Spontaneous breaking of multipole symmetries

Charles Stahl,[1] Ethan Lake,[2] and Rahul Nandkishore[1]

[1] *Department of Physics and Center for Theory of Quantum Matter, University of Colorado Boulder, Boulder CO 80309, USA*

[2] *Department of Physics, Massachusetts Institute of Technology, Cambridge, MA, 02139*

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Multipole symmetries are of interest both as a window on fracton physics and as a crucial ingredient in realizing new universality classes for quantum dynamics. Here we address the question of whether and when multipole symmetries can be *spontaneously broken* in thermal equilibrium or at zero temperature. We derive generalized Mermin-Wagner arguments for the total or partial breaking of multipolar symmetry groups and generalized Imry-Ma arguments for the robustness of such multipolar symmetry breaking to disorder. We present both general results and explicit examples. Our results should be directly applicable to quantum dynamics with multipolar symmetries and also provide a useful stepping stone to understanding the robustness of fracton phases to thermal fluctuations, quantum fluctuations, and disorder.

I. INTRODUCTION

Hamiltonians invariant under polynomial symmetry transformations conserve not only charge, but also various multipole moments of charge. Such 'multipolar' symmetries are known to offer a robust route to ergodicity breaking [1–6], and also to exotic universality classes of quantum dynamics [7–13]. They are known to arise in 'fracton' phases of quantum matter [14–19], the key dynamical properties of which are known to descend from conservation laws on multipole moments of charge [20–22]. They are also known to arise (in a prethermal sense) in various ultracold atom platforms [2, 23, 24]. There are thus multiple reasons for thinking about systems with multipolar symmetries. However, just because a symmetry is present in the *Hamiltonian* does not mean that it will be present in the *state*; there is always the possibility of spontaneous symmetry breaking (SSB).

For conventional symmetries, there exist general theorems which constrain the settings in which SSB can occur. In clean systems, the relevant theorem is due to Mermin and Wagner [25], and involves the physics of (thermal or quantum) fluctuations of the Goldstone modes associated with SSB, whereas in disordered systems the key results are due to Imry and Ma [26, 27], and also Aizenman and Wehr [28], and involve the physics of order parameter deformation for local alignment with disorder. Multipolar symmetries, however, allow for a much richer pattern of possible symmetry breakings (including breaking some but not all of the multipolar symmetries), and the analogous theorems have not yet been derived, except in the special case of isotropic clean systems with total breaking of the symmetry [29].

In this work, we place generalized Mermin-Wagner and Imry-Ma constraints on the total and/or partial breaking of multipolar symmetries, in both clean and disordered systems. Along the way we also discuss the exotic Goldstone modes associated with total or partial SSB of multipolar symmetries. We will also provide explicit models of multipole symmetry breaking, to give intuition for these unusual forms of SSB.

Throughout, we consider only multipole groups where the underlying internal group is continuous and abelian. For concreteness, we will say it is $U(1)$. Multipole groups with a nonabelian underlying symmetry suffer a cascade effect where the dynamics in at least one direction must be trivial [9] and we shall not discuss them here. We note that specific examples of spontaneous symmetry breaking of multipolar symmetries have been discussed (sometimes in a dual language) in [30–36]. Our goal here is to place general constraints on when certain symmetry breaking phase transitions involving multipolar symmetries can occur.

This paper is organized as follows. In Section II we introduce the multipole group, how to build multipole-invariant field theories, and the generalized Mermin-Wagner argument for total breaking of the maximal multipole symmetry. In Section III we explore phases in which multipole symmetries are partially broken. Then, we discuss generalized Imry-Ma arguments in the presence of quenched disorder in Section IV. Finally, we consider an explicit lattice model that illustrates some of these ideas in Section V before concluding with a discussion of open questions in VI.

II. THE MULTIPOLE GROUP

The multipole group is well-explained in Ref. [22]. For concreteness, and since we will be interested in situations where a multipolar symmetry group is spontaneously broken, we will describe the multipole group in terms of its action on a compact scalar field $\phi(x) \approx \phi(x) + 2\pi$. ethan: *are there any examples where ϕ is noncompact?* The multipole group generalizes the internal shift symmetry $\phi(x) \to \phi(x) + c$ by allowing a shift by some set of polynomials involving the spatial coordinates, viz. $\phi(x) \to \phi(x) + \lambda_\alpha P^\alpha(x)$. The variables λ_α are symmetry parameters, while α labels the set of polynomials $P^\alpha(x)$. These so-called polynomial shift symmetries [37] all commute with each other. It is helpful to limit our-

selves to homogeneous polynomials. We can label these as $P_a^{I_a}$, where a is the degree of the polynomial and I_a is an abstract index that runs over the polynomials of degree a.

The full structure of the multipole group comes into play when we also include spatial symmetries. For example, translation in the x_1 direction will fail to commute with any polynomial shift where the polynomial is a function of x_1. Thus, if we want to consider a collection of polynomial shift symmetries, we must consider whether that collection closes under conjugation by translations and rotations. If it does not, we must either exclude the offending translations or rotations, or expand the set of polynomial shift symmetries. The result is a multipole symmetry group [22].

A. Examples

Reference [22] includes discussion of some multipole groups; we will review a few here. The simplest case is the maximal multipole group $\mathcal{M}^a_{\text{max}}$, which includes all shifts by polynomials of degree a or less. Individual polynomials can be written as

$$P_c^{I_c} = \mu^{I_c}_{i_1 \dots i_c} x^{i_1} \dots x^{i_c}, \tag{1}$$

where each matrix $\mu^{I_c}_{i_1 \dots i_c}$ is fully symmetric and $c \leq a$. This group also includes all translations and rotations.

