What does a FOGM look like in an Allan Variance Chart?

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Here, we try to analytically derive what shape a FOGM will make in an Allan Variance chart.

The FOGM can be completely expressed by a 2×2 covariance matrix that expresses how one sample is related to the next sample:

$$\begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}, \tag{1}$$

where σ^2 is the variance of any given sample in the time series, while ρ gives the correlation between the two samples. To implement a FOGM in a time series, you would have:

$$x_{k+1} = \rho x_k + \nu \quad : \; \nu \sim \mathcal{N}(0, \sigma_\rho^2), \tag{2}$$

where $\sigma_\rho^2 = \sigma^2\sqrt{1-\rho^2}$.

Any given entry in the Allan Variance chart requires us to know the relative 2×2 covariance matrix between two *averaged* components. We introduce here the notation

$$y_{i,m} = \frac{1}{m}\sum_{j=0}^{m-1} x_{j+i} \tag{3}$$

where m is the number of elements that have been averaged together and i is the index of the starting element. When computing an Allan Variance entry (a), we are trying to find

$$a_m = \frac{1}{2}\mathtt{E}[(y_{i,m} - y_{i+m,m})^2], \tag{4}$$

where $\mathtt{E}$ is the expected value operator and i is an arbitrary index, and the $\frac{1}{2}$ is part of the definition of Allan Variance.

To solve Equation (4), we first form a covariance matrix for all the values of x that will go into computing the Allan Variance. For a given value of m, this covariance matrix will be a $2m \times 2m$ Toeplitz matrix of the following form:

$$\boldsymbol{\Sigma}_{x_{2m}} = \sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 & \cdots & \rho^{2m-1} \\ \rho & 1 & \rho & \rho^2 & \cdots & \rho^{2m-2} \\ \rho^2 & \rho & 1 & \rho & \cdots & \rho^{2m-3} \\ \vdots & & & \ddots & \cdots & \vdots \\ \rho^{2m-2} & \cdots & \rho^2 & \rho & 1 & \rho \\ \rho^{2m-1} & \cdots & \rho^3 & \rho^2 & \rho & 1 \end{bmatrix}. \tag{5}$$

Note that given an input with covariance ($\boldsymbol{\Sigma}$), the output covariance of a linear process ($\boldsymbol{A}$) acting on the input can be computed as $\boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^\top$. Using this property, the covariance of an Allan Variance term can be computed from $\boldsymbol{\Sigma}_{x_{2m}}$ as

$$\begin{aligned} a_m &= \boldsymbol{u}^\top \boldsymbol{\Sigma}_{x_{2m}} \boldsymbol{u} \\ \boldsymbol{u} &= \frac{1}{m} \begin{bmatrix} 1 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 \end{bmatrix}^\top, \end{aligned} \tag{6}$$

where the vector $\boldsymbol{u}$ is the representation of the averaging and differencing operation used to find the Allan Variance.

To compute the results of Equation (6), we subdivide $\boldsymbol{\Sigma}_{x_{2m}}$ into four, $m \times m$ matrices (ignoring the σ^2 in front of the matrix for now). By adding all the elements together in the top-left and bottom-right submatrices (the diagonal sub-matrices), and subtracting all elements of the top-right and bottom-left submatrices (the off-diagonal sub-matrices), we can compute a_m to within a scale factor. In the next two sections, we derive equations for the summation of all the elements in each of these sub-matrices.

1 Off-diagonal Submatrices

For the bottom left and top-right submatrices, note that every column can be expressed as a constant multiple of the vector:

$$\begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{m-1} \end{bmatrix}. \tag{7}$$

Note that the sum of this vector is

$$\frac{1-\rho^m}{1-\rho}. \tag{8}$$

To obtain the columns of the bottom left (or top right) sub-matrix, the following constant multiples are applied to the values in Equation (7):

$$\begin{bmatrix} \rho & \rho^2 & \cdots & \rho^m \end{bmatrix}, \tag{9}$$

which has a sum equal to

$$\rho \frac{1-\rho^m}{1-\rho}. \tag{10}$$

Multiplying the sums in (8) and (10) together, we obtain the sum of all the elements in the bottom left submatrix to be

$$\rho\left(\frac{1-\rho^m}{1-\rho}\right)^2. \tag{11}$$

2 Diagonal Submatrices

The summation for the diagonal submatrices is a bit more complex. Rather than finding a common column representation, we find an equation for all of the elements to the right and down from a particular entry on the diagonal. In this case, every diagonal element has a geometric sum associated with it (times two, one for the entries to the right, one for the entries going down), while the next element in the diagonal has the same geometric sum *less one element.* Therefore, starting with a diagonal element at location i (where indexing starts with 1, not 0), the sum of all elements to the right of that diagonal entry will be

$$\rho\frac{1-\rho^{m-i}}{1-\rho}. \tag{12}$$

Using this equation, we have the total summation of elements in a diagonal sub-matrix as

$$\begin{aligned}
& m+2\rho\sum_{i=1}^{m}\frac{1-\rho^{m-i}}{1-\rho} && \text{move } 1-\rho \text{ out of summation} \\
& m+2\frac{\rho}{1-\rho}\sum_{i=1}^{m-1}1-\rho^{m-i} && \text{move 1 out of summation} \\
& m+2\frac{\rho}{1-\rho}\left(m-\sum_{i=1}^{m}\rho^{m-i}\right) && \text{evaluate summation} \\
& m+2\frac{\rho}{1-\rho}\left(m-\frac{1-\rho^m}{1-\rho}\right).
\end{aligned} \tag{13}$$

