Multipole Mermin-Wagner Constraints

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Outline

Ordinary Mermin-Wagner theorem

Multipole symmetries

Multipole field theories and Mermin-Wagner

Mermin-Wagner for partial breaking of multipole symmetries

Mermin-Wagner theorem

Consider a system with an internal $U(1)$ symmetry

Assume the symmetry is broken $\rightarrow$ Goldstone boson

$$\mathcal{L} = (\partial_t \phi)^2 + (\partial_i \phi)^2$$

Correlation functions go as

$$\begin{aligned} \langle \phi(x,t)\phi(0,0) \rangle &= \int \frac{d\omega dk\, k^{d-1}}{\omega^2 + k^2} \\ &= \int \frac{dk}{k} k^{d-1}, \end{aligned}$$

which diverges (in the IR) when $d \leq 1$

Interpretation

When the correlation function does not diverge ($d > 1$), there can be a phase distinguished by symmetry properties

At the critical dimension ($d = 1$), there can still be a phase distinguished by a non-vanishing compressibility

Luttinger liquid

Quasi-long-range order

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Multipole symmetries

Recall that an ordinary symmetry acts as $\phi(x) \to \phi(x) + c$

Gauge "symmetries" are not physical symmetries

Generalize to $\phi(x) \to \phi(x) + P(x)$

$P(x) = a_i x^i + b$, $c(x^2 + 1)$, etc.

Gauging these symmetries can lead to fractons [Pretko 2018]

Multipole groups

Choose an internal group $G_{\rm int}$ and a set of polynomials

The internal group has to be Abelian [Glorioso et. al. 2020], stick with $G_{\rm int} = U(1)$

There is a generator corresponding to each polynomial [Gromov 2019]

$\mathcal{P}_0, \mathcal{P}_1^i, \mathcal{P}_2^{ij}$, etc.

$$\exp(c\mathcal{P}_0)\phi(x) = \phi(x) + c$$

$$\exp(a_{ij}\mathcal{P}_2^{ij})\phi(x) = \phi(x) + a_{ij}x^i x^j$$

(restriction to monomials makes indices simpler)

Multipole groups

Generators might not commute with space symmetries [Gromov 2019]

$$[T_j, \mathcal{P}_a^{i_1 \dots i_a}] = \delta_j^{i_m} \mathcal{P}_{a-1}^{i_1 \dots i_{m-1} i_{m+1} \dots i_a}$$
$$[R_{jk}, \mathcal{P}_a^{i_1 \dots i_a}] = \delta_{[j}^{(i_1} \mathcal{P}_a^{k] i_2) \dots i_a}$$

For example, $[T_1, \mathcal{P}_1^1] = \mathcal{P}_0$ and $[R_{12}, \mathcal{P}_1^1] = \mathcal{P}_1^2$

Include all resulting polynomials, or exclude some translations or rotations

Maximal multipole group

$\mathcal{M}^a_{\max}$ contains all polynomials of degree $< a$
Note: this is NOT the definition in the literature, but it makes formulas nicer

Any multipole group is therefore a subgroup of $\mathcal{M}^a_{\max}$ for some a

Also contains all translations and rotations

$\mathcal{M}^1_{\max}$ is ordinary $U(1)$, $\mathcal{M}^0_{\max}$ is the trivial group

Outline

Multipole-invariant field theories

Example: $\mathcal{L} = (\partial_t \phi)^2 + g(\partial_i \partial_j \phi)^2$ is invariant under $\mathcal{M}^2_{\max}$

For a generic multipole group, it is not possible to find such a theory [Gromov 2019]

Interesting examples do exist, and can lead to fracton phases after gauging [Pretko 2018; Gromov 2019]

A special example leads to Haah's code [Bulmash, Barkeshli 2018]

Multipole-invariant field theories

Stick with rotationally invariant theory

$$\mathcal{L} = (\partial_t \phi)^2 + g(\partial_{I_a}\phi)^2,$$

where $I_a = i_1 \dots i_a$ is a multi-index and

$$\partial_{I_a} = \partial_{i_1} \dots \partial_{i_a}$$

This is invariant under $\mathcal{M}^a_{\mathrm{max}}$

Interesting things to say about total derivatives and rotational invariance...