An example of a multipole group that contains all translations and rotations but is *not* the maximal multipole group is the group generated by shifts of the form

$$\phi \to \phi + \lambda_0 P_0^0 + \lambda_i P_1^i + \lambda_{I_2} P_2^{I_2}, \tag{2}$$

where the degree-0 polynomial is $P_0^0 = 1$. The other polynomials are

$$P_1^i = x^i, \qquad P_2^{I_2} = \mu^{I_2}_{ij} x^i x^j, \tag{3}$$

where μ^{I_2} are a basis for the rank-2 traceless symmetric $d \times d$ matrices. Let us call this group $\mathcal{M}^2_{\text{sym}}$. Recall that the maximal quadrupole group $\mathcal{M}^2_{\text{max}}$ is already only built from symmetric matrices μ. The tracelessness condition thus only removes one polynomial from the set.

This set of polynomial shift symmetries is compatible with all rotations because no rotation will generate a traceful matrix from a traceless one. The set of symmetric matrices, seen as a representation of the group of rotations, decomposes into two independent irreducible representations. One of these is the set of traceless matrices while the other is the single matrix δ_{ij}. In fact, the set of polynomial shift symmetries consisting of constant and linear shifts along with shifts of the form $\delta\phi \propto x^i x^i$ is also compatible with all rotations. We could call this group $\mathcal{M}^2_{\text{tr}}$.

There is one multipole group worth mentioning that does not include all rotations. This is the multipole group corresponding to Haah's U(1) code [21, 22, 38]. We will explain the correspondence in the next subsection. The group itself consists of all translations, a single rotation about the $(1,1,1)$ axis (on the cubic lattice), and shifts by five polynomials [22]. These are

$$\begin{aligned} P_0^0 &= 1, \\ P_1^1 &= x_1 - x_2, \\ P_1^2 &= x_1 + x_2 - 2x_3 \\ P_2^1 &= (x_1 - x_2)(x_1 + x_2 - 2x_3) \\ P_2^2 &= (2x_1 - x_2 - x_3)(x_2 - x_3). \end{aligned} \tag{4}$$

Although this looks complicated, we can simplify the presentation by first choosing a new spatial basis and then also new basis for the polynomials.

If we define new variables $x = (x_1 - x_2)/\sqrt{2}$, $y = (x_1 + x_2 - 2x_3)/\sqrt{6}$, and $z = (x_1 + x_2 + x_3)/\sqrt{3}$, then we can write the Haah group as all translations, a single rotation in the $x - y$ plane, and

$$\begin{aligned} P_1^1 &= \sqrt{2}x, \\ P_1^2 &= \sqrt{6}y, \\ P_2^1 &= \sqrt{12}xy \\ P_2^2 &= 6y^2 + 2\sqrt{12}xy - 6x^2. \end{aligned} \tag{5}$$

Finally, a basis change and redefinition for the polynomials allows us to write

$$\begin{aligned} P_1^1 &= x, \\ P_1^2 &= y, \\ P_2^1 &= xy \\ P_2^2 &= y^2 - x^2. \end{aligned} \tag{6}$$

Note that no polynomials depend on z.

The Haah multipole group thus has a product structure, $\mathcal{M}_{\text{Haah}} = \mathbf{R} \times \mathcal{M}^2_{\text{sym}}$, where the $\mathbf{R}$ corresponds to translations along z. The second group $\mathcal{M}^2_{\text{sym}}$ is the submaximal quadrupole group in 2 dimensions, only containing quadrupole shifts corresponding to symmetric traceless tensors.

B. Multipole field theories

We can now consider building IR field theories for phases in which a multipolar symmetry is completely spontaneously broken, with no residual unbroken subgroup. Reference [22] describes the process in detail, so in the following we will be somewhat succinct. We will continue to write things down in terms of the compact scalar ϕ, which (in the cases for which SSB is not preempted by strong fluctuations) is to be viewed as the Goldstone for the sponatneously broken symmetry.

Since we are only interested in the low-energy physics of the putative symmetry-broken phase, constructing an appropriate IR field theory amounts to nothing more

than constructing an appropriate kinetic term for ϕ. To find a kinetic term invariant under a given multipole group $\mathcal{M}$, we need to find operators D built out of spatial derivatives that annihilate all the polynomials in $\mathcal{M}$. If $a_{\max}$ is the highest degree of the polynomials in $\mathcal{M}$, the simplest derivative operators which generically do the job are of the form

$$D = q^{i_1 \dots i_{a_{\max}+1}} \partial_{i_1} \dots \partial_{i_{a_{\max}+1}}, \tag{7}$$

for some symmetric tensor $q^{i_1 \dots i_{a_{\max}+1}}$. Although it is not generically possible to do so [22], we can sometimes find a set of D_α (α is an abstract index) with $s \leq a_{\max}$, where $a_{\max}$ is the highest degree of the polynomials. In this case, the effective field theory will be invariant under some non-maximal multipole symmetry.

