Asymmetric butterfly velocities in local time-independent Hamiltonians

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The butterfly velocity v_B is the velocity at which initially local operators spread. In many 1-D systems this velocity is independent of the direction of spreading. This need not be the case. In fact, with arbitrarily nonlocal Hamiltonians, or arbitrarily deep circuit models, the ratio of the two butterfly velocities may be made arbitrarily large. We provide a class of circuits whose limiting behavior shows this arbitrarily large ratio. We also describe a local Hamiltonian with an asymmetric v_B, presenting various methods to measure the asymmetry.

INTRODUCTION

Thermalization is important because...

In this paper we will...

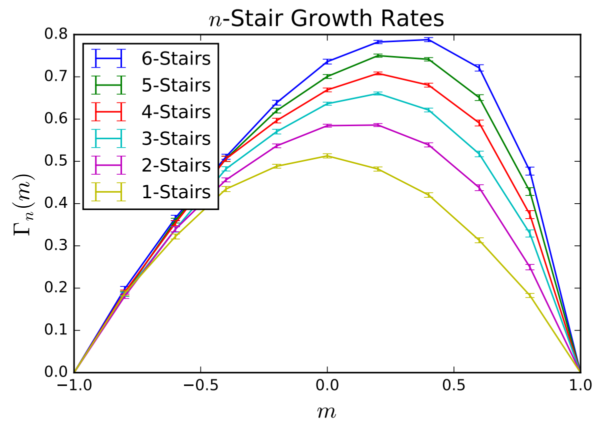

FIG. 1. Empirical growth rate as a function of slope for n-stair circuits. The right/forward and left/backward butterfly velocities are the slopes of these curves at their endpoint, indicating that as the left v_B stays constant, the right v_B increases. The appendix includes an argument that the right v_B is unbounded in the large-n limit.

CIRCUIT MODELS WITH ASYMMETRIC v_B

In a 1-D circuit, how asymmetric can the spreading be?

On the edge of a 2-D system, spreading can be chiral even with a finite circuit depth [**?**].

To be completely chiral with only 1 dimension, the circuit will have to be of infinite depth.

Given a constraint on the depth, how asymmetric can the spreading be?

How much of the circuit model, with $\Gamma(ds/dx)$, etc. should we describe here?

For the growth rate curves of n-stair circuits for $n \leq 6$ see Fig. 1.

For small n, we can simulate the circuit directly. This is particularly easy in the large q limit, where q is the dimension of the Hilbert space at each site. Again, how in-depth should this section be?.

For large n, approaching the size of the system, we

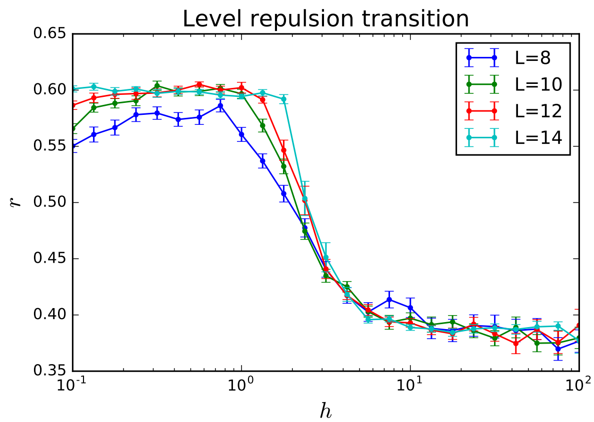

FIG. 2. Phase transition for the model, with level repulsion parameter plotted against field strength. Note that in the thermalizing phase the ratio is 0.6 instead of 0.53 because the statistics are GUE instead of GOE. I think I remember Vedika saying this but I can't find where.

can approximate the entanglement curve as being uncorrelated. In that limit, the growth rate is $\Gamma(ds/dt) = ds/dt + 1$, so that for spreading to the left $v_B = 1$ and for spreading to the right $v_B = \infty$.

LOCAL HAMILTONIANS

Motivate triple product: Has to be asymmetric-can't have 2-site interactions. Impose SU(2) as a constraint? Then our only option is the triple product.

Alone, this model is not very general. Large degeneracy at $E = 0$. Show parts of this degeneracy can be broken with various fields. A random Z field breaks all the degeneracy. This phase change can be seen in Fig. 2.

We'll want to cite a bunch of people [18, 19, 21–24, 26–46].

