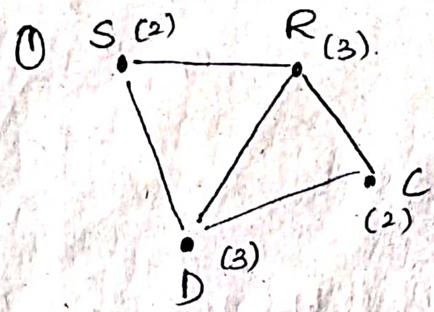
7. GRAPH

Graph:- Represented with nodes or vertices and edges.

→ edges - to represent relation b/w nodes.

$\boxed{G(V, E)}$

Eg:- ①



Vertices $V = \{S, R, C, D\}$

Edges $E = \{SR, RC, RD, CD, SD\}$

$G = (V, E)$

Given a vertex set, there can be any no. of edge sets

S has ② frnds, C has ② frds, R has ③ frnds, D has ③ frnds.

Degree:- no. of edges eminating from the vertex.

from above graph

degree(S) = 2

degree(R) = 3

degree(C) = 2

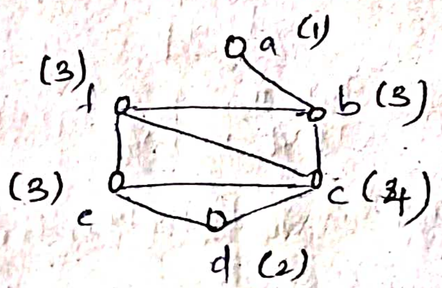
degree(D) = 2

Nodes - Vertices

Lines - edges

2, 3, 2, 3 → degree sequence.

② Identify degree of each node in the below graph.

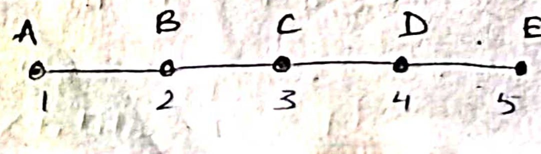


degree sequence:

1 3 4 2 3 3

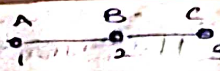
③ path graph (P_K)

P_5 :



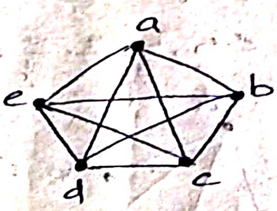
where K is the no. of nodes

ii) P_3 :



2. Complete Graph – K_n

→ In this Graph every node is connected to remaining other nodes.



→ K_5

→ denoted with K_n

n – no. of nodes

Note :- A path with n vertices, P_n, will always have n-1 edges.

Note:- A Complete graph with n vertices K_n will always have $\boxed{{}^nC_2}$ edges.

→ total degree = 2 × no. of edges

$$\sum_{v \in V} (\text{degree of } v) = 2\,(\text{no. of edges})$$

→ every edge contribute a degree in two different vertices.

→ Sum of even no. of odd number = even number

Eg: $\underbrace{1+3}_{\text{Even no. of odd numbers}} = 4$ → even

→ sum of odd no. of odd numbers = odd number

Eg: $\underbrace{1+3+5}_{\text{odd no. of odd number}} = 9$ → odd

$$\underbrace{\sum_{v \in V} (\text{degree of } v)}_{\text{even number}} = \underbrace{2\,(\text{no. of edges})}_{\text{even number}}$$

→ total degree is always even.

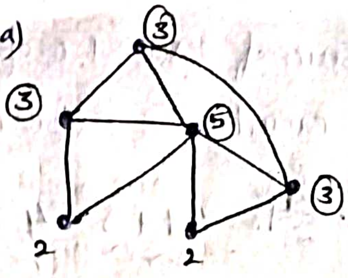
→ $\underbrace{d_1 + d_2 + \cdots + d_n}_{\text{even number}} = 2m$

All are even numbers (or) There are even no. of odd numbers.

i.e., there are even number of odd degree vertices in the graph

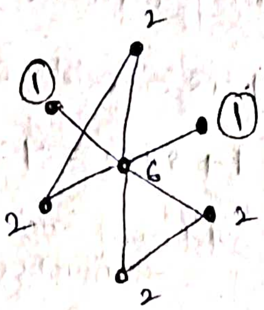
→ Every graph has even no. of odd degree vertices.

Eg:- a)



4 vertices with odd degree

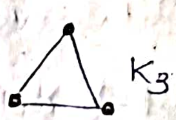
b)



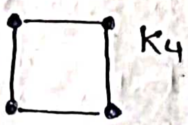
2 vertices with odd degree

draw graph for the following degree sequence?

i) <2, 2, 2>



ii) <2, 2, 2, 2>



<1, 1, 1> → it is impossible to construct a graph

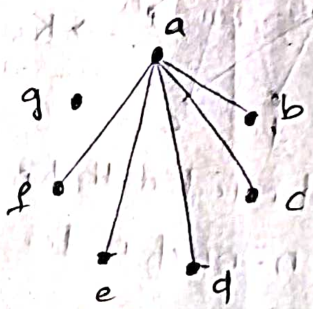
Note :- Given a graph, we can write a degree sequence. But given a degree sequence, you may not have a graph with that degree sequence.

