1. let $A \in M_n(F)$ $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

• main diagonal $\{a_{ii}\}$, main subdiagonal $\{a_{i,i-1} \mid i=2,\dots,n\}$, main superdiagonal $\{a_{i-1,i} \mid i=2,\dots,n\}$

• trace $:= \sum a_{ii}$

$\sum (rA_{ii}+sB_{ii}) = r\sum A_{ii} + s\sum B_{ii}$

let $A\,B\,C \in M_n(F)$, $r,s \in F$, $tr(rA+sB) = r\,tr(A)+s\,tr(B)$ (linearity)

(commutative) $tr(AB) = tr(BA)$ $\quad \sum_{i=1}^{n}(AB)_{ii} = \sum_i \sum_k A_{ik}B_{ki} = \sum_k \sum_i B_{ki}A_{ik} = \sum_k (BA)_{kk}$ OR by $\boxed{15}$. $P_{⑥}$ 和 $\boxed{24}$ 证 1

$tr(ABC) = tr(BCA)$ (Cyclic) $\quad \sum_t (ABC)_{tt} = \sum_t \sum_k A_{tk}(BC)_{kt} = \sum_t \sum_k A_{tk} \sum_j B_{kj} C_{jt}$

$= \sum_k \sum_t \sum_j B_{kj} C_{jt} A_{tk} = \sum_k \sum_t BC_{kt} A_{tk} = \sum_k (BCA)_{kk}$

注 一般 $tr(AB) \neq tr(A)\,tr(B)$

$tr(A^T) = tr(A)$

$tr(A^TB) = \sum_i \sum_j A_{ij}B_{ij}$, $tr(u \otimes v) = u^Tv$ $\quad tr\begin{pmatrix} u_1v_1 & & \\ & \ddots & \\ & & u_nv_n \end{pmatrix}$

2. 对 Vec Space V 的 nonempty subset S. span(S) 为 L. comb. of vectors in S, 令 $span(\phi) = \{\underline{0}\}$ $\in V$

Algebra: 之比是为何 span(S) 可记为 $\langle S \rangle$, 为 subspace, P_{11}. 4 eg 1 的 + 很, 非线而 set, 代数无 L. comb, 用 generator 概念

$\mathcal{T}_1$ 若 $S \subseteq V$, span(S) 为 V 的 subspace; 事实上, 若 V 的 subspace W contains S, 也 contains span(S)

□ i) 若 $S = \phi$, $\{0\}$ 为 任何 vector space 的 trivial subspace; 若 $S \neq \phi$, $\underset{\in F}{0}\,\underset{\in span(S)}{x} = \underline{0} \in V$, 而 $x,y \in span(S)$ $\quad a_i b_i \in F$

$x = a_1v_1 + \dots + a_nv_n$

$y = b_1u_1 + \dots + b_mu_m$

易证 $x+y$, cx 者为 span(S) 中

ii) 即: $s_1 \dots s_n \in W$, 对 $a_i \in F$, $\underline{a_1s_1 + \dots + a_ns_n} \in W$ induction on n. 用 V.S. 的两 closure

3. DEF $S \subseteq$ vector space V. S generates/spans V iff $span(S) = V$

eg. 1. $\{(1\,1\,0)(1\,0\,1)(0\,1\,1)\}$ spans $\mathbb{R}^3$ $\quad$ ∵ 对 $\forall\, x = (a\,b\,c) \in \mathbb{R}^3$, $(a\,b\,c) = x(1\,1\,0) + y(1\,0\,1) + z(0\,1\,1)$

$x = \frac{a+b-c}{2}$ $\quad y = \frac{a-b+c}{2}$ $\quad z = \frac{-a+b+c}{2}$

$\mathcal{T}_2$: $W \leq V \Leftrightarrow span(W) = W$

□ ⇒, $span(W) \subseteq W$: 因 $W \subseteq W$, $\underline{span(W) \subseteq W}$ (据 $\mathcal{T}_1$ ii)) ; $span(W) \supseteq W$: let $w_i \in W$, $w_i = 0w_1 + \dots + 1w_i + \dots \in span(W)$

⇐ $span(\phi) = \{0\}$, ∴ W nonempty; $\forall\, w_1 w_2 \in W$, $w_1 + w_2 \in W$ (L. comb 相加 仍 L. comb); scalar multiplication trivial

4. P i) (span 的传递) S_1, S_2 为 V 的 subset, $S_1 \subseteq S_2 \Rightarrow span(S_1) \subseteq span(S_2)$; 若 $span(S_1) = V$, $span(S_2) = V$

ii) $S_1, S_2 \subseteq V$, $span(S_1 \cup S_2) = span(S_1) + span(S_2)$

□ $\subseteq$: $e \in$ LHS 写为 $\underbrace{\sum^n a_i s_{1_i}}_{\in span(S_1)} + \underbrace{\sum^m b_i s_{2_i}}_{\in span(S_2)}$ $\quad \supseteq$: $e \in$ RHS 写为 $\underset{span(S_1)}{u} + \underset{span(S_2)}{v}$

iii) $S_1, S_2 \subseteq V$, $span(S_1 \cap S_2) \subseteq span(S_1) \cap span(S_2)$ 注 非等

□ $e \in$ LHS 为 $\sum a_i s_i$, $\underset{\in S_1 \cap S_2}{}$ ∴ $e \in span(S_1)$, $e \in span(S_2)$ $\quad$ 如 $V = \mathbb{R}^2$, $S_1 = \{(1,0)\}$, $S_2 = \{(0,1)\}$ 如 P_{11}.4.eg.1 $\quad$ LHS $= span(\phi) = \{(0,0)\}$, RHS $= \{(0,0)\}$, LHS = RHS

又: $S_1 = \{(1,0)(1\,1)\}$, $S_2 = \{(1\,0)(0\,1)\}$ $\quad$ LHS $= \underline{\langle (1,0) \rangle}$, RHS $= \mathbb{R}^2$

iv) 对 $S \subseteq V$. 若 S L.Ind. 则 $\forall\, s \in \underline{span(S)}$ (即 — L.Ind 的 span), s 可由 S $\underline{unique}$ 表示

□ let $s = \sum a_i s_i = \sum b_i s_i$ $\quad a_i\,b_i$ 不全相同, 则 $s - s = 0 = \sum (a_i - b_i) s_i$, 由 L.Ind, $a_i - b_i = 0$, $\forall i$ ∴ $a_i = b_i$

v) (消去) 对 $span(v_1 \dots v_n)$ 若 v_i 可被其余 L. comb, 则 $span(v_1 \dots v_n) = span(v_1 \dots v_{i-1}, v_{i+1} \dots v_n)$

注 1. $x^Ty \cdot y^Tx = tr(xx^T\,yy^T) = \sum_{ij} x_ix_j \cdot y_iy_j$

□ 令 $u = x$, $v = yy^Tx$ $\quad$ 由 1. $u^Tv = tr(\underset{A^T}{x}\,x^T\,y\,\underset{B}{y^T}) = \sum_{ij} (xx^T)_{ij}\,(yy^T)_{ij}$

Date. /

例1 求 $\det\left(\begin{bmatrix}1&2&3&\cdots&n\\2&3&4&\cdots&n&1\\3&4&5&\cdots&1&2\\ & & \vdots\\ n&1&2&\cdots&n-2&n-1\end{bmatrix}\right)$,

□ note 各列之和相同, ∴ 各列全加于 col 1, 提出 col 1: $=\sum_{i=1}^{n} i\begin{vmatrix}1&\cdots&n\\1&3\cdots&n&1\\ \vdots\\1&\cdots&n-1\end{vmatrix}$ note 相邻行相差 1, 依次上一行 $\times(-1)$ 加于下一行

$=\sum i\begin{vmatrix}1&\cdots&n\\0&1\cdots&1&1-n\\0&1\cdots&1-n&1\\0&1-n&\cdots&1\end{vmatrix}$ col 1 展开, $=\sum i\underbrace{\begin{vmatrix}1&\cdots&1-n\\ \vdots&1-n&1\\1-n&\cdots&1\end{vmatrix}}$ $\}n-1$ note 各行之和 $=-1$ 仿造一开始, 各列全加于 col 1 $=\sum i\begin{vmatrix}-1&\cdots&1-n&1\\ \vdots\\-1&1-n&\cdots&1\\-1&1&\cdots&1\end{vmatrix}$

