SYK Model and Its Thermodynamics

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1. Introduction

Before the SYK model came into being, Sachdev and Ye's SY model came first. The model that they proposed consisted of Dirac Fermions, and had a Hamiltonian that was defined as,

$$H = \frac{1}{2N^{\frac{3}{2}}} \sum_{i,j,k,l} J_{ij;kl} c_i^{\dagger} c_j^{\dagger} c_k c_l$$

$$J_{ji;kl} = -J_{ij;kl}\,, \qquad J_{ij;lk} = -J_{ij;kl}\,, \qquad J_{kl;ij} = {J^*}_{ij;kl}\,, \qquad \overline{{J_{ij;kl}}^2} = J^2$$

Kitaev later introduced the following model that consists of N Majorana fermions that randomly interact with each other. This came to be called, the SYK model, and it's Hamiltonian was given as follows:

$$\mathrm{H} = (i)^{\frac{q}{2}} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} j_{i_1 i_2 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q}$$

$$\langle j^2_{i_1 i_2 \cdots i_q} \rangle = \frac{\mathrm{J}^2 (q-1)!}{N^{q-1}} = \frac{2^{q-1}}{q} \frac{J^2 (q-1)!}{N^{q-1}}$$

The SYK model is currently deemed important for several reasons. First, its entropy per degree of freedom does not go to zero at low temperatures. It has been found that the out of time order correlator of this model decays in a manner that suggests that chaotic dynamics is present down to the low temperatures. Furthermore, at low temperatures, this model is critical; it develops an approximate conformal symmetry, which is believed to go along with the existence of a holographic dual. These attributes of the SYK model – its approximate conformal symmetry, chaotic dynamics, and macroscopic entropy at low temperature – make the model an interesting model in the study of gravity, and in condensed matter physics [1].

Extensive work has been done on both the SY and the SYK model. They exhibit quite similar behaviors, and in this paper, we study both models, albeit with focus on SYK model at $q = 4$.

We first discuss the thermodynamics of the SY model at large N, and find that the entropy per degree of freedom of the system converges to a nonzero value at low temperatures. And then, we look at the reparametrization and conformal symmetry of the SYK model in the infrared. Then, by studying Fu and Sachdev's result on the decay of out of time order correlator, we find that chaos is present in the SY model. Last, we give our results from our numerical study of the SYK model. We find that the Eigenstate Thermalization Hypothesis works in the SYK model, which is further proof of the SYK model's underlying chaos. Also, we find that at infinite temperature with increasing quadratic perturbation, an increase in chaos seems to exist by studying the distribution of an eigenstate's expectation value of an eigenstate, and by observing the difference in decay of the out of time order correlator.

2. Thermodynamics at Large N

We first study the thermodynamics of the SY model, using the method introduced in Fu and Sachdev's paper. The partition function and the action satisfies,

$$\mathrm{Z} = \int Dc^{\dagger}Dc\,\exp(-S)$$

$$\mathrm{S} = \int_0^{\beta} d\tau\,(c^{\dagger}\partial_{\tau}c + H)$$

We now use the replica trick, $\ln Z = \lim_{n\to 0}\frac{1}{n}(Z^n - 1)$

$$Z^n = \int \prod_{i,j,k,l} dJ_{ijkl}\sqrt{\frac{N^3}{12\pi J^2}}\exp\left(-\frac{N^3 {J_{ijkl}}^2}{12J^2}\right)\left(\int Dc_{ia}{}^{\dagger}Dc_{ia}\exp(-S)\right)^n$$

By integrating J_{ijkl} first (disorder averaging), one gets

$$S_n = \sum_{ia}\int_0^{\beta} d\tau\, c_{ia}^{\dagger}\left(\frac{\partial}{\partial\tau} - \mu\right)c_{ia} - \frac{J^2}{4N^3}\sum_{ab}\int_0^{\beta} d\tau\, d\tau'\left|\sum_i c_{ia}^{\dagger}(\tau)c_{ib}(\tau')\right|^4$$

$$Z^n = \int \prod_a Dc_a{}^{\dagger}Dc_a\,\exp(-S_n)$$

It is possible to decouple the interaction by a Hubbard Stratonovich transformations. Let's introduce a real field Q such that $Q_{ab}(\tau,\tau') = Q_{ba}(\tau',\tau)$ (This is required as the action is invariant under the reparametrization $\mathrm{a},\tau \leftrightarrow \mathrm{b},\tau'$)

$$S_n = \sum_{ia}\int_0^{\beta} d\tau\, c_{ia}^{\dagger}\left(\frac{\partial}{\partial\tau} - \mu\right)c_{ia}$$
$$+\sum_{ab}\int_0^{\beta} d\tau\, d\tau'\left\{\frac{N}{4J^2}[Q_{ab}(\tau,\tau')]^2 - \frac{1}{2N}Q_{ab}(\tau,\tau')\left|\sum_i c_{ia}^{\dagger}(\tau)c_{ib}(\tau')\right|^2\right\}$$

Introducing a second complex field $P_{ab}(\tau,\tau')$ that obeys $P_{ab}(\tau,\tau') = {P_{ba}}^{*}(\tau',\tau)$,

$$S_n = \sum_{ia}\int_0^\beta d\tau\, c_{ia}^\dagger\left(\frac{\partial}{\partial\tau} - \mu\right)c_{ia}$$

$$+\sum_{ab}\int_0^\beta d\tau\, d\tau' \left\{\frac{N}{4J^2}[Q_{ab}(\tau,\tau')]^2 + \frac{N}{2}Q_{ab}(\tau,\tau')|P_{ab}(\tau,\tau')|^2\right.$$

$$\left. - Q_{ab}(\tau,\tau')P_{ba}(\tau',\tau)\sum_i c_{ia}^\dagger(\tau)c_{ib}(\tau')\right\}$$

