Asymmetric butterfly velocities in local time-independent Hamiltonians

The butterfly velocity v_B is the velocity at which initially local operators spread. In many 1-D systems this velocity is independent of the direction of spreading. This need not be the case. In fact, with arbitrarily nonlocal Hamiltonians, or arbitrarily deep circuit models, the ratio of the two butterfly velocities may be made arbitrarily large. We provide a class of circuits whose limiting behavior shows this arbitrarily large ratio. We also describe a local Hamiltonian with an asymmetric v_B, presenting various methods to measure the asymmetry.

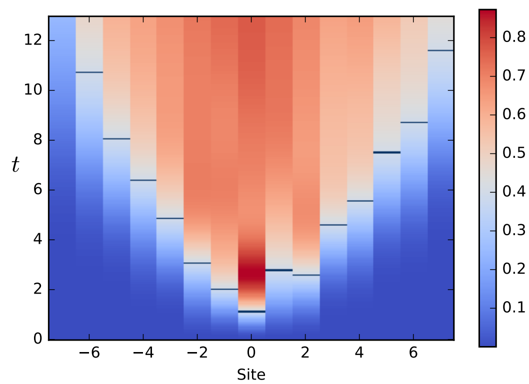

FIG. 1. Illustration of the initially local operator. The bars indicate the time at which the OTOC passes 0.4, to emphasize the asymmetry.

I. INTRODUCTION

Thermalization is important because...

Asymmetric transport is seen in "staircase" and "glider" circuits, but we wanted to show it is also possible in time-independent Hamiltonians.

In a 1-D circuit, how asymmetric can the spreading be?

On the edge of a 2-D system, spreading can be chiral even with a finite circuit depth [**?**].

To be completely chiral with only 1 dimension, the circuit will have to be of infinite depth.

Given a constraint on the depth, how asymmetric can the spreading be?

In this paper we will start by discussing a local Hamiltonian with asymmetric spreading. We show that it is a general Hamiltonian, and provide multiple methods for measuring v_B for left and right spreading. We then discuss staircase circuits in the small- and large-staircase limit and show that in the latter limit the circuit is completely chiral.

Throughout this paper we will use different methods to measure v_B. For the Hamiltonian system, we use two methods, both directly related to the spreading of operators. The first is non-local, measuring the velocity of the peak in the right- and left-weights. The other defines velocity-dependent Lyapunov exponents from the early-time OTOCs. For the circuit we extract v_B from the growth rate of the entanglement entropy.

CONTENTS

II. LOCAL HAMILTONIANS

In order to define a local Hamiltonian with asymmetric spreading, we have to move away from 2-site interactions because these will have to be symmetric. The space of 3-site Hamiltonians is large ($q^6 = 64$) so we restrict to SU(2)-symmetric terms.

This space is still large Does it matter how large?, but we know we want Hamiltonians that are different in opposite directions. If we restrict further to Hamiltonians antisymmetric under inversion of the spin chain, we are left with only one option, the triple product of spins. The Hamiltonian on the full chain is then

$$H = \sum_{i=1}^{L-2} \mathbf{S}_i \cdot (\mathbf{S}_{i+1} \times \mathbf{S}_{i+2})$$

A. Degeneracy and Generality

As is, the model is not general, presenting a large degeneracy at $E = 0$. This can be traced to the inversion antisymmetry, along with other related antisymmetries in the model [**?**]. Should we go in-depth to show that various parts of the antisymmetry can be broken,

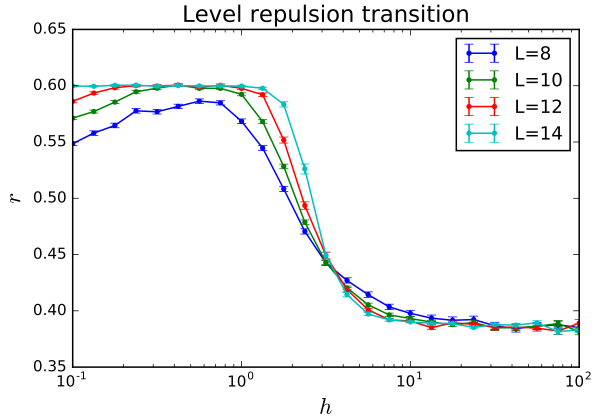

FIG. 2. Phase transition for the model, with level repulsion parameter plotted against field strength. Note that in the thermalizing phase the ratio is 0.6 instead of 0.53 because the statistics are GUE instead of GOE. I think I remember Vedika saying this but I can't find where.