3 The Full equation

Putting all the pieces back together, the Allan Variance for a FOGM process is going to be

$$a_m=\frac{\sigma^2}{m^2}\left(m+2\frac{\rho}{1-\rho}\left(m-\frac{1-\rho^m}{1-\rho}\right)-\rho\left(\frac{1-\rho^m}{1-\rho}\right)^2\right). \tag{14}$$

Examples of the Allan variances are plotted in Fig. 1.

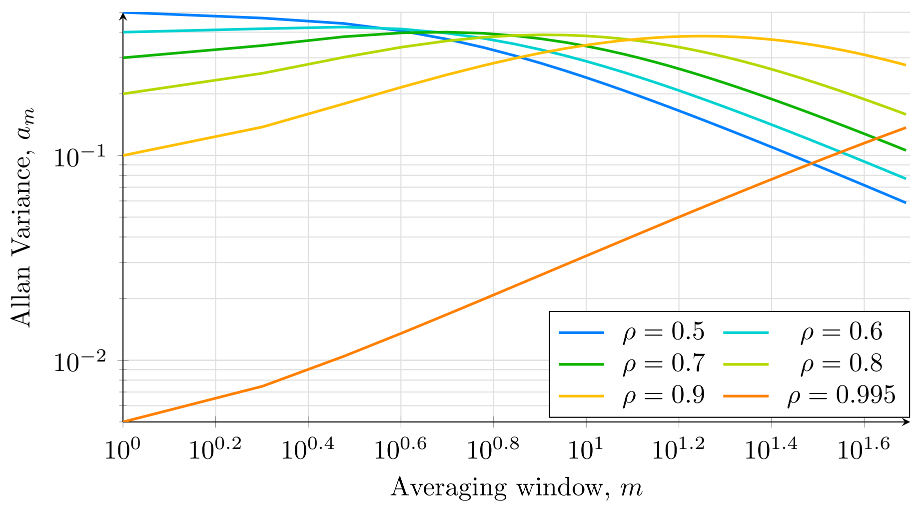

Figure 1: Allan variances for various values of ρ. The σ value is set to 1 in all cases.

4 Analysis

For a given FOGM, it might be nice to know at which point it is at it highest and how high it goes. To start, we will take the derivative of (14) with respect to m, set it equal to 0, and solve for m to find where its maximum point is.

$$\frac{\partial a_m}{\partial m} = -2\frac{\sigma^2}{m^3}\left(m + 2\frac{\rho}{1-\rho}\left(m - \frac{1-\rho^m}{1-\rho}\right) - \rho\left(\frac{1-\rho^m}{1-\rho}\right)^2\right) + \frac{\sigma^2}{m^2}\left(1 + 2\frac{\rho}{1-\rho}\left(1 + \frac{\rho^m \ln\rho}{1-\rho}\right) + 2\rho\left(\frac{1-\rho^m}{1-\rho}\right)\frac{\rho^m \ln\rho}{1-\rho}\right) \quad (15)$$

Unfortunately, this equation is fairly complex and does not appear to be easily solved for the actual peak location. However, using a numerical solver to find when $\frac{\partial a_m}{\partial m} = 0$, we can quickly find the "peak location" for a given FOGM process in the Allan Variance chart.[1] In Table 1, we show the peak values for several different ρ values with $\sigma^2 = 1$. The peak is usually close to $\frac{2}{1-\rho}$, but seems to settle at a ratio of .9463 (i.e. slightly smaller) than the value of $\frac{2}{1-\rho}$. Furthermore, the value of a_m also seems to converge to a value close to 0.3811. The downward slope after the peak also converges to -1 in the Allan Variance (log-log) chart

[1]See the Jupyter notebook at https://github.com/cntaylor/FOGMs-and-Allan-Variance to view an example.

	0.9	0.99	0.999	0.9999	0.99999
Peak Location	1.7822e+01	1.8830e+02	1.8917e+03	1.8925e+04	1.8926e+05
Ratio to $\frac{2}{1-\rho}$	0.8911	0.9415	0.9458	0.9463	0.9463
a_m at peak	0.3827	0.3812	0.3811	0.3811	0.3811

Table 1: With $\sigma^2 = 1$, information on the peak for different ρ values. Generated by https://github.com/cntaylor/FOGMs-and-Allan-Variance