Thermalization in Hamiltonians and random unitary circuits

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Overview

Closed quantum systems

In quantum mechanics we considered closed quantum systems, but then we became sophisticated and considered open quantum systems.

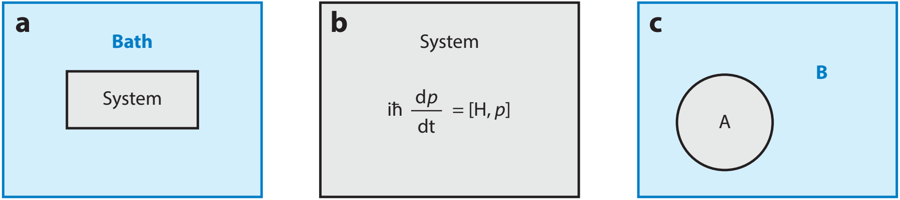


Figure: Closed and open quantum systems [Nandkishore, Huse, 2015]

Now we'll go back to studying closed quantum systems, with a twist. But then how can we still have thermalization? (Subsystem looks thermal at late time)

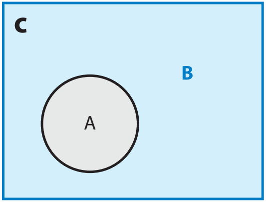
Thermalization in quantum systems

Contradiction? Unitarity vs. Information Loss

- Quantum mechanics
- Fluid dynamics
- Thermodynamics
- Black holes

Resolution: Information Spreading

- Information not destroyed but spread
- Not accessible to local measurements



Thermalization in quantum systems

System acts as a bath, but for what quantity?

- In thermodynamics, any (exchanged) conserved quantities
- In our systems, even energy may not be conserved
- Entropy is conserved $\rightarrow$ exchanging entanglement

Eigenstate thermalization hypothesis.

- Full-system eigenstates don't evolve
- But subsystem must look thermal at late time
- Then subsystem must look thermal at all time
- Applies to any system that thermalizes

Measures of thermalization

We will look at systems with discrete space, and either discrete or continuous time. We will be studying the spreading of operators, so we will use the Heisenberg picture,

$$\mathcal{O}_0(t) = e^{iHt}\mathcal{O}_0 e^{-iHt}$$

Furthermore, our operators will in general be initially local.

Teaser

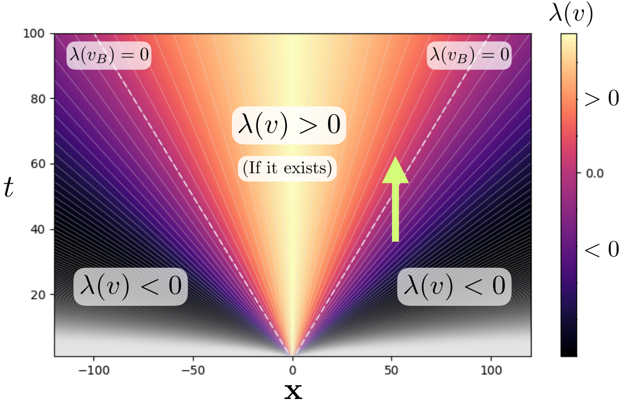


Figure: Initially local information, spreading [Khemani, Huse, Nahum, 2018].

Measures of thermalization

We want to measure the spreading of "information"
But how can we quantify this information and see how it spread?

- Operator right weight, $\rho_R(i,t)$
- Out-of-time-order commutator (OTOC), $C(i,t)$
- Bipartite entanglement entropy, $S(x,t)$

Operator right weight, $\rho_R(i,t)$

The Pauli matrices (X, Y, Z, I) form a basis for single site operators.
Pauli strings $(I_1 \otimes Z_2 \otimes X_3 \otimes I_4$, etc.$)$ form a basis for single site operators.
How many of these basis operators (weighted) end on site i?
If we write $\mathcal{O} = \sum c_j \mathcal{O}_j,\ j = 1, \ldots, q^{2L}$, then

$$\rho_R(i,t) = \sum_{\mathcal{O} \text{ ends on } i} |c_i|^2.$$

By unitarity, $\sum_i \rho_R(i,t) = 1$.
Not a density operator.

Operator right weight, $\rho_R(i,t)$

Example:

$$\begin{aligned}
\mathcal{O} &= \sqrt{\frac{1}{5}} IIXZI + \sqrt{\frac{2}{5}} IZXIII + \sqrt{\frac{2}{5}} XIIIZ, \\
\rho_R(1) &= \rho_R(2) = 0, \\
\rho_R(3) &= \frac{2}{5}, \\
\rho_R(4) &= \frac{1}{5}, \\
\rho_R(5) &= \frac{2}{5}.
\end{aligned}$$

In general, these will depend on time.

