Tension in the Hubble Constant

Charles Stahl

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Abstract

The Hubble constant H_0 measures the rate at which distant objects are receding away from us, as a function of their distance. In effect, it measures how fast our Universe is expanding. Although H_0 can be measured in our local region, it can also be inferred using the Cosmic Microwave Background (CMB) and the current standard model of cosmology, the Lambda-Cold Dark Matter (ΛCDM) model. Together, the measurement and the inference can provide a check of the validity of the model. Recently, errors bars on both methods of arriving at H_0 have grown small enough to put them in tension. Here we review the tension and some attempts at relieving it.

1 Introduction and History

When Einstein published the general theory of relativity in 1916 [1], it did not take long for the scientific community to realize it implied the Universe was evolving. The next year, he showed that the Universe could be static if the equations for GR included an extra constant, now called the cosmological constant Λ [2].

Throughout the 1920s scientists such as Friedmann and Lemaître continued to explore the possibility of an expanding universe [3, 4]. In 1929 Edwin Hubble published his observation that the Universe was in fact expanding [5], leading Einstein to call the introduction of the cosmological constant a blunder.[1] Einstein would never again use the cosmological constant in a published paper.

Hubble's measurements showed a linear relationship between an object's distance from Earth and its velocity, which implied that space itself is expanding. Although Hubble's data

[1] There is some controversy over George Gamow's claim that Einstein called it his "greatest blunder".

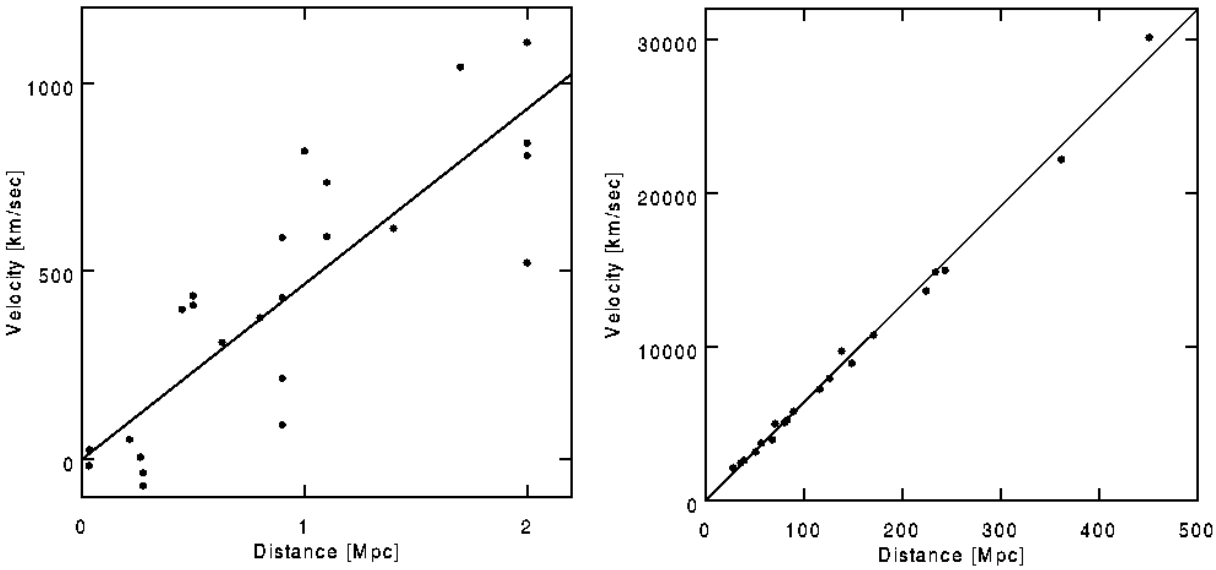

Figure 1: Comparison of Hubble's original data with more modern observations, from Ref. [6]. Note the much larger distance scale on the right, and the improved linear fit.

were far from perfect, observations have improved in the decades since. A comparison can be seen in Fig. 1. The slope of this graph, in units of $\mathrm{km\,s^{-1}\,Mpc^{-1}}$, is called the Hubble constant, H_0.

