to 0.

Fig. 5 shows the VDLEs for the right-going and left-going OTOCs. Finite-size effects slightly perturb $\lambda(v)$ around v_B, but we can see that $v_{B,l} \sim 0.7$ and $v_{B,r} \sim 0.85$.

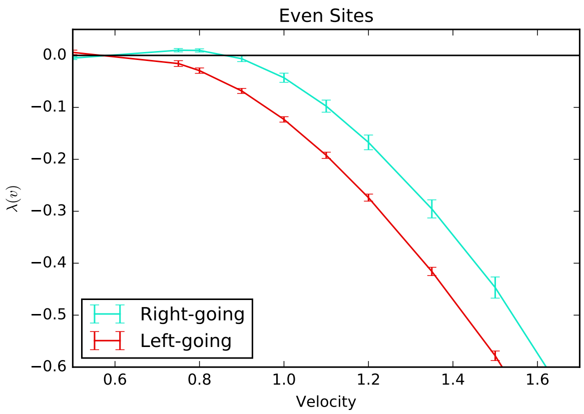

FIG. 5. Velocity-dependent Lyapunov exponents extracted from the OTOC on even sites. Since $\lambda_r(v) > \lambda_l(v)$, we know $v_{B,r} > v_{B,l}$.

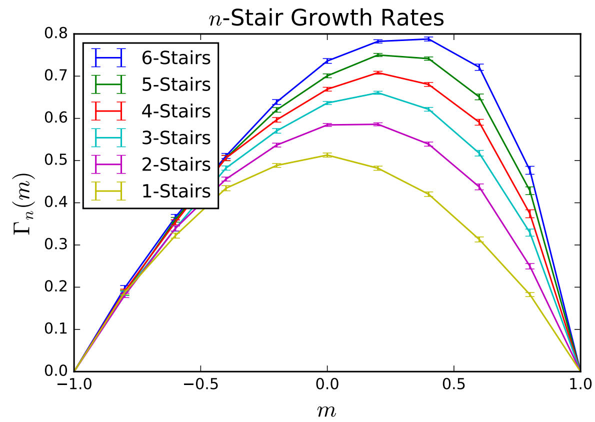

FIG. 6. Empirical growth rate as a function of slope for n-stair circuits. The right/forward and left/backward butterfly velocities are the slopes of these curves at their endpoint, indicating that as the left v_B stays constant, the right v_B increases. The appendix includes an argument that the right v_B is unbounded in the large-n limit.

CIRCUIT MODELS WITH ASYMMETRIC v_B

I'm going to introduce staircase circuits first and then the $\Gamma(s)$ formalism, but I don't know if I should do it the other way around.

Consider a spin chain of N sites, each with dimension q. Sites are labeled by $i = 1, \ldots, N$, while the bonds between sites are labeled by $x = 1, \ldots, N1$. Define the entropy function $S(x)$ as the bipartite entanglement entropy across bond x. Subadditivity tells us $|S(x+1) - S(x)| \leq S_1$, where S_1 is the entropy at a single site. If we take our logarithms with base q, then $S_1 \leq 1$.

If a gate acts on bond x, it can increase the bipartite entanglement entropy $S(x)$, up to the constraint $|S(x+1) - S(x)| \leq 1$. In the large-q limit, a Haar-randomly chosen gate will, with probability 1, maximally increase the entanglement across the bond it acts on [70]. Given the previous constraint, this means that if a gate acts at bond x at time t, then $S(x, t+1) = \min\{S(x-1, t) + 1, S(x+1, t) + 1\}$. Should we explain why? It suffices to consider integer-valued $S(x)$ with $|S(x) - S(x-1)| = 1$ for all x.

Staircase circuits of length n are defined by always having strings of n gates act on sites x through $x + n - 1$ in succession. For $n = 1$ this is just a random architecture, but large n results in more asymmetric circuits. Would a figure of the staircase circuit, as in Fig. 38 or 39 in my thesis?

After course-graining, the entanglement becomes a continuous function $S(x, t)$ with maximal slope 1. Given a circuit architecture, the entanglement growth rate is to first order only a function of the slope, so we can write [59]

$$\frac{\partial S}{\partial t} = \Gamma\left(\frac{\partial S}{\partial x}\right). \tag{3}$$

It is useful to define the entropy density $s = \partial S/\partial x$, which is so-called because the equilibrium entropy is $S(x, t) = s_{\mathrm{eq}} \min\{x, L - x\}$. In our models $s_{\mathrm{eq}} = 1$.