Mermin-Wagner for multipole theories

$$\mathcal{L} = (\partial_t \phi)^2 + g(\partial_{I_a}\phi)^2$$

Assume the ϕ field exists. Is long-range order possible?

$$\begin{aligned}\langle \phi(x,t)\phi(0)\rangle &\sim \int \frac{d\omega dk\, k^{d-1}}{\omega^2 + k^{2a}} \\ &= \int \frac{dk}{k} k^{d-a}\end{aligned}$$

IR divergence for $d \leq a$ [Griffin 2015]

Critical dimension at $T = 0$ is $d_{\rm c} = a$

Mermin-Wagner for multipole theories

What other possibilities are there?

Non-maximal groups, eg. Haah group

Partial breaking from $\mathcal{M}^a_{\max} \to \mathcal{M}^c_{\max}$

Combine both?

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Mermin-Wagner for partial breaking, I

Naive method to break from $\mathcal{M}^a_{\max}$ to $\mathcal{M}^c_{\max}$ with $c < a$

Define field $\phi_{i_1 \ldots i_c} = \phi_{I_c}$

$\exp(\beta_{I_b} \mathcal{P}_b^{I_b}) \phi_{J_c}(x) = \phi_{J_c} + \beta_{J_c I_{b-c}} x^{I_{b-c}}$

Invariant under polynomials of degree $< c$

ϕ_{I_c} develops LRO: $\mathcal{M}^a_{\max} \to \mathcal{M}^c_{\max}$

Example: ϕ_i would break the dipole group $\mathcal{M}^2_{\max}$ to the monopole group $\mathcal{M}^1_{\max}$

[CNS, Lake, Nandkishore]

Mermin-Wagner for partial breaking, I

Naive result: can ϕ_{I_c} develop LRO?

Consider $\mathcal{L} = (\partial_t \phi_{I_c})^2 + (\partial_{J_{a-c}} \phi_{I_c})^2$. Then

$$\begin{aligned} \langle \phi_{I_c}(x,t)\phi_{I_c}(0)\rangle &\sim \int \frac{d\omega dk\, k^{d-1}}{\omega^2 + k^{2a-2c}} \\ &= \int \frac{dk}{k} k^{d-a+c} \end{aligned}$$

It looks like the critical dimension should be $d_\mathrm{c} = a - c \ldots$

Mermin-Wagner for partial breaking, II

The operator $\partial_{I_{c-b}}\phi_{J_b}$ is breaks $\mathcal{M}^a_{\max} \to \mathcal{M}^c_{\max}$

Let's look at the theory $\mathcal{L} = (\partial_t\phi_{J_b})^2 + (\partial_{I_{a-b}}\phi_{J_b})^2$

First b compressibilities are all 0 $\to$ specifies theory

$\kappa_e \equiv \frac{d\langle n_e^{I_e}\rangle}{d\mu_e^{I_e}} = 0$ for $e \leq b$

Now the correlation function becomes

$$\langle \partial_{I_{c-b}}\phi_{J_b}(x,t)\partial_{I_{c-b}}\phi_{J_b}(0,0)\rangle = \int \frac{d\omega dk\, k^{d-1}\, k^{2(c-b)}}{\omega^2 + k^{2(a-b)}}$$
$$= \int \frac{dk}{k} k^{d-a-b+2c}$$

and $\partial_{I_{c-b}}\phi_{J_b}$ can only order if $d \geq a + b - 2c$

Interpreting the critical dimension

a and b define the theory, c defines the preserved subgroup

At fixed a, b, c

- In this theory, $\mathcal{M}_{\max}^{c+1}$ can only be broken if $d > a + b - 2c$
- Doesn't tell you what the preserved subgroup actually is

At fixed a, b, d: What is the preserved subgroup?

- $\mathcal{M}_{\max}^{c}$ is broken for $c > (a + b - d)/2$
- Preserved subgroup is $\mathcal{M}_{\max}^{\lfloor (a+b-d)/2 \rfloor}$

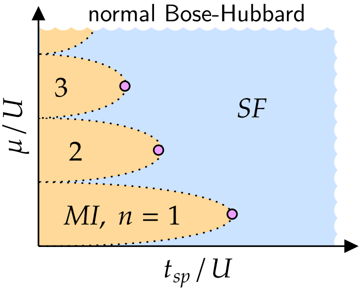
At fixed a, c: When is this SSB possible?