用第一列的 -1 cancel 其余所有 1 列 $\sum i\begin{vmatrix}-1&0&\cdots&-n\\ \vdots\\-1&-n&\cdots&0\\-1&0&\cdots&0\end{vmatrix}$ last row 展开 $=\sum i(-1)^{n-1+1}\cdot -1\cdot\overbrace{\begin{vmatrix}0&&-n\\-n&&0\end{vmatrix}}^{n-2}=\frac{n(n+1)}{2}\cdot(-1)^{n+1}\cdot\left[(-1)^{n-2+\frac{(1+n-2)(n-2)}{2}}(-n)^{n-2}\right]$

例2 $\begin{vmatrix}a_1&a_2&\cdots&a_n\\b_2&1&&0\\ \vdots&&\ddots\\b_n&0&&1\end{vmatrix}\xlongequal[\text{加入col 1}]{\text{各列适当乘}}\begin{vmatrix}x&a_2&\cdots&a_n\\0&1\\ \vdots&&\ddots\\0&&&1\end{vmatrix}$ $x=a_1-a_2b_2-\cdots-b_na_n$ $=a_1-\sum_{i=2}^{n}a_ib_i$

例3 $\begin{vmatrix}a_1+b_1&a_1+b_2&\cdots&a_1+b_n\\a_2+b_1&a_2+b_2&\cdots&a_2+b_n\\ &\vdots\\a_n+b_1&a_n+b_2&\cdots&a_n+b_n\end{vmatrix}=\begin{vmatrix}a_1&a_1+b_2&\cdots&a_1+b_n\\a_2\\ \vdots\\a_n&a_n+b_2&\cdots&a_n+b_n\end{vmatrix}+\begin{vmatrix}b_1&a_1+b_2&\cdots&a_1+b_n\\b_1\\ \vdots\\b_1&a_n+b_2&\cdots&a_n+b_n\end{vmatrix}$ 用各列去掉 col 1 $=\begin{vmatrix}a_1&b_2&\cdots&b_n\\a_2&b_2&&b_n\\ \vdots&\vdots\\a_n&b_2&\cdots&b_n\end{vmatrix}$ $\times\frac{b_2}{b_n}$ 后为 col 2, col 两相同, $\det=0$

例4 $\begin{vmatrix}a_1&b_1&0&\cdots&0\\c_1&a_2&b_2\\0&c_2&\ddots\\ &&&&b_{n-1}\\0&\cdots&&c_{n-1}&a_n\end{vmatrix}=$ 第 i 列 $-\frac{b_i}{b_1}$ 与 1 col $+\begin{vmatrix}b_1&a_1&\cdots&a_1\\b_1&a_2&\cdots&a_2\\ \vdots\\b_1&a_n&\cdots&a_n\end{vmatrix}=0+0=0$ 两 col 相同

□ 令其为 D_n, 对 last row expand, $=a_nD_{n-1}-c_{n-1}\begin{vmatrix}a_1&&0\\&\ddots&\vdots\\0&\cdots c_{n-2}&b_{n-1}\end{vmatrix}$

$=a_nD_{n-1}-c_{n-1}\cdot\underset{\text{expand last col}}{b_{n-1}}\cdot D_{n-2}$

例5 $\begin{vmatrix}x&y&\cdots&y\\z&\ddots&&\vdots\\ \vdots&&\ddots&y\\z&\cdots&z&x\end{vmatrix}=$ last col 拆为 $\binom{0+y}{x-y+y}$, 按 P_i split $\underbrace{\begin{vmatrix}x&\cdots&0\\ &\ddots&\vdots\\ &&x-y\end{vmatrix}+\begin{vmatrix}x&\cdots&y\\ &\ddots&y\\ &&y\end{vmatrix}}$ $=(x-y)\overbrace{\begin{vmatrix}x&\cdots&y\\ \vdots&\ddots\\z&\cdots&x\end{vmatrix}}^{n-1}+\underbrace{\begin{vmatrix}x&y&\cdots&y\\z-x&x-y&0&\cdots&0\\ &&\ddots\\0&\cdots&0&z-x&0\end{vmatrix}}$ 每行上行 $\times-1$ 加此行

$=(x-y)D_{n-1}-(-1)^{1+n}y(z-x)^{n-1}$ expand last col

例6 求 k st. $\begin{vmatrix}b_1+c_1,&b_2+c_2,&b_3+c_3\\a_1+c_1,&a_2+c_2&a_3+c_3\\a_1+b_1,&a_2+b_2&a_3+b_3\end{vmatrix}=k\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}$

□ $=-\begin{vmatrix}-(b_1+c_1)&\cdots&-(b_3+c_3)\\ &\cdots\end{vmatrix}$ r2, r3 加于 r1 $=-\underbrace{\begin{vmatrix}2a_1&\cdots&2a_3\\ &\cdots\end{vmatrix}}$ 提出 r1 的 2 且加于 r2, r3 消去 a $=-2\underbrace{\begin{vmatrix}a_1&a_2&a_3\\c_1&c_2&c_3\\b_1&b_2&b_3\end{vmatrix}}=2\begin{vmatrix}a_1&\cdots\\b_1&\cdots\\c_1&\cdots\end{vmatrix}$

例7 □

determinant 1 (det 为 minor)

1. 令 $A_{n\times n}$, DEF $\tilde{A}_{ij}$ 为 $(n-1)\times(n-1)$, 为 A 中去掉 i row j col.

Date. /

可 recursively def $\begin{cases} \det(A)=\sum_{j=1}^{n}\underbrace{(-1)^{1+j}\det(\tilde{A}_{1j})}_{\text{cofactor } C_{1j}} A_{1j}, & \text{cofactor expansion (1st row)} \\ \text{当 } A_{2\times2},\ \det\begin{pmatrix} a & b\\ c & d\end{pmatrix}=ad-bc \end{cases}$

cofactor C_{ij}, scalar, 称 A 的 adjugate 为 $\mathrm{adj}(A)=C^T$

P_1 i) (det 为 linear function of row)

$$\det\begin{pmatrix} -a_1- \\ -u+kv- \\ -a_n- \end{pmatrix}=\det\underbrace{\begin{pmatrix} -a_1- \\ -u- \\ -a_n- \end{pmatrix}}_{B}+k\det\underbrace{\begin{pmatrix} -a_1- \\ -v- \\ -a_n- \end{pmatrix}}_{C} \leftarrow r \text{ row}$$

□ induction on n. 假设 n-1 成立, 对 n, A_r 为 $(b_1+kc_1,\dots)=u+kv$, $u=(b_1,\dots)$, $v=(c_1,\dots)$, 当 $r=1$, trival. 当 $r>1$, 考虑对各 j, $\tilde{A}_{1j}$ 的 r-1 row 为 $(b_1+kc_1,\dots,b_{j-1}+kc_{j-1},b_{j+1}+kc_{j+1},\dots)$ (删除 j 列, 往上移), 即 $\tilde{B}_{1j}$ 的 row r-1 + $\tilde{C}_{1j}$ 的 r-1 row, 且其余 row $\tilde{A}_{1j}$ 同 $\tilde{B}_{1j}$ 同 $\tilde{C}_{1j}$. 而这三个 n-1 大矩阵. ∴ 由 inductive hypothesis, $\det(\tilde{A}_{1j})=\det(\tilde{B}_{1j})+k\det(\tilde{C}_{1j})$

$\det(A)=\sum_j(-1)^{1+j}A_{1j}\det(\tilde{A}_{1j})=\sum(-1)^{1+j}A_{1j}\cdot\det(\tilde{B}_{1j})+k\sum_j(-1)^{1+j}A_{1j}\det(\tilde{C}_{1j})=\det B+k\det C$ ($r\neq1$, 1st row 相同)

ii) 一行全 0, $\det=0$ □ 由 i) $\det\begin{pmatrix} -a_1- \\ -0- \\ -a_n- \end{pmatrix}=\det\begin{pmatrix} -a_1- \\ -0\cdot z- \\ -a_n- \end{pmatrix}=0\cdot\det\begin{pmatrix} -a_1- \\ -z- \\ -a_n- \end{pmatrix}=0$ (z 为任意 row)

iii) A row $i=e_k=(0\cdots\overset{k}{1}\cdots0)$, $\det A=(-1)^{i+k}\det(\tilde{A}_{ik})$