In addition, our understanding of cosmology has improved. The current standard model of cosmology is still based on the fundamental work of Friedmann and Lemaître, along with Robertson and Walker. Called the Lambda-cold dark matter (ΛCDM) model, it describes a Universe that started with a Big Bang and evolved under general relativity with contents consisting of matter, radiation, and some from of cosmological constant Λ.

A key confirmation of the Big Bang came in 1965 when Penzias and Wilson tried to make a really good radio antenna. They kept trying to remove some background in the microwave range that was present no matter where they looked in the sky. Finally, they realized this background was actually predicted by the Big Bang model, as leftover radiation from the time of recombination. [7].

Soon after the Big Bang the universe was very hot. Eventually it reached a state in which the dominant form of matter was protons and electrons. At this point, the matter and was coupled to the radiation and the temperature was high enough to keep the matter ionized. Thus, the universe was a plasma, and the plasma was opaque. In this phase the entire universe was in thermal equilibrium. Note that this fact alone is surprising, since we know that there were points in thermal equilibrium that were outside each other's cosmic horizons. This is one of the motivations for inflation.

Eventually (300 000 years after the Big Bang), the Universe cooled down enough to let atoms form (recombination). At this point the matter and radiation lost contact and the photons were able to free-stream (decoupling). This moment in time is called the surface of last scattering. The photons from last scattering are still traveling through the now-transparent Universe, and we see them as a background microwave signal coming from every direction.

Although this cosmic microwave background (CMB) is extremely uniform, it has some anisotropies. In the 1990s cosmologists figured out how to use these perturbations to confirm the ΛCDM model and also make predictions. In particular, it is possible to use the CMB to predict the value of H_0. Initially, the value predicted from the CMB and the value measured by observations did not agree, but had large enough error bars that the two values were compatible.

In recent years, both methods have drastically decreased their error bars without converging. The result is that today the two methods are in tension with each other with a statistical significance of $4 - 6\sigma$ [8]. It is possible that one method suffers from systematic bias, but if not then the ΛCDM model needs to be updated. This disagreement, called the Hubble tension, has been the topic of many recent conferences [9, 10]. In this paper we will describe how both methods arrive at a value for H_0 and summarize some possible resolutions to the tension.

The primary source for this paper will be Ref. [8], which reviews various classes of resolutions to the tension. We will refer to sections in that review as ([8] Sec. 1), etc, so that the interested reader can find more detailed information. We will start, however, by reviewing some helpful background material in order to provide the reader with a more gentle introduction to the Hubble tension. Throughout, we will set $c = 1$, but convert back to useful units when reporting values for H_0.

2 Background: Cosmology

In order to understand the tension in the Hubble constant, we will take a step back and review the history of the universe. Our tool here will be relativistic cosmology, which describes the evolution of the universe on the largest scales. All material in this section is taken from Ref. [6], unless otherwise noted. We will take only the bare bones for understanding the CMB prediction of H_0, but the interested reader can find more detail in that reference.

The Robertson-Walker (or Friedmann-Lemaître-Robertson-Walker or FLRW) metric is the most general spacetime metric consistent with homogeneity and isotropy. It is given by

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \tag{1}$$

where k is the curvature index. Our Universe is very close to flat, so we will set $k = 0$ for the rest of this paper. In these equations t is called the cosmic time, which is the time measured by a comoving observer. The function $a(t)$ is called the scale factor. Moving backward in time we might find a value $t_{\rm min}$ such that $a(t_{\rm min}) = 0$. This is the Big Bang. It is convenient to define $t_{\rm min} = 0$.

From the scale factor we can define the redshift z. If light is emitted with wavelength λ when the scale factor is a, then if it is observed when the scale factor is a_0 it will have wavelength $\lambda_0 = \lambda\, a_0/a$. If we place the observer at our timeslice so that $a_0 = 1$, then we can define the redshift

$$z = \frac{1}{a} - 1, \tag{2}$$

so that $\Delta\lambda = z\lambda$.

