Fracton-Elasticity Duality

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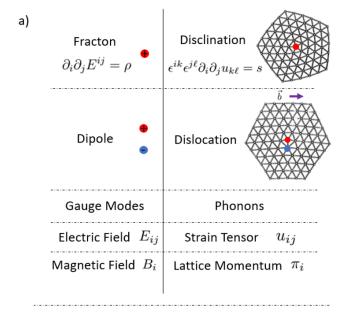*Department of Physics and Center for Theory of Quantum Matter*
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(Dated: November 29, 2017)

Motivated by recent studies of fractons, we demonstrate that elasticity theory of a two-dimensional quantum crystal is dual to a fracton tensor gauge theory, providing a concrete manifestation of the fracton phenomenon in an ordinary solid. The topological defects of elasticity theory map onto charges of the tensor gauge theory, with disclinations and dislocations corresponding to fractons and dipoles, respectively. The transverse and longitudinal phonons of crystals map onto the two gapless gauge modes of the gauge theory. The restricted dynamics of fractons matches with constraints on the mobility of lattice defects. The duality leads to numerous predictions for phases and phase transitions of the fracton system, such as the existence of gauge theory counterparts to the (commensurate) crystal, supersolid, hexatic, and isotropic fluid phases of elasticity theory. Extensions of this duality to generalized elasticity theories provide a route to the discovery of new fracton models. As a further consequence, the duality implies that fracton phases are relevant to the study of interacting topological crystalline insulators.

Introduction. There has been a recent surge of theoretical interest in a new class of quantum phases of matter featuring excitations of restricted mobility. The archetypal examples of this phenomenon are models that exhibit "fracton" excitations, particles which are strictly immobile in isolation, but which can move through interaction with other particles. More generally, there are particles which move freely only along certain subspaces while being immobile in the transverse directions, exhibiting subdimensional behavior. Fractons and other subdimensional particles were first seen in the context of certain exactly solvable lattice models.[1–8] It was later realized that these exotic phases of matter have a natural theoretical description in the language of tensor gauge theories, which feature higher moment charge conservation laws restricting the motion of particles.[9–11] There has been rapid recent progress in the field, establishing connections with quantum Hall systems,[12,13] gravity,[14] and glassy dynamics,[15,16] among many other theoretical developments.[17–30]

Despite extensive studies of their exotic properties, fracton models have so far been lacking concrete physical realizations. To this end, in this Letter we explicitly demonstrate that, intriguingly, a two-dimensional quantum crystal realizes a fracton model described by a noncompact rank-two tensor gauge theory. This duality is a direct tensor analogue of the familiar particle-vortex duality relating a two-dimensional superfluid to a noncompact $U(1)$ gauge theory.[31,32] As summarized in Fig.1a, the longitudinal and transverse phonons of a crystal map onto the two gapless gauge modes of the tensor gauge theory, with the phonon momentum and strain tensor mapping onto the magnetic and electric tensor fields. Concomitantly, the topological lattice defects map directly onto the gauge charges. Specifically, disclinations and dislocations correspond to fractons and dipoles, respectively. In this way, the constrained mobility of fracton models is demystified in terms of the well-known restrictions on motion of a crystal's topological
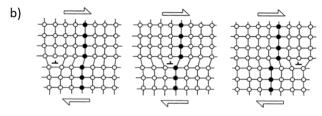

FIG. 1: a) The Fracton-Elasticity Dictionary: Excitations and operators of the scalar-charge tensor-gauge theory are in one-to-one correspondence with those of a two-dimensional quantum crystal. (Pictures of lattice defects adapted from Ref. 33.) b) A dislocation can only freely move by gliding along its Burgers vector $\vec{b}$, while dislocation climb (motion perpendicular to $\vec{b}$) requires the presence of vacancy/interstitial defects.

defects. Dislocations can only glide along their Burg-

ers vector, as shown in Fig.1b, while tranverse motion (dislocation climb) requires the absorption or emission of vacancies and interstitials. Similarly, any motion of a disclination creates a "scar" of extra dislocations in the crystal.[34–37] There are no local processes which move a single disclination, which is therefore immobile in isolation, a manifestation of the fracton phenomenon.

Utilizing this duality, we make numerous predictions about the phases and phase transitions of the fracton gauge theory by mapping onto established results in elasticity theory. For example, the fracton system will exhibit natural gauge theory analogues of the commensurate (vacancy/interstitial-free) crystal, supersolid, hexatic, and isotropic fluid phases. We can thereby also determine the critical properties of transitions between these phases. By generalizing the duality to elasticity theories of other physical systems, such as three-dimensional crystals, magnetic Wigner crystals and liquid crystals, our arguments provide a route to the discovery of new fracton phases. In turn, the conservation laws of fracton gauge theories provide a convenient and systematic tool for encoding and analyzing the dynamics of crystal defects. As a further application of the duality, we discuss the relevance of fracton theories to the study of interacting topological crystalline insulators (TCIs).