Eg:- <1, 1, 1> → no graph

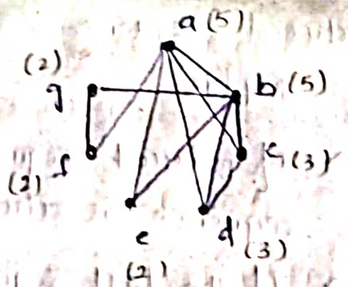
Havel Hakimi theorem

→ Given a degree sequence there may or may not exist a graph.

draw a graph for the degree sequence <5, 5, 3, 3, 2, 2, 2>
a b c d e f g



$S_1 = \langle 5_a, 5_b, 3_c, 3_d, 2_e, 2_f, 2_g \rangle$



$S_2 = \langle *, 4_b, 2_c, 2_d, 1_e, 1_f, 2_g \rangle$

$S_2' = \langle *_a, 4_b, 2_c, 2_d, 2_g, 1_e, 1_f \rangle$

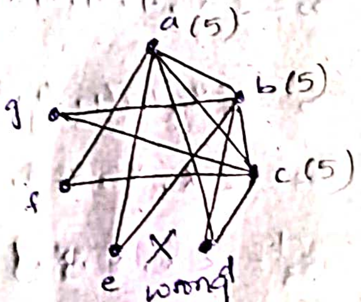
$S_3 = \langle *_a, *_b, 1_c, 1_d, 1_g, 0_e, 1_f \rangle$

$S_3' = \langle *_a, *_b, 1_c, 1_d, 1_g, 1_f, 0_e \rangle$

$S_4 = \langle *_a, *_b, *_c, *_d, 1_g, 1_f, 0_e \rangle$

$S_5 = \langle *_a, *_b, *_c, *_d, *_g, *_f, 0_e \rangle$

$S_1 = \langle 5_a, 5_b, 5_c, 5_d, 2_e, 2_f, 2_g \rangle \rightarrow$ Graphic



$S_2 = \langle *_a, 4_b, 4_c, 4_d, 1_e, 1_f, 2_g \rangle$

$S_2' = \langle *_a, 4_b, 4_c, 4_d, 2_g, 1_e, 1_f \rangle$

NOT a Graphic

$S_3 = \langle *_a, *_b, 3_c, 3_d, 1_g, 0_e, 1_f \rangle$

$S_3' = \langle *_a, *_b, 3_c, 3_d, 1_g, 1_f, 0_e \rangle$

$S_4 = \langle *_a, *_b, *_c, 2_d, 0_g, 0_f, 0_e \rangle$ $S_4 = \langle *, *, *, 2_d, 0, 0, 0 \rangle$

$\langle *, *, *, *, -1, -1, 0 \rangle$

A degree sequence is called graphic if a simple graph can be drawn with that sequence.

$\langle 5, 5, 3, 3, 2, 2, 2 \rangle$ — Graphic sequence

$\langle 2, 2, 2 \rangle$ — Graphic sequence

A degree sequence $S = d_1 \geq d_2 \geq d_3 \geq \dots \geq d_n$, where $d_1 \leq n-1$, is graphic if and only if the reduced sequence $S' = \{ *, d_2-1, d_3-1, \dots d_n \}$ is graphic

→ If reduced sequence is graphic then the original sequence is also graphic.

→ If the last sequence contains all 0s or *s, then it is graphic.

$\underbrace{\text{Sequence} \xrightarrow{\text{reduce}} \text{sequence} \xrightarrow{\text{reduce}} \text{sequence} \dots}$ Sequence contains all 0's

HAVEL HAKIMI Theorem

$S_1 = \langle 5, 5, 5, 5, 2, 2, 2 \rangle$ → Not graphic

$S_2 = \langle *, 4, 4, 4, 1, 1, 2 \rangle$

$S_2' = \langle *, 4, 4, 4, 2, 1, 1 \rangle$

$S_3 = \langle *, *, 3, 3, 1, 0, 1 \rangle$

$S_3' = \langle *, *, 3, 3, 1, 1, 0 \rangle$

$S_4 = \langle *, *, *, 2, 0, 0, 0 \rangle$

$S_5 = \langle *, *, *, *, -1, -1, 0 \rangle$ → Degree cannot be negative

Regular graph and irregular graph

Regular graph:

Degree of all the nodes happens to be the same.

Irregular graph: A graph that is not regular i.e., if you find atleast 2 vertices such that both of them have different degrees, then graph cannot be regular.

Can you Construct a graph on 5 nodes, where degree of all nodes are different?

→ It is impossible to Construct a graph with above data.

partial ordering and total ordering.

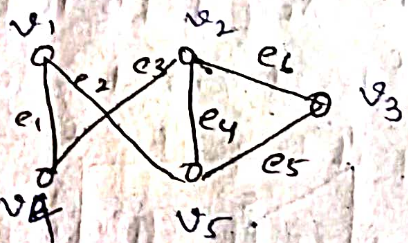
Graph Representation

- Adjacency List
- Adjacency matrix
  1. Adjacency of vertices
  2. Incident matrix Representation

$G = (V, E)$ $\quad |V| \times |E|$

$M = [m_{ij}]$ where

$$m_{ij} = \begin{cases} 1 & \text{edge } e_j \text{ ~~locked~~ incident with vertex } v_i \\ 0 & \text{otherwise.} \end{cases}$$

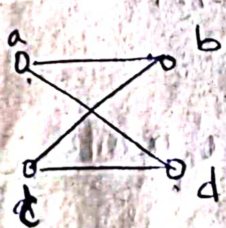


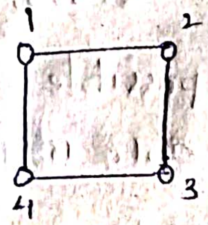
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Dense graph – matrix

sparse graph – list

Isomorphism





definition :- The sample graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function f from V_1 to V_2 with property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2 for all a & b in V_1