Writing the invariant derivative operators as D_α, the most general kinetic term is [22]

$$K[\phi(x)] = g_{\alpha\beta}(D_\alpha \phi)(D_\beta \phi) \tag{8}$$

for some symmetric tensor $g_{\alpha\beta}$. We will often write the Fourier transform of the kinetic term as $\phi_{-k} K_k \phi_k$. Requiring some spatial symmetries restricts the choices of $g_{\alpha\beta}$. For the maximal multipole group, enforcing all rotation symmetries results in the kinetic term

$$K_a[\phi(x)] = (\partial_{i_1} \dots \partial_{i_{a+1}} \phi)(\partial_{i_1} \dots \partial_{i_{a+1}} \phi), \tag{9}$$

which is the kinetic terms studied in Ref. [37].

For $a > 0$, K_a can be split into multiple terms while still remaining rotationally invariant. For example, the kinetic term for $\mathcal{M}^1_{\max}$ is

$$K_1[\phi(x)] = g_1(\partial^2 \phi)^2 + g_2 \sum_{ij} (\partial_i \partial_j \phi)^2. \tag{10}$$

For the special cases of $g_2 = 0$ or $g_2 = -g_1$, the symmetry group expands to $\mathcal{M}^2_{\text{sym}}$ or $\mathcal{M}^2_{\text{tr}}$, respectively. However, for generic g_1, g_2 the symmetry group is $\mathcal{M}^1_{\max}$. Similar statements can be made for larger a.

Finally, some of these multipole symmetries can be gauged to arrive at effective field theories for fracton phases. Of course, after gauging the theory will have gapless excitations due to the U(1) symmetry, but they can be Higgsed to arrive at a gapped phase. This is the sense in which the Haah group mentioned earlier corresponds to a field theory for Haah's code [21]. See Ref. [21, 22] for the full story.

C. Generalized Mermin-Wagner argument

We are now ready to describe the generalized Mermin-Wagner argument for an arbitrary maximal multipole group. This already appears in Ref. [37], so we are simply reviewing it here in preparation for the more generic cases to follow. Here, we will discuss both thermal and $T = 0$ systems.

Consider first a clean classical system at $T > 0$ with a spontaneously broken maximal multipole symmetry of degree a. Then the kinetic term is proportional to

$$K[\phi(x)] = (\partial_{i_1} \dots \partial_{i_{a+1}} \phi)^2. \tag{11}$$

ethan: *really it's $\sim e^{i\phi}$ which is the order parameter* Now consider nucleating domains of linear size L in which the expectation value of the order parameter — namely $\langle e^{i\phi} \rangle$ — varies by an amount of order 1. Since the order parameter is continuous there will be a thick domain "wall" over which the order parameter changes gradually from its value in the background phase to its value in the domain. The thickness will be of order L. Equation 11 tells us the energy cost of the field changing its value will be minimized if it varies as a polynomial with degree $a+1$. In that case the energy density will scale as $K_k \sim k^{2(a+1)} \sim L^{-2(a+1)}$. Integrating that energy density over a region of size L gives a total energy cost to nucleating domains of $L^{d-2(a+1)}$.

In clean systems there is no energy gain from domain nucleation. When $d \leq 2(a+1)$, the energy cost is bounded, so the entropy gain favors domain creation. Thus, ordered phases are unstable and SSB cannot occur for $d \leq 2(a+1)$.

We can also reproduce this argument by considering correlation functions. As a warm-up, consider the standard case where a monopole U(1) symmetry is spontaneously broken, leading to a Goldstone boson $\phi(x)$. The correlation function

$$C(x) = \langle \phi(x)\phi(0) \rangle = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik\cdot x}}{k^2}, \tag{12}$$

diverges when $d \leq 2$. Correlation functions should not diverge, so the interpretation is that the Goldstone boson fluctuates strongly enough that the field $\phi(x)$ cannot be well defined. In turn, this tells us the symmetry could not have been broken. This is the classical Mermin-Wagner argument. We should note that, since we only care about long-distance divergences, we don't need to include the complex exponential as long as we only look for divergence at small k. We will thus drop this dependence in future calculations.

When we consider a (maximal) multipole symmetry, the dispersion changes to $K_k = k^{2(a+1)}$. Now the correlation functions will scale as

$$C \sim \int \frac{d^d k}{K_k} \sim \int \frac{d^d k}{k^{2(a+1)}}, \tag{13}$$

which diverges for $d \leq 2(a+1)$. As before, we interpret the divergence of the correlation function as a sign that the symmetry cannot be spontaneously broken. This allows us to reproduce our scaling argument that SSB of the multipole group $\mathcal{M}^a_{\max}$ cannot occur for $d \leq 2(a+1)$.

If we look at the zero temperature setting, the energy scaling no longer tells us where the critical dimension is. This is because even when there is no energy cost to forming domains, there is no entropy gain. Instead, quantum

fluctuations must be the motivation for domain nucleation. To find the quantum critical dimension, we calculate the correlation function by way of the (imaginary-time) IR Lagrangian

$$\mathcal{L}[\phi] = (\partial_\tau \phi)^2 + K[\phi]. \tag{14}$$

This gives

$$C(x) = \langle \phi(x)\phi(0)\rangle \sim \int \frac{d\omega d^d k}{\omega^2 + k^{2(a+1)}}, \tag{15}$$

which now diverges at $d \le a+1$ [37], where d is the number of spatial dimensions. We have halved the critical dimension, so that SSB cannot occur at $d \le a+1$. Note that the dynamical critical exponent in these theories is $z = a+1$, so that as expected, the classical and quantum critical dimensions are related by $d_{\rm cl} = d_{\rm q} + z$.