Duality. We begin by presenting a streamlined derivation of fracton-elasticity duality, relegating a more detailed derivation and discussion to a companion paper.[38] Dual gauge formulations of elasticity theory have been investigated in the past[39], though with different focus and without making physical connection with fracton theories, which is the aim of the present work. The theory of elasticity is conveniently formulated in terms of a phonon vector field $u_i(\mathbf{x})$ representing the displacement of an atom from its equilibrium position. The low-energy action for the displacement is given by[40–42],

$$S = \int d^2x dt \, \frac{1}{2} \left[(\partial_t u^i)^2 - C^{ijk\ell} u_{ij} u_{k\ell} \right], \qquad (1)$$

where $u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the linear part of the symmetric strain tensor and $C^{ijk\ell}$ is a matrix of elastic constants, with its components determined by the underlying lattice. It is useful to separate the displacement field into its singular and smooth phonon pieces, in terms of which we write $u_{ij} = u_{ij}^{(s)} + \frac{1}{2}(\partial_i \tilde{u}_j + \partial_j \tilde{u}_i)$, where $\tilde{u}_i$ is a smooth single-valued function, and the singular strain component $u_{ij}^{(s)}$ is sourced by topological defects via

$$\epsilon^{ik}\epsilon^{j\ell}\partial_i\partial_j u_{k\ell} = \epsilon^{ik}\epsilon^{j\ell}\partial_i\partial_j u_{k\ell}^{(s)} = s. \qquad (2)$$

The disclination density $s = \epsilon^{ij}\partial_i\partial_j\theta_b$ is defined as a singularity of the bond angle, $\theta_b = \epsilon^{k\ell}\partial_k u_\ell$, giving $s = \epsilon^{ij}\partial_i\partial_j(\epsilon^{k\ell}\partial_k u_\ell)$. Dislocations are also implicitly accounted for in this treatment, since a dislocation can be regarded as a bound state of two disclinations,[33,40,41,43] as we will see explicitly.

We now introduce two Hubbard-Stratonovich fields, a momentum vector π_i and a symmetric stress tensor σ_{ij}.

In terms of these variables, we rewrite the action as,

$$S = \int d^2x dt \left[\frac{1}{2}C_{ijk\ell}^{-1}\sigma^{ij}\sigma^{k\ell} - \frac{1}{2}\pi^i\pi_i \right.$$
$$\left. - \sigma^{ij}(\partial_i\tilde{u}_j + u_{ij}^{(s)}) + \pi^i\partial_t(\tilde{u}_i + u_i^{(s)}) \right], \qquad (3)$$

with the original action recovered by integrating out the fields π_i and σ_{ij}. The smooth displacement field $\tilde{u}_i$ can now be integrated out, thereby enforcing the constraint,

$$\partial_t\pi^i - \partial_j\sigma^{ij} = 0, \qquad (4)$$

which is simply the continuum form of the Newton's equation of motion, relating the stress imbalance to the rate of change of lattice momentum. To solve this constraint it is convenient to first introduce rotated field redefinitions, $B^i = \epsilon^{ij}\pi_j$ and $E_\sigma^{ij} = \epsilon^{ik}\epsilon^{j\ell}\sigma_{k\ell}$, which transforms the Newton equation constraint (4) into the generalized Faraday equation, appearing in fracton tensor gauge theories,[10]

$$\partial_t B^i + \epsilon_{jk}\partial^j E_\sigma^{ki} = 0. \qquad (5)$$

The label σ on the field E_σ^{ij} indicates its relation to the rotated stress tensor.