This function encodes the butterfly velocity as the derivative $\Gamma'(\partial S/\partial x)|_{s_{\mathrm{ext}}}$, where s_{ext} is either extremal entropy density. Should we say why? It follows that any symmetry $\Gamma(s)$ will have symmetric butterfly velocities. Then any circuit with asymmetric $\Gamma(s)$ will have asymmetric butterfly velocities.

For small n, we can simulate the circuit directly. For the growth rate curves of n-stair circuits for $n \leq 6$ see Fig. 6.

For large n, approaching the size of the system, we can approximate the entanglement curve as being uncorrelated. In that limit, the growth rate is $\Gamma(ds/dt) = ds/dt + 1$, so that for spreading to the left $v_B = 1$ and for spreading to the right $v_B = \infty$. This is of course maximally asymmetric.

CONCLUSION

Advantages of this model: Time-independent Hamiltonian. Only ingredients are chains of two-level systems.

Further work: How do the velocities depend on h? What happens at the phase transition? Maximally asymmetric three-site Hamiltonians? 2-D systems?

ACKNOWLEDGEMENTS

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Note somewhere about arXiv:1809.02614v1

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[69] The expression on the left-hand side of (**??**) becomes a "partition function" for two paths. The local weights $\partial u(\mathbf{y}_{i+1}, i+1)/\partial u(\mathbf{y}_i, i)$ depend not only on $\mathbf{y}_{i+1}$ and $\mathbf{y}_i$ but also on the configuration $u(\mathbf{y_i}, i)$. The chaotic time-dependence of $u(\mathbf{y}_i, i)$ means that the configurational average has a similar effect to averaging over weakly correlated randomness in the weights. Since we are averaging the "partition function", rather than its logarithm, this

is an annealed average, and $-\lambda(\mathbf{v})t$ is an annealed "free energy" for the pair of paths. The quenched free energy, in which we take the logarithm before averaging, would give the more conventional definition of the Lyapunov exponent [66–68].

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[72] In random circuits related random walk pictures underlie the calculation of both the OTOC and the second Renyi entropy [43, 44]. In these random systems this yields a relation between $\lambda(v)$ and the "entanglement line tension" defined in [59], specifically the line tension $\mathcal{E}_2(v)$ for the second Renyi entropy. This motivates the conjecture, for non-random systems, that $\lambda(v)|_{\rm cont} = -s_{\rm eq}(\mathcal{E}_2(v) - v)$, where $s_{\rm eq}$ is the thermal entropy density. The left hand side denotes the analytic continuation of $\lambda(v)$ from $v > v_B$ to values $v < v_B$. In random circuits we must distinguish different kinds of averages. The line tension extracted from a calculation of $\overline{e^{-S_2}}$ determines $\lambda(v)$ for the average OTOC $\overline{C(x,t)}$ by the above formula. It is natural to expect that the line tension determined by the more natural direct average $\overline{S_2}$ determines $\lambda(v)$ for the typical value of the OTOC, $\exp \overline{\ln C(x,t)}$. The average and typical values of the OTOC are parametrically close in the region close to the front, but they may differ significantly in the far-front regime where both are exponentially small.

[73] In some circuit models in $d > 1$ (which do not have continuous spatial rotation symmetry) some sections of the operator's front can be "glued" to the strict lightcone defined by the discrete time circuit [43]. This is a peculiar case where $v_B(\hat{\mathbf{n}}) = v_{\rm LC}(\hat{\mathbf{n}})$ for some directions $\hat{\mathbf{n}}$ in space, so that no nontrivial $\lambda(\mathbf{v})$ can be defined for these directions of $\mathbf{v}$.

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[82] Let the probability distribution for weak-link "waiting times" be $P(\tau) \sim \tau^{-a-2}$. At weak disorder ($1 < a$) the broadening of the operator's front [50] is diffusive, as in the clean system. At intermediate disorder ($0 < a < 1$) the front broadens more strongly, giving $\lambda(v) \sim -(v - v_B)^{(a+1)/a}$. For strong disorder ($-1 < a < 0$) the butterfly speed vanishes: in this regime $\lambda(v) \sim -|v|^{1-|a|}$. In the disordered system the definition of $\lambda(v)$ depends on whether we consider e.g. the mean or the typical value of the OTOC, but this should not change these exponents.

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