□ 数归 n. 若 n-1 成立, 对 n, 若 $i=1$, 由定义得证. 当 $i>1$, 考虑 $\forall j$, $\tilde{A}_{1j}$ 的 i-1 row 为 $\begin{cases} e_{k-1} & j<k \\ 0 & j=k \\ e_k & j>k \end{cases}$ (shift 上, $\in\mathbb{F}^{n-1}$, k-1 位为 1; 1 在 k 位, 在 k 位为 1)

因 $\tilde{A}_{1j}$ (n-1)阶方阵, inductive hypothesis, $\det(\tilde{A}_{1j})=\begin{cases} (-1)^{i-1+k-1}\det(C_{ij}) & j<k \\ 0 \ (P_1\ ii)) & j=k \text{ 行全 0} \\ (-1)^{i-1+k}\det(C_{ij}) & j>k \end{cases}$ ↑满足 induction 条件, row 有 e

C_{ij} 为 A 删去 row 1, i, col j, k (对 $j<k$, $\tilde{A}_{1j}$ 删 i-1 row k-1 col; $j>k$, $\tilde{A}_{1j}$ 删 i-1 row k col)

$\therefore \det A=\sum_j(-1)^{1+j}A_{1j}\det(\tilde{A}_{1j})=\sum_{j<k}(-1)^{1+j}A_{1j}(-1)^{i+k}\det(C_{ij})+\sum_{j>k}(-1)^{1+j}(-1)^{i+k-1}A_{1j}\det(C_{ij})$

提取共同项

$=(-1)^{i+k}\left[\sum_{j<k}(-1)^{1+j}A_{1j}\det(C_{ij})+\sum_{j>k}(-1)^{1+j-1}A_{1j}\det C_{ij}\right]$ 即为 $\tilde{A}_{ik}$ 的 1st row cofactor expansion ∴ $=\det\tilde{A}_{ik}$

(在为 $1+(j-1)$ 因删了 k 列), shift 上, A_{1j} 变为 $(\tilde{A}_{ik})_{1,j-1}$; 且因 k col 删, ∴ 无 gap

iv) (Laplace Expansion) cofactor expansion 可按 ∀ row 展开

□ 对 $\forall i$, WTS $\det A=\sum_{j=1}^{n}(-1)^{i+j}A_{ij}\det\tilde{A}_{ij}$. $i=1$ 即为 def. $i>1$, row i 表为 $A_{i1}e_1+A_{i2}e_2+\dots+A_{in}e_n$ eg $(1\,3\,1)=1\cdot(100)+3\cdot(010)+1\cdot(001)$

由 P_1 i), $\det A=A_{i1}\det\begin{pmatrix} -a_1- \\ -e_1- \\ -a_n- \end{pmatrix}+\dots+A_{in}\det\begin{pmatrix} -a_1- \\ -e_n- \\ -a_n- \end{pmatrix}=\sum_j A_{ij}\det\begin{pmatrix} -a_1- \\ -e_j- \\ -a_n- \end{pmatrix}=\sum_j A_{ij}(-1)^{i+j}\det\tilde{A}_{ij}$ 由 iii)

v) 二行相同, $\det=0$

□ induction n. 若 n-1 成立, 对 n, 令 row r, s 相同 选 $i\neq r\neq s$, $\det A=\sum_j(-1)^{i+j}A_{ij}\det\tilde{A}_{ij}$ → $n-1\times n-1$, 且有相同行, $\therefore\det=0$

vi) 对 ... 的 row op, 1) row scaling $\det A\to\alpha\det A$
2) row exchange $\det A\to-\det A$
3) row adjustment $\det A\to\det A$

□ 1) 由 i) 2) 考 i, j row. $0=\det\begin{pmatrix} -a_1- \\ -a_j+a_i- \\ -a_i+a_j- \\ -a_n- \end{pmatrix}=\det\begin{pmatrix} -a_1- \\ -a_j- \\ -a_i+a_j- \\ -a_n- \end{pmatrix}+\det\begin{pmatrix} -a_1- \\ -a_i- \\ -a_j+a_i- \\ -a_n- \end{pmatrix}$ (两行相同; apply i)) $\to\dots$ 拆后 $=\det\begin{pmatrix} -a_1- \\ -a_j- \\ -a_i- \\ -a_n- \end{pmatrix}+\det\begin{pmatrix} -a_1- \\ -a_j- \\ -a_j- \\ -a_n- \end{pmatrix}$ (=0, v)) $+\det\begin{pmatrix} -a_1- \\ -a_i- \\ -a_i- \\ -a_n- \end{pmatrix}$ (=0, v)) $+\det\begin{pmatrix} -a_1- \\ -a_i- \\ -a_j- \\ -a_n- \end{pmatrix}$ 原 matrix

3) $\det\begin{pmatrix} -a_1- \\ -a_i- \\ -a_j+\alpha a_i- \\ -a_n- \end{pmatrix}=\det(\text{原 matrix})+\alpha\det\begin{pmatrix} -a_1- \\ -a_i- \\ -a_i- \\ -a_n- \end{pmatrix}$ 两行相同, $=0$

vii) 两行成比例, $\det=0$ □ 结合 i) 和 v)

viii) $\det(kA)=k^n\det(A)$ □ $\det\begin{pmatrix} -ka_1- \\ -ka_n- \end{pmatrix}=k\det\begin{pmatrix} -a_1- \\ -ka_2- \\ -ka_n- \end{pmatrix}=\dots=k^n\det\begin{pmatrix} -a_1- \\ -a_n- \end{pmatrix}$

∴ 只当 n = even, $\det(-A)=\det A$ OR $\det kA=\det kI\cdot\det A=k^n\det A$

Date. /

例1 $A, B \in M_n(\mathbb{C})$, 若 B invertible, $\exists\, k \in \mathbb{C}$ s.t. $A+kB$ not invertible

□ $\det(A+kB) = \overset{\neq 0,\ 可逆}{\underline{\det B}} \cdot \underbrace{\det(B^{-1}A + kI_n)}_{可以看作\ B^{-1}A\ 的\ \text{characteristic polynomial}\ \boxed{25}.2}$ 由 fundamental thm of algebra $\exists$ zero(root) st. polynomial $= 0$

det 2

($\det A=0$ 若可逆，有 $AB=I$，但 $\det I=1=\det A\cdot\det B=0$)

实为 bicondition，$\det A\neq 0 \Leftrightarrow \operatorname{rnk} A=n$，可逆

不可逆，$\operatorname{rnk}A=\operatorname{rnk}L_A$，而 L_A bijection 才可逆

1 T_1 若 $A_{n\times n}$，$\operatorname{rnk}A<n$，$\det A=0$

[23]. P_{22})

□ rows L. dep. ∴有 row $a_r=\sum_{i\neq r} k_i\cdot a_i$ 引 #2 ；row op type 3, add $-k_i\cdot a_i$ 在 a_r，Ps. P_1 vi) 3)

$\det\begin{pmatrix}-a_1-\\ -a_r-\\ -a_n-\end{pmatrix}=\det\begin{pmatrix}-a_1-\\ -k_1a_1+\dots+k_na_n-\\ -a_n-\end{pmatrix}=\det\begin{pmatrix}-a_1-\\ -0-\\ -a_n-\end{pmatrix}=0$

行 = 0, det = 0

T_2 对 $A,B\in M_n(\mathbb{F})$，$\det AB=\det A\cdot\det B$

□ lemma：对 elementary E，$\det EB=\det E\cdot\det B$. 回. T_1

△ 若 E type 1)，LHS = 等效于 B row scale，$=\alpha\cdot\det B$

RHS = $\underset{=}{\alpha}\cdot\det B$ …

$\det(\text{“}\alpha\text{”}_1)=\alpha$

当 $\operatorname{rnk}A=n$. A 可逆 ∴ $A=E_m\cdots E_1$ [23]. P_{iii}

∴ $\det AB=\det(E_m\cdots E_1B)=\det(E_m)\cdot\det(E_{m-1}\cdots E_1B)$ lemma

$=\dots=\det(E_m)\cdots\det(E_1)\det B=\det(E_m\cdots E_1)\det B$

lemma 反过来看，聚合

$=\det A\cdot\det B$

若 $\operatorname{rnk}A<n$，$\overset{T_1}{\det A=0}$，∴ RHS = 0. 而对 AB，$\operatorname{rnk}AB\le\operatorname{rnk}A<n$，∴ LHS = 0 [23]. T_2 c)