The saddle point of this action satisfies $\frac{\delta S}{\delta P_{ba}} = 0$.

$$P_{ab}(\tau,\tau') = \frac{1}{N}\sum_i c_{ia}^\dagger(\tau)c_{ib}(\tau')$$

At large N, $P_{ab}(\tau,\tau') = \frac{1}{N}\langle c_{ia}^\dagger(\tau)c_{ib}(\tau')\rangle$

Now, the saddle point also satisfies $\frac{\delta S}{\delta Q_{ab}} = 0$, therefore,

$$Q_{ab}(\tau,\tau') = J^2|P_{ab}(\tau,\tau')|^2$$

The self-energy of the system $\sum(\tau,\tau') = -\,Q_{ab}(\tau,\tau')P_{ba}(\tau',\tau)$

Also, the Green's function is defined by $G(\tau,\tau') = -\langle Tc^\dagger(\tau)c(\tau')\rangle = P(\tau',\tau)$

Therefore, the saddle point solution is,

$$G(i\omega_n) = \frac{1}{i\omega_n + \mu - \sum(i\omega_n)}, \Sigma(\tau) = -J^2G^2(\tau)G(-\tau)$$

The free energy of the system is,

$$F = -\frac{1}{\beta}\ln(Z_{eff}) = -\frac{1}{\beta}\ln\left(\int Dc^\dagger Dc\, \exp(-S)\right)$$

Now, for the free energy density, we can drop the index i in the action. Also we can substitute the Green's function and self energy

$$S = \int_0^\beta d\tau d\tau' \, c^\dagger(\tau) \left(\frac{\partial}{\partial\tau} \delta(\tau-\tau') - \mu\delta(\tau-\tau') + \Sigma(\tau,\tau') \right) c(\tau')$$

$$+ \int_0^\beta d\tau \, d\tau' \left\{ \frac{N}{4J^2} [\Sigma(\tau,\tau')/G(\tau,\tau')]^2 - \frac{1}{2}\Sigma(\tau,\tau')G(\tau,\tau') \right\}$$

Integrating,

$$S = -\mathrm{Tr} \ln[(\partial\tau - \mu)\delta(\tau-\tau') + \Sigma(\tau,\tau')]$$

$$+ \int_0^\beta d\tau \, d\tau' \left\{ \frac{N}{4J^2} [\Sigma(\tau,\tau')/G(\tau,\tau')]^2 - \frac{1}{2}\Sigma(\tau,\tau')G(\tau,\tau') \right\}$$

At large N, we can use the saddle point solution we have calculated, and then the free energy density is,

$$\frac{F}{N} = T \sum_n \ln\big(-\beta G(i\omega_n)\big) - \int_0^\beta d\tau \frac{3}{4} \Sigma(\tau) G(-\tau)$$

The entropy density can now be obtained as $\frac{S}{N} = -\frac{1}{N}\frac{\partial F}{\partial T}$

At high temperatures, by iteration starting at $G_0 = \frac{1}{i\omega_n}$, we can find the series expansion of G and Σ.

$$\Sigma_4(i\omega_n) = J^8 \left[\frac{561}{256(i\omega_n)^7} + \frac{75}{512T^2(i\omega_n)^5} - \frac{1}{256T^4(i\omega_n)^3} \right]$$

$$G_4(i\omega_n) = J^8 \left[\frac{5}{2(i\omega_n)^9} + \frac{81}{512T^2(i\omega_n)^7} - \frac{1}{256T^4(i\omega_n)^5} \right]$$

The entropy becomes

$$\frac{S}{N} = \ln 2 - \frac{1}{64}\frac{J^2}{T^2} + \frac{1}{512}\frac{J^4}{T^4} - \frac{1}{36864}\frac{J^6}{T^6} + \frac{599}{11796480}\frac{J^8}{T^8} + \cdots$$

This result agrees well with the exact entropy at high temperature, as the following graph shows.

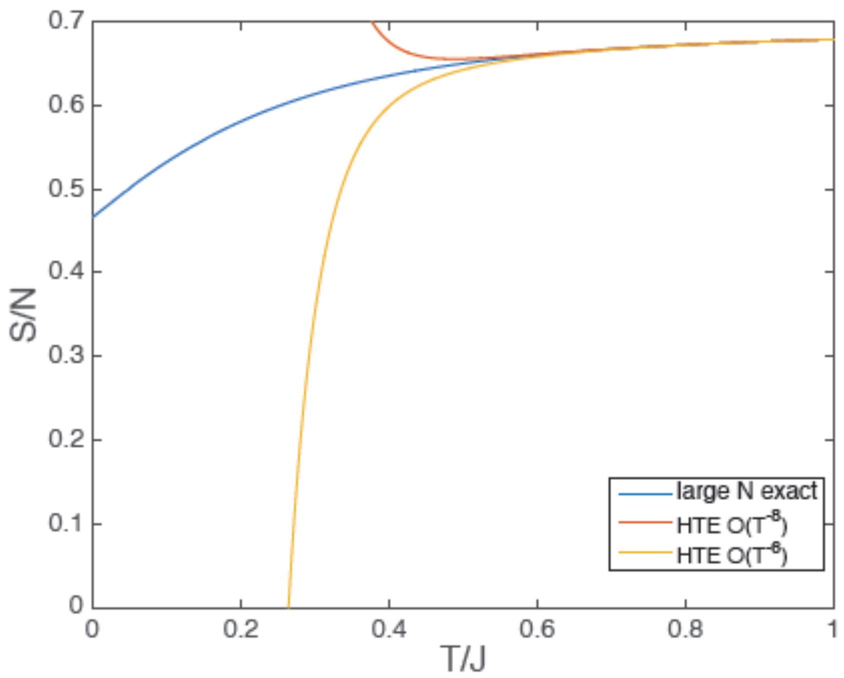

Figure 1. Temperature vs Entropy

The blue line is the result from numerical calculations at large N. The red line and the yellow line, which are the above result till the T^8 term and T^6 term, agree well with the blue line at high temperatures. Observe that as $\mathrm{T} \rightarrow 0$, the $\frac{\mathrm{S}}{\mathrm{N}}$ does not go to 0, which indicates that even at low temperatures, chaos is present [1].