Furthermore, the structure of the quantum correlation function,

$$\begin{aligned} C &\sim \int \frac{d\omega d^d k}{\omega^2 + K_k} \\ &\sim \int \frac{d^d k}{\sqrt{K_k}}, \end{aligned} \tag{16}$$

suggests that, at least in some broad class of theories, the quantum critical dimension will be half the classical critical dimension. This is not true in general, though, as we will show for theories with quenched disorder.

D. Non-maximal multipole group

We can also consider the fate of symmetry breaking for multipole groups other than the maximal multipole group. In general this will not match the critical dimension for the maximal group.

For concreteness, we will consider some examples. First, recall the group $\mathcal{M}^2_{\rm sym}$ from Sec. II A. The group contains polynomials of degree 2. However, since the polynomials are all traceless, the most relevant derivative is $D = \sum_i \partial_i^2$. We can immediately see the dispersion is $(k^2)^2$, so that the critical dimension is $d_{\rm cl} = 4$ or $d_{\rm q} = 2$.

We can also consider an anisotropic multipole group. Let the conserved charges be the monopole moment and d_2 components of the dipole moment. Then the dispersion will be $k^4 + p^2$, where k has d_2 components, p has d_1 components, and $d = d_1 + d_2$. Classically, the correlation function is

$$\begin{aligned} C &\sim \int \frac{d^{d_2}k\, d^{d_1}p}{k^4 + p^2} \\ &\sim \int x^{d_2/2-1} p^{d_1-1} \frac{dx\, dp}{x^2 + p^2} \\ &\sim \int q^{d_2/2+d_1-2} \frac{dq}{q}, \end{aligned} \tag{17}$$

where $q^2 = x^2 + p^2 = k^4 + p^2$. If $d_1 = 2$ the symmetry can be broken for any $d_2 > 0$, and if $d_1 = 1$ then $d_2 = 2$ is critical. Of course, if $d_1 = 0$ we have an isotropic quartic dispersion and $d = d_2 = 4$ is critical.

We can recover this result in the energy-scaling argument. Since the system is anisotropic, the domains will have different sizes in different directions. Consider forming a domain of linear size L_p in the direction with quadratic dispersion and L_k in the quartic direction. When the gradient of the order parameter is in the quadratic direction, the order parameter has to change over a length L_p so the energy density is L_p^{-2}. Similarly, when the order parameter is changing in the quartic direction its energy density is L_k^{-4}.

Both of these energy densities will be integrated over domain "walls" with volume $L_k^{d_2} L_p^{d_1}$. As a result, the total energy cost is

$$E \sim L_k^{d_2} L_p^{d_1-2} + L_k^{d_2-4} L_p^{d_1}, \tag{18}$$

so that $L_p \sim L_k^2$ to make the terms match, and $E \sim L_p^{d_2/2+d_1-2}$. Recall that when E does not increase for larger domains, the entropy gain will cause them to nucleate, destroying the ordered phase. We see that the critical dimension $d = d_2 + d_1$ can be 2, 3, or 4, depending on how many directions have quadratic or quartic dispersions.

The quantum correlation function behaves as

$$\begin{aligned} C &\sim \int \frac{d\omega d^{d_2}k\, d^{d_1}p}{\omega^2 + k^4 + p^2} \\ &\sim \int k^{d_2-2} p^{(d_1-1)} \frac{k\, dk\, dp}{\sqrt{k^4 + p^2}} \\ &\sim \int x^{d_2/2-1} p^{d_1-1} \frac{dx\, dp}{\sqrt{x^2 + p^2}} \\ &\sim \int q^{d_2/2-1} q^{d_1-1} dq, \end{aligned} \tag{19}$$

where again $q^2 = x^2 + p^2 = k^4 + p^2$. This diverges when $d_2/2 + d_1 \le 1$.

The previous analysis suggests a procedure for finding the critical dimension in clean anisotropic systems for breaking the multipole symmetry to the trivial group. First, sort each dimension by the degree of its dispersion relation, so that there are d_n dimensions with dispersion k^{2n}. Then define the effective dimension as

$$d_{\rm eff} = \sum_n \frac{d_n}{n}. \tag{20}$$

In a classical system, we then conclude that SSB cannot occur if $d_{\rm eff} \le 2$, while in a quantum system the critical dimension is $d_{\rm eff} = 1$.

We should emphasize that the statements in this subsection were framed in terms of the dispersion of the Goldstone modes rather than the structure of the symmetry group. It is always possible to find the dispersion given the multipole symmetry group, but it may require some basis changes (as in the Haah multipole group).

III. PARTIAL BREAKING OF MULTIPOLE SYMMETRIES

In this section we examine what happens when a multipolar symmetry is spontaneously broken to a subgroup. Studying this problem will reveal a few ways in which our general expectations for symmetry breaking and universality need to be revised in the context of multipolar symmetries.

A. Breaking to a subgroup

In the previous section we assumed we had an order parameter field that fully broke the multipole symmetry. Let us now consider the case where the order parameter breaks the symmetry from the original group G to some subgroup H. For simplicity, we will let G be the maximal multipole group $\mathcal{M}^a_{\text{max}}$ and will let H be $\mathcal{M}^b_{\text{max}}$, with $b < a$.