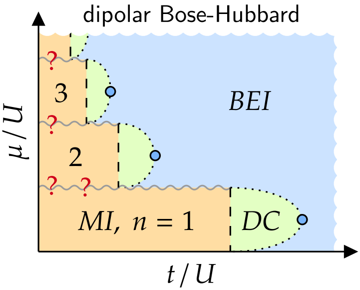
- Critical dimension is $d = \min_b a + b - 2c = a - 2c$
- Symmetry breaking is possible at $d = 1$!

Example: dipolar Bose-Hubbard model

Bose-Hubbard model with dipole constraint imposed

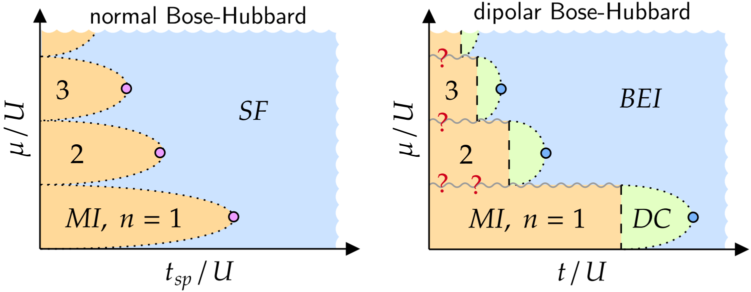
$$H = -t \sum_{i,\mu,\nu} b_i^\dagger b_{i+\mu} b_{i+\mu+\nu}^\dagger b_{i+\nu} - \mu \sum_i n_i + \frac{U}{2} n_i (n_i - 1)$$





[Lake, Hermele, Senthil]

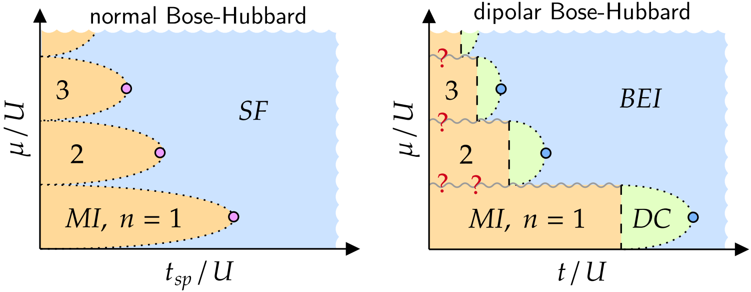
Example: dipolar Bose-Hubbard model



Dipole Condensate: $(\partial_t \phi_i)^2 + (\partial_j \phi_i)^2 \quad (b = 1)$

Bose-Einstein Insulator: $(\partial_t \phi)^2 + (\partial_j \phi)^2 \quad (b = 0)$

Example: dipolar Bose-Hubbard model

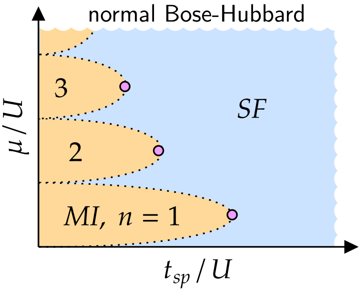


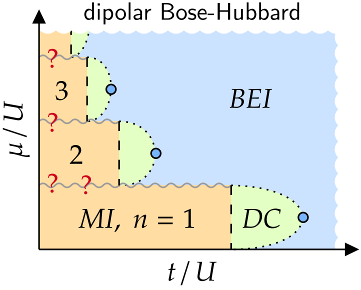
Normal Bose-Hubard model

$d \geq 2$: SF has SSB $\mathcal{M}^1_{\max} \to \mathcal{M}^0_{\max}$

$d = 1$: SF has no SSB, instead has QLRO

Example: dipolar Bose-Hubbard model





Dipolar Bose-Hubard model

$d \geq 3$: BEI:$\mathcal{M}^2_{\max} \to \mathcal{M}^0_{\max}$, DC: $\mathcal{M}^2_{\max} \to \mathcal{M}^1_{\max}$

$d = 2$: BEI:$\mathcal{M}^2_{\max} \to \mathcal{M}^1_{\max}$, DC: $\mathcal{M}^2_{\max} \to \mathcal{M}^1_{\max}$

$d = 1$: BEI:$\mathcal{M}^2_{\max} \to \mathcal{M}^1_{\max}$, DC: $\mathcal{M}^2_{\max} \to \mathcal{M}^2_{\max}$