The general solution to this equation is conveniently expressed in terms of a symmetric rank-2 tensor gauge field, A^{ij}, and a scalar potential, ϕ, (in analogy with the potential formulation of Maxwell's vector electrodynamics)

$$B^i = \epsilon_{jk}\partial^j A^{ki}, \qquad E_\sigma^{ij} = -\partial_t A^{ij} - \partial_i\partial_j\phi, \qquad (6)$$

with ϕ playing the role of the Airy stress function of static elasticity theory. Note that the fields E_σ^{ij} and B^i are invariant under the generalized gauge transformation on the potentials,

$$A_{ij} \to A_{ij} + \partial_i\partial_j\alpha, \qquad \phi \to \phi + \partial_t\alpha \qquad (7)$$

for arbitrary function $\alpha(\mathbf{x}, t)$. The potential formulation has therefore introduced a gauge redundancy into the problem. Expressing the action (3) in terms of electric and magnetic fields, using the potentials in (6) inside the last two terms, integrating by parts and utilizing the definition of the disclination density (2), we obtain,

$$S = \int d^2x dt \left[\frac{1}{2}\tilde{C}_{ijk\ell}^{-1}E_\sigma^{ij}E_\sigma^{k\ell} - \frac{1}{2}B^i B_i - \rho\phi - J^{ij}A_{ij} \right], \qquad (8)$$

where $\tilde{C}^{ijk\ell} = \epsilon^{ia}\epsilon^{jb}\epsilon^{kc}\epsilon^{\ell d}C_{abcd}$ is a function of the elastic coefficients, $\rho = s$ is the disclination density, and $J^{ij} = \epsilon^{ik}\epsilon^{j\ell}(\partial_t\partial_k - \partial_k\partial_t)u_\ell$ is the current tensor capturing the motion of dislocations and disclinations, as introduced in Ref. 10,44. For a dislocation with Burgers vector b^ℓ moving at velocity v^j, this tensor takes the

form $J^{ij} = \epsilon^{(i\ell}v^{j)}b_\ell$,[45] with the trace $J^i{}_i$ describing dislocation climb.[44,46] The action of Eq.8 is in precisely the form of the scalar-charge tensor gauge theory, allowing for anisotropy in the electric field term, with disclinations playing the role of fracton charges.[10,12] This action leads to two gapless gauge modes, corresponding to the longitudinal and transverse phonon modes of elasticity theory.

We note in passing that this gauge theory does not support instanton events in the path integral, which would correspond to terms in the elasticity Hamiltonian which explicitly break translational symmetry and gap out the phonon modes, as could arise via coupling to a substrate. For a conventional crystal, which breaks an underlying translational symmetry *spontaneously*, instantons are forbidden and the gauge field is noncompact, as discussed further in the companion paper.[38]

It will also be useful to introduce a canonical conjugate electric tensor field, $E_{ij} = -\partial \mathcal{L}/\partial \dot{A}^{ij} = \tilde{C}^{-1}_{ijk\ell}E^{k\ell}_\sigma$, in terms of which the tensor gauge theory Hamiltonian is given by

$$H = \int d^2x \left(\frac{1}{2}\tilde{C}^{ijk\ell}E_{ij}E_{k\ell} + \frac{1}{2}B^iB_i + \rho\phi + J^{ij}A_{ij} \right). \quad (9)$$

Note that the scalar potential ϕ does not have a conjugate field, but rather acts as a Lagrange multiplier enforcing the scalar Gauss's law constraint,

$$\partial_i\partial_j E^{ij} = \rho. \quad (10)$$

This constraint is the dual formulation of Eq.(2), defining the disclination density. We see that the duality maps E^{ij} to a rotated strain tensor via $E^{ij} = \epsilon^{ik}\epsilon^{j\ell}u_{k\ell}$, while the closely-related "velocity"-like field, E^{ij}_σ is mapped to a rotated stress tensor via $E^{ij}_\sigma = \epsilon^{ik}\epsilon^{j\ell}\sigma_{k\ell}$. The relation $E^{ij}_\sigma = \tilde{C}^{ijk\ell}E_{k\ell}$ between the two electric field tensors exactly mirrors the relation $\sigma^{ij} = C^{ijk\ell}u_{k\ell}$ between the stress and strain tensors. The Gauss's law (10) is notable for leading to conservation of both charge and dipole moment[9]

$$Q = \int d^2x\,\rho = \text{const.}, \quad \mathbf{P} = \int d^2x\,(\rho\mathbf{x}) = \text{const.} \quad (11)$$

The conservation of dipole moment has the dramatic consequence that an isolated charge is strictly locked in place, since a motion of a fracton charge proceeds by a creation of a dipole moment, and thus would violate dipole charge conservation. The presence of this extra conservation law therefore directly encodes the fractonic behavior of disclinations.