It is to be mentioned that similar characteristics are observed in the SYK model. Maldacena and Stanford demonstrates that at the $q \rightarrow \infty$ limit, as $\mathrm{T} \rightarrow 0$, the $\frac{\mathrm{S}}{\mathrm{N}}$ does not go to 0 [2].

3. Conformal Limit and Reparametrization Symmetry

For a free Majorana fermion [2],

$$G_{free}(\tau) = \langle \mathrm{T}\psi(\tau)\psi(0)\rangle = \frac{1}{2}\, sgn(\tau), \qquad G_{free}(\omega) = -\frac{1}{i\omega}$$

Now, let's compute the correction to the propagator due to the interaction. The relevant Feynman diagrams are shown in the figures below. In Figure 1, the dashed line represents averaging over disorder. In Figure 2, a black circle represents the one particle irreducible term, and a grey circle represents the full Green's function [2].

Figure 2. One Particle Irreducible Terms in the Propagator

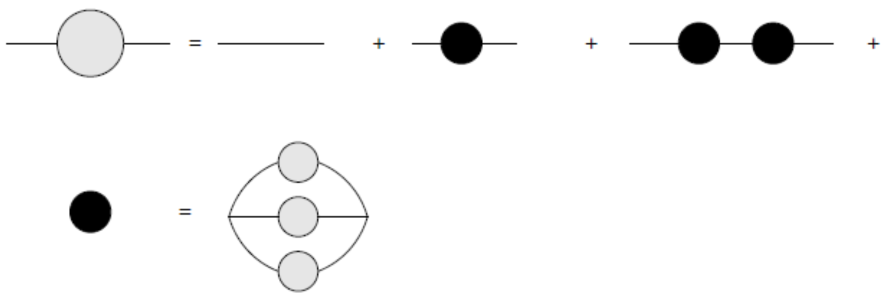

Figure 3. The Self Energy

Based on these diagrams,

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega), \;\; \Sigma(\tau) = J^2[G(\tau)]^3$$

At strong coupling or low temperatures, $i\omega$ can be ignored in the above equation, which gives $\frac{1}{G(\omega)} = -\Sigma(\omega)$ This is equivalent to,

$$\int d\tau' G(\tau,\tau')\Sigma(\tau',\tau'') = -\delta(\tau-\tau''), \;\; \Sigma(\tau,\tau') = J^2[G(\tau,\tau')]^3$$

Now, these equations are invariant under the reparametrizations of,

$$G(\tau,\tau') \to [f'(\tau)f'(\tau')]^{\frac{1}{4}}G\big(f(\tau),f(\tau')\big),\ \Sigma(\tau,\tau') \to [f'(\tau)f'(\tau')]^{\frac{3}{4}}\,\Sigma\big(f(\tau),f(\tau')\big)$$

Hence, as we discussed earlier, the SYK model exhibits reparametrization invariance at low temperatures.

To solve the original equation $\frac{1}{G(\omega)} = -\Sigma(\omega),\ \Sigma(\tau) = J^2[G(\tau)]^3$,

Let's try, $G(\tau) = \frac{b}{|\tau|^{\frac{1}{2}}}sgn(\tau)$. Using the following, Fourier transformations,

$$\int d\tau e^{i\omega\tau}\frac{sgn(\tau)}{|\tau|^{1/2}} = (1+i)|\omega|^{\frac{1}{2}}(sgn(\omega)+1)/2$$

$$\int d\tau e^{i\omega\tau}\frac{sgn(\tau)}{|\tau|^{3/2}} = -(1-i)|\omega|^{\frac{1}{2}}(1+sgn(\omega))$$

We get,

$$J^2 b^q \pi = \frac{1}{4}$$

Observe that $G(\tau)$ has a power law form, which is a characteristic of a critical point.

In addition, by reparametrization symmetry, one can set $f(\tau) = \tan\frac{\tau\pi}{\beta}$, and hence get the following solution as well.

$$G(\tau) = b\left[\frac{\pi}{\beta\sin\frac{\pi\tau}{\beta}}\right]^{1/2} sgn(\tau)$$

This is the finite temperature version [1].

4. Dynamics: The Out of Time Order Correlation Function

As discussed previously, the SYK model (and the SY model) exhibits quantum chaos. A way to study this is by using the out of timer order correlator, $\langle A(t)B(0)A(t)B(0)\rangle$. By letting $|x\rangle = A(t)B(0)|0\rangle$, $|y\rangle = B(0)A(t)|0\rangle$,

$$\langle A(t)B(0)A(t)B(0)\rangle = \langle y|x\rangle$$

At $\mathrm{t} = 0$, A usually commutes with B, and hence $|x\rangle = |y\rangle$ and this results in,

$$\langle A(0)B(0)A(0)B(0)\rangle = \langle x|x\rangle$$

However, as t increases, the more chaotic the system, the faster $\langle y|x\rangle$ averages out to 0. Therefore, by studying how fast the out of time order correlator decays, one can compare the amount of chaos inherent in a system.

Maldacena and Stanford found, $F(t_1, t_2) = Tr\left[y\psi_i(t_1)y\psi_j(0)y\psi_i(t_2)y\psi_j(0)\right]$, $\mathrm{y} = \rho(\beta)^{\frac{1}{4}}$ satisfies the following equation in the conformal limit.

$$F(t_1, t_2) = \frac{e^{-h\frac{\pi}{\beta}(t_1+t_2)}}{[\cosh\frac{\pi}{\beta}(t_1 - t_2)]^{\frac{1}{2}-h}}$$

When $t_1 = t_2$, the equation simplifies to $F(t_1, t_1) = e^{\frac{2\pi}{\beta}t_1}$ [2].