→ from above graphs

4 vertices

4 edges

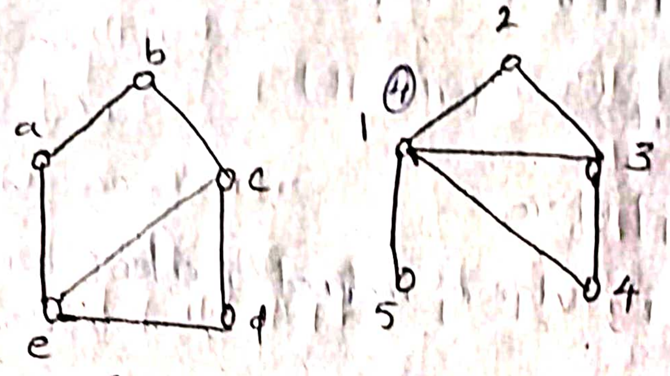
→ degree of each node is 2   2, 2, 2, 4 – degree sequence

→ mapping: $f(a)=1$ $f(c)=3$

$f(b)=2$ $f(d)=4$

→ In Compiler design (application)

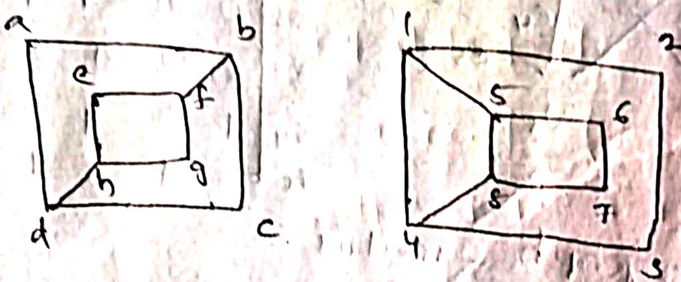
Eg-2:



↳ No vertex of degree 4

↳ Not isomorphic

Eg-3:



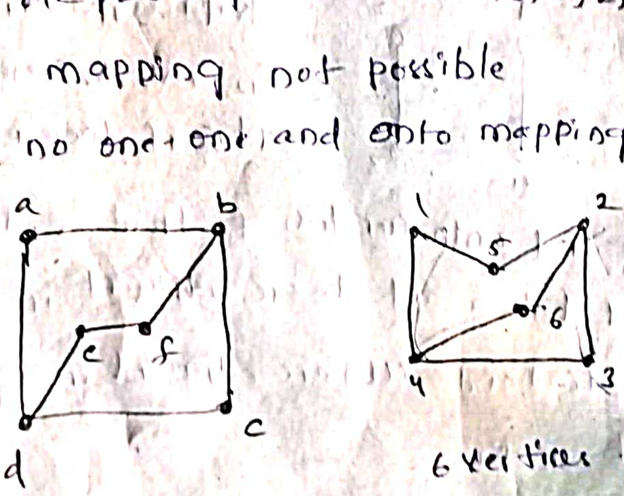
Total no. of vertices = 8

no. of edges = 10

degree sequence. 3, 3, 3, 3, 2, 2, 2, 2

mapping not possible

no one-one and onto mappings possible



6 vertices

7 edges

3, 3, 2, 2, 2, 2

$f(a)=3$

$f(b)=2$

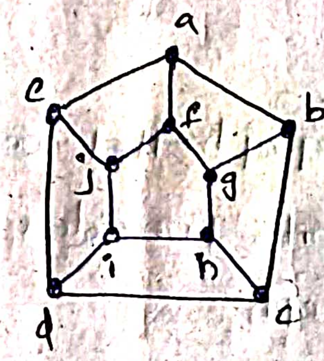
$f(c)=6$

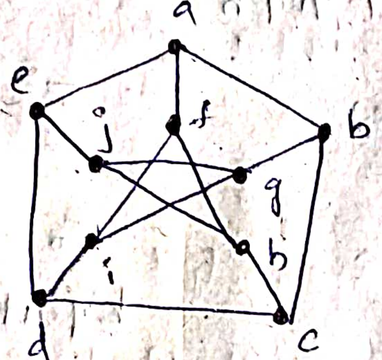
$f(d)=4$

$f(e)=1$

$f(g)=5$

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \\ f \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \quad \begin{array}{c|c} & \begin{matrix} 3 & 2 & 6 & 4 & 1 & 5 \end{matrix} \\ \begin{matrix} 3 \\ 2 \\ 6 \\ 4 \\ 1 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

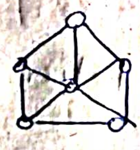




Not isomorphic

Wheel graph W_n

$n \rightarrow$ represents no. of outer nodes

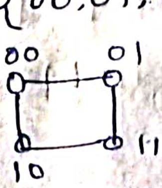
 $\rightarrow W_5$

 $\rightarrow W_3$

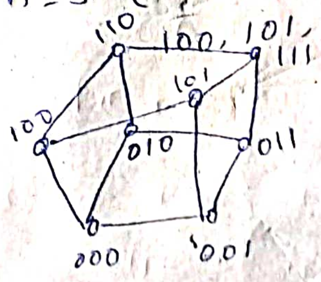
n-Cube graph Q_n

draw a graph with n bit string edge if two strings differ by 1 bit

$n=2$ {00, 01, 10, 11}



$n=3$ {000, 001, 010, 011, 100, 101, 110, 111}



$n=3$ Cubic graph

$n>3$

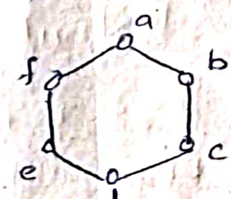
hyper cubic graph

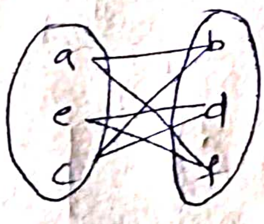
⑤ Bipartite graph

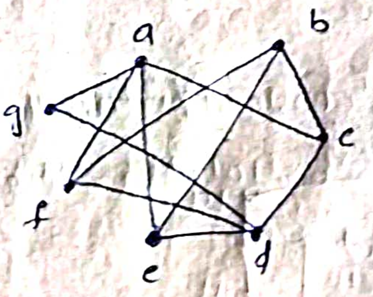
→ If we can partition the graph into two partition sets v_1 & v_2 so that there is no edge within the vertices of v_1 & v_2 but b/w v_1 & v_2 there are some edges we call it as bipartite graph