Out-of-time-order commutator

Another option, the OTOC:

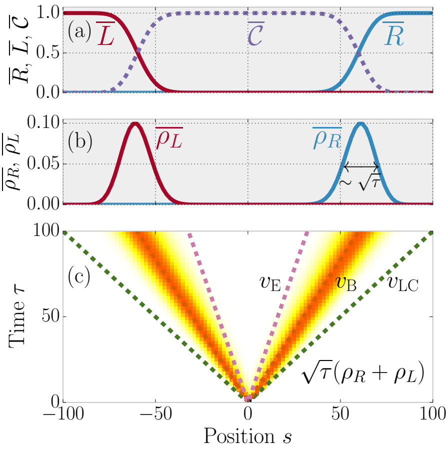
$$C(i,t) = \frac{1}{2}\left\langle |[\mathcal{O}_0(t), W_i]|^2 \right\rangle,$$

with $\mathcal{O}_0$ initially local, $W_i = I_1 \cdots I_{i-1} W_i I_{i+1} \cdots I_L$ and $W = X, Y$, or Z at site i.

Connection

Average the OTOC over choices of W. This then measures the weight of non-identity operators at site i.
OTOC saturates at late time (generically), while right weight has a traveling 'pulse.'

Connection



Compare the integrated right (and left) weights and the OTOC (properly scaled).

The right weight is a pulse that diffuses with width $\sim \sqrt{t}$.

Right and left weights spreading and diffusing in time [von Keyserlingk, Rakovszky, Pollmann, Sondhi, 2017]

Time-independent Hamiltonian

- Sites with finite Hilbert space, dimension q
- Not relativistic: no limit to information speed
- *Most* information still travels at finite speed: Lieb Robinson bounds [Lieb, Robinson, 1972]
- OTOC is exponentially small outside "lightcone"

Velocity-dependent Lyapunov exponent

For a set velocity, $C_W(i = vt, t) \sim e^{\lambda(v)t}$

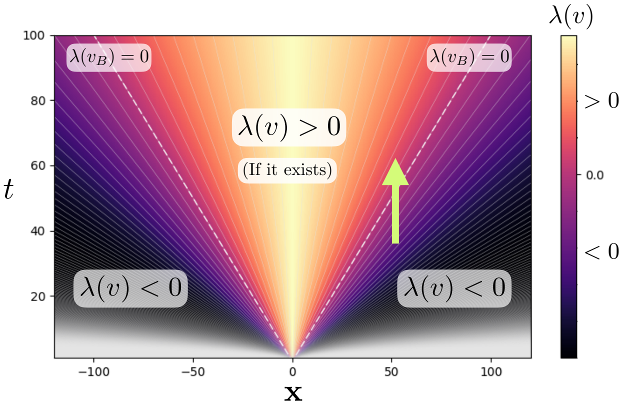


Figure: Lyapunov exponent at various velocities [Khemani, Huse, Nahum, 2018].

Velocity-dependent Lyapunov exponent

Behavior of VDLEs

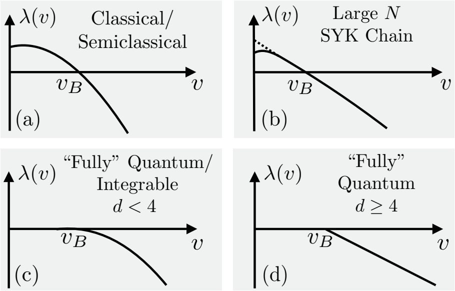


Figure: VDLEs for various systems, [Khemani, Huse, Nahum, 2018].

$\lambda(v) \sim -(v - v_B)^\alpha$, then ρ_R broadens as t^β, with $\beta = \frac{\alpha - 1}{\alpha}$.

Quantum circuits

- Chain of q-dimensional sites
- Apply unitary operators at discrete times
- Information speed is set by circuit architecture

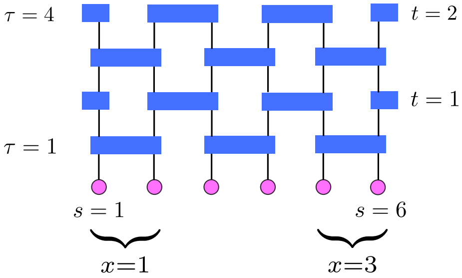


Figure: Quantum circuit, [von Keyserlingk, Rakovszky, Pollmann, Sondhi, 2017]

Operator hydrodynamics (circuit)

After one gate [Nahum, Vijay, Haah, 2018],

$$\begin{aligned}\rho(i,\tau+1) &= (1-p)\left[\rho(i,\tau)+\rho(i+1,\tau)\right],\\ \rho(i+1,\tau+1) &= p\left[\rho(i,\tau)+\rho(i+1,\tau)\right],\end{aligned}$$

where p is the probability of moving forwards,

$$p = 1 - p_{\mathsf{back}} = 1 - \frac{q^2-1}{q^4-1} = \frac{q^2}{q^2+1}.$$

The problem is that gates don't act everywhere at every time step.