T_3 $\det A^T=\det A$

□ $\operatorname{rnk}A<n$，但 $\operatorname{rnk}A^T=\operatorname{rnk}A$，∴ det 都为 0 ；$A$ invertible，$A=E_m\cdots E_1$，而 $\det E_i=\det E_i^T$

回 (对各 type 1&2 symmetry)

∴有 $\det A^T=\det(E_1^T\cdots E_m^T)\overset{T_2}{=}\det(E_1^T)\cdots\det(E_m^T)=\det(E_m)\cdots\det(E_1)=\det(E_m\cdots E_1)=\det A$

回 例1

图 直接用 Laplace expansion 也可沿 ∀ col 展开，增强 Ps. P_1 iv)

T_4 $\det A=\prod_{i=1}^{n}\lambda_i$ λ_i = eig val

有 n root

□ characteristic func $f(\lambda)=(-1)^n(\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)=-(\lambda_1-\lambda)\times-(\lambda_2-\lambda)\times\dots\times-(\lambda_n-\lambda)=(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$

因 λ 为变量，令 $\lambda=0$ 则 $f(0)=\prod_{i=1}^{n}\lambda_i=\det(A-0\cdot I)$ NOTE，即 [5]. T_{14} 的是 NOTE，$k_0=\prod\lambda_i$

OR 若可 eig decomposition $A=Q\Lambda Q^{-1}$，$\det A\overset{T_2}{=}\det Q\det\Lambda\det Q^{-1}=\det\Lambda=\prod^{n}\lambda_i$

Date.

Vector Space

~~n维向量空间 (*队了们 实斜和 实实空格)~~ n-dim 向量空间

let K field, (Vector Space 特例, $K^n = \{(a_1; \dots ; a_n) \mid a_i \in K\}$)

V over K satisfy 以下 即 V vector space

$+ : V \times V \to V$, $\cdot : K \times V \to V$ (operation 未包括 closure)

$(V,+)$ abelian group:

(A1) $\forall x, y \in V$, $x+y = y+x$ commutative

(A2) $\forall x, y, z \in V$, $(x+y)+z = x+(y+z)$ associative

(A3) $\exists!\ \underline{0} \in V$, $\forall x \in V$, $x+0 = x$ identity ($(0; \dots ; 0) = \vec{0}$)

(A4) $\forall x \in V$, $\exists!\ y \in V$, $x+y = 0$ inverse (let $x = (x_1; \dots ; x_n) \in K^n$, $y = -x = (-x_1; \dots ; -x_n)$)

monoid:

(M1) $\forall a, b \in K$, $\forall x \in V$, $(ab)x = a(bx)$ associative

(M2) $\exists!\ \underline{1} \in K$, $\forall x \in V$, $1 \cdot x = x$ identity

(D1) $\forall x, y \in V$, $\forall c \in K$, $c(x+y) = cx + cy$

(D2) $\forall x \in V$, $\forall c, d \in K$, $(c+d)x = cx + dx$

! 不代表 $ax = bx \Rightarrow a = b$ $(x=0)$ 或 $ax = ay \Rightarrow x = y$ $(a=0)$

1\. i) cancel law: $\forall x, y, z \in V$, $x+z = y+z \Rightarrow x = y$

ii) A3, A4, M2 都 unique 如 A3 $0+0' = 0$ ($0'$ unique), $0+0' = 0'$ (0 unique) $\therefore 0 = 0'$

iii) $\underset{\in K,\ \text{identity for addition}}{\underline{0}}x = \underset{\in V}{\underline{0}}$ $\forall x \in V$. $0x = \underline{(0+0)x} = 0x + 0x$ (D_2), $\therefore 0x = 0$ (1. i) cancel law)

iv) $x\underset{\in V}{\underline{0}} = \underset{\in V}{\underline{0}}$ $\forall x \in K$ $x0 = \underline{x(0+0)} = x0 + x0$ D_1

任意 set!!

2\. $\phi \subset \underline{\underline{S}}$, F field, $\mathcal{F}(S, F)$ set of all $f : S \to F$. let $f, g \in \mathcal{F}(S, F)$, $c \in F$

$f = g$ iff $f(s) = g(s)$ $\forall s \in S$, $+ : (f+g)(s) = f(s) + g(s)$, $\cdot : (cf)(s) = cf(s)$, 0 为 $f(x) = 0$

3\. F field, $\mathbb{P}[F]$ 为 coeff $\in F$ 的 polynomial space. $P_n(x) = a_0 + \dots + a_n x^n \in \mathbb{P}[F]$, degree n

degree 为 n 不是! $x^2 - x^2 = 0$, 非 degree 2 $\qquad$ 0 为 $0x^0 = 0 \in \mathbb{P}[F]$

事实上 $\mathbb{P}_n[F]$ 为 $\mathbb{P}[F]$ 中 degree at most n, 也为 vector space over F. ($\mathbb{P}_n(F)$ 为 subspace of $\mathbb{P}(F)$)

且 $P_n(x) = a_0 + \dots + a_n x^n \in \mathbb{P}_n[F]$ 可写为 $\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$

4\. • 最trivial 的 vec. space over any field F is $\{0\}$, $0+0 = 0$, $a0 = 0$

• (seq space) $\mathrm{Seq}(F)$ 为 ele in F 的 seq. $\{s_n\} + \{t_n\} = \{s_n + t_n\}$, $c\{s_n\} = \{c \cdot s_n\}$

• (coordinate space) $(a_1, \dots a_n)$, $a_i \in F$: F^n

• (Complex) $\mathbb{C}$ over $\mathbb{C}$ basis $\{1\}$, $\mathbb{C}$ over $\mathbb{R}$ basis $\{1, i\}$, 但 $\mathbb{R}$ over $\mathbb{C}$ 非 VS. $i \cdot 1 = i \notin \mathbb{R}$

vs2

Date. / 7

1 $\mathbb{R}^+$ all positive reals. $\mathbb{R}^+$ over $\mathbb{R}$ VS, addition : $x \boxplus y = xy$
scalar multi : $\alpha \odot y = y^{\alpha}$

dim = 1, basis = $\{10\}$ ($\forall x \in \mathbb{R}^+$, $x = (\log_{10} x) \odot 10$ 且 = ele 一定 L.dep)

例 1 判断 : 1° $V = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{F}\}$, +: coord 加法
·: $c(a_1, a_2) = (a_1, 0)$

2° $V = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$, +: $(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$
·: $c(a_1, a_2) = (ca_1, ca_2)$

口 1° ✗ ∵ 因 $0 \cdot (a_1, a_2) = (a_1, 0)$ zero vec. 不unique 2° ✗ $[(a_1, a_2) + (b_1, b_2)] + (c_1, c_2) \neq (a_1, a_2) + [(b_1, b_2) + (c_1, c_2)]$

1. subspace $W \subseteq$ vector space V　　subspace 不一定将 x+y 包括

定满足 vector space 的性质. P_8 的 8 点只有 (A3), (A4), 和 "+"/"·" 的 closure 不可推证 subspace 同性质 (未标本)

而 回.5 的 3 条 ("+"/"·" closure + (A3)) 可证 (A4)

□ let x inverse 为 $(-1)\cdot x$, $x + (-1)\cdot x = (1+(-1))\cdot x = 0\cdot x = 0$　　M2; $x = 1\cdot x$ & D2　P_8. 1. iii)

field multi identity 的 add inverse

~~① $0 \in W \wedge$ ② $\forall a \in F, x, y \in W$ $ax+y \in W$ 也可证 W 为 V subspace~~

NOTE, 由 $\forall x \in V, 0\cdot x = 0$ (P_8. 1. iii)) 可知: $0 \in V$ 总在任意 V 的满足 "·" closure 的 subset

但 subspace 的 "zero vector in W" 还保障了至少 W 非空 (对 ϕ, closure trivally true)

eg 1. W 为 set of symmetric matrix in $M_{n\times n}(F)$, W 为 $M_{n\times n}(F)$ 的 subspace.　□ $(A+B)^T = A^T + B^T = A+B \in W$

2. $F(R,R)$ 的 subspace 为 all continuous func: $R \to R$　□ sum of cont. cont.