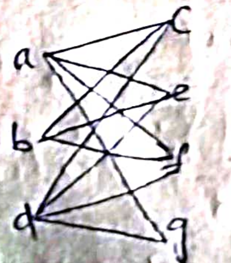
is C_6 bipartite?

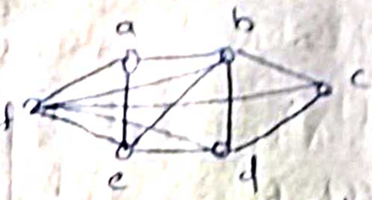
Ans:- yes











Not bipartite bcz f is connected to other vertices.

Graph coloring

Color a graph in such a such that no two adjacent vertices are of same col...

→ Note

- Any graph that we can colored with two colors is a bipartite graph

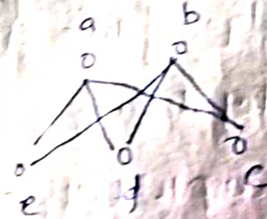
→ as no. of colors increases then that graph is not a bipartite

Complete bipartite graph

$K_{m,n}$ notation

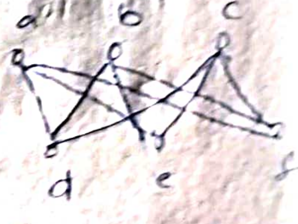
$K_{2,3}$

$m=2$ $n=3$



no. of edges $= m \times n$

$K_{3,3}$



no. of edges

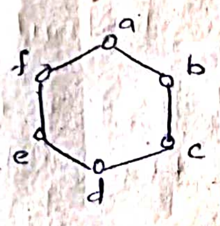
$3 \times 3 = 9$

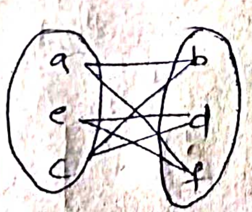
(5) Bipartite graph

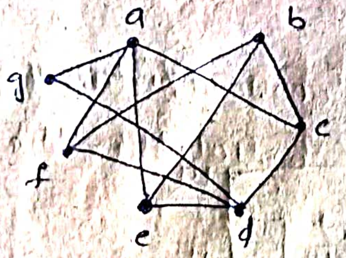
→ If we can partition the graph into two partition sets V_1 & V_2 so that there is no edge within the vertices of V_1 & V_2 but b/w V_1 & V_2 there are some edges we call it as bipartite graph

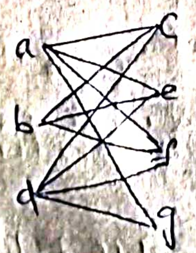
is C_6 bipartite?

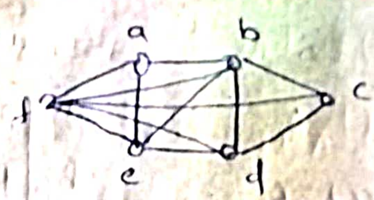
Ans: yes











Not bipartite bcz f is connected to other vertices

Graph Coloring

Color a graph in such a way such that no two adjacent vertices are of same color

→ Note

- Any graph that we can colored with two colors is a bipartite graph

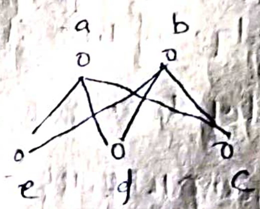
→ as no. of colors increases then that graph is not a bipartite

Complete bipartite graph

$K_{m,n}$ notation

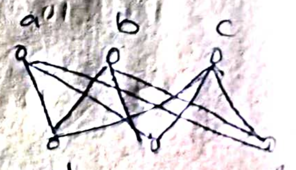
$K_{2,3}$

$m=2 \quad n=3$



no. of edges = $m \times n$

$K_{3,3}$

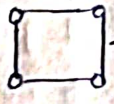


no. of edges

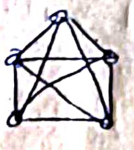
$3 \times 3 = 9$

Regular graph [Rg]

In regular graph Every vertex has same degree

Eg: C_4  → Cycle graph

K_5 (Complete graph)

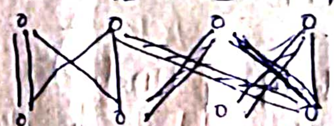


- Application

(a) Job assignment

model the problem using bipartite graph

P_1 P_2 P_3 P_4



J_1 J_2 J_3 J_4

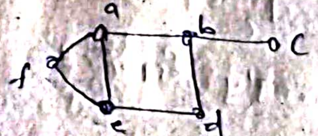
matching

- maximal matching
- maximum matching

→ Given a graph $G = (V, E)$ matching M in G is set of pairwise non adjacent edges

i.e., no two edges share a common vertex.

maximum matching



for above graph

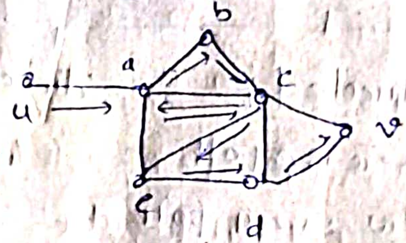
maximum matching

a — f
b — c
e — d
} best possible match

maximal matching

a — b
f — e
} we cannot add any more match.