On the other hand, Fu and Sachdev studied the evolution of the out of time order correlator at infinite temperature. They defined the out of time order correlator as follows.

$$\mathrm{OTOC} = -\frac{\langle A(t)B(0)A(t)B(0)\rangle + \langle B(0)A(t)B(0)A(t)\rangle}{2\langle AA\rangle\langle BB\rangle}$$

They chose A and B as $A = c_1 + c_1^\dagger$, $B = c_2 + c_2^\dagger$. Figure 4 is their result at infinite temperature with different Js, and Figure 5 is their result at different temperatures, with $J = 1$ [1].

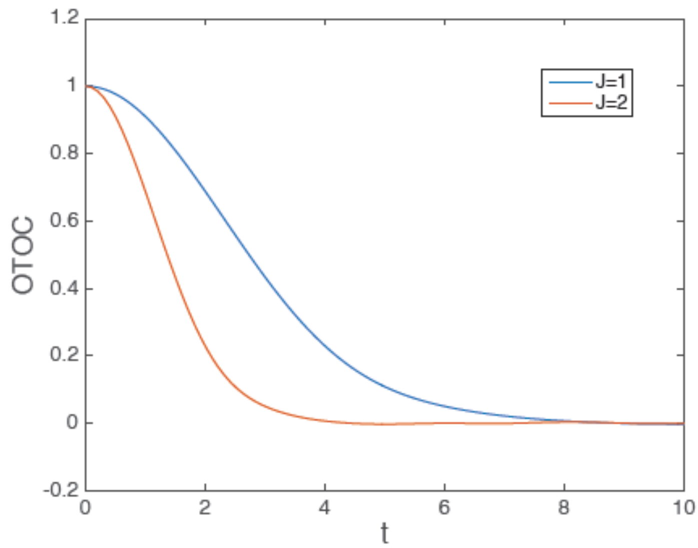

Figure 4. Time vs Out of Time Order Correlator (Infinite Temperature)

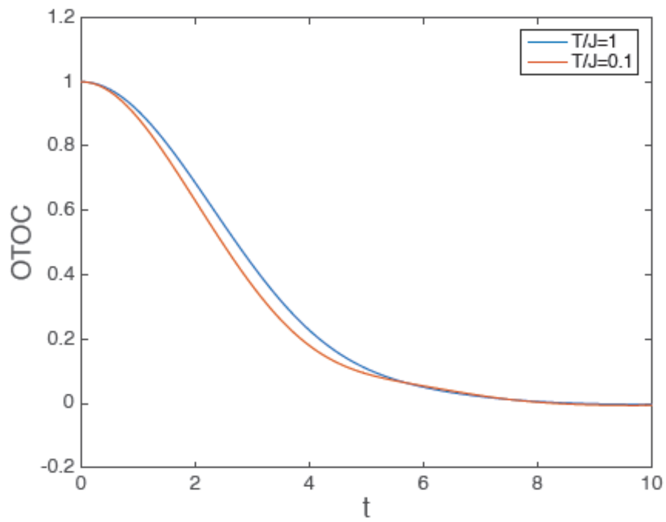

Figure 5. Time vs Out of Time Order Correlator (Finite Temperature)

It is evident that the decay depends on J the most. The decay rate is not dependent on β as much, but this is somewhat explained by the fact that we are very far away from the conformal limit.

5. Eigenstate Thermalization Hypothesis

According to the Eigenstate Thermalization Hypothesis, if the system satisfies

$$\left|A_{(n+1)(n+1)} - A_{nn}\right| \leq e^{-\alpha N} \ \& \ |A_{mn}| \leq e^{-\beta N}$$

where $A_{mn} = \langle m|A|n\rangle$, with m, n being mth and nth eigenstates of the Hamiltonian, then the for any $|\psi\rangle$, $\langle\psi|A|\psi\rangle$ reaches thermal equilibrium at some $A_{th}(E)$, where E is $|\psi\rangle$'s energy expectation value [3, 4].

Therefore, by studying whether the SYK model satisfies the above conditions or not for an operator, one can see whether or not the system has chaos, for thermalization signifies the existence of chaos.

Hence, we tested the hypothesis by looking at the distribution of $A = \langle\psi_0\psi_2\psi_4\psi_6\rangle$ at different Ns.

First, we created a series of matrices with different dimensions. The following figures show their general structure. The left is $\mathrm{N} = 18$, and the right is $\mathrm{N} = 20$. The more navy it is, the closer the value is to 0.

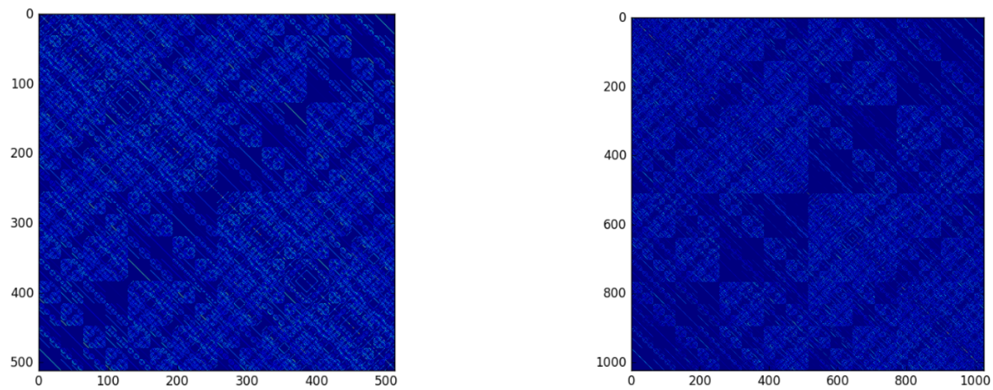

Figure 6. The Hamiltonian of N = 18, N = 20

The eigenstates of the Hamiltonian shows the following distribution at $\mathrm{N} = 24$.