$M_{ij} = 0$ 当 $i \neq j$　　set of $M_{n\times n}(F)$ trace $= 0$

3. set of diagonal matrix subspace of $M_n(F)$ / set of upper triangular matrix subspace of $M_n(F)$

2. P_1 intersection of subspace of V.S. V 也为 V 的 subspace　对 partial order with $\subseteq$, $\cap C_i$ 即 meet, $\wedge C_i$　因 intersection 是 largest subset of V

□ let C 为 V 的所有 subspace, W 为 C 的 ele 的 intersection ($\cap C_i$).　$0 \in W$　$x+y, cx \in W$ (x, y 在所有 C_i 中)

NOTE. union 非 subspace, 如 $V = \mathbb{R}^2$, $W_1 = \{[\begin{smallmatrix} x \\ 0 \end{smallmatrix}] : x \in \mathbb{R}\}$, $W_2 = \{[\begin{smallmatrix} 0 \\ x \end{smallmatrix}] : x \in \mathbb{R}\}$, 见 P_2

例 1 V 为 V.S. 且 $W \subseteq V$ 也为 V.S., W 为 V subspace ✗　除非 "+", "·" 也完全一致　如 $V = \mathbb{R}$ on field $\mathbb{R}$ 但 $W = \mathbb{Q}$ 为 V.S. on $\mathbb{Q}$ 而非 $\mathbb{R}$

W 为 $\mathbb{R}^3$ 的 xy-plane, 即 $W = \{(a_1, a_2, 0)\}$, 则 $W = \mathbb{R}^2$ ✗　$W \cong \mathbb{R}^2$, $w \in W$ 有 3 coord, 但 $x \in \mathbb{R}^2$ 只有 2 个

isomorphism

P_2. W_1, W_2 为 V 的 subspace. $W_1 \cup W_2$ 为 V subspace $\Leftrightarrow W_1 \subseteq W_2 \vee W_2 \subseteq W_1$

□ $\Leftarrow$ trival, $\Rightarrow$, 若两者都不对, 则 $\exists x \in W_1 - W_2$, $y \in W_2 - W_1$. 则 $x, y \in W_1 \cup W_2$, $x+y$ 也 $\in W_1 \cup W_2$ 那么若 $x+y \in W_1$, $x+y+(-x) = y \in W_1$ 矛盾　$x+y \in W_2$ 同理

对 $W_1, \ldots, W_n \subseteq V$. 若 $\cup W_i \subseteq V \Leftrightarrow$ 有 W_k contain all. 见右.5.

($\subseteq V$)

(V subspace) 即 $w_1 + w_2 = 0 \Rightarrow w_1 = w_2 = 0$

3. 对 W_1, W_2, 令 $W_1 + W_2 = \{(x+y) \mid x \in W_1, y \in W_2\}$, if $W_1 \cap W_2 = \{0\} \wedge W_1 + W_2 = V$, 则 $V = W_1 \oplus W_2$

T_1 若 W_1, W_2 为 V 的 subspace, $W_1 + W_2$ 也为 subspace; any subspace of V contains both W_1, W_2 must also contains $W_1 + W_2$

□ i) 如 $0+0 \in W_1 + W_2$　且: $x_1 + y_1 + x_2 + y_2 \in W_1 + W_2$　ii) 因 W subspace, 令 $x \in W_1$, $y \in W_2$, $x+y$ 必在 W. $\therefore W_1 + W_2 \subseteq W$

例 2. 证 $\mathbb{F}^n = \{(a_1, \ldots, a_n) \mid a_n = 0\} \oplus \{(a_1, \ldots, a_n) \mid a_1 = \cdots = a_{n-1} = 0\}$

□ $W_1 \cap W_2 = \{0\}$, 而 $\begin{cases} W_1 + W_2 \subseteq V: & \text{易证 } W_1, W_2 \text{ subspace of } \mathbb{F}^n. \therefore W_1 + W_2 \text{ 也为 subspace (} T_1\text{). subspace 本为 subset} \\ V \subseteq W_1 + W_2: & \text{let } v = (v_1, \ldots, v_n) \in V,\ v = (v_1, \ldots, v_{n-1}, 0) + (0, \ldots, 0, v_n) \text{ 为 subspace, 见 eg 1} \end{cases}$

例 3 证当 F 非 characteristic 2, $M_n(F) = \{$set of skew-symmetric$\} \oplus \{$set of symmetric$\}$

□ $W_1 \cap W_2 = \{0\}$, 而 $M_n(F) = \{A : A \in M_n(F)\} = \{(A + A^T) + (A - A^T) \mid A \in M_n(F)\} = W_1 + W_2$

若 F characteristic 2, $F \cong \mathbb{Z}_2$　$M^T = M = -M^T$ ($\begin{smallmatrix} -1 = 1 \\ -0 = 0 \end{smallmatrix}$)　　因为首, Left symmetric, Right skew symmetric

事实上, F 非 characteristic 2 时, $M_n(F) = \{$set of symmetry$\} \oplus \{$set of A, $A_{ij} = 0$ $i \leq j\}$

□ $W_1 \cap W_2 = \{0\}$, 而 $M_n(F) = \{A : A \in M_n(F)\} = \{T + A - T\} = W_1 + W_2$. $T_{ij} = T_{ji} = A_{ij}$ 选择 $T \in W_1$ 且 $A-T$ 的上三角 $= 0$, $\in W_2$

NOTE, 由例 3, 3. 的 W_1, W_2 不一定 unique.

T_2 W_1, W_2 subspace of V, $W_1 \oplus W_2 = V \Leftrightarrow \forall v \in V$, $v = x_1 + x_2$ uniquely, $x_1 \in W_1$, $x_2 \in W_2$

□ $\Rightarrow$ 若不同 present, $v = x_1 + x_2 = x_1' + x_2'$　$\underbrace{x_1' - x_1}_{\in W_1} = \underbrace{x_2 - x_2'}_{\in W_2}$ 但 $W_1 \cap W_2 = \{0\}$ $\therefore x_1 = x_1'$, $x_2 = x_2'$

$\Leftarrow$ 若 $W_1 \cap W_2 \neq \{0\}$, $\exists t \in W_1 \cap W_2$ 而 $t \neq 0$, 则 $t = t + 0 = 0 + t$, 不 unique

VS4

4. eg1. 对 vec spa V over F, 令 $v\in V$, $\langle v\rangle=\{av \mid a\in F\}\le V$, 为包含 v 的最小 subspace

类似 group theory 的 generator, 但不像 coset, aH 中 a 为 G 的 ele, 而此处 $a\in F\notin V$

事实上 $\langle v\rangle=\mathrm{span}(\{v\})$ 见 $P_2\,T_1$, $\langle v\rangle\le V$

eg2. $W\le V$, $v+W=\{v+w \mid w\in W\}$ 为 coset. (L. Alg 称为 affine subspace)

类似 group theory. $v+W\le V \Leftrightarrow v\in W$; $v_1+W=v_2+W \Leftrightarrow v_1-v_2\in W$

□ i) ⇒ $0\in v+W$, $v+?=0$, $?=-v\in W$, ∴ $v\in W$. ⇐ W closed

ii) ⇒ $v_1+w=v_2+w'$, $v_1-v_2=\underline{w'-w}\in W$. ⇐ $\forall e=v_1+w\in v_1+W$, since $t=v_1-v_2\in W$, $v_1=t+v_2$, $e=v_2+t+w\in v_2+W$

而 $v_2+W\le v_1+W$ 同理

定义 coset 的两 operation: $(v_1+W)+(v_2+W)=(v_1+v_2)+W$ 可见 quotient space V/W 为 V.S.

$a(v_1+W)=(av_1)+W$

5. P_{10} P_2 的脚注: ⇒. 反证, 若无 subspace contains all. 则 $\exists\, u\in S_1\setminus(S_2\cup\cdots\cup S_n)$

且 $\exists\, v\in(S_2\cup\cdots\cup S_n)\setminus S_1$ 若 $au+v\in S_1$, 因 $u\in S_1$, $-au\in S_1$, $au+v-au=v\in S_1$, 矛盾.

若 $au+v\in S_k$ for some k, 则必有 $b\ne a$, $bu+v\in S_k$, 则 $(a-b)u\in S_k$ 但 $u\in S_1$ 矛盾.