Walk: no constraint (no restriction)



walk: sequence of edges and vertices

u — a — b — c — a — c — e — d — v

Trail: A walk with constraint no repeated edges.

Eg: u — a — b — c — b — a — e — d — v

Path: No repetition of edges no repetition of vertices (more constraint)

u — a — b — c — v

u — a — b — c — e — d — v

Close: beginning and end vertex is same

closed trail : Circuit

a — c — b — a

c — b — a — c — v — d — c

Closed path /cycle (no vertex is repeated)

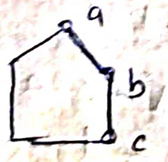
a — c — b — a

→ Cycle can be a part of ckt but ckt not a cycle

Connectedness

Undirected

— there is a path b/w every pair of



a & c

path a—b—c

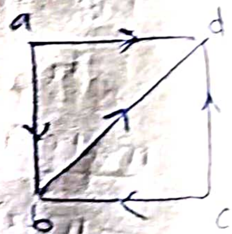
directed

i) Weekly connected

↳ the underlying undirected graph is connected

↓

path may not be there b/w every pair of distinct vertices

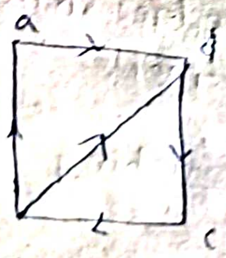


↓

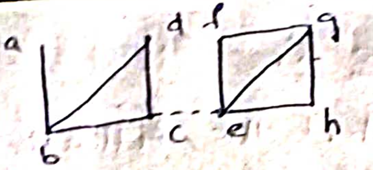
there is no path from a to c

ii) Strongly connected

— the underlying undirected graph is connected & also path exist b/w two distinct pair of vertices



Strongly connected.



Cutvertex / Articulation point

Removing a Cut vertex makes the graph disconnected

c, e, b

Cut edge / Bridge

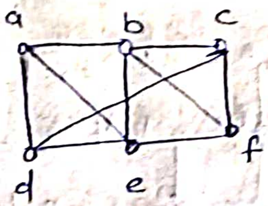
for above graph ce edge makes the graph disconnected

Edge Cut

A subset E' of edgeset E of $G=(V,E)$ such that $G-E'$ is a disconnected graph minimum edge cut

$|\{ad, dc, de\}|$

Vertex Cut

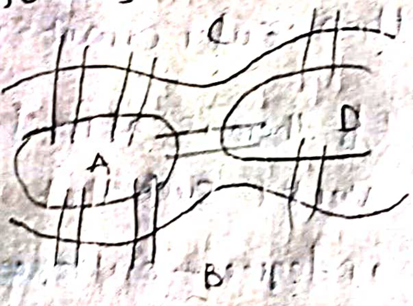


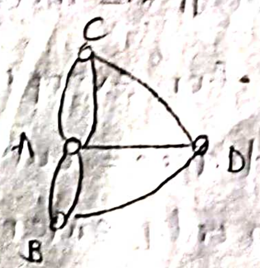
A subset V' of vertex set V of $G=(V,E)$ such that $G-V'$ is a disconnected graph

minimum vertex Cut

$|\{b, d, e\}| = 3$

24/01/2023 (Imp topics)
Tuesday





Euler Circuit : (Start & end point is same)

A simple ckt in G Containing every edge of G exactly once (closed)

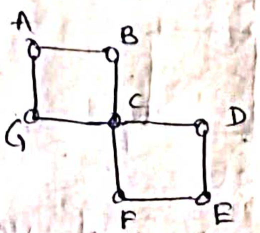
Euler path (trail) :

(no edge is repeated)

A trail in G contain every edge exactly once

→ If Every vertex is of even degree the Euler's Ckt Exists

→ If there exists exactly two vertices of odd degree then Euler path exists. (ckt doesnot exist)

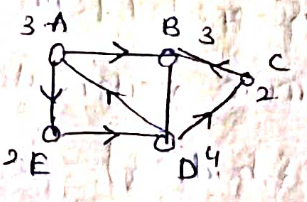


Each node has even degree

∴ Euler's ckt exists.

The ckt. starts from A

A — B — C — D — E — C — G — A

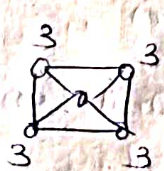


Euler's path exists but not exists Euler's ckt

. path

A — E — D — C — B — D — A — B

→ Euler trail exists but no Euler Circuit



more than two vertices with odd degree

therefore neither Euler's trail nor Euler ckt exists.

Gate Q:

which of the following graph has an Euler ckt.

regular disconnected

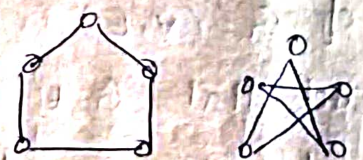
(A) A K-regular graph where K is an even number ✗

(B) Complete graph with (odd) vertices ✗
each vertex has degree 89 (odd)

(C) The Complement of a cycle of 25 vertices ✓

(D) none

Complement of a graph



For directed graph

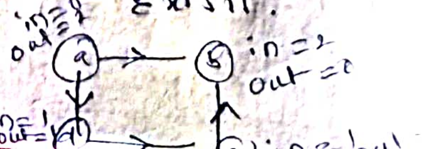
① if for every vertex indegree = outdegree then Euler Circuit exists

② If there is at most one vertex such that indegree – outdegree = 1

Outdegree – indegree = 1

and for every other vertex indegree = outdegree

then only Euler path exists.



for a and b indegree ≠ outdegree

∴ Euler ckt does not exist

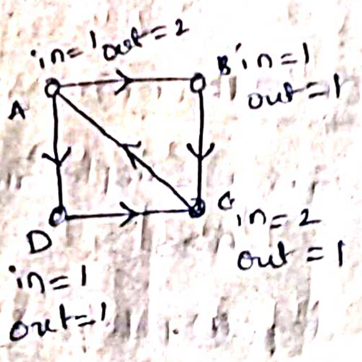
degree diff for a & b 2-0=2

∴ Euler path also doe.