The dipole moment conservation law also implies that a dipole is a topologically stable excitation, since it cannot decay directly into the vacuum. In elasticity language, this corresponds to a bound state of two equal and opposite disclinations, known as the dislocation defect.[40,41] We can check this correspondence explicitly by studying the total dipole moment contained in a region V. Assuming the region has zero net charge (so that dipole moment is independent of origin), we can write the dipole moment in the form,

$$P^i = \int_V d^2x\,(\rho x^i) = \oint_{\partial V} ds^j \partial_j(\epsilon^{ik}u_k),$$
$$= \epsilon^{ik}\Delta u_k = \epsilon^{ik}b_k\ , \quad (12)$$

where Δu_k is the net change in the displacement u_k going around the boundary of a region V, which is precisely the definition of a Burgers vector $\mathbf{b}$. From this, we see that the dipole matches explicitly with a dislocation defect, $\mathbf{P} = \hat{\mathbf{z}} \times \mathbf{b}$, with the dipole vector perpendicular to the Burgers vector. With this correspondence in place, the fracton-elasticity dictionary is now complete, as summarized in Fig.1.

One important additional property of a crystal that dual gauge theory must capture is that in the absence of vacancies and interstitials a dislocation can only move along its Burgers vector, *i.e.*, can glide but is unable to climb. On the other hand, by itself a conservation of a dipole moment does not place any fundamental restriction on the motion of a dipole. To see how the one-dimensional constrained dipole dynamics arises in the tensor gauge theory, we consider a particular component of the quadrupole moment. Following the standard analysis of fracton theories,[9] it is straightforward to derive the following conservation law,

$$\int d^2x\,(\rho x^2 - 2E^i{}_i) = \text{const.} \quad (13)$$

Any longitudinal motion of a dipole requires a change of this quadrupole moment, which, as we see from the above constraint is necessarily accompanied by a change in $E^i{}_i$. To understand the physical meaning of this conservation law, we rewrite the trace in elasticity language,

$$E^i{}_i = \partial_i u^i = n_d + \partial_i\tilde{u}^i, \qu(14)$$

where we have broken up the divergence into n_d, the number of vacancies minus the number of interstitial defects, and a smooth elastic piece $\tilde{u}^i$. We can then write our conservation law as,

$$\int d^2x\,(\rho x^2 - 2n_d) = \text{const.} \quad (15)$$

In other words, the longitudinal motion of a dipole (corresponding to a dislocation climb) requires the absorption or creation of vacancies or interstitial defects. This provides a kinetic and energetic barrier, which, in the absence of vacancies and interstitials constrain dislocations and the corresponding fracton dipoles into quasi-one-dimensional particles, as expected.

To see more formally and explicitly that longitudinal dipole motion (equivalently, dislocation motion transverse to its Burgers vector, *i.e.*, a climb) creates vacancy/interstitial defects, we examine the Ampere equation of motion, $\delta H/\delta A_{ij} = 0$, which takes the form,

$$\partial_t E^{ij} + \frac{1}{2}(\epsilon^{ik}\partial_k B^j + \epsilon^{jk}\partial_k B^i) = -J^{ij}. \quad (16)$$
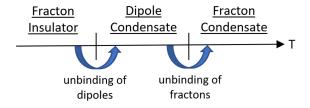
FIG. 2: The duality with elasticity theory predicts that the fracton gauge theory exhibits two finite-temperature phase transitions, corresponding to unbinding of dipoles and fractons, respectively.

The piece of this equation that is relevant for our purposes is the trace, which can be written as[44]

$$\partial_t n_d + \partial_i J_d^i = -J^i_{\ i}\ , \qquad (17)$$

where $J_d^i = \pi^i = \epsilon^{ji} B_j$ is the current density of vacancies and interstitials, and we have used the fact that $E^i_{\ i} \approx n_d$, since $\partial_i \tilde{u}^i \ll 1$. The above equation represents a continuity equation for the vacancy/interstitial number, sourced by a dislocation current transverse to the Burgers vector, $J^i_{\ i} = \hat{\mathbf{z}} \cdot (\mathbf{v} \times \mathbf{b})$, indicating that dislocation climb creates vacancy/interstitial defects.[44]

Dual fracton superconductor. The duality has mapped a 2d crystal onto a rank-two gauge theory coupled to fracton matter, with the dual gauge theory action (8) naturally describing a fracton insulator. However, to access finite density fracton phases, it is convenient to introduce more explicit coupling to matter fields[38]

$$\begin{aligned} S &= \int d^2x dt \Big[\frac{1}{2} \tilde{C}^{-1}_{ijk\ell} E^{ij}_\sigma E^{k\ell}_\sigma - \frac{1}{2} B^i B_i \\ &\quad - \frac{1}{2} g_1 (\partial_i \partial_j \theta - A_{ij})^2 + \frac{1}{2} g_0 (\partial_t \theta - \phi)^2 \Big], \quad (18) \end{aligned}$$

where θ is the phase of the fracton field and the g_i are determined by core energies of the defects. This action is capable of describing a dual fracton "superconductor" (*i.e.*, a condensate of fractons), with the normal phase (*i.e.*, the fracton insulator) corresponding to the crystal. This action also supports a third phase between the fracton superconductor and insulator, as we discuss next.