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[64] Here $\tilde{v}_B(\hat{\mathbf{n}})$, with a tilde, denotes the normal propagation speed of a straight front whose normal is parallel to $\hat{\mathbf{n}}$. In Ref. [43] this was denoted $v_B(\hat{\mathbf{n}})$, but here we use $v_B(\hat{\mathbf{n}})$ to denote the speed at which an initially local operator spreads away from the origin in the direction $\hat{\mathbf{n}}$. These differ because in the absence of rotational symmetry the operator's front is not in general perpendicular to the radial vector, but they are related by a geometrical construction known from classical droplet growth [43, 83, 84].

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[69] The expression on the left-hand side of (**??**) becomes a "partition function" for two paths. The local weights $\partial u(\mathbf{y}_{i+1}, i+1)/\partial u(\mathbf{y}_i, i)$ depend not only on $\mathbf{y}_{i+1}$ and $\mathbf{y}_i$ but also on the configuration $u(\mathbf{y_i}, i)$. The chaotic time-dependence of $u(\mathbf{y}_i, i)$ means that the configurational average has a similar effect to averaging over weakly correlated randomness in the weights. Since we are averaging the "partition function", rather than its logarithm, this is an annealed average, and $-\lambda(\mathbf{v})t$ is an annealed "free energy" for the pair of paths. The quenched free energy, in which we take the logarithm before averaging, would give the more conventional definition of the Lyapunov exponent [66–68].

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[71] Inside the light cone there is a large deviation form governing convergence to the saturation value: $C^{\mathrm{rc}}_{1d}(x,t) \sim 1 - \exp\left(-\frac{(v-v_B)^2}{2D}t\right)$. The exponent here is the continuation of $\lambda(v)$ outside the front. However, in the higher dimensional examples, the large deviation form inside the front scales with a distinct power of t, t^d in d spatial dimensions [74]. In the presence of additional conserved densities (like energy or charge), the late time saturation of the OTOC is a power-law in time instead of exponential [45, 46].

[72] In random circuits related random walk pictures underlie the calculation of both the OTOC and the second Renyi entropy [43, 44]. In these random systems this yields a relation between $\lambda(v)$ and the "entanglement line tension" defined in [59], specifically the line tension $\mathcal{E}_2(v)$ for the second Renyi entropy. This motivates the conjecture, for non-random systems, that $\lambda(v)|_{\mathrm{cont}} = -s_{\mathrm{eq}}(\mathcal{E}_2(v) - v)$, where s_{eq} is the thermal entropy density. The left hand side denotes the analytic continuation of $\lambda(v)$ from $v > v_B$ to values $v < v_B$. In random circuits we must distinguish different kinds of averages. The line tension extracted from a calculation of $\overline{e^{-S_2}}$ determines $\lambda(v)$ for the average OTOC $\overline{C(x,t)}$ by the above formula. It is natural to expect that the line tension determined by the more natural direct average $\overline{S_2}$ determines $\lambda(v)$ for the typical value of the OTOC, $\exp \overline{\ln C(x,t)}$. The average and typical values of the OTOC are parametrically close in the region close to the front, but they may differ significantly in the far-front regime where both are exponentially small.

[73] In some circuit models in $d > 1$ (which do not have continuous spatial rotation symmetry) some sections of the operator's front can be "glued" to the strict lightcone defined by the discrete time circuit [43]. This is a peculiar case where $v_B(\hat{\mathbf{n}}) = v_{\mathrm{LC}}(\hat{\mathbf{n}})$ for some directions $\hat{\mathbf{n}}$ in space, so that no nontrivial $\lambda(\mathbf{v})$ can be defined for these directions of $\mathbf{v}$.

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[82] Let the probability distribution for weak-link "waiting times" be $P(\tau) \sim \tau^{-a-2}$. At weak disorder ($1 < a$) the broadening of the operator's front [50] is diffusive, as in the clean system. At intermediate disor-

der ($0 < a < 1$) the front broadens more strongly, giving $\lambda(v) \sim -(v - v_B)^{(a+1)/a}$. For strong disorder ($-1 < a < 0$) the butterfly speed vanishes: in this regime $\lambda(v) \sim -|v|^{1-|a|}$. In the disordered system the definition of $\lambda(v)$ depends on whether we consider e.g. the mean or the typical value of the OTOC, but this should not change these exponents.

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