From the scale factor we can also define the Hubble parameter $H = \dot{a}/a$, where $\dot{a} = da/dt$. This value, often called the Hubble constant, is constant in space but not in time. The current value, H_0, is the value of $H(t)$ at our time slice. The Hubble parameter is often reported in the literature using the standard but strange units $H = 100h\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$.

We could replace the coordinate time t with conformal time τ which obeys $dt/d\tau = a$, so the metric becomes

$$ds^2 = a^2(\tau)\left[-d\tau^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{3}$$

where we have set $k = 0$. Conformal time is convenient because the speed of light is 1. It will be useful to use

$$dt = a\ d\tau = \frac{da}{Ha} = -\frac{a\,dz}{H} \tag{4}$$

to convert between the variables.

Horizons and distance scales are of paramount importance in observational cosmology. If an emitter emits a light signal at time $t_e = t(\tau_e)$ and an observer observes the signal at $t_o = t(\tau_o)$, then the comoving distance between them is

$$r_{oe} = \tau_o - \tau_e = \int_{t_e}^{t_o} \frac{dt}{a(t)}. \tag{5}$$

Both the observer and the emitter will agree on this value. If we instead want the proper distance as measured by the observer, we get $d_o = a(t_o)r_{oe}$. The emitter measures a proper distance of $d_e = a(t_e)r_{oe}$. Since we define the scale factor such that $a(t_0) = 1$ at the present time, comoving and proper distances agree for us. Eqn. 5 will be necessary for calculating horizons and comoving distances. For example, in a universe with a Big Bang at $\tau = t = 0$, signals can only have traveled a comoving distance of $\tau(t)$ by time t.

In addition to the kinematic variables we need some dynamical variables. Usually, we assume the universe is made a collection of perfect fluids labeled by i, each defined by an energy ρ_i, a pressure p_i, and an equation of state $p_i = w_i \rho_i$ (no sum). The variable w_i is called the equation of state.

Having defined our metric and some useful variables, we can use the Einstein equation of general relativity to arrive at the Friedmann equations. The first Friedmann equation is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i + \frac{\Lambda}{3}, \tag{6}$$

which constrains the expansion of the universe given its matter content. The famous cosmological constant Λ is just a constant of integration. It can be absorbed as a source of the equations if we let $\rho_\Lambda = \frac{\Lambda}{8\pi G}$. Under these conventions, the energy content of the universe is matter ρ_m (including baryonic matter ρ_b and cold dark matter ρ_c), radiation ρ_r, and dark energy ρ_Λ.

The second Friedmann equation is the evolution equation,

$$\dot{H} + \frac{3}{2}H^2 = \frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi G \sum_i p_i + \frac{\Lambda}{2}, \tag{7}$$

where we can again absorb Λ by letting $p_\Lambda = \frac{-\Lambda}{8\pi G}$. For the remainder of this paper we will absorb Λ as a source in both equations. In that case, the second Friedmann equation tells us $\sum_i \rho_i = \frac{3H^2}{8\pi G}$. We refer to this value as the critical density ρ_c and often report energy densities as $\Omega_i = \rho_i/\rho_c$ [11].

If we combine the Friedmann equation we can see that each fluid evolves as

$$\rho_i(t) \propto a(t)^{-3(1+w_i)}, \tag{8}$$

which reproduces some reasonable intuition. Matter dilutes as a^{-3} because it is in an expanding volume, radiation dilutes as a^{-4} because it cools in addition to being in an expanding volume, and the cosmological constant does not dilute (remains constant). The evolution of the fluids and Eqn. 5 should be all we need to understand the Hubble tension.

3 Tension

Now that we have introduced the necessary cosmological prerequisites, we can see what the tension in the Hubble constant really is. First we will see how measurements of the early universe through the CMB can "predict" the current value H_0. Then, we will review some methods of directly measuring the value of H_0 in our local universe, using methods that are more accurate than those of Hubble.