Applications. The field of fractons is still in the early stages of development, and thus lacks much of the basic machinery used in the study of symmetry breaking systems and conventional topological phases. As such, much less is known about the various phases and phase transitions of fracton models (though recent progress has been made on this subject).[13] For the specific fracton model discussed here, however, we can obtain the entire phase diagram and characterize the nature of phase transitions by the above mapping onto a two-dimensional crystal, which has been studied in great detail.[47–49] The duality thereby gives key features of phases and phase transitions of the above scalar charge fracton model, which we expect to also provide insight into more general fracton systems.

More specifically, in addition to the above established correspondence between a crystal and gauged fracton insulator, two fracton-proliferated phases emerge as duals of the orientationally ordered (*e.g.*, hexatic) and isotropic fluids. On the elasticity side, these appear at finite temperature as a result of two-stage BKT-like melting transitions: (i) a crystal-to-hexatic fluid transition, at which dislocations (that are logarithmically bound in a crystal) proliferate,[47–49] followed by (ii) a hexatic-to-isotropic fluid BKT transition[50–52], at which disclinations (bound quadratically in the crystal phase, but screened down to logarithmic binding in the hexatic) entropically proliferate. We thus predict a finite temperature fracton phase diagram with three distinct phases, distinguished by the proliferation of dipoles and fractons, as summarized in Fig.2. The proliferated phases can be regarded as a dipole condensate and a fracton condensate, respectively, with implications for the quantum theory of melting, discussed elsewhere.[38] These transitions are all captured by the tensor dual "superconductor" model, (18), that at finite temperature reduces to a classical 2d tensor sine-Gordon model. We leave the more detailed analysis of these fracton phases and transitions on the gauge theory side to future research.[38]

We also note that at zero temperature two qualitatively distinct quantum crystal phases are allowed. A "commensurate crystal" (with the weight of Bragg peaks commensurate with the number of particles) is characterized by long-range positional and orientational orders and a vacuum of gapped vacancies and interstitials, *i.e.*, a Mott insulator. With increased quantum fluctuations (*e.g.*, reduced mass), vacancies and interstitials condense at finite density into an "incommensurate crystal", that is a supersolid[53–55] in the case of bosonic atoms. The fracton-elasticity duality thus predicts two distinct zero-temperature fracton insulating phases on the tensor gauge theory side, distinguished by gapped and condensed quadrupole excitations.

We conclude by noting that fracton-elasticity duality draws an intriguing connection to a seemingly unrelated subject of classification of interacting crystal symmetry protected topological insulators (TCIs).[56–66] In classifying interacting symmetry-protected topological (SPTs) phases, one particularly powerful tool is gauging the symmetry protecting the SPT phase.[67] The result is a topologically ordered state[68], described by a gauge theory with a gauge group equivalent to the symmetry group of the original SPT phase, with different interacting SPT phases corresponding to distinct topological phases.

For internal symmetry groups, this gauging procedure is fairly straightforward, done by coupling to a dynamical flux of the symmetry group. However, for the case of spatial symmetries, the notion of flux insertion is less clear. As recently demonstrated[69], flux of a crystal symmetry is equivalent to a lattice defect, with a dislocation and a disclination respectively corresponding to a flux of translational and rotational symmetries. A resulting

model with a fully gauged crystalline symmetry exhibits dynamical lattice defects, i.e., it is a quantum elasticity theory. Fracton-elasticity duality then allows us to map the gauged system onto a fracton theory. Hence, the result of gauging a two-dimensional crystalline symmetry is a fracton phase, as opposed to the more conventional topological phases obtained by gauging an internal symmetry. We expect that a more detailed understanding of fracton phases thus obtained by gauging crystal symmetries may prove useful for classifying interacting TCIs, a quest that is still being actively pursued.[70–76] We leave the details of implementing this program as a task for the future.

Conclusions. In this manuscript, we have explicitly demonstrated a duality between two-dimensional quantum elasticity and a fracton tensor gauge theory, in a natural tensor generalization of conventional particle-vortex duality. The topological defects of a 2d crystal map directly onto the charges and dipoles of the gauge theory, while phonons and elastic strain tensor respectively correspond to the gapless gauge modes and the tensor electric field. This duality demystifies the constrained mobility of fractons and dipoles by mapping them onto known properties of disclinations and dislocations, respectively. As a result, we made predictions about fracton phases and phase transitions by mapping onto the phase diagram of quantum crystals. Our work opens the door for the future exchange of ideas between the emerging field of fractons and the well-established study of elasticity theory.