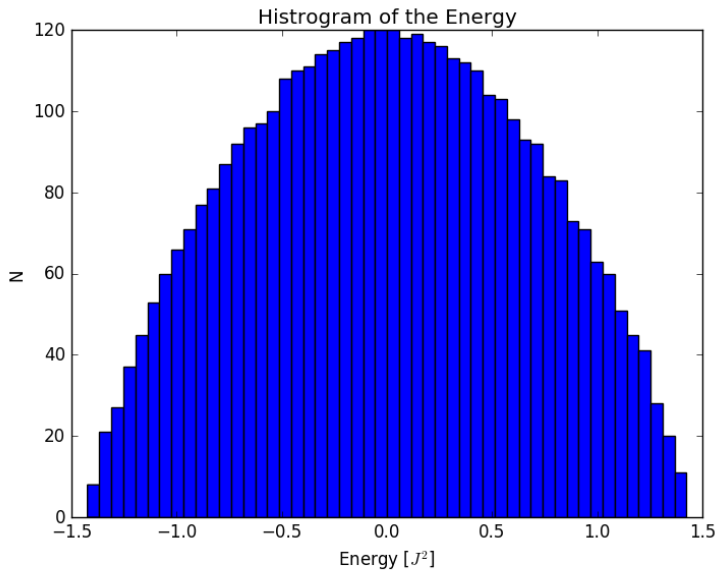

Figure 7. Histogram of the Energy of Eigenstates

The result is in agreement with [2].

The result was as in the following graph. Red dots represents $\mathrm{N} = 20$, green, $\mathrm{N} = 22$, blue, $\mathrm{N} = 24$, and cyan, $\mathrm{N} = 26$

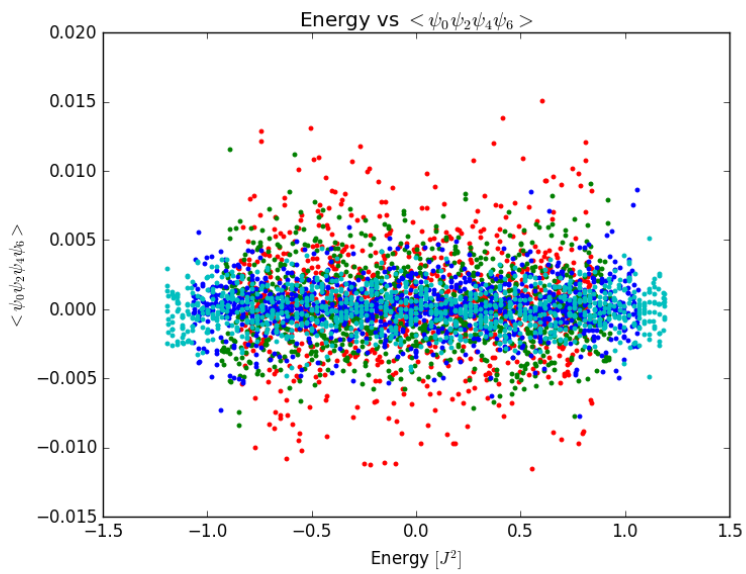

Figure 8. Distribution of $\langle\psi_0\psi_2\psi_4\psi_6\rangle$

Clearly, as N increases, the dispersion of $\langle\psi_0\psi_2\psi_4\psi_6\rangle$ decreases. This suggests that the Eigenstate Thermalizaiton Hypothesis holds in the SYK model.

Analyzing more closely, we find indeed that the Eigenstate Thermalization Hypothesis works. Below is a graph of N vs log $\langle|A_{(n+1)(n+1)} - A_{nn}|^2\rangle$.