3.1 CMB Prediction

As mentioned in the introduction, the CMB is very close to uniform, with perturbations a factor of 10^4 smaller than the uniform temperature. These perturbations have been measured by WMAP, *Planck*, and others in order to probe the evolution of the early universe. We will refer to the *Planck* 2013 results [11] and the *Planck* 2018 results [12] in this analysis.

The perturbations are decomposed into spherical harmonics, and their amplitudes $a_{\ell m}$ are averaged to obtain the power spectrum [13, 14],

$$C_l = \langle a^*_{lm} a_{lm} \rangle_m = \frac{1}{2l+1} \sum_{m=-l}^{l} |a_{lm}|^2, \tag{9}$$

which is independent of a choice of coordinate system. Fig. 2 shows the spectrum from *Planck* [12].

To see how the temperature anisotropies can be used to infer cosmological parameter we need to understand where the anisotropies come from. Before recombination, in the opaque plasma, the sound speed is

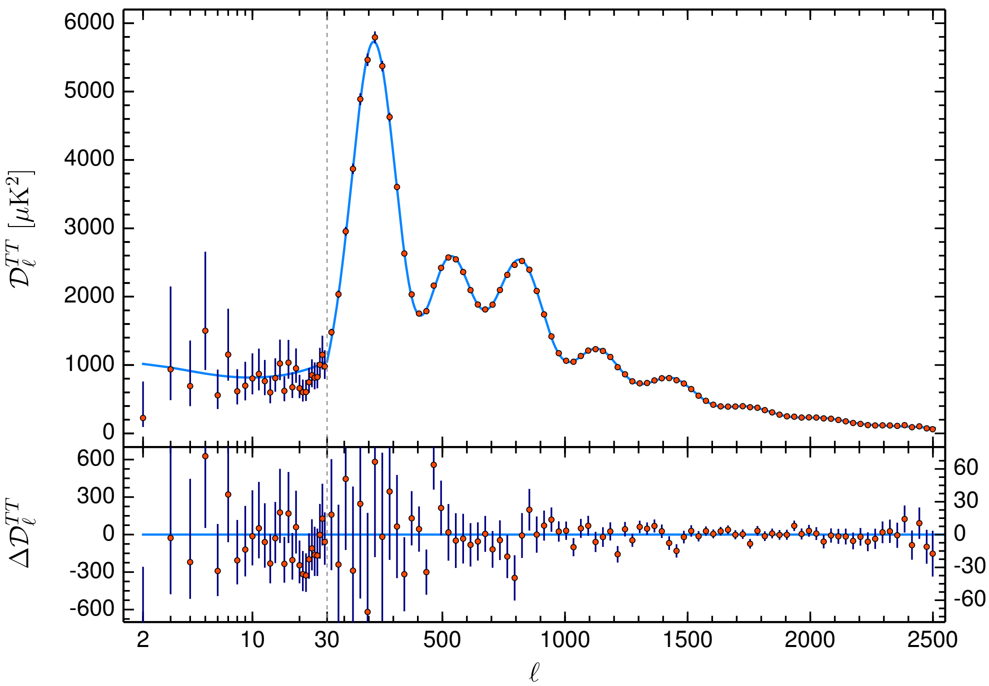

Figure 2: The (normalized) CMB spectrum. The top panel shows the numerical prediction (from best-fit parameters) in blue and observations in red, while the bottom panel shows the difference between prediction and observation.

$$c_s = \frac{1}{\sqrt{3(1+\frac{3\rho_b}{\rho_\gamma})}}, \tag{10}$$

where ρ_b is the baryon density and ρ_γ is the density of photons. Neutrons and dark matter do not affect the sound speed because they are only weakly coupled. In analogy with Eqn. 5 we can define the comoving sound horizon [11]

$$r_s(\tau) = \int_0^\tau c_s d\tau' = \int_{z(\tau)}^\infty \frac{c_s\, dz'}{H}, \tag{11}$$

which defines how far density waves can have traveled by conformal time τ.