[1] C. Chamon, *Quantum glassiness in strongly correlated clean systems: An example of topological overprotection.* Phys. Rev. Lett. 94 040402 (2005), arXiv:cond-mat/0404182v2

[2] S. Bravyi, B. Leemhuis, and B. M. Terhal, *Topological order in an exactly solvable 3D spin model.* Ann. Phys., Vol. 326:4, 839 (2010), arXiv:1006.4871

[3] J. Haah, *Local stabilizer codes in three dimensions without string logical operators.* Phys. Rev. A 83, 042330 (2011), arXiv:1101.1962v2

[4] C. Castelnovo and C. Chamon, *Topological quantum glassiness.* Philosophical Magazine Vol. 92, Iss. 1-3 (2012), arXiv:1108.2051v2

[5] B. Yoshida, *Exotic topological order in fractal spin liquids.* Phys. Rev. B 88, 125122 (2013), arXiv:1302.6248v4

[6] S. Bravyi and J. Haah, *Quantum self-correction in the 3D cubic code model.* Phys. Rev. Lett. 111, 200501 (2013), arXiv:1112.3252

[7] S. Vijay, J. Haah, L. Fu, *A new kind of topological quantum order: A dimensional hierarchy of quasiparticles built from stationary excitations.* Phys. Rev. B 92, 235136 (2015), arXiv:1505.02576

[8] S. Vijay, J. Haah, L. Fu, *Fracton topological order, generalized lattice gauge theory and duality.* Phys. Rev. B 94, 235157 (2016), arXiv:1603.04442

[9] M. Pretko, *Subdimensional particle structure of higher rank $U(1)$ spin liquids.* Phys. Rev. B 95, 115139 (2017), arXiv:1604.05329v3

[10] M. Pretko, *Generalized electromagnetism of subdimensional particles.* Phys. Rev. B 96, 035119 (2017), arXiv:1606.08857v2

[11] A. Rasmussen, Y.-Z. You, and C. Xu, *Stable gapless bose liquid phases without any symmetry.* arXiv:1601.08235 (2016)

[12] M. Pretko, *Higher-spin Witten effect and two-dimensional fracton phases.* Phys. Rev. B 96, 125151 (2017), arXiv:1707.03838v3

[13] A. Prem, M. Pretko, and R. Nandkishore, *Emergent phases of fractonic matter.* arXiv:1709.09673 (2017)

[14] M. Pretko, *Emergent gravity of fractons: Mach's principle revisited.* Phys. Rev. D 96, 024051 (2017), arXiv:1702.07613v3

[15] A. Prem, J. Haah, R. Nandkishore, *Glassy quantum dynamics in translation invariant fracton models.* Phys. Rev. B 95, 155133 (2017), arXiv:1702.02952

[16] M. Pretko, *Finite-temperature screening of $U(1)$ Fractons.* Phys. Rev. B 96, 115102 (2017), arXiv:1706.01899v2

[17] D. J. Williamson, *Fractal symmetries: Ungauging the cubic code.* Phys. Rev. B 94, 155128 (2016), arXiv:1603.05182v3

[18] S. Vijay, *Isotropic layer construction and phase diagram for fracton topological phases.* arXiv:1701.00762 (2017)

[19] H. Ma, E. Lake, X. Chen, M. Hermele, *Fracton topological order via coupled layers.* Phys. Rev. B 95, 245126 (2017), arXiv:1701.00747v2

[20] T. H. Hsieh, G. B. Halász, *Fractons from partons.* Phys. Rev. B 96, 165105 (2017), arXiv:1703.02973

[21] K. Slagle, Y. B. Kim, *Fracton topological order from nearest-neighbor two-spin interactions and dualities.* Phys. Rev. B 96, 165106 (2017), arXiv:1704.03870v2

[22] B. Shi, Y.-M. Lu, *Decipher the nonlocal entanglement entropy of fracton topological orders.* arXiv:1705.09300 (2017)

[23] S. Vijay, L. Fu, *A generalization of non-abelian anyons in three dimensions.* arXiv:1706.07070 (2017)

[24] G. B. Halász, T. H. Hsieh, L. Balents, *Fracton topological phases from strongly coupled spin chains.* arXiv:1707.02308 (2017)