→ but not Hamiltonian circuit exists

∵ when we start from A we can't come back to A.

③

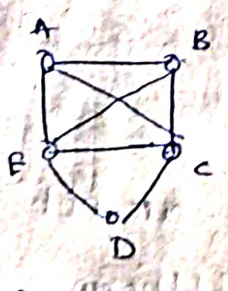


A – B – C – A – D – C

* Hamiltonian Circuit and path

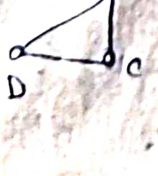
Hamiltonian path: A simple path in Graph 'G' that passes through every vertex exactly once is called a Hamiltonian path.

Hamiltonian ckt: closed hamiltonian path



A – B – C – D – E – A

↳ Hamiltonian ckt exists



A – B – C – D

↳ Hamiltonian path exists

31/01/2022
Tuesday

Hamiltonian Ckt & path

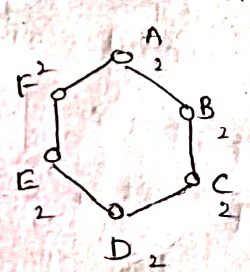
Path: visit Every Vertex exactly once [no need of visit each & every edge].

Ckt: Closed path.

Dirac's theorem	Ore's theorem
G is a simple Graph	G is a Simple Graph
$n \geq 3$	$n \geq 3$
Every vertex in G has degree atleast n/2 then G has hamiltonian Ckt.	$deg(u) + deg(v) \geq n$ for every pair of non-adjacent vertices u, v in G, then G has hamiltonian Ckt.
→ But Converse need not be true.	

Note: when both of these theorems are not satisfied then we cannot say anything about non-existence of hamiltonian ckt.

Eg:-

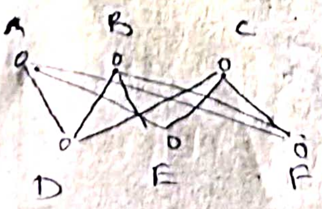


C_6 [Cycle Graph]

Ckt A-B-C-D-E-F

When Ore's thm & Dirac's thm are not applicable even though there exist hamiltonian Ckt. for C_6

for $K_{3,3}$ [Complete bipartite]



Ckt: ~~AE~~

A-E-C-F-B-D

n = 6 n/2 = 6/2 = 3

deg of non adjacent nodes

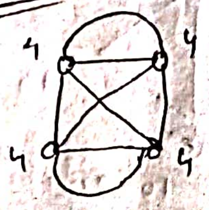
deg (A) + deg (B) = 3+3 = 6

∴ two thms are satisfied.

∴ There exist hamiltonian

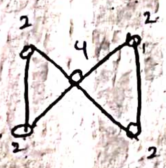
ckt for $K_{3,3}$.

Example problems

① 

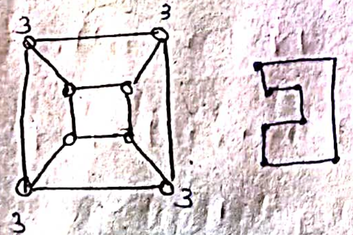
→ Eulerian ckt

→ Hamiltonian

② 

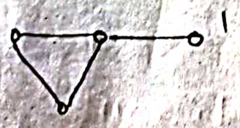
→ Eulerian

→ not hamiltonian

③ 

→ Not Eulerian

→ hamiltonian

④ 

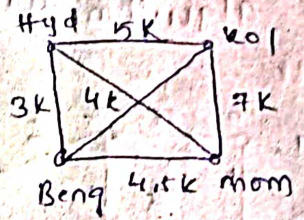
→ not Eulerian

→ not hamiltonian

Shortest path problem

— weighted graph

→ edges are assigned with weights

Eg:- 

— Computer network Graph

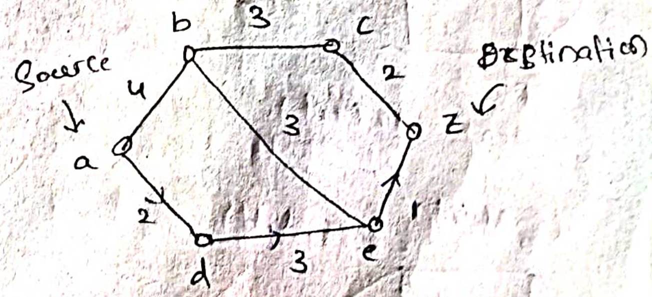
→ Response time

→ memory &
Storage.

— Map
— distance

Shortest path

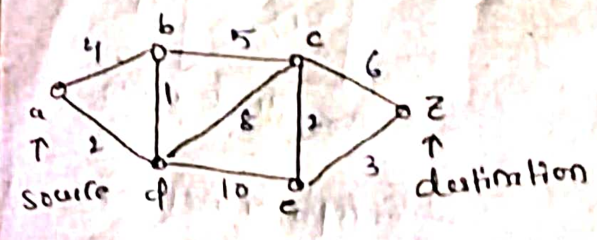
Q:- Find the shortest path b/w two vertices in a weighted graph.