若拓展至 infinite. 对 partial ordered set $(\mathcal{P}(V),\subseteq)$ 的特例 $(\mathrm{sub}(V),\subseteq)$, 其 infinite chain

$C=\{W_1,\dots\}$ st. $W_1\subseteq W_2\subseteq\cdots$, $\underset{\text{union}}{\underline{\cup}}C$ 为 subspace of V. P_2

□ 若 $u,v\in\cup C$, $u\in W_i$, $v\in W_j$, 令 $j>i$, 则 $\underline{u+v\in W_j}$

6. 用 direct product 增加维度, 对 $V_1,\dots,V_n$ over $\underline{\text{same}}$ field F,

$\prod_{i}^{n} V_i=\{(v_1,\dots,v_n) \mid v_i\in V_i\}$ 为 V.S. over F. addition: $(v_1,\dots,v_n)+(u_1,\dots,u_n)=(v_1+u_1,\dots,v_n+u_n)$
multiplication: 才能 st. $\alpha v_i\in V_i$
$\alpha(v_1,\dots,v_n)=(\alpha v_1,\dots,\alpha v_n)$

eg. $\mathbb{R}^3=\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ over $\mathbb{R}$

7 T_3 对 V.S. V, $W<V$, $\exists\, T<V$ st. $W\oplus T=V$

□ let F family of all sub(V) st. disjoint from W. 对 order operation $\subseteq$, let C 为 F 的 chain,

类5. $\underline{\cup C\le V}$, 且 $\cup C\cap W=\{0\}$, 因 V 的 upper bound, 由 Zorn's lemma, $\exists$ maximal of F: M

if $W\oplus M<V$, 则 $\exists\, e\in V-(W\oplus M)$, $e\ne 0$, ???

(可由 P_{12} T_{20} 证明)

NOTE, 掌握了 basis 后, 可用 basis 证明: 易证 当 $W,T\le V$, $\beta_W\cap\beta_T=\{0\}$, $V=W\oplus T \Leftrightarrow \beta_W\cup\beta_T=\beta_V$

则对 W 的 basis β_W, def $T=\mathrm{span}(\beta_V-\beta_W)$ 由 lemma, $W\oplus T=V$

In fact T 不唯一, 如 $\mathbb{R}^2=V$, $W=\{(a_1,0) \mid a_1\in\mathbb{R}\}$, T 可取 $\{(0,a_2) \mid a_2\in\mathbb{R}\}$ 也可取 $\{(a_2,a_2) \mid a_2\in\mathbb{R}\}$

例1 W_1: {set of odd func}, W_2: {set of even} $\overset{\text{连续函数}}{C(\mathbb{R})}=W_1\oplus W_2$

□ 对 $\forall f\in C(\mathbb{R})$ $f(x)=\underbrace{\tfrac{1}{2}(f(x)-f(-x))}_{\text{odd},\ g(-x)=-g(x)}+\underbrace{\tfrac{1}{2}(f(x)+f(-x))}_{\text{even},\ g(-x)=g(x)}$

例1. 对 P10. eg1.3, 求 dim. & symmetric & ~~skew~~-symmetric 同见 ***10 dim

□ diagonal: $\{E_{ii}\}$. dim = n, trace = 0: 因为 $\begin{bmatrix} -a_{22}-a_{33}-\cdots-a_{nn} & & \\ & a_{22} & \\ & & \ddots \\ & & & a_{nn}\end{bmatrix} = \{E_{ij} \mid i \neq j\} \cup \{[\ldots], [\ldots], \ldots, [\ldots]\}$

~~upper triangle: $\{E_{ij} \mid i \le j\}$ dim = $\frac{(1+n)n}{2}$~~ dim = n^2-1, tr 可视为 L.T. $M_{n\times n}(F) \to F$, nullity = n^2-1, 非 1-1 但 onto, rank = 1 ... 同 $\{E_{11}-E_{ii} \mid 2 \le i \le n\}$

symmetric: 同 upper triangle

skew-symmetric: $\begin{bmatrix} 0 & & ? \\ & \ddots & \\ -? & & 0\end{bmatrix}$, $\{[\ldots], [\ldots], \ldots, [\ldots]\} = \{E_{ij}-E_{ji} \mid i<j\}$, dim = $\frac{(1+n)(1+n)}{2} \ldots = \frac{n^2-n}{2}$

1. T_1 $W_1, W_2 \le V$, dim V finite. $W_1 \subseteq W_2 \Leftrightarrow \dim(W_1 \cap W_2) = \dim(W_1)$

□ $\Rightarrow W_1 \cap W_2 = W_1$. $\Leftarrow$ let $\dim W_1 = n$, β 为 basis for $W_1 \cap W_2$. note $W_1 \cap W_2 \subseteq W_1$,

$\beta \subseteq W_1$, 且因 $\dim W_1 = n$, β 本为 n 个 L.Ind, ∴ β 为 W_1 basis ***3. P_{13} 同 basis, 生成空间相同, $W_1 \cap W_2 = W_1$ ∴ $W_1 \subseteq W_2$

OR: $W_1 \cap W_2 \le W_1$ 且两者 dim 相同, $W_1 \cap W_2 = W_1$ (***3. T_3 来证明) (W_1 本为 subspace cp-vector space, ∴ $W \le W_2$)

T_{2a} $W_1, W_2 \le V$ 且 finite dim 之 subspace $W_1 + W_2$ 有 $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

□ let $\{u_1, \ldots, u_k\}$ basis for $W_1 \cap W_2$, since $W_1 \cap W_2 \subseteq W_1, W_2$, extend to (***3. P_{13}) $\{u_1, \ldots, u_k, v_1, \ldots, v_m\}$ basis for W_1, $\{u_1, \ldots, u_k, w_1, \ldots, w_p\}$ basis for W_2

WTS $\{u_1, \ldots, u_k, v_1, \ldots, v_m, w_1, \ldots, w_p\}$ basis for W_1+W_2 LHS 表示 $\in W_1$ (basis 为 u_i, v_i), RHS 表示 $\in W_2$, ∴ 同时存在 both

i) L.Ind. let $a_i, b_i, c_i \in F$, $\underline{\sum a_i u_i + \sum b_i v_i = -\sum c_i w_i \in W_1 \cap W_2}$

∴ $-\sum c_i w_i$ 可由 $\{u_i\}$ 表示: $-c_1 w_1 - \cdots - c_p w_p = t_1 u_1 + \cdots + t_k u_k$, 但这为 W_2 basis 的 L.Comb. ∴ L.Ind. $c_i = t_i = 0$

同理 $\sum a_i u_i + \sum c_i w_i = -\sum b_i v_i$, $b_i = 0$, ∴ $\sum a_i u_i + 0 + 0 = 0$, 由 $W_1 \cap W_2$ basis, $a_i = 0$

ii) span: $x \in W_1 + W_2$ 可写为 $(\sum a_i u_i + \sum b_i v_i) + (\sum c_i u_i + \sum d_i w_i)$

T_{2b} $V = W_1 + W_2$, 则 $V = W_1 \oplus W_2 \Leftrightarrow \dim V = \dim W_1 + \dim W_2$

□ $W_1 \cap W_2 = \phi \Leftrightarrow \dim W_1 \cap W_2 = 0 \Leftrightarrow \dim V = \dim W_1 + \dim W_2$ (T_{2a})

例2. $W_1, W_2 \le V$, $\dim(W_1 \cap W_2) \le \min(\dim W_1, \dim W_2)$
~~$\dim(W_1+W_2) \le \dim(W_1) + \dim(W_2)$~~ ***3. T_3 非负

□ let $\dim W_1 \le \dim W_2$, note $\underline{W_1 \cap W_2 \le W_1}$ ($W_1 \cap W_2$ v.s. (P_{10} P_3) & subset of W_1), ∴ $\underline{\dim LHS \le \dim W_1}$; $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \underline{\dim(W_1 \cap W_2)}$

Note: 例2 i) 等号: $W_1 = \{[\begin{smallmatrix}a\\0\\0\end{smallmatrix}]\}$ $W_2 = \{[\begin{smallmatrix}b\\0\\0\end{smallmatrix}]\}$ $W_1 \cap W_2 = \{[\begin{smallmatrix}a\\0\\0\end{smallmatrix}]\}$ dim = 1

ii) 等号 $W_1 = \{[\begin{smallmatrix}a\\b\\0\end{smallmatrix}]\}$ $W_2 = \{[\begin{smallmatrix}0\\0\\c\end{smallmatrix}]\}$ $W_1 + W_2 = \{[\begin{smallmatrix}a\\b\\c\end{smallmatrix}]\}$ dim = 3