As an illustrative example, consider the case when a dipolar symmetry is spontaneously broken to the monopolar subgroup, so that $a = 1$ and $b = 0$. In the spirit of Goldstone's theorem, the most natural thing to do would be to write down an action involving a set of fields θ_j which transform linearly under the dipolar part of the symmetry group, but which are invariant under the remaining monopole subgroup (i.e. under a transformation parameterized by $P(x) = \alpha + \beta_j x^j$, $\theta_j \mapsto \theta_j + \beta_j$). In this case, the minimal IR action for the putative symmetry-breaking phase would be

$$\mathcal{L} = \sum_j (\partial_\tau \theta_j)^2 + \sum_{i,j} g_{ij} (\partial_i \theta_j)^2. \tag{21}$$

By calculating correlation functions of the θ_j fields using the above action, we would conclude that SSB would be possible only for $d > 1$ at $T = 0$, and $d > 2$ at $T > 0$. However, this argument does not actually give the correct lower critical dimension. To see this, consider instead the single scalar field theory

$$\mathcal{L} = (\partial_\tau \phi)^2 + g_{ij} (\partial_i \partial_j \phi)^2 \tag{22}$$

where ϕ transforms under the symmetry as $\phi \mapsto \phi + \alpha + \beta_j x^j$. In a classical system at $T > 0$ the correlation functions of this operator are

$$\langle \partial_i \phi(x) \partial_j \phi(0) \rangle \sim \delta_{ij} \int \frac{d^d k \, k^2}{k^4}, \tag{23}$$

so that the critical dimension is $d = 2$, as in the standard Mermin-Wagner theorem for monopole symmetries. This allows us to interpret the dipole charges of the monopole symmetry as monopole charges of the dipole part of the symmetry group.

However, this interpretation does not hold for quantum systems. The correlation function scales as

$$\begin{aligned} \langle \partial_i \phi(x) \partial_j \phi(0) \rangle &\sim \delta_{ij} \int \frac{d\omega \, d^d k \, k^2}{\omega^2 + k^4} \\ &= \delta_{ij} \int \frac{d^d k \, k^2}{k^2}, \end{aligned} \tag{24}$$

so that SSB is possible in any $d > 0$. Thus, we cannot view this as analogous to ordinary monopole symmetry breaking.

It is interesting to note that condensing an object that carries dipole charge must break rotation symmetry. In Sec. V we show how to break the dipole symmetry while preserving a lattice rotation symmetry by considering multiple species of dipole. We are not aware of a way to condense a dipole while preserving continuous rotation symmetry.

In more generality, for $G = \mathcal{M}^a_{\text{max}}$ and $H = \mathcal{M}^b_{\text{max}}$ there will be a charged operator $\partial_{i_1} \dots \partial_{i_{b+1}} \phi$ with a dispersion as in Eqn. 11. Because G is a maximal multipole group, we can still write the polynomials as in Eqn. 1. Recall that $P_c^{I_c}$ is a degree-c monomial. As long as ϕ transforms as

$$\delta\phi = \lambda_{I_c} P_c^{I_c}, \tag{25}$$

then $\partial_{i_1} \dots \partial_{i_{b+1}} \phi$ is invariant under $H = \mathcal{M}^b_{\text{max}}$.

Classical correlation functions for these operators take the form

$$\begin{aligned} C &= \left\langle \partial_{i_1} \dots \partial_{i_{b+1}} \phi(x) \partial_{j_1} \dots \partial_{j_{b+1}} \phi(0) \right\rangle \\ &\sim \int \frac{d^d k \, k^{2(b+1)}}{k^{2(a+1)}}, \end{aligned} \tag{26}$$

and diverge when $d \leq 2(a-b) \equiv d_{\text{cl}}$. On the other hand, the $T = 0$ correlation functions behave as

$$\begin{aligned} C &= \left\langle \partial_{i_1} \dots \partial_{i_{b+1}} \phi(x) \partial_{j_1} \dots \partial_{j_{b+1}} \phi(0) \right\rangle \\ &\sim \int \frac{d\omega \, d^d k \, k^{2(b+1)}}{\omega^2 + k^{2(a+1)}} \\ &\sim \int \frac{d^d k \, k^{2(b+1)}}{k^{a+1}}, \end{aligned} \tag{27}$$

so the quantum critical dimension is $d_{\text{q}} = a - 2b - 1$. We recover the maximal generalized Mermin-Wagner argument if we let $b = -1$ denote the trivial group ($b = 0$ denotes the monopole group). Furthermore, the critical dimensions again satisfy $d_{\text{cl}} = d_{\text{q}} + z$.

B. Different phases for the same subgroup

The previous subsection shows that when the kinetic term in the effective action is $K \sim (\partial_i \partial_j \phi)^2$, there are intermediate dimensions ($d = 2$ for $T > 0$ or $d = 1, 2$ for $T = 0$) where the symmetry breaking pattern preserves the monopole group. For smaller dimensions there is no symmetry breaking, while for larger dimensions

(with this effective action) the group is fully broken. We can also write down theories where the kinetic term will be $(\partial_i\theta_j)^2$. The latter type of theory will preserve the monopole group in any large enough dimension. Since the latter theory is quadratic, it behaves like an ordinary theory, with a MW theorem prohibiting symmetry breaking for $d \leq 2$ at $T > 0$ or $d \leq 1$ at zero temperature.

We can organize these different theories that have the same SSB structure by their compressibilities. Since we are considering theories that conserve the dipole group, there will be dipole charges and monopole charges. Call the compressibility for the monopole charges μ_0 and the compressibility for the dipole charges μ_1. In any number of dimensions, there will be a Mott-insulator-like phase where $\mu_0 = \mu_1 = 0$. The kinetic term $(\partial_i\theta_j)^2$ assumes $\mu_0 = 0$ and $\mu_1 > 0$, while the term $(\partial_i\partial_j\phi)^2$ assumes both compressibilies are nonzero.