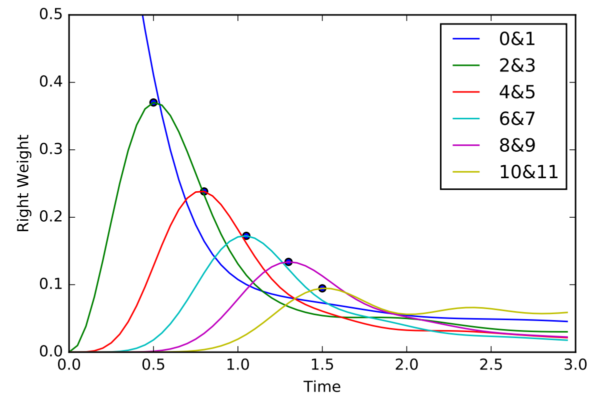

FIG. 3. Right weight at even sites for $L = 13$. The peak travels ballistically. Later peaks are smaller Is this due to broadening?

etc? We can fully break this degeneracy within each U(1) block by introducing a random field in the Z direction, so the total Hamiltonians is

$$H = \sum_{i=1}^{L-2} \mathbf{S}_i \cdot (\mathbf{S}_{i+1} \times \mathbf{S}_{i+2}) + \sum_{i=1}^{L} h_i S_i^z, \qquad (1)$$

where each h_i has a uniform probability distribution on $[-h, h]$. This field breaks the SU(2) symmetry but leaves the U(1) subgroup intact.

For sufficiently large h the model becomes localized. In the large-L limit the transition from ergodic to localized is a phase transition, described in [?]. The transition for the present model can be seen in Fig. 2, showing the ratio of adjacent energy gaps. Note that at smaller L the model also drifts away from GUE statistics at very small h, when the field is no longer large enough to fully lift the $E = 0$ degeneracy.

B. Right-weight peaks

We will measure the asymmetry using two metrics. The first is the weight of all operators with right (left) endpoint on site i, which we will call the right (left) weight. The other is the OTOC. should we define these in this section? Make sure to point out use of initial operators as being on site 0 or $L-1$.

We use the definition of the right weight from [?]. An arbitrary operator $\mathcal{O}$ can be decomposed into Pauli strings $\mathcal{O} = \sum_\nu c_\nu \sigma^\nu$ where each string contains one of $\{I, X, Y, Z\}$ acting on each site. As the operator evolves in time, so do the c_ν. The right weight is then

$$\rho_r(i,t) = \sum_\nu |c_\nu(t)|^2 \, \delta(\mathrm{RHS}(\nu) = i), \qquad (2)$$

where the delta function ensures that we only count Pauli strings that have their right-most non-identity operator on site i. The left weight $\rho_l(i,t)$ is defined analogously. If $\mathcal{O}$ is initially local on site j then $\rho_r(i,0) = \rho_l(i,0) = \delta_{ij}$. As the operator spreads, the support of ρ_r moves right at $v_{B,r}$ and ρ_L moves left at $v_{B,l}$. Operator broadening manifests itself in the support of both weights increasing in size. At late times both weights should vanish near j.

In the thermalizing phase, the right weights peak as the information front passes. Because of the three-site nature of each term in the Hamiltonian, the right weight and OTOC exhibit an "odd-even" effect. It is possible to account for these by averaging judiciously, or by only looking at even (or odd) sites. At $L = 13$, there are enough even sites that the asymmetry can be seen. For a picture of the rights weights with their successive peaks, see Fig. 3.

Fig. 4 shows the peaks traveling ballistically. The peaks reach equivalent sites at later times for the left-moving wave, implying $v_{B,l} < v_{B,r}$. We can extract $v_{B,l}$ and $v_{B,r}$ from these curves by fitting linear functions to the peak timings.