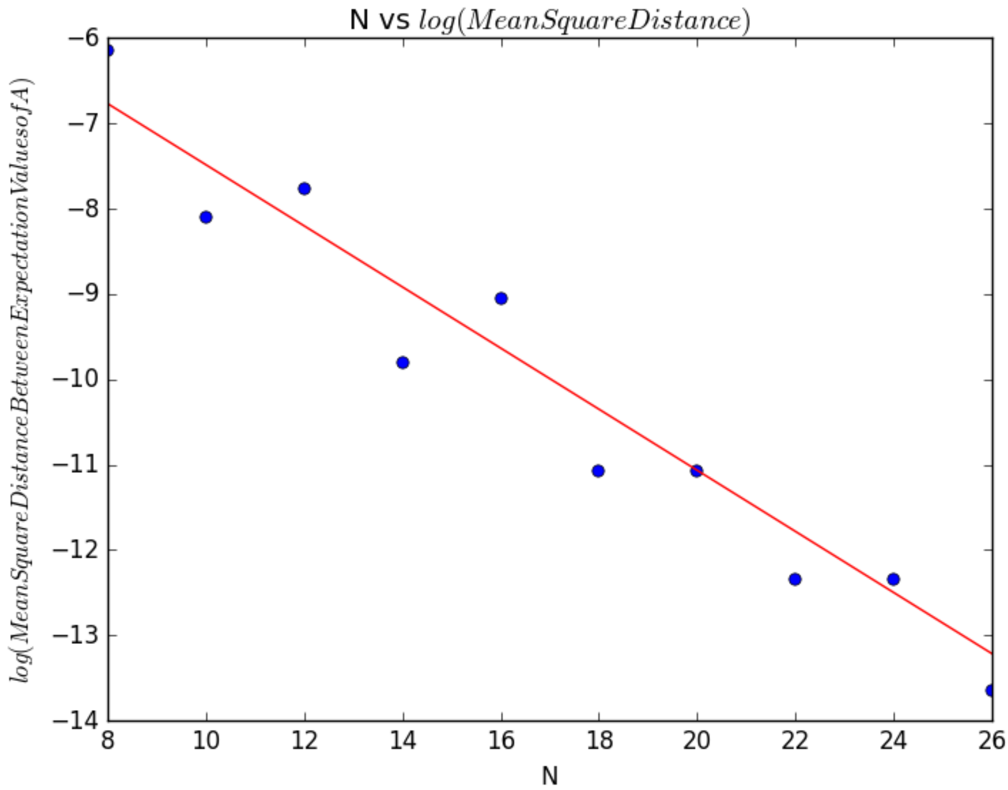

Figure 9. N vs log $\langle|A_{(n+1)(n+1)} - A_{nn}|^2\rangle$

Clearly, as N increases, log $\langle|A_{(n+1)(n+1)} - A_{nn}|^2\rangle$ decreases linearly. That is, $|A_{(n+1)(n+1)} - A_{nn}|$ decreases exponentially.

$|A_{mn}|, m \neq n$ follows the same trend. Figure 10 is a graph of N vs log $(\langle|A_{mn}|^2\rangle)$

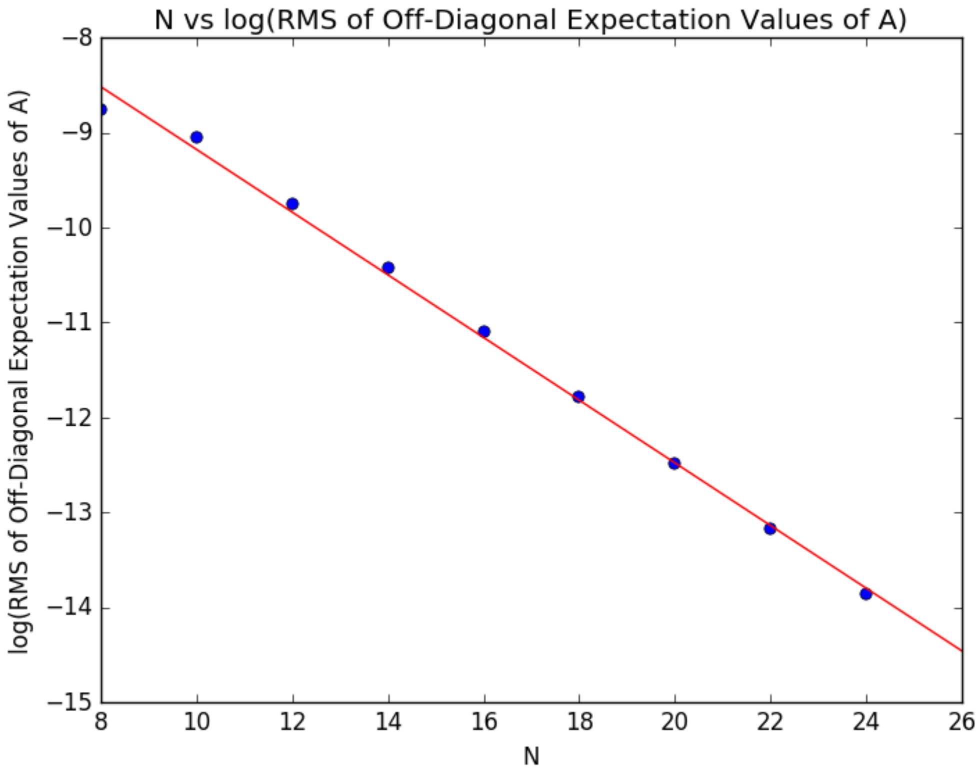

Figure 10. $\mathbf{N}$ **vs** $\mathbf{log}\left(\langle|A_{mn}|^2\rangle\right)$

Figure 9, and Figure 10 confirm that the Eigenstate Thermalization Hypothesis works with the SYK model, and supports the claim that the model is chaotic.

6. Quadratic Perturbation

We tested how the SYK model at $q = 4$ behaves when subjected to quadratic perturbations, which is by itself, integrable. The Hamiltonian that we used is as follows, and t's value was between 0 and 1, and $\mathrm{N} = 20$

$$H = (1 - \mathrm{t}) \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le N} j_{i_1 i_2 i_3 i_4} \psi_{i_1} \psi_{i_2} \psi_{i_3} \psi_{i_4} + \mathrm{t} \sum_{1 \le i_1 < i_2 \le N} j'_{i_1 i_2} \psi_{i_1} \psi_{i_2}$$

$$\langle j^2_{i_1 i_2 i_3 i_4} \rangle = \frac{6 J^2}{N^3}, \quad \langle j'^2_{i_1 i_2} \rangle = \frac{J^2}{N}$$

Similar to when we were testing the Eigenstate Thermalization Hypothesis, we tested the amount of perturbation by studying how the distribution of $\langle \psi_0 \psi_2 \psi_4 \psi_6 \rangle$ changes according to t.