The list of compressibilies only tells us which fields are available for writing down an effective action. As in the previous subsection, which operators develop long-range order depends on the dimension. The calculations are analogous to previous correlation functions, and the results are as follows. At nonzero temperature, the order b of the preserved subgroup $\mathcal{M}^b_{\text{max}}$ is

	$d = 1, 2$	$d = 3, 4,$	$d > 4$
$(\partial_i\partial_j\phi)^2$	1	0	-1
$(\partial_i\theta_j)^2$	1	0	0
Mott	1	1	1

(28)

where we are, as always, denoting the trivial group by $b = -1$. Because the correlation calculations do not have any ω integral, the critical dimensions for a given symmetry-breaking pattern are the same in different theories.

The same is not true at $T = 0$. Similar correlation function calculations show that the symmetry-breaking patterns are

	$d = 1$	$d = 2,$	$d > 2$
$(\partial_i\partial_j\phi)^2$	0	0	-1
$(\partial_i\theta_j)^2$	1	0	0
Mott	1	1	1

(29)

The interesting difference from finite temperature is that the theory with a ϕ field can preserve $\mathcal{M}^0_{\text{max}}$ at $d = 1$, while the theory with a θ_j field cannot. Both the $T > 0$ and the $T = 0$ table can easily be generalized to higher values of a, the order of the preserved multipole group.

Whenever there are distinct theories with the same symmetry-breaking pattern, some of the theories have operators with quasi-long-range order. QLRO vs SRO For example, the $(\partial_i\theta_j)^2$ theory at $d = 1$ acts like a Luttinger liquid of dipoles. A full analysis of quasi-long-range ordered fields is beyond the scope of this paper. Furthermore, we have only considered free fixed points. There certainly may be interacting theories with distinct phases and symmetry-breaking patterns [39].

IV. SYSTEMS WITH QUENCHED DISORDER

Let us now add some quenched disorder to our systems. In particular, we will consider disorder that explicitly breaks the symmetry locally but does not break the symmetry on average. Spatial disorder will also break translation and rotation symmetry, but again not on average. Of course, strong enough disorder can always destabilize the ordered phase, so will not consider that case. Weak disorder can discourage the ordered phase and raise the critical dimension at which SSB is impossible. Theorems of this type originate with Imry and Ma [26] and were proved by Aizenman and Wehr [28].

In a disordered classical system, the energy scaling argument gives a nice explanation for why the critical dimension changes. Consider the formation of a domain in an otherwise ordered phase fully breaking the maximal multipole group $\mathcal{M}^a_{\text{max}}$. There are $\sim L^d$ disorder samples in the new domain. Since each sample is taken independently, the typical energy gain from forming the domain will be $\sim L^{d/2}$, by the central limit theorem. Comparing this to the cost of domain formation we calculated previously, the ordered phase will be unstable to domain nucleation when $d \leq 4(a + 1)$.

In clean systems, the transition from classical to quantum brought down the critical dimension because the argument depended on entropy. The Imry-Ma argument only appeals to energy considerations, so the quantum critical dimension in the presence of disorder remains the same as the classical critical dimension [27]. In disordered quantum systems, SSB is impossible for $d \leq 4(a + 1)$.

To reproduce this argument using a correlation function calculation [26] we need to calculate the response to disorder. Call the disorder field $h(x)$ and redefine $\phi(x)$ to be the variation away from the average value ϕ_0. The relevant part of the Hamiltonian is

$$\begin{aligned} H &\sim \int d^dx \left[\frac{1}{2}K[\phi(x)] - \phi(x)h(x)\right] \\ &\sim \int d^dk \left[\frac{1}{2}\phi_{-k}K_k\phi_k - \phi_{-k}h_k\right] \\ &\sim \int d^dk \left[\frac{1}{2}\phi'_{-k}K_k\phi'_k - \frac{1}{2}h_{-k}K_k^{-1}h_k\right], \end{aligned} \tag{30}$$

where we have written $\phi'_k = \phi_{-k} - K_k^{-1}h_{-k}$. Then, from the expectation of ϕ_k,

$$\begin{aligned} \langle\phi_k\rangle &\sim \frac{\delta Z}{\delta h_{-k}} \\ &\sim K_k^{-1}h_k, \end{aligned} \tag{31}$$

we can see that the disorder produces fluctuations mediated by the susceptibility.

We can then compute the correlation function for $\phi(x)$,

$$\langle\phi(0)\phi(x)\rangle \sim \int d^dk\, K_k^{-2}\langle h_{-k}h_k\rangle e^{ik\cdot x}, \tag{32}$$

where $\langle h_{-k} h_k \rangle e^{ik\cdot x}$ does not affect the divergence at small k, assuming the disorder is short range correlated. We can compare to Eqn. 13 to see that in the presence of disorder that couples linearly to an order parameter fully breaking multipole symmetry, the critical dimension for a disordered system is twice the critical dimension for having direct full multipole symmetry breaking in a clean classical system.