Operator hydrodynamics (circuit)

After two applications (and some new variables),

$$\rho(x, t+1) = p^2 \rho(x-1, t) + 2p(1-p)\rho(x, t) + (1-p)^2 \rho(x+1, t).$$

This is a biased random walk with

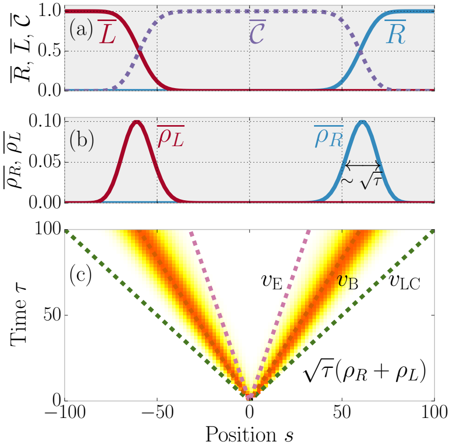
$$v_B(q) = p^2 - (1-p)^2 = \frac{q^2+1}{q^2-1}, \quad D(q) = \frac{q/2}{q^2+1},$$

where

$$\langle x \rangle - x_0 = v_B t, \quad \left\langle x^2 \right\rangle - \langle x \rangle^2 = 2Dt.$$

Exact limit as $q \to \infty$.

Recall

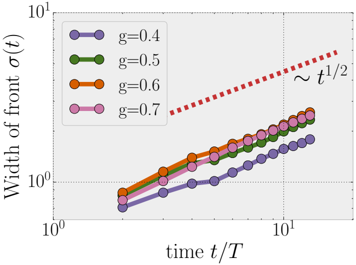


(a)
$\overline{L}$
$\overline{\mathcal{C}}$
$\overline{R}$
$\overline{R}, \overline{L}, \overline{\mathcal{C}}$
1.0
0.5
0.0
(b)
$\overline{\rho_L}$
$\overline{\rho_R}$
$\sim \sqrt{\tau}$
$\overline{\rho_R}, \overline{\rho_L}$
0.10
0.05
0.00
(c)
v_{E}
v_{B}
v_{LC}
$\sqrt{\tau}(\rho_R + \rho_L)$
Time τ
100
50
0
−100
−50
0
50
100
Position s

Operator hydrodynamics (Hamiltonian)

Floquet system: "kicked Ising model"

$$U = e^{-i\frac{T}{2}h\sum_s X_s} e^{-i\frac{T}{2}\sum_s [Z_s Z_{s+1} + g Z_s]}.$$



Diffusive broadening of operator front exists at various values of g [von Keyserlingk, Rakovszky, Pollmann, Sondhi, 2017].

More cool stuff

Wait a minute! That wasn't a time independent Hamiltonian!

- Conserved quantities
- Any Hamiltonian conserves energy
- Leads to more complicated (slower) dynamics
- But you can still model this with circuits!
- With enough conserved quantities, the circuit can localize [Pai, Pretko, Nandkishore, 2019]

Summary

- Concept of thermalization
- Measures of thermalization
- Thermalization in circuits
- thermalizations in Hamiltonians (kinda...)

Alternative: localization*

Maybe some information doesn't spread out.

- Quantum phase transition [Pal, Huse, 2010]

$$H = \sum_{i=1}^{L} [h_i S_i^z + J \vec{S}_i \cdot \vec{S}_i]$$

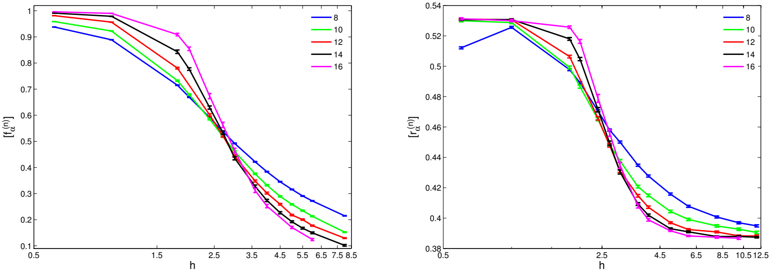
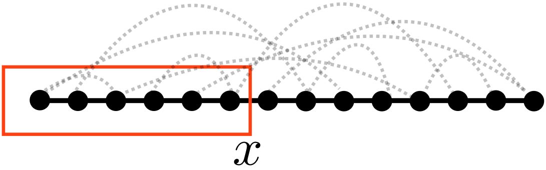


Figure: Spin mobility (L) and relative gaps between energy eigenvalues (R)

Bipartite entanglement entropy*

Another useful measure of information spreading, this time spreading of entanglement (Fig. from [Nahum, Ruhman, Vijay, Haah, 2016])



Entropy across a cut at bond x.

$$S(x,t) = -\operatorname{Tr}\rho_x \log \rho_x,$$

where ρ_x is the reduced density matrix on one side of x,

$$\rho_x = \operatorname{Tr}_{\text{left}} \rho.$$