[25] K. Slagle and Y. B. Kim, *Quantum field theory of X-cube fracton topological order and robust degeneracy from geometry.* Phys. Rev. B 96, 195139 (2017), arXiv:1708.04619v3

[26] V. V. Albert, S. Pascazio, and M. H. Devoret, *General*

phase spaces: from discrete variables to rotor and continuum limits. J. Phys. A: Math. Theor. 50, 504002 (2017), arXiv:1709.04460v2

[27] T. Devakul, S. A. Parameswaran, and S. L. Sondhi, *Correlation function diagnostics for type-I fracton phases.* arXiv:1709.10071 (2017)

[28] O. Petrova and N. Regnault, *A simple anisotropic three-dimensional quantum spin liquid with fracton topological order.* arXiv:1709.10094 (2017)

[29] H. Ma, A. T. Schmitz, S. A. Parameswaran, M. Hermele, and R. M. Nandkishore, *Topological entanglement entropy of fracton stabilizer codes.* arXiv:1710.01744v2 (2017)

[30] H. He, Y. Zheng, B. A. Bernevig, and N. Regnault, *Entanglement entropy from tensor network states for stabilizer codes.* arXiv:1710.04220 (2017)

[31] C. Dasgupta and B. I. Halperin, *Phase transition in a lattice model of superconductivity.* Phys. Rev. Lett. 47, 1556 (1981)

[32] M. P. A. Fisher and D. H. Lee, *Correspondence between two-dimensional bosons and a bulk superconductor in a magnetic eld.* Phys. Rev. B 39, 2756 (1989)

[33] H. S. Seung and D. R. Nelson, *Defects in flexible membranes with crystalline order.* Phys. Rev. A 38, 1005 (1988)

[34] M. Y. Gutkin, I. A. Ovid'ko, and N. V. Skiba, *Transformations of grain boundaries due to disclination motion and emission of dislocation pair.* Materials Science and Engineering: A, 339 73 (2003)

[35] R. C. Desai and R. Kapral, *Dynamics of self-organized and self-assembled structures.* Cambridge University Press (2009)

[36] C. Fressengeas, V. Taupin, and L. Capolungo, *An elasto-plastic theory of dislocation and disclination fields.* International Journal of Solids and Structures 48 25 3499 (2011)

[37] H. Altenbach, S. Frost, and A. Krivtsov, *Generalized continua as models for materials: with multi-scale effects or under multi-field actions.* Springer (2013)

[38] M. Pretko and L. Radzihovsky, forthcoming.

[39] Aron J. Beekman et al., *Dual gauge field theory of quantum liquid crystals in two dimensions*, Physics Reports 683, 1-110 (2017), arXiv:1603.04254v2

[40] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics.* Cambridge University Press (2000)

[41] L. D. Landau and L. P. Pitaevskii, *Theory of Elasticity.* Course of Theoretical Physics: Volume 7. 3rd Edition. Butterworth-Heinemann (1986)

[42] M. Kardar, *Statistical Physics of Fields.* Cambridge University Press (2007)

[43] S. Gopalakrishnan, J. C. Y. Teo, and T. L. Hughes, *Disclination classes, fractional excitations, and the melting of quantum liquid crystals.* Phys. Rev. Lett. 111, 025304 (2013), arXiv:1302.3617

[44] M. C. Marchetti and L. Radzihovsky, *Interstitials, vacancies and dislocations in flux-line lattices: A theory of vortex crystals, supersolids and liquids.* Phys. Rev. B 59, 12001 (1999), arXiv:cond-mat/9811193v2

[45] This tensor will also have a contribution from processes in which a disclination (fracton) moves while emitting or absorbing a dislocation (dipole).

[46] V. Cvetkovic, Z. Nussinov, and J. Zaanen, *Topological kinematic constraints: Quantum dislocations and the glide principle.* Philos. Mag. 86, 2995 (2006), arXiv:cond-mat/0508664

[47] B. I. Halperin and D. R. Nelson, *Theory of two-dimensional melting.* Phys. Rev. Lett. 41, 121 (1978)

[48] D. R. Nelson and B. I. Halperin, *Dislocation-mediated melting in two dimensions.* Phys. Rev. B 19, 2457 (1979)

[49] A. P. Young, *Melting and the vector Coulomb gas in two dimensions.* Phys. Rev. B 19, 1855 (1979)

[50] V. L. Berezinskii, *Destruction of long-range order in one-dimensional and two-dimensional systems having a continuous symmetry group: I. Classical systems.* Sov. Phys. JETP, 32 3: 493 (1971)