C. Velocity-dependent Lyapunov exponents

It is also possible to extract butterfly velocities from the the velocity-dependent Lyapunov exponents, which in turn rely on the OTOC. We define the OTOC as

$$\begin{aligned} C(i,t) &= \frac{1}{2}\langle |[Z_j(t), Z_i(0)]|^2 \rangle_{\beta=0} \\ &= 1 - \frac{1}{2^L} \mathrm{Re}\ \mathrm{Tr}\,[Z_j(t) Z_i(0) Z_j(t) Z_i(0)] \qquad (3) \end{aligned}$$

where j is the site of the initial operator and the expectation value in the top row is with respect to a thermal ensemble at infinite temperature. We set $j = 1$ to measure $v_{B,r}$ and $j = L$ to measure $v_{B,l}$. The OTOC should

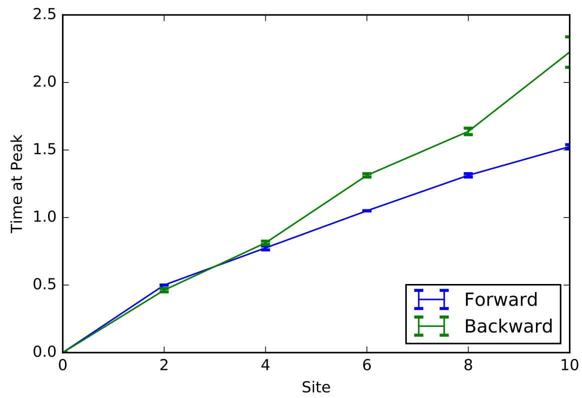

FIG. 4. Time of peak vs. site. Since this is plot of time as a function of distance, the larger slope in the left weight means that v_B is larger for propagation to the right. This has the wrong normalization but I'm working on it.

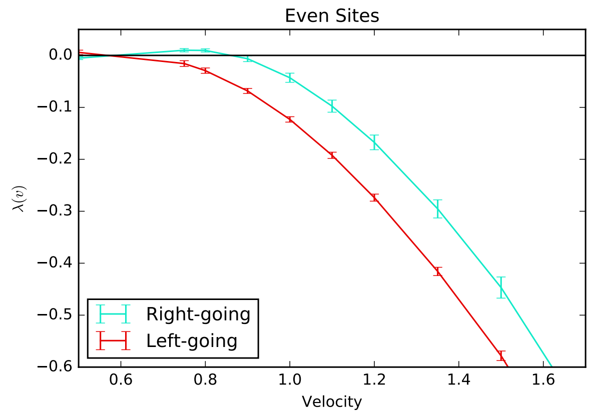

FIG. 5. Velocity-dependent Lyapunov exponents extracted from the OTOC on even sites. Since $\lambda_r(v) > \lambda_l(v)$, we know $v_{B,r} > v_{B,l}$.

be order-1 inside the lightcone and exponentially small outside the lightcone defined by v_B.

From conservation of S_z^{tot}, the Hamiltonian and all relevant operators are block-diagonal, with the size of the i^{th} block being $\binom{L}{i}$. For smaller blocks we can compute the trace directly, but for larger blocks this becomes computationally difficult. We then rely on quantum typicality to approximate the trace in the large blocks. For each disorder realization we replace the trace with an average over expectation values in pure states [**?**]. The pure states are chosen Haar-randomly, and we find that using 5 vectors gives relative errors around 0.05 for all but the smallest blocks.

The VDLEs quantify how fast signals decay along constant-velocity trajectories outside the lightcone. In particular, if the OTOC is measured along the ray defined by each site i at time $t_i = i/v$ for some v, then it should decay exponentially,

$$C(i,t) \sim e^{\lambda(v)t} \quad \text{for} \quad i = vt. \tag{4}$$

Ref. [**?**] gives a thorough explanation of VDLEs. The name comes from the fact that the Lyapunov exponent defines how fast a signal grows inside a lightcone in a classically chaotic system.

In the current system, the OTOCs are influenced by the previously-mentioned odd-even effects. We can once again look only at even sites for sufficiently large L to calculate $\lambda(v)$. Then v_B is the point at which $\lambda(v)$ smoothly goes to 0.

Fig. 5 shows the VDLEs for the right-going and left-going OTOCs. Finite-size effects slightly perturb $\lambda(v)$ around v_B, but we can see that $v_{B,l} \sim 0.7$ and $v_{B,r} \sim 0.85$.

III. CIRCUIT MODELS

I'm going to introduce the $\Gamma(s)$ formalism first and then staircase circuits, but I don't know if I should do it the other way around.

Before discussing asymmetric circuits we will explain how v_B can be extracted from the growth of entanglement. We will show that this method is particularly tractable in the large-q limit before applying this method to staircase circuits. We will see that the butterfly velocities can be made arbitrarily asymmetric by considering long staircases.