Since we want to make measurements of the CMB, we care about the sound horizon at last scattering $r_s^* = r_s(\tau_*)$. Since this value depends only on baryon and photon densities, r_s^* is independent of CMB measurements. From the CMB, we can instead extract the angular scale of the sound horizon at last-scattering, θ_s^*. Unfortunately, there is not a simple formula for θ_s^*. Instead, it is found using the positions of as many peaks in the spectrum as possible. For the *Planck* mission this means seven peaks [11, 12].

In Euclidean geometry, an object of size r at distance $d >> r$ subtends an angle $\theta = r/d$. Similarly, the sound horizon and its angular scale are related by

$$\theta_s^* = \frac{r_s^*}{D_A^*}, \tag{12}$$

where D_A is the (coming) angular diameter distance.[2] We can once again appeal to Eqn 5 to calculate the angular diameter distance as [14, 8]

$$D_A^* = \int_{\tau_*}^{\tau_0} d\tau = \int_0^{z_*} \frac{dz}{H(z)}, \tag{13}$$

so that a value of D_A^* can be used to predict $H(z)$. In particular, we can use the CMB pipeline to predict a value of $H(0) = H_0$.

Actually, the whole process is slightly more complicated than that. A more careful analysis shows that the above procedure most directly predicts the reduced density parameters $\omega_b = \Omega_b h^2$ and $\omega_c = \Omega_c h^2$. These predictions can then be combined with local measurements of Ω_b and Ω_c to predict the derived parameter h or H_0. Ref. [12] has nice graphs showing how h and other derived parameter co-vary with the different parameters like ω_b that are directly predicted by *Planck*.

Furthermore, the *Planck* predictions are based not only on the temperature anisotropy spectrum but also on polarization spectra. The gold standard result from *Planck* uses temperature-temperature, temperature-E-mode, and E-mode-E-mode spectra, plus low-E and lensing data. This pipeline provides a prediction of $H_0 = 67.27 \pm 0.6\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

[2] Although all sources use the comoving angular diameter distance for this calculation, there is considerable disagreement over what to name the variable. Ref. [14] uses d_A while Refs. [11] and [8] use D_A. Both Ref. [12] and Wikipedia [15] use D_A to refer to the proper angular diameter distance, which is equal to the scale factor times the comoving distance.

Further addition of B-mode polarization and BAO data can further change the prediction, but only slightly.

3.2 Measurement

Of course, all that being said, we can follow Hubble's lead and just measure the distance and velocity of objects in our local region. This is complicated by the fact that nearby objects have proper motion, meaning that their velocity is not entirely determined by the expansion of the universe. That being said, if we look far enough away we should be able to see stars or other objects whose motion is dominated by the Hubble flow. These allow us to measure H_0 to high accuracy.

The goal then becomes to find many stars and measure both their velocity and their distance. It is relatively easy to measure velocity using the Doppler effect and the redshift in spectrum lines. For nearby stars it is possible to measure distance using parallax. For further distances we rely on types of stars that have very predictable intrinsic luminosities. Then, from the observed velocity, we can infer the distance. The primary types of stars to use are pulsating Cepheid variables and exploding type Ia supernovae [8]. Together, all these measurements make up the so-called distance ladder.

Cepheid variable stars are stars that pulsate, with a predictable relationship between luminosity and period. This relationship was discovered in 1908 by Henrietta Swan Leavitt [16] and allows observers to infer the intrinsic luminosity, and thus distance, from period. They are prevalent and bright enough to reach distances of $10 - 40\,\mathrm{Mpc}$. Supernovae, while brighter, are much rarer. They are used only to measure velocity at the furthest distances [8], and are the last rung on the ladder.

The SH0ES project [17] is the current gold standard for observational measurements of the Hubble constant, with a value of $73.2 \pm 1.3\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. Other projects have varying estimates, but they generally agree with the stated value.