We will now consider disorder that breaks the symmetry to a subgroup. For simplicity, let the symmetry of the system be the maximal dipole group and let the disorder break the dipole part of the group but not the monopole part. The kinetic term is $K[\phi(x)] = (\partial_i \partial_j \phi)^2$, while the coupling to disorder will be something like $(\partial_i \phi - \zeta_i)^2$, where ζ_i is the disorder field. Define the vector field $\xi_i = \partial_i \phi$, with kinetic term $K[\xi(x)] = (\partial_i \xi_j)^2$. In the ordered phase ξ_i will have some constant value. It need not vanish because of the dipole symmetry. In the presence of weak disorder, it will want to follow the ζ_i field where possible. We can now follow the argument of the Imry-Ma theorem to say that the critical dimension will be $d = 4$.

A more general case is a system with symmetry $G = \mathcal{M}^a_{\max}$ and disorder that breaks the symmetry to $\mathcal{M}^b_{\max}$ but preserves G on average. In this system the critical dimension for long-range order of a field that breaks $\mathcal{M}^a_{\max}$ but preserves $\mathcal{M}^b_{\max}$ should be $d = 4(a-b)$ for both classical and quantum systems. Again, let $b = -1$ denote the trivial group in order to recover our previous results.

V. EXPLICIT LATTICE EXAMPLE

We now present a transparent quantum lattice example to illustrate the spontaneous breaking of multipolar symmetry. To this end, we take seriously the idea that "dipole charges are monopole charges of a dipole symmetry," and include explicit secondary degrees of freedom that couple to dipole moments of the original degrees of freedom. This allows us to access phases corresponding to partial or complete symmetry breaking.

Consider a d-dimensional hypercubic lattice with sites labeled by j and directions labeled by $\mu = 1, \ldots, d$. Let there be bosonic quantum degrees of freedom n_j on the sites and $n'_{j,j+\mu}$ on the edges. Thus, we can say there are d species of edge-type bosons.

Define the symmetry operators

$$\begin{aligned} \mathcal{O}^{(1)}(\xi) &= \prod_j e^{i\xi n_j} \\ \mathcal{O}^{(2)}_\mu(\xi) &= \prod_j e^{i\xi (j_\mu n_j)} e^{i\xi n'_{j,j+\mu}}, \end{aligned} \tag{33}$$

where j_μ is the μ-th component of the site label j. The first operator corresponds to conservation of the total number of site bosons, while the μ-th component of the second operator corresponds to the sum of the μ-th component of the total dipole moment of site bosons and the total number of the μ-th type of edge bosons. These symmetries allow us to exchange a dipole of site bosons for an edge boson.

The most relevant Hamiltonian obeying these symmetries is

$$\begin{aligned} H_0 &= H_b + H_d + H_{\text{int}} \\ H_b &= \sum_j \left[\frac{U_b}{2} n_j (n_j - 1) - \mu_b \, n_j \right] \\ &\quad - \sum_{j,\mu,\nu} t_b \left[b_j b^\dagger_{j+\mu} b^\dagger_{j+\nu} b_{j+\mu+\nu} + H.C. \right] \\ H_d &= \sum_{j,\mu} \left[\frac{U_d}{2} n'_{j,j+\mu} (n'_{j,j+\mu} - 1) - \mu_d \, n'_{j,j+\mu} \right] \\ &\quad - \sum_{j,\mu,\nu} t_d \left[d^\dagger_{j,j+\mu} d_{j+\nu,j+\mu+\nu} + H.C. \right] \\ H_{\text{int}} &= \sum_{j,\mu} g \left[b_j d_{j,j+\mu} b^\dagger_{j+\mu} + H.C. \right], \end{aligned} \tag{34}$$

where sums are taken over sites j and lattice directions μ and ν. The operator $b^\dagger_j$ creates a boson on site j while $d^\dagger_{j,j+\mu}$ creates an boson on the edge between j and $j+\mu$. For simplicity, let us set

$$\frac{\mu_b}{U_b} = \frac{\mu_d}{U_d} = \frac{1}{2}, \tag{35}$$

so there is a robust insulating phase at weak hopping t_b, t_d [40].

The Hamiltonian H_b controls condensation of the site bosons. When t_b is small, the site bosons are in number eigenstates while when t_b is large the site bosons condense. Similarly, the edge bosons condense when t_d is large. Note that all d species of edge boson will condense equally, so that lattice rotation symmetry is not broken. The term H_{int} is allowed by the symmetry because it simultaneously removes an edge boson and creates a dipole of site bosons. When the edge bosons are condensed, H_{int} gives the site bosons an effective single particle hopping. Relevance of cosine terms and stability to crystal phase?

The phases of this model will be those in Eqn. 29. The values of t_b and t_d will determine which fields are available for the IR action. When the site bosons are condensed the ϕ field exists but might not be long-range ordered. Similarly when the edge bosons are condensed the θ_j field exists but might not be long-range ordered. For $d < 2$ any symmetry-breaking pattern is possible. For $0 < d \leq 2$ the symmetry can not be completely broken, but the dipole part can be broken. What is the effect of H_{int}?

Let us now consider the same system, but with disorder. The disorder Hamiltonian is

$$H_{\text{dis}} = \left[h_b \sum_j \sigma^*_j b_j + h_d \sum_{j,\mu} \sigma^*_{j,\mu} d_{j,j+\mu} \right] + H.C., \tag{36}$$

where h_b and h_d control the magnitude of the disorder and each instance of σ is a random phase. We will always consider $h_b, h_d << 1$.

For $h_b, h_d \neq 0$, we can fully rely on the Imry-Ma argument. No symmetry breaking can occur for $d \leq 4$, the dipole part of the symmetry may be broken for $4 < d \leq 8$, and any symmetry-breaking phase can occur for $d > 8$.