2. T_{2c} $\dim(V) = \dim(W) + \dim(V/W)$ P114.

□ 令 $\beta_W = \{v_1 \ldots v_k\}$, extend by β_W ***3. P_{13} $\beta_V = \{v_1, \ldots v_n\}$, WTS $\{v_{k+1}+W, \ldots, v_n+W\}$ basis for V/W

i) L.Ind $\sum_{i=k+1}^{n} k_i(v_i + W) = \underbrace{\sum_{i=k+1}^{n}(k_i v_i + W)}_{\text{scalar multiplication, addition def}} = \sum_{i=k+1}^{n}(k_i v_i) + W = \underbrace{0 + W}_{\text{zero vector}}$ ∴ $\sum_{i=k+1}^{n} k_i v_i \in W$ P114 性质

但这部分 v_i 不在 W 中, ∴ $\sum k_i v_i$ 只能 = 0, 而 v_i L.Ind. ∴ $k_i = 0$

ii) span. $\forall x + W \in V/W$, $x + W = (\sum^{k} a_i v_i + \sum_j a_j v_j) + W$. 因 $\sum a_i v_i \in W$, ∴ = $\sum a_j v_j + W$ P114 性质

~~NOTE, 以 L.T. 的角度证明: let $\eta: V \to V/W$, $v \to v+W$, 易证 η linear~~

~~$Null(\eta) = \{v \in V \mid v + W = 0 + W\}$ that is, $v \in W$, ∴ $Null(\eta) = W$~~

∴ 由 rank Thm, $\dim V = \text{rank}\,\eta + \text{nullity}\,\eta = \dim V/W + \dim W$ 由 def 得

3. 有 $T: V \to Z$ linear 且 onto. def $\bar{T}: V/Nul(T) \to Z$, $v + Nul(T) \to T(v)$

证 $\bar{T}$ well-defined 且为 isomorphism, 且 $T = \bar{T}\eta$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\eta"'] & Z \\ V/Nul(T) \arrow[ur, "\bar{T}"'] & \end{tikzcd}$$

□ well-defined: $v + N(T) = v' + N(T) \Rightarrow T(v) = T(v')$ map 到同处 P114 性质

{因 $v - v' \in Nul(T)$, $T(v - v') = 0$, $T(v) = T(v')$}

~~linear 易证~~, onto: 因 T onto 对 $z \in Z$, 有 $T(x) = z$ ∴ 有 $\bar{T}(x + Nul(T)) = T(x) = z$

1-1: 令 $\bar{T}(x + N(T)) = T(x) = 0$, $\underline{x \in Nul(T)}$ def. $x + Nul(T) = Nul(T) = \bar{0}$ coset zero vector P114 性质

对 $\forall x \in V$ $T(x) = \bar{T}(x + Nul(T)) = \bar{T}\eta(x)$

例1. $V_1, V_2 \le V$, 若 $\dim(V_1+V_2) = \dim(V_1\cap V_2)+1$, 证 $V_1+V_2 = V_i$ (V_1, V_2中一个)
$V_1\cap V_2 = V_{1-i}$ (另一个)

□ 由 $V_1\cap V_2 \le V_i \le V_1+V_2$ $\quad \dim(V_1\cap V_2) \le \dim V_i \le \dim(V_1+V_2)$ 3. T_3

∴ 有 要么 $\dim V_i = \dim(V_1\cap V_2)$ or $\dim(V_i) = \dim(V_1+V_2)$ → 同理 ∴ 两种情况都成立

→ 则 $V_{1-i} \supseteq V_i$, 则 $V_{1-i} = V_1+V_2$

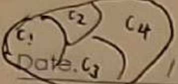
R_2 T_1

$\dim(V_1\times V_2) = \dim(V_1)+\dim(V_2)$

$\dim(V_1\otimes V_2) = \dim(V_1)\times\dim(V_2)$

13.

1. 对 ~ equivalent relation on S. 因 equivalent relation 可记为 ~~partition~~ (Algebra P17. 2. Lemma)



为了快速了解 $\forall s \in S$, s 属于哪个 equivalent class, 在各 equivalent class E_i 中选一个 c_i, 将 s 与 c_i 比较:

$C \subseteq S$ 为 system of distinct representatives for ~ on S iff $\forall E_i$ contains 1 member in C

C 也称为 set of canonical form

eg1. let $A, B \in M_{m,n}(F)$, $A \equiv B$ (称 A, B equivalent, 事实上为 equivalent relation) iff $\exists$ invertible $P \in M_m(F)$, $Q \in M_n(F)$, $B = PAQ$

即: A 可由 row/col operations reduce 为 B

事实上, $\forall$ matrix R 会被 reduce 为 $J_r = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$.

且 no distinct $J_s \equiv J_r$ $\square$ if $J_s = PJ_rQ$, $\mathrm{rank}(J_s) = s = \mathrm{rank}(PJ_rQ) = \mathrm{rank}(J_r) = r$

这样 $C = \{J_r \mid r \geq 0\}$ 为 set of canonical form, $A \equiv B$ iff A, B 有同 canonical form J_s

2. 用 func 了解 equivalent class

$f: S \to X$ 为 invariant under ~ (on S) iff $a \sim b \Rightarrow f(a) = f(b)$ (constant on equivalent class)

complete invariant under ~ iff $a \sim b \Leftrightarrow f(a) = f(b)$

eg2. 对 equivalent relation $\equiv$ (见 eg1) (见 eg1. 的 $\equiv$), rank 为 complete invariant func

$\square$ $A \equiv B \Leftrightarrow \mathrm{rank}(A) = \mathrm{rank}(B)$

row operation

eg3. $A, B \in M_n(F)$, $A\ B$ similar ($A \sim B$) iff $\exists$ invertible P, $B = P^{-1}AP$

可证 $A \sim B$ 为 equivalent relation on $M_n(F)$

Jordan

rational canonical form

matrix similar 有多种 invariants 但无 complete invariants

如 det: $\square$ $\det(B) = \det(P^{-1})\det(A)\det(P) = \det(A)$, $\therefore$ invariant, 但不同 matrix 会有同 det

trace: $\square$ $tr(B) = tr(P^{-1}AP) = tr(AP^{-1}P) = tr(A)$, 但 $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ trace 同 $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ 但非 similar

NOTE: complete invariant 可完全决定 ele 属于哪个 equivalent class, 只用施加 func

但 complete invariant 很少, 一般用 noncomplete 的逆否来判断 ele 不属于哪个 class

Date. /

此时 $inv = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$

证1. (Matrix Inverse Lemma) 对可逆 A, $A+uv^T$ 可逆 $\Leftrightarrow v^TA^{-1}u \neq -1$.

$A+uv^T$ & A^{-1} 可逆, 若 $=0$, 可逆则 $u=0=u$

□ $\Rightarrow$ $[(A+uv^T)A^{-1}]u \neq 0 \Leftrightarrow u + uv^TA^{-1}u \neq 0 \Leftrightarrow u(1+v^TA^{-1}u) \neq 0$

$\Leftarrow$ Note $P = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$ 是 $Q = A+uv^T$ 的逆. By check $PQ = QP = I$

当 $A=C=I$ reduce 为证4. lemma 2, 并从 lemma 2 令 $U=A^{-1}X$, $V=CY$ 还原

证2 (Sherman-Morrison-Woodbury) A_{nn} 可逆, 有维数的 U_{nk} V_{kn} C_{kk}

代入证=I $A+UCV^T$ 可逆 $\Leftrightarrow$ k×k 的 $(C^{-1}+V^TA^{-1}U)$ 可逆, 此时 $(A+UCV^T)^{-1} = A^{-1} - A^{-1}U(C^{-1}+V^TA^{-1}U)^{-1}V^TA^{-1}$

证3 $(I+P)^{-1} = I-(I+P)^{-1}P$ □ 令 $U = I+P$ $U(I-U^{-1}P) = U-P = I$ 证4. lemma 2 令 $A=I$, $B=P$

证4 (Push-Through) $U^T_{nd}(I_n+V_{nd}U^T)^{-1} = (I_d+U^TV)^{-1}U^T$

(lemma 1) $A(I+BA)^{-1} = (I+AB)^{-1}A$ $A(I+BA) = (I+AB)A \Leftrightarrow A(I+BA)^{-1} = (I+AB)^{-1}A$