3.3 Summary

In order to quantify the tension between SH0ES and the *Planck* measurement, we want to calculate to what level of uncertainty the difference between the measurements vanishes [18]. Explicitly,[3] using the values for h to avoid units and letting subscripts distinguish the *Planck* and SH0ES data,

$$\begin{aligned} \Delta &= h_1 - h_2 = .6727 - .732 = 5.93, \\ \sigma &= \sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{.0060^2 + .013^2} = .0143, \\ \frac{|\Delta|}{\sigma} &= 4.15, \end{aligned} \tag{14}$$

which is to say that the two measurements disagree at 4.2σ. Other transformations of the data from either the CMB or local measurements can result in slightly different tensions, but they generally lie in the 4σ to 6σ range [8].

There are other methods of measuring, such as those relying on gravitational waves. Currently, they have high enough uncertainty that they agree with the CMB prediction and the SH0ES observation. If they can significantly decrease uncertainty they may come out of agreement with one or the other. New methods will not relieve tension if they agree with observations, but can say whether or not SH0ES data has systematics. Fig. 3 shows the value and uncertainty of H_0 using many different methods. Note that many pairs of error bars do not overlap.

4 Possible resolutions

The simplest possible resolution to the Hubble tension would be that we are still not measuring velocity far enough out in the universe, so that our local value of H_0 is not the correct value for our entire timeslice. This possibility is highly unlikely. It would require that our

[3]This may be basic statistics, but the author was unable to find the calculation in any of the reference papers, and thought it might be useful to include.