[51] V. L. Berezinskii, *Destruction of long-range order in one-dimensional and two-dimensional systems having a continuous symmetry group: II. Quantum systems.* Sov. Phys. JETP, 34 3: 610 (1972)

[52] J. M. Kosterlitz and D. J. Thouless, *Ordering, metastability and phase transitions in two-dimensional systems.* Journal of Physics C: Solid State Physics, 6 7: 1181 (1973)

[53] A. F. Andreev and I. M. Lifshitz, *Quantum theory of defects in crystals.* Sov. Phys. JETP 29, 1107 (1969)

[54] E. Kim and M. H. W. Chan, *Probable observation of a supersolid helium phase.* Nature 427, 225 (2004)

[55] J.-R. Li et al., *A stripe phase with supersolid properties in spinorbit-coupled BoseEinstein condensates.* Nature 543, 91 (2017), arXiv:1610.08194v2

[56] C. L. Kane and E. J. Mele, Z_2 *topological order and the quantum spin Hall effect.* Phys. Rev. Lett. 95, 146802 (2005), arXiv:cond-mat/0506581v2

[57] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, *Quantum spin Hall effect and topological phase transition in HgTe quantum wells.* Science, 314, 1757 (2006), arXiv:cond-mat/0611399

[58] M. König et al., *Quantum spin Hall insulator state in HgTe quantum wells.* Science, 318, 766 (2007)

[59] L. Fu and C. L. Kane, *Topological insulators with inversion symmetry.* Phys. Rev. B 76, 045302 (2007), arXiv:cond-mat/0611341v2

[60] M. Z. Hasan and C. L. Kane, *Topological insulators.* Rev. Mod. Phys. 82, 3045 (2010), arXiv:1002.3895v2

[61] L. Fu, *Topological crystalline insulators.* Phys. Rev. Lett. 106, 106802 (2011), arXiv:1010.1802v3

[62] Y. Ando and L. Fu, *Topological crystalline insulators and topological superconductors: From concepts to materials.* Annual Review of Condensed Matter Physics 6, 361-381 (2015), arXiv:1501.00531

[63] C. Wang, A. C. Potter, and T. Senthil, *Classification of interacting electronic topological insulators in three dimensions.* Science 343, 6171 (2014), arXiv:1306.3238v4

[64] C. Wang and T. Senthil, *Interacting fermionic topological insulators/superconductors in three dimensions.* Phys. Rev. B 89, 195124 (2014), arXiv:1401.1142v3

[65] T. Senthil, *Symmetry protected topological phases of matter.* Annual Review of Condensed Matter Physics 6, 299 (2015), arXiv:1405.4015

[66] X. Chen et al., *Symmetry protected topological orders and the group cohomology of their symmetry group.* Phys. Rev. B 87, 155114 (2013), arXiv:1106.4772v6

[67] M. Levin and Z.-C. Gu, *Braiding statistics approach to symmetry-protected topological phases.* Phys. Rev. B 86, 115109 (2012), arXiv:1202.3120v2

[68] More precisely, the system will have long-range entanglement. For a discrete symmetry group, the resulting gauge theory will be gapped, and we can rigorously use the term topological order. For a continuous symmetry group, however, the resulting gauge theory will be gapless.

[69] R. Thorngren and D. V. Else, *Gauging spatial symme-*

tries and the classification of topological crystalline phases. arXiv:1612.00846v6 (2016)

[70] H. Isobe and L. Fu, *Theory of interacting topological crystalline insulators.* Phys. Rev. B 92, 081304 (2015), arXiv:1502.06962v2

[71] H. Song, S.-J. Huang, L. Fu, and M. Hermele, *Topological phases protected by point group symmetry.* Phys. Rev. X 7, 011020 (2017), arXiv:1604.08151v2

[72] S.-J. Huang, H. Song, Y.-P. Huang, and M. Hermele, *Building crystalline topological phases from lower-dimensional states.* Phys. Rev. B 96, 205106 (2017), arXiv:1705.09243

[73] M. Cheng, *Microscopic theory of surface topologi-cal order for topological crystalline superconductors.* arXiv:1707.02079v2 (2017)

[74] S. Hong and L. Fu, *Topological order and symmetry anomaly on the surface of topological crystalline insula-tors.* arXiv:1707.02594 (2017)

[75] L. Zou, *Bulk characterization of topological crys-talline insulators: Stability under interactions and rela-tion to symmetry enriched U(1) quantum spin liquids.* arXiv:1711.03090 (2017)

[76] S.-J. Huang and M. Hermele, *Surface field theories of point group symmetry protected topological phases.* arXiv:1711.03649 (2017)