VI. DISCUSSION

Should we say anything about Ref. [41], which doesn't consider multipole symmetry, but does look at restricted mobility and crystal-like phases from abnormal kinetic terms?

In this paper we analyzed the spontaneous symmetry breaking of various multipole groups, and discussed generalized Mermin-Wagner theorems, which (we argued) should be best understood as constraints on when a direct transition fully breaking a multipole group can occur, and when a symmetry unbroken phase and a (monopole) symmetry breaking phase must be separated by intermediate phases breaking higher multipole symmetries. We also considered multipole groups that are not the maximal multipole group, and the effect of quenched disorder. The disorder that we considered explicitly broke the symmetry, either fully or to a subgroup.

Of course, we could consider further combinations of effects. We could spontaneously break a symmetry from a group G to a subgroup H where either G or H are non-maximal. Or we could consider non-maximal groups with disorder. While the number of potential examples to consider is unlimited, they should all be analyzable using the ideas introduced herein.

We should be clear that the arguments in this paper are Mermin-Wagner or Imry-Ma arguments. These essentially amount to stability analyses about the Gaussian symmetry breaking fixed point. In principle, even when the Gaussian symmetry breaking fixed point is unstable, a non-trivial fixed point with long range order could arise (see e.g. [39]). Whether and when such non-trivial fixed points can be realized in models with multipolar symmetry is an important problem for future work.

We also emphasize that when the ordered phase does not exist, we did not provide any argument for what phase should replace it. For example, in 2 spatial dimensions there can be no ordered phases of continuous monopole (ordinary) symmetries. For $O(n)$ models with $n > 2$, the result is that the disordered phase is the only phase [42]. For $n = 2$, there can in addition be quasi-long-range-ordered phases. Determining what kind of phase *can* obtain in the absence of long range ordered symmetry breaking is beyond the scope of this work, and would at a minimum require understanding the nature and role of topological defects in the symmetry breaking order parameter, akin to vortices in the XY model. The range of possible symmetry unbroken phases could be even richer in the presence of disorder, where various glassy phases could also come into play [40]. Our discussion of disorder physics was also limited to quenched short-range correlated disorder. Extensions to disorder with long-range correlations, or annealed disorder, are left to future work.

We should also emphasize that our discussion has utilized standard concepts from statistical physics, which in turn amounts to assuming ergodicity. However, quantum dynamics with multipolar symmetries can break ergodicity [2, 3], in which case our analysis would not straightforwardly apply. It is however believed that the strict ergodicity breaking is limited to systems with strictly short range interactions (below some critical range) and that systems in which the interactions have long range tails (whether power law or exponential) should generically obey ergodicity at long times (although see [43]). Since long range tails are generic in physical systems, we believe our arguments should generically apply.

Another setting for generalized Mermin-Wagner-type arguments is higher form global symmetries [44–46]. It would also be interesting to see what sort of subtleties could exist in the spontaneous breaking of those symmetries, through partial symmetry breaking or disorder. Since the order parameters for higher-form symmetries are nonlocal, it is difficult to couple disorder directly to the order parameters. In the case of arbitrary perturbations the symmetry becomes broken microscopically, but emerges at long wavelengths. Could disorder have any effect on the Mermin-Wagner behavior of higher-form symmetries? We leave these questions for future work.

Finally, there exists a body of work on generalized Mermin-Wagner arguments in systems with subsystem symmetries [47–52]. Subsystem symmetries are rather different in character to the multipolar symmetries discussed herein, but are also related to fracton phases via duality [17]. There can be theories with subsystem symmetries where symmetry breaking cannot occur even above the critical dimension, due to the UV/IR mixing [51]. However, it is always possible to write down theories that saturate the generalized Mermin-Wagner bound [52]. Exploration of connections between the present work and the literature on subsystem symmetries would also be a fruitful topic for future work.

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Appendix A: No mediated condensation

In Sec. III we saw that the generalized Mermin-Wagner argument provides critical dimensions for partial breaking that are lower than the critical dimensions for full symmetry breaking. Can we use this behavior to try to evade the Mermin-Wagner argument for the full symmetry group?

To be explicit, consider a monopole field $\phi(x)$ and a dipole field $\varphi_i(x)$. Under the dipole symmetry they transform as

$$\begin{aligned}\phi(x) &\to \phi(x) + a_i x^i + b,\\ \varphi_i(x) &\to \varphi_i(x) + a_i.\end{aligned} \tag{A1}$$

The leading terms in the Lagrangian will be

$$\mathcal{L} = (\partial_t\phi)^2 + (\partial_t\varphi_i)^2 + (\partial_i\partial_j\phi)^2 + (\partial_i\varphi_j)^2, \tag{A2}$$

so that ϕ has a k^4 dispersion and φ_i has a k^2 dispersion. We can also include an interaction term of the form $g(\partial_i\phi - \varphi_i)^2$, which does not break the symmetry. The φ_i field is well-defined whenever the dipole symmetry is broken.

We can then redefine the φ_i field to $\tilde{\varphi}_i = \varphi_i - \partial_i\phi$, so that it is neutral under the symmetry. The interaction term becomes $g\tilde{\varphi}_i^2$, so that the $\tilde{\varphi}_i$ field becomes massive. The only remaining massless field is ϕ, which remains charged under the full dipole group and keeps its k^4 dispersion. As a result, the critical dimension for breaking the full dipole group is not affected by the possibility of partially breaking. What else to say here? Compare to Griffin/Grosvenor? Compare to fracton superfluids