Consider a spin chain of N sites, each with dimension q. Sites are labeled by $i = 1, \dots, N$, while the bonds between sites are labeled by $x = 1, \dots, N1$. Define the entropy function $S(x)$ as the bipartite entanglement entropy across bond x.

After course-graining, the entanglement becomes a continuous function $S(x,t)$ with maximal slope 1. Given a circuit architecture, the entanglement growth rate is to first order only a function of the slope, so we can write [59]

$$\frac{\partial S}{\partial t} = \Gamma\left(\frac{\partial S}{\partial x}\right). \tag{5}$$

It is useful to define the entropy density $s = \partial S/\partial x$, which is so-called because the equilibrium entropy is $S(x,t) = s_{\mathrm{eq}} \min\{x, L-x\}$. In our models $s_{\mathrm{eq}} = 1$.

This function encodes the butterfly velocity as the derivative $\Gamma'(\partial S/\partial x)|_{s_{\mathrm{ext}}}$, where s_{ext} is either extremal entropy density. Should we say why? It follows that any symmetry $\Gamma(s)$ will have symmetric butterfly velocities. Then any circuit with asymmetric $\Gamma(s)$ will have asymmetric butterfly velocities.

A. Staircase circuits

Subadditivity tells us $|S(x+1) - S(x)| \leq S_1$, where S_1 is the entropy at a single site. If we take our logarithms

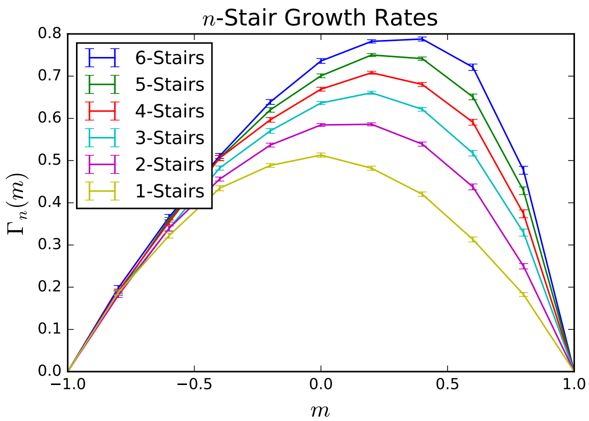

FIG. 6. Empirical growth rate as a function of slope for n-stair circuits. The right/forward and left/backward butterfly velocities are the slopes of these curves at their endpoint, indicating that as the left v_B stays constant, the right v_B increases. The appendix includes an argument that the right v_B is unbounded in the large-n limit.

with base q, then $S_1 \leq 1$.

If a gate acts on bond x, it can increase the bipartite entanglement entropy $S(x)$, up to the constraint $|S(x+1) - S(x)| \leq 1$. In the large-q limit, a Haar-randomly chosen gate will, with probability 1, maximally increase the entanglement across the bond it acts on [70]. Given the previous constraint, this means that if a gate acts at bond x at time t, then $S(x, t+1) = \min\{S(x-1, t) + 1, S(x+1, t) + 1\}$. Should we explain why? It suffices to consider integer-valued $S(x)$ with $|S(x) - S(x-1)| = 1$ for all x.

Staircase circuits of length n are defined by always having strings of n gates act on sites x through $x + n - 1$ in succession. For $n = 1$ this is just a random architecture, but large n results in more asymmetric circuits. Would a figure of the staircase circuit, as in Fig. 38 or 39 in my thesis?

B. Asymmetric v_B

For small n, we can simulate the circuit directly. For the growth rate curves of n-stair circuits for $n \leq 6$ see Fig. 6. It is possible to approximate these growth rates by assuming the up and down steps are uncorrelated. Then the probability that a randomly placed gate will result in growth is $(1 - s^2)/2$. Any asymmetry comes from the fact that for longer staircases, a gate acting on site x is more likely to have been preceded by a gate on site $x - 1$ so the step from $x - 1$ to x is more likely to be a down step.

Under these assumptions the growth rate is

$$\Gamma_n(s) = \frac{\gamma}{n} \frac{1+s}{1-s} \left((1+s) \left[\left(\frac{1+s}{2}^2 \right)^n - 1 \right] + n(1-s) \right). \tag{6}$$

This produces successively more asymmetric growth rates as n increases. However, the correlation in the true steady state, and therefore the error of this approximation, also increases. It is possible to correct for the correlation term-by-term in correlation length, but this quickly becomes tedious. For 2-stairs, including the nearest-neighbor correlation removes most of the error in $\Gamma(s)$.