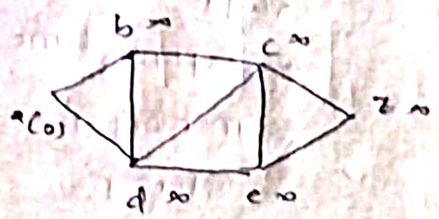


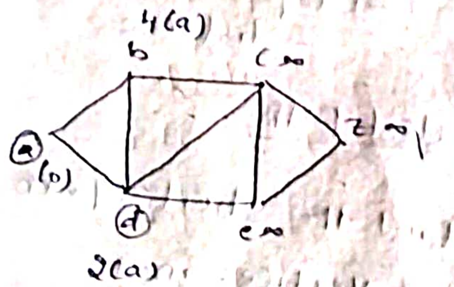
Shortest path with cost, 2+3+1=6

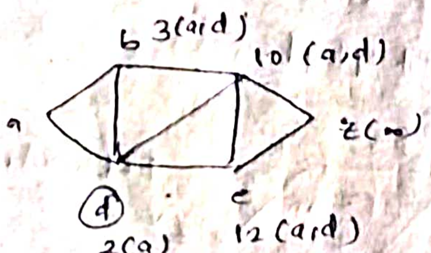
Dijkstra's Algorithm

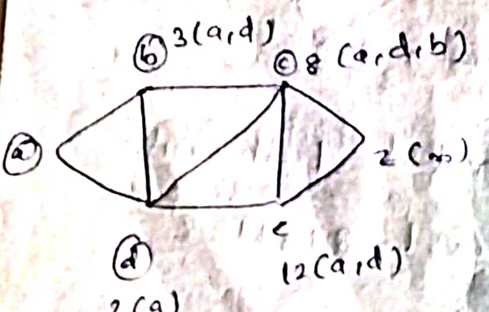
→ edge weight must be +ve [if it is -ve the algorithm will not work]

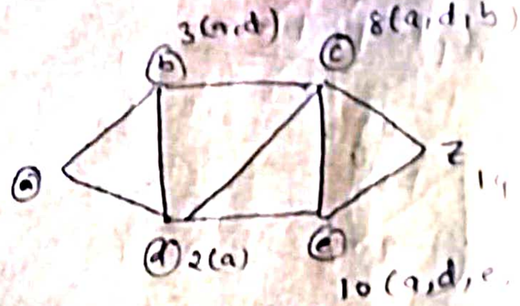


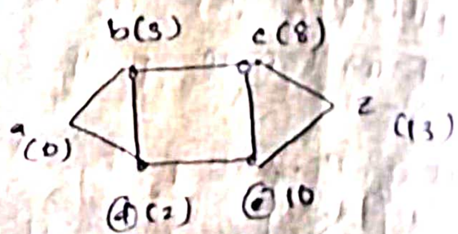






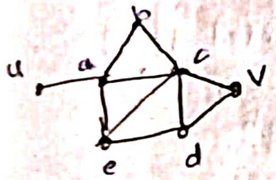






videos

Walk



i) u-a-b-c-V

ii) u-a-c-e-$\overset{a?}{c}$-V

iii) u-a-e-d-c-V

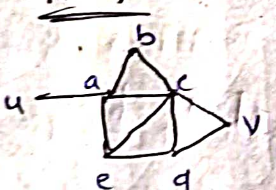
Walk: Sequence of vertices & edges

w(u,v)={u-a-b-c-v}

w(u,v)={u-a-c-e-a-c-v}

→ In a walk no restriction is required we can go though one vertex several times.

Trail



Constraint: We cannot go through an edge more than once we can repeat vertices.

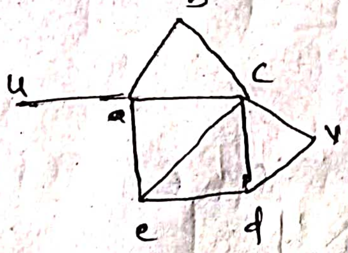
Trail : {u-a-c-v}
↳ trail

-A walk is a $\boxed{\text{trail}}$ if the edges are all distinct, but vertices need not be distinct.

{u-a-c-e-d-c-v}
↳ no edge is repeated

Path

Vertices cannot be repeated and also edges.

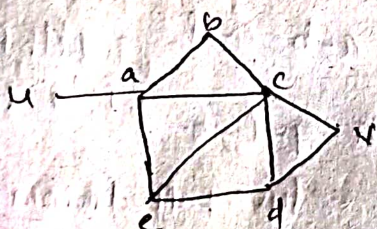


{u-a-c-v}

{u-a-e-d-v}

Path : where vertices are not repeated.

Closed path : A path which starts at a vertex and ends in the same vertex.

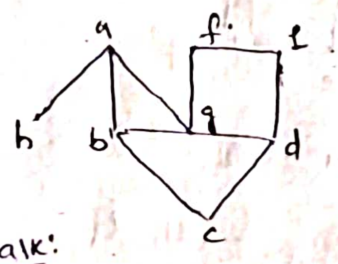


Starts from e.