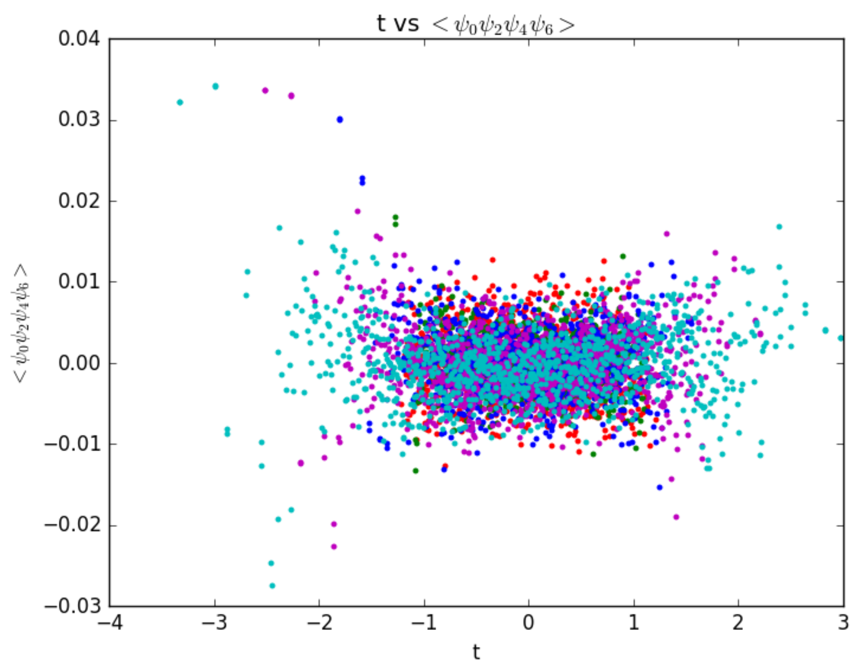

Figure 11. Expectation Value of $\langle \psi_0 \psi_2 \rangle$ according to t

Red is $\mathrm{t} = 0$, green, $\mathrm{t} = 0.25$, blue, $\mathrm{t} = 0.5$, purple, $\mathrm{t} = 0.75$, cyan, $\mathrm{t} = 1$. It seems as though the dispersion of $\langle\psi_0\psi_2\psi_4\psi_6\rangle$ decreases slightly with increasing t.

Let's study whether the decrease of the dispersion is actually due to chaos or not. Below is a graph of the distribution of the expectation values of $\langle\psi_0\psi_2\psi_4\psi_6\rangle$ at different Ns. Red is $N = 10$, green, $N = 12$, blue, $N = 14$, purple, $N = 16$, cyan, $N = 18$, light green, $N = 20$.

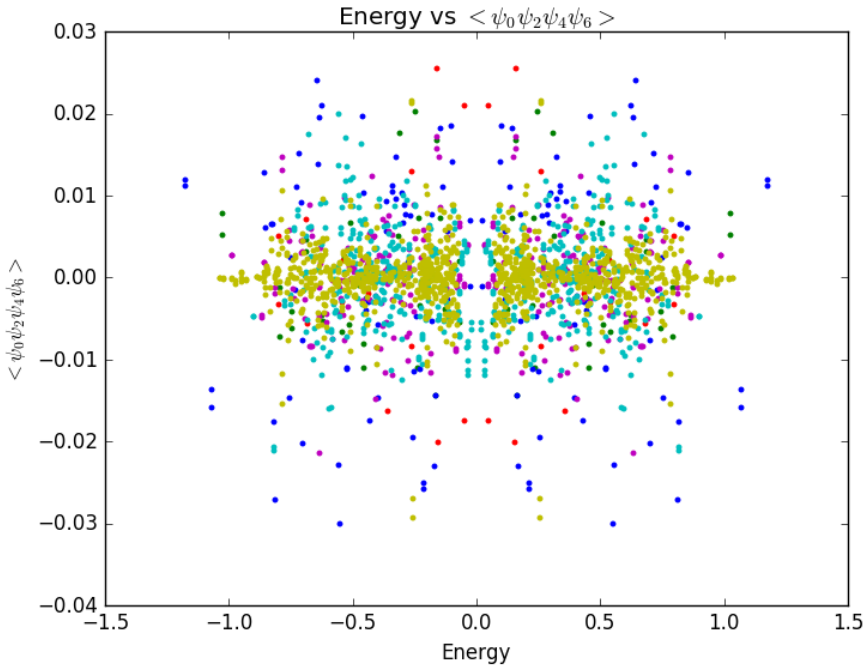

Figure 12. $t = 1$, Distribution of $\langle\psi_0\psi_2\psi_4\psi_6\rangle$ According to Different Ns

Although it seems as though the dispersion decreases as N increases, this system actually does not satisfy the Eigenstate Thermalization Hypothesis. The following graph of N vs $\langle|A_{(n+1)(n+1)} - A_{nn}|^2\rangle$ clearly illustrates this point, for $\langle|A_{(n+1)(n+1)} - A_{nn}|^2\rangle$ does not decrease exponentially.

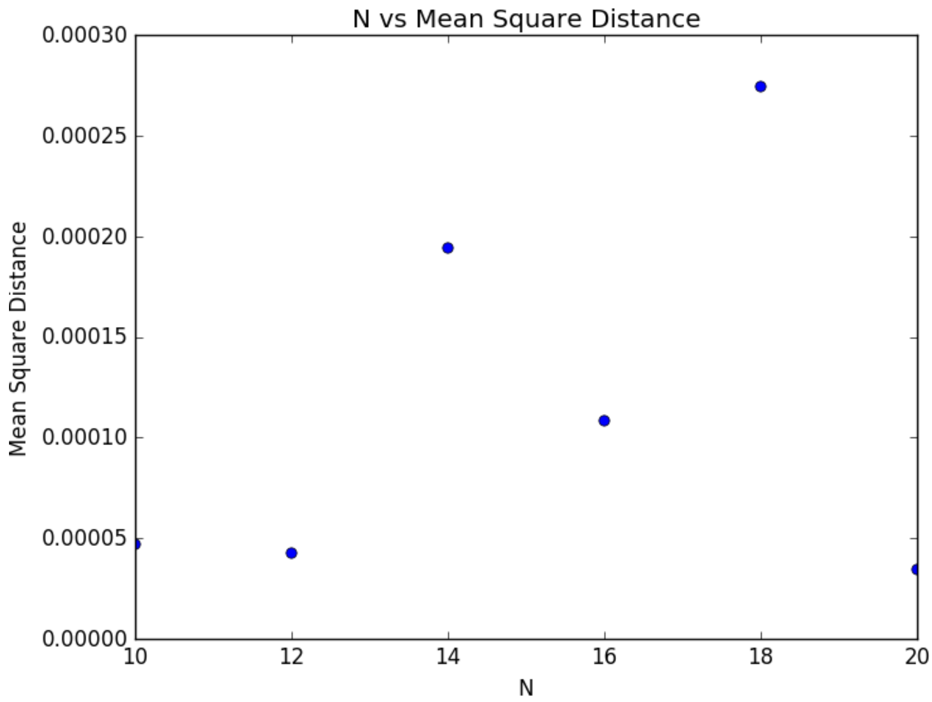

Figure 13. N vs log $\langle |A_{(n+1)(n+1)} - A_{nn}|^2 \rangle$

We also studied the evolution of the out of time order correlator. The out of time order correlator that we studied was,