Luckily, as n becomes very large or approaches the size of the system, the correlations again become unimportant. To see this we can consider the growth rate at $s = -1, 0$, and 1.

The growth rate approaches $\Gamma_\infty = \gamma(s+1)$, so that for spreading to the left $v_B = 1$ and for spreading to the right $v_B = \infty$. This is of course maximally asymmetric.

IV. CONCLUSION

Advantages of this model: Time-independent Hamiltonian. Only ingredients are chains of two-level systems.

Further work: How do the velocities depend on h? What happens at the phase transition? Maximally asymmetric three-site Hamiltonians? 2-D systems?

ACKNOWLEDGEMENTS

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Note somewhere about arXiv:1809.02614v1

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[69] The expression on the left-hand side of (**??**) becomes a "partition function" for two paths. The local weights $\partial u(\mathbf{y}_{i+1}, i+1)/\partial u(\mathbf{y}_i, i)$ depend not only on $\mathbf{y}_{i+1}$ and $\mathbf{y}_i$ but also on the configuration $u(\mathbf{y_i}, i)$. The chaotic time-dependence of $u(\mathbf{y}_i, i)$ means that the configurational average has a similar effect to averaging over weakly correlated randomness in the weights. Since we are averaging the "partition function", rather than its logarithm, this is an annealed average, and $-\lambda(\mathbf{v})t$ is an annealed "free energy" for the pair of paths. The quenched free energy, in which we take the logarithm before averaging, would give the more conventional definition of the Lyapunov exponent [66–68].

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[71] Inside the light cone there is a large deviation form governing convergence to the saturation value:

$C^{\rm rc}_{1d}(x,t) \sim 1 - \exp\left(-\frac{(v-v_B)^2}{2D}t\right)$. The exponent here is the continuation of $\lambda(v)$ outside the front. However, in the higher dimensional examples, the large deviation form inside the front scales with a distinct power of t, t^d in d spatial dimensions [74]. In the presence of additional conserved densities (like energy or charge), the late time saturation of the OTOC is a power-law in time instead of exponential [45, 46].

[72] In random circuits related random walk pictures underlie the calculation of both the OTOC and the second Renyi entropy [43, 44]. In these random systems this yields a relation between $\lambda(v)$ and the "entanglement line tension" defined in [59], specifically the line tension $\mathcal{E}_2(v)$ for the second Renyi entropy. This motivates the conjecture, for non-random systems, that $\lambda(v)|_{\rm cont} = -s_{\rm eq}(\mathcal{E}_2(v) - v)$, where $s_{\rm eq}$ is the thermal entropy density. The left hand side denotes the analytic continuation of $\lambda(v)$ from $v > v_B$ to values $v < v_B$. In random circuits we must distinguish different kinds of averages. The line tension extracted from a calculation of $\overline{e^{-S_2}}$ determines $\lambda(v)$ for the average OTOC $\overline{C(x,t)}$ by the above formula. It is natural to expect that the line tension determined by the more natural direct average $\overline{S_2}$ determines $\lambda(v)$ for the typical value of the OTOC, $\exp \overline{\ln C(x,t)}$. The average and typical values of the OTOC are parametrically close in the region close to the front, but they may differ significantly in the far-front regime where both are exponentially small.

[73] In some circuit models in $d > 1$ (which do not have continuous spatial rotation symmetry) some sections of the operator's front can be "glued" to the strict lightcone defined by the discrete time circuit [43]. This is a peculiar case where $v_B(\hat{\mathbf{n}}) = v_{\rm LC}(\hat{\mathbf{n}})$ for some directions $\hat{\mathbf{n}}$ in space, so that no nontrivial $\lambda(\mathbf{v})$ can be defined for these directions of $\mathbf{v}$.

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[82] Let the probability distribution for weak-link "waiting times" be $P(\tau) \sim \tau^{-a-2}$. At weak disorder ($1 < a$) the broadening of the operator's front [50] is diffusive, as in the clean system. At intermediate disorder ($0 < a < 1$) the front broadens more strongly, giving $\lambda(v) \sim -(v - v_B)^{(a+1)/a}$. For strong disorder ($-1 < a < 0$) the butterfly speed vanishes: in this regime $\lambda(v) \sim -|v|^{1-|a|}$. In the disordered system the definition of $\lambda(v)$ depends on whether we consider e.g. the mean or the typical value of the OTOC, but this should not change these exponents.

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