{e-a-b-c-d-e}

{e-a-b-c-e}

	Repetition of vertices	Repetition of edges
Walk	✓	✓
Trail	✓	✓
Path	✓	✓



walk:
a-b-g-d-e-f-g-b-c

Q: starting from a and ending at e. find a path.

a-g-d-c-b-g-f-e

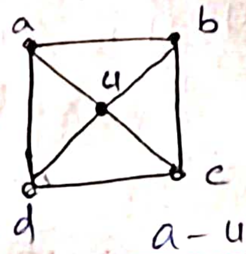
a-b-c-d
a-g-f-e-d } path from a to d.

close path from a

i] a-g-f-e-d-c-b-a

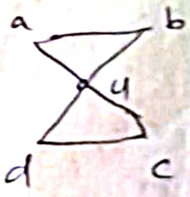
ii] a-b-g-a

Cycle and CKT



a-u-c-d-u-b (trail)

a-u-c-d-u-b
↳ closed tr



↳ This is CKT

path from a to b

a-u-c-b

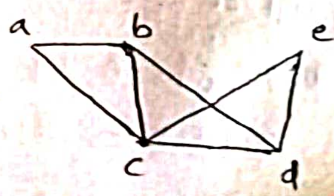
closed path from a

a-u-c-b-a

Closed path: It is Called a cycle

Note

A cycle is part of a circuit But a circuit can ot be a cycle.



a-b-c-d-e-c-a

[closed circuit]

a-b-c-a
↳ cycle

cycle starting from c:

c – d – e – c
↓
part of ckt

If there is a path from u, v then there is a walk from u to v

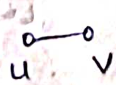
→ A walk from u to v might have ckts if we remove such ckts, we will get a path from u to v.

P(n) → If there exists a uv-walk there exists a uv-path in G.

Induct on length of the walk.

Basic step:

length of the walk = 1



Basic step is true

Induction hypothesis:

Assume the length of the walk ≤ K.

To prove:

The thm holds true for a walk of length K+1

- Assume $w(u,v)$ is a walk of length K+1, where no vertex is repeated.

→ Assume a vertex gets repeated in $w(u,v)$

$w(u,v) = u\, e_1\, a, \ldots \boxed{e_i\, x \ldots x\, e_j}$

$e_1, e_2, \ldots e_j$ → edges

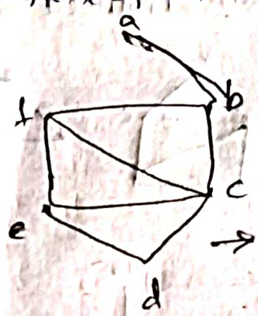
$u, a, \ldots x$ → vertices

$w'(u,v) = u\, e_1\, a\, e_2 \ldots e_i\, x\, e_j \ldots v$

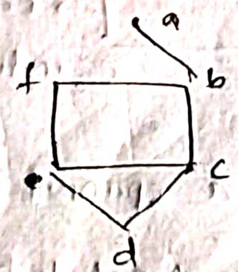
↳ walk of length ≤ K

There exists a uv path

Hence proved



→ original graph



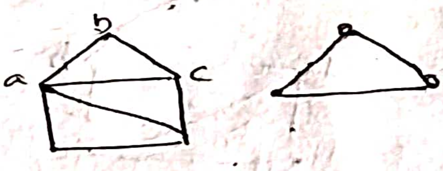
↳ Subgraph

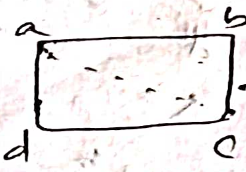
Subgraph: Given a Graph G, $V' \subset V$, $E' \subset E$ forms the subgraph

→ The subgraph which has same vertex set as the original graph is called as spanning subgraph.

Spanning subgraph   original graph

→ A subgraph which has all the edges corresponding to those set of vertices is called an Induced subgraph.





→ not an induced subgraph

→ a spanned subgraph need not to be an induced subgraph.

→ subgraph which is both spanning and induced is the graph itself.

→ A Tree is a connected acyclic graph.

↓

no cycles

no. of edges = no. of vertices − 1

$$\boxed{|E| = |V| - 1}$$

Graph

- Connected
- disconnected → have more than one component

Property of a cycle

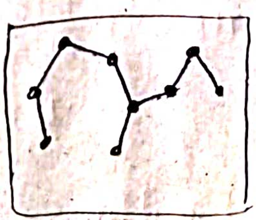
→ In a graph G, if there is a cycle and if an edge is remove from the cycle, the graph will still remains connected.

→ Given any graph G, that is connected with n vertices, the no. of edges $> n-1$

$|E| > n-1$

proof:-

G, if is a tree, it has n-1 edges If G is not a tree, there is a cycle in the graph.



→ edge is removed from a cycle the graph does not disconnected.

Graph is not disconnected;

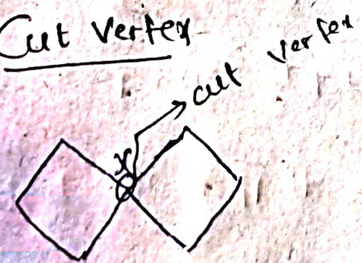
Graph has no cycles.

∴ Graph is a tree, having atleast n-1 edges

Connected graph

A Graph in which there exists a path from node a to node b, for any pair of nodes a and b
→ any pair of vertices

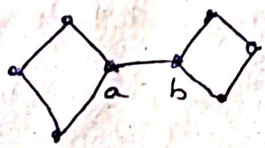
Cut vertex



→ If we removed x then the graph becomes disconnected graph

Such a vertex is called Cut vertex.

Cut edge:-

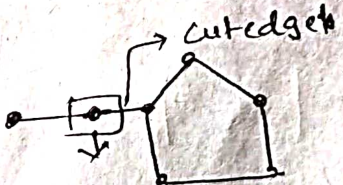


If we remov a-b edge then the above graph is disconnected

Cut edge
→ It is an edge on whose removal graph becomes disconnected

Cut vertex:-

If the removal of a vertex, makes the graph disconnected, such a vertex is a Cut vertex



Cut vertex and