$$\frac{\langle \psi_0(\mathrm{t})\psi_2(0)\psi_0(\mathrm{t})\psi_2(0)\rangle + \langle \psi_2(0)\psi_0(\mathrm{t})\psi_2(0)\psi_0(\mathrm{t})\rangle}{\langle \psi_0\psi_0\rangle\langle \psi_2\psi_2\rangle}$$

The out of time order correlator was evaluated at infinite temperature. The following graph is the result that we obtained.

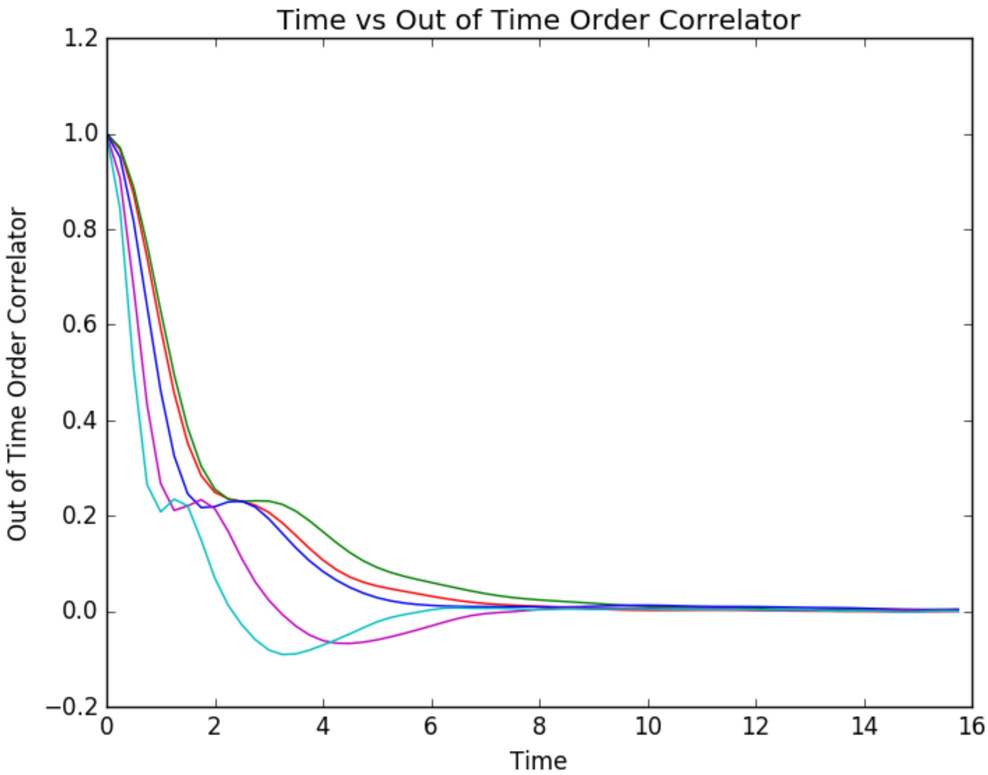

Figure 14. Time vs Out of Time Order Correlator

Red is $t = 0$, green, $t = 0.25$, blue, $t = 0.5$, purple, $t = 0.75$, cyan, $t = 1$. In general, with increasing t the rate of decay increases.

The increase of the rate of decay suggests that as the amount of quadratic perturbation increases, the chaos of the system increases, for the out of time order correlator decays faster in a more chaotic system.

However, the system at $t = 1$ is integrable, and hence is expected to show the least of amount of chaos. This contradictory result is a subject for further study, and suggests that the result of the out of time order correlator should be interpreted with caution.

7. Conclusion

In this paper, we studied the thermodynamics of the SY model. We found that the entropy per degree of freedom converges to a nonzero value at low temperatures, which is a characteristic that the SYK model shares. We also showed that the SYK model exhibits reparametrization and conformal symmetry at low temperatures. Furthermore, we looked at Fu and Sachdev's result on out of time order correlators. Observing how the out of time order correlator decays at different temperatures and with different J values. Although Fu and Sachdev's result was that the decay does not depend much on T, this can be explained by the fact that their analysis was done quite far away from the conformal limit.

By numerical calculation, we found that the SYK model satisfies the Eigenstate Thermalization Hypothesis. This was studied by finding the relation between N and $\langle |A_{(n+1)(n+1)} - A_{nn}|^2 \rangle$, and N and A_{mn}. This is further proof of the chaos underlying the SYK model.

We also studied how the SYK model behaves under a quadratic perturbation. We found that the dispersion of the expectation value of the operator $\langle \psi_0 \psi_2 \psi_4 \psi_6 \rangle$ somewhat decreases according to the increase of perturbation. Furthermore, we found that with increasing perturbation the decay occurs at a faster pace. These finding implies that with increasing perturbation, more chaos ensues. However, the fact at $t = 1$, the system is integrable and hence has no chaos (which has been proven by studying whether the ETH works at $t = 1$) suggests that this result should be interpreted with caution.

Further work, however, is still needed to understand the SYK model more fully. For one, looking at Figure 8, it looks like $N \equiv 2 \pmod 4$ behaves differently than $N \equiv 0 \pmod 4$. This is certainly an interesting result that needs more scrutiny.

8. Acknowledgements

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