(lemma 2) $(I+AB)^{-1} = I - A(I+BA)^{-1}B$ 考虑 $(I-(I+AB)^{-1}AB)(I+AB) = I+AB-(I+AB)^{-1}AB(I+AB)$

$= I+AB-(I+AB)^{-1}(I+AB)(AB) = I+AB-AB = I$, 即 $I-A(I+BA)^{-1}B$, lemma 1

∴ 有 lemma 1 ⟹ Push-Thru

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例1 inv A 10×10 $A = \begin{pmatrix} 2 & & 1 \\ & 2 & \\ 1 & & \ddots\, 2 \end{pmatrix}$ ∵ $A = I + uu^T$, $u = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ 故 $= I - \dfrac{uu^T}{1+10} = \begin{cases} 10/11 & \text{diag} \\ -1/11 & \end{cases}$

norm 3 ← P52

Date. /

1 Frobenius/energy $\|A\|_F = \|A^T\|_F = \sqrt{\sum_{ij} a_{ij}^2} = \|\mathrm{vec}(A)\|_{\ell_2}$ B.1.

note $\|A\|_F^2 = tr(AA^T) = tr(A^TA)$ □ $(A^TA)_{ij} = \sum_k A^T_{ik}A_{kj} = \sum_k A_{ki}A_{kj}$

P1a) 为 matrix norm

□ i ii) trivl (iii) 令$\hat{A}\hat{B}$=vec(A), vec(B) 由 vec norm 的 iii) (iv) $\|AB\|_F^2 = \sum_{ij}(\bar{a}_i b_j)^2 \le \sum_{ij}\|\bar{a}_i\|^2\|b_j\|^2$ Cauchy B2. P1)

$= (\sum_i\|\bar{a}_i\|^2)(\sum_j\|b_j\|)^2 = \|A\|_F^2\|B\|_F^2$

1b) 对 vec $\vec{x}\ \vec{y} \in \mathbb{R}^n$ $\|\vec{x}\otimes\vec{y}\|_F = \|x\|_{\ell_2}\|y\|_{\ell_2}$ ④.1

□ $x\otimes y = xy^T$ LHS $= \sqrt{\sum_{ij} x_iy_j^2} = \sqrt{\sum x_i^2 \sum y_j^2}$ = RHS

1c) energy preserve in orthogonal transform 对 A and P orthogonal $\|AP\|_F = \|A\|_F$

□ $\|AP\|_F^2 = tr(APP^TA) = tr(AA^T) = \|A\|_F$

d) AB Frobenius orthogonal iff $tr(AB^T)=0$ 此时 $\|A+B\|_F^2 = \|A\|_F^2 + \|B\|_F^2$

□ $tr((A+B)(A^T+B^T)) = tr(AA^T + AB^T + BA^T + BB^T) \overset{P2.1}{=} tr(AA^T) + \underbrace{tr(AB^T) + tr(BA^T)}_{0} + tr(BB^T)$

e) 对 $x \perp y$, $\forall a, b$ 有 $\underset{=0}{xa^T}$ Frobenius orthogonal yb^T

□ $tr((yb^T)^Txa^T) = tr(by^Txa^T) = 0$

f) $\|A\|_F = \sqrt{\sum_{i=1}^{\min(n,m)} \sigma_i^2(A)}$ □ SVD $tr(A^TA) = tr(V\Sigma^T\Sigma V^T) \overset{cyclic}{=} tr(\Sigma^T\Sigma) = \sum\sigma_i^2(A)$

应用1 of perturbed $(A+\boxed{\delta A})\hat{x} = b + \boxed{\delta b}$ (Ax=b err 在 A & b) (matrix, vec) (δ 引起的 $\hat{x}$ 解), 有 $\frac{\|\hat{x}-x\|}{\|x\|} \le \frac{cond(A)}{(1-cond(A)\frac{\|\delta A\|}{\|A\|})}\left(\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|}\right)$

□ 有 $A^{-1}(A+\delta A)\hat{x} = A^{-1}(b+\delta b)$, $\hat{x} + A^{-1}\delta A\hat{x} = x + A^{-1}\delta b$, $\hat{x}-x = A^{-1}\delta b - A^{-1}\delta A(\hat{x}-x) - A^{-1}\delta Ax$

取 $\|\cdot\|$ $\Longrightarrow \|\hat{x}-x\| \overset{def}{\le} \|A^{-1}\|\cdot\|\delta b\| + \|A^{-1}\|\cdot\|\delta A\|\cdot\|\hat{x}-x\| + \|A^{-1}\|\cdot\|\delta A\|\cdot\|x\|$

$\le cond(A)\frac{\|\delta b\|}{\|b\|}\underbrace{\frac{\|b\|}{\|A\|}}_{\le\|x\|} + cond(A)\frac{\|\delta A\|}{\|A\|}\|\hat{x}-x\| + cond(A)\frac{\|\delta A\|}{\|A\|}\cdot\|x\|$

当 > 0 移 v 项

$(1 - cond A\frac{\|\delta A\|}{\|A\|})\frac{\|\hat{x}-x\|}{\|x\|} \le cond(A)\left(\frac{\|\delta b\|}{\|b\|} + \frac{\|\delta A\|}{\|A\|}\right)$

应用2. 对 x^* st. $\|Ax^*\|_2 \le \|Av\|_2$, $\forall v$, (x^*, v unit) 对 $\Delta = \inf_{A+\delta A\ singular}\|\delta A\|$, 有 $\Delta = \frac{1}{\|A^{-1}\|}$

□ 1° inf 可取到: ∃ δA 满. $\|\delta A\| = \frac{1}{\|A^{-1}\|}$. st. A + δA singular

B1. 取 $V = V \oplus V^\perp$, 令 $z = \alpha x^* + w$, 即 令 $x^* \in V$, $w \in V^\perp$, ie. $x^* \perp w$

Consider $\delta A = -yx^{*T}$, where $y = Ax^*$ $\|\delta A\| \overset{def\ of\ norm}{=} \max_{\|z\|=1}\|yx^{*T}z\| = \max_{\|\alpha x^*+w\|=1}\|y\underbrace{x^{*T}\alpha x^*}_{=1\ 因\ unit} + y\underbrace{x^{*T}w}_{=0}\|$

w 不影响 y 长度, 只有 unit αx^* 的 本位; ∴ w=0

$= \max_{\|\alpha x^*+w\|=1}\|\alpha y\| = \|y\|$ 而 $\|x^*\|=1$, ∴ α=1 , 而 $\|A^{-1}\| = \max_{y\neq0}\frac{\|A^{-1}y\|}{\|y\|} = \max_{x\neq0}\frac{\|A^{-1}Ax\|}{\|Ax\|} = \max_{x\neq0}\frac{\|x\|}{\|Ax\|} \overset{放缩 1}{=} \max_{\|x\|=1}\frac{1}{\|Ax\|} \overset{即 \|Ax\| 最小}{=} \frac{1}{\|Ax^*\|} = \frac{1}{\|y\|}$

$\therefore = \frac{1}{\|A^{-1}\|}$

且, A+δA singular: 对 x^*, $Ax^* + \delta Ax^* = y - yx^{*T}x^* = 0$, ∴ nullity ≠ 0,

2° $\frac{1}{\|A^{-1}\|}$ smallest, 即 $\frac{1}{\|A^{-1}\|} \le \forall\|\delta A\|$ consider $\underline{z}$ st. $(A+\delta A)z = 0$ (必 ∃ 因 singular), $Az = -\delta Az$, $z = -A^{-1}\delta A\cdot z$. $\|z\| \le \|A^{-1}\|\cdot\|\delta A\|\cdot\|z\|$

$1/\|A^{-1}\| \le \|\delta A\|$

norm 4

综证明 $\|A\|=?$ i. $\forall \|x\|=1, \|Ax\| \le ?$
ii. construct $\|x^*\|=1$ s.t. $\|Ax^*\|=?$ (obtainable)

Date. /

$\max \sum$ row

例1. $\|\vec{x}\|_\infty = \max_i |x_i|$ induced matrix norm 为 $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$

□ P49.4 取 $\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty$，令 $\|Ax\|_\infty = \max_i \underbrace{\left|\sum_k a_{ik} x_k\right|}_{\le \sum |a_{ik}||x_k|} \le \max_i \sum_k |a_{ik}|$ ∴ 找 $\|x\