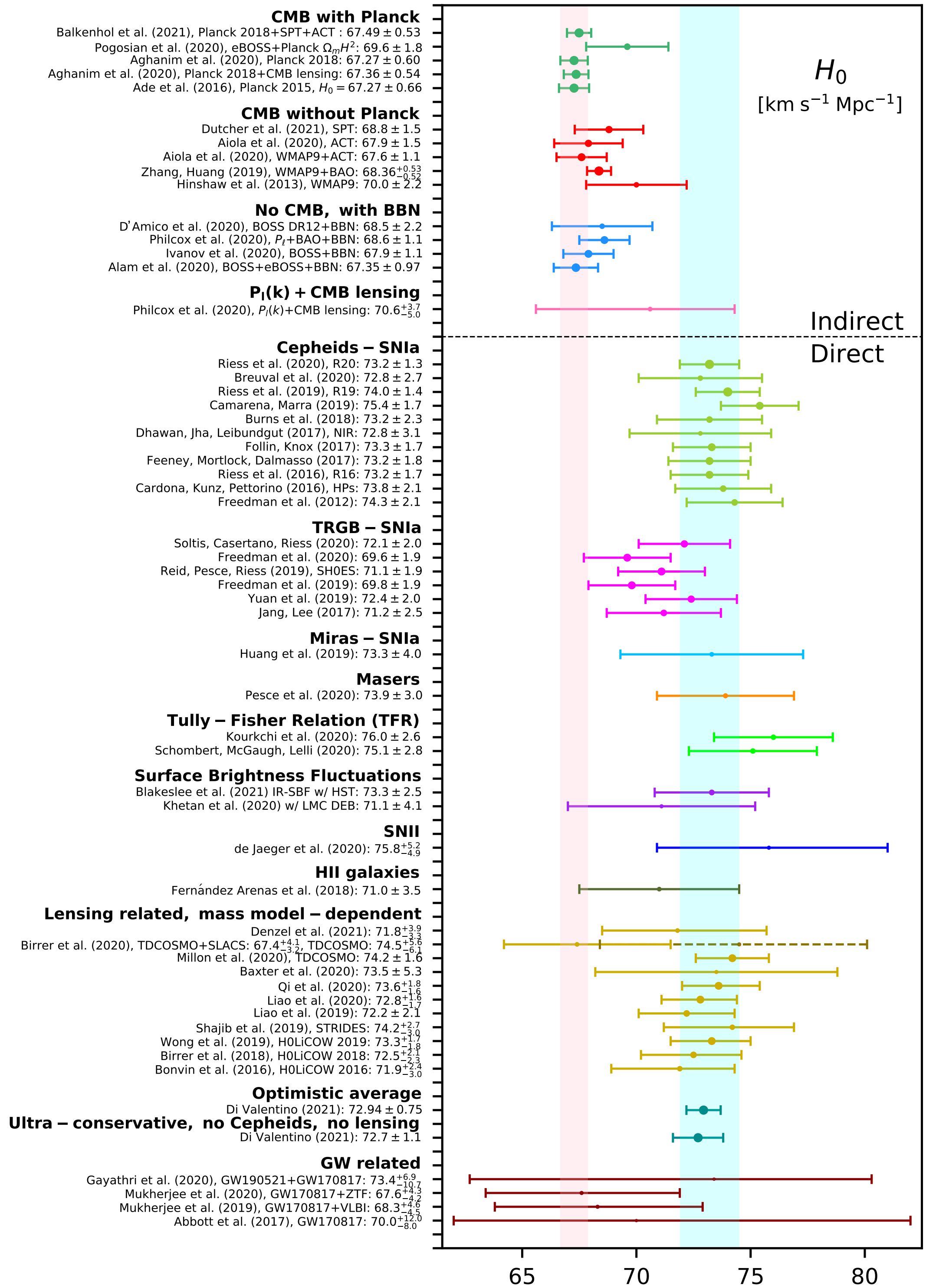

Figure 3: A summary of many different indirect and direct measurements of the Hubble constant, taken from Ref. [8]. The values quoted in the current paper are the third from the top and the first in the "indirect" section. This figure is included to show that there are many methods in addition to those described in the current paper for measuring H_0. Direct measurements in general favor a higher value, although sometimes with very large error bars.

local Universe is about 20 times more underdense than statistical fluctuations suggest is likely [8]. There are many more proposed ways to resolve the Hubble tension. In this section we will review a few methods that invoke modified dark energy, followed by a quick rundown of alternative methods.

4.1 Modified dark energy

In some sense, the most phenomenological element of the ΛCDM model is dark energy. If we claim that ρ_Λ is equal to the vacuum energy of the effective field theory of the Standard Model, it is off by a stunning 120 orders of magnitude [6]. Thus, it is tempting to modify its contribution to the Friedmann equations before any other component. The first type of modification we will consider is Early Dark Energy (EDE, [8] Sec. 4). These models have an extra flavor of dark energy that existed at early times but does not exist now. This changes evolution in the pre-reionization plasma, decreasing r_s.

Our first example of EDE is a scalar field ϕ performing sitting in an anharmonic potential, $V(\phi) = [1 - \cos(\phi/f)]^n$. The scalar field sits still at the bottom of the potential until a critical redshift z_c when it begins to oscillate. Its equation of state as a function of a is

$$w_\phi(a) = -1 + \frac{1 + w_n}{1 + (a_c/a)^{3(1+w_n)}}, \tag{15}$$

where $w_n = (n-1)/(n+1)$ and $a_c = a(z_c)$ is the scale factor at the critical redshift. Thus, at early times phi behaves like a cosmological constant with $w_\phi \approx -1$ and at late times it has an effective equation of state $w_\phi = w_n$. This would correspond to matter for $n = 1$ and radiation for $n = 2$, while for $n = 3$ it can help decrease the Hubble tension to 2σ.

A popular fad in the world of EDE is Rock 'n' Roll, which is also a model of a scalar field ϕ. Now, the field lives in a potential

$$V(\phi) = V_0 \left(\frac{\phi}{M_{\rm Pl}} \right)^{2n} + V_1, \tag{16}$$

where V_0 and V_1 are constants, n is an index, and $M_{\rm Pl} = (8\pi G)^{-1/2}$ is the Planck mass. Once this field begins to oscillate, it either oscillates for all time (rocking), or asymptotes to a rolling solution. Models from this class claim to resolve the tension to 1.9σ, but more careful analysis shows they only decrease it to 3.3σ.

Late dark energy (LDE, [8] Sec. 5) instead changes the equation of state for the component of dark energy that we do see. A different equation of state for dark energy changes the angular diameter distance (Eqn. 13) by modifying the first Friedman equation (Eqns. 6 and 8). The simplest LDE model is the wCDM model, where dark energy has a constant equation of state $w_{\rm DE} \neq -1$. A wCDM model can completely alleviate the Hubble tension, at the cost of introducing phantom-like dark energy, $w_{\rm DE} < -1$. From Eqn. 8 we can see that such a flavor of dark energy would have an energy density that increases with time. Then the first Friedmann equation tells us that the scale factor a will diverge in finite time, a situation called the Big Rip.

EDE and LDE models all alleviate the Hubble tension in part by increasing the number of free parameters compared to the ΛCDM model. Since all these models are phenomenological anyway, we should try to match the number of parameters in ΛCDM. This class of models with six degrees of freedom ([8] Sec. 6) includes Phenomenologically Emergent Dark Energy (PEDE). The equation of state is

$$w_{\rm PEDE} = -1 - \frac{1}{3\ln 10}[1 + \tanh \log_{10}(1+z)], \tag{17}$$

and the model can resolve the Hubble tension to within 1σ.

4.2 Other possible resolutions

There are many other possible resolutions to the Hubble tension. Here we will very briefly review a handful of categories of theories from Ref. [8] before concluding. Recall that the ingredients of the ΛCDM model are General Relativity with the FLRW metric, some form of dark energy Λ, cold dark matter, the Standard Model of matter we can see, and a history of the Universe. Since we already focused on theories that modify dark energy, the remaining theories will change some other ingredient.

The first class contains theories with extra relativistic degrees of freedom ([8] Sec. 7), in the form of a higher effective number of neutrinos, N_{eff}. This shifts the peaks in the CMB by increasing ρ_r and Ω_r. Alternatively, we can consider extra interactions between dark energy and dark matter or between either of the two dark fluids and ordinary matter ([8] Sec. 8). These allow the different forms of energy density and pressure to interconvert, modifying Eqn. 8.

A fun example of an extra interaction model is the $L_\mu - L_\tau$ model [19], which includes a massive Z' boson that decays into neutrinos. The authors claim that this form of extension to the Standard Model, while largely unconstrained, is able to solve both the Hubble tension and the longstanding and recently-confirmed tension in measurements of the muon magnetic moment [20].

The interaction models are taken to the extreme in so-called unified dark fluid models ([8] Sec. 9). In these theories, there is a single dark fluid, with a time-dependent equation of state, that behaves like dark matter in the early Universe and dark energy in the late Universe.

Finally, there are theories of modified gravity, inflation, modified behavior at recombination, phase transitions in dark energy, and other possibilities ([8] Sec. 10-14). At the time of the publication of Ref. [8] many of these models had not been analyzed using the most recent *Planck* 2018 data, so reanalysis might be useful. For all of these models, the quantitative reduction in tension can be found in Fef. [8].

5 Conclusion

The goal of this paper has been to introduce the Hubble tension, while splitting the difference between cosmology reviews that do not explain how to measure H_0 and Hubble tension reviews that do not introduce the relevant cosmology. To that goal, we have tried to not shy away from some of the complicated aspects. That being said, the story here has still been highly simplified. For example, the actual way that H_0 is inferred from CMB data involves heavy numerical modeling and statistical work.

In addition there are many confounding factors, from lensing of the CMB to foreground sources to instrumental noise. All of these are discussed in detail in the many *Planck* data releases.

Furthermore, there are other cosmological sources of data such as Baryon Acoustic Oscillations (BAO) that can provide alternative methods of measuring H_0. These methods can be combined with those mentioned in this paper to paint a picture of the Universe for which we do not have a single dominant consistent model. Observational cosmology will certainly be a useful testbed of fundamental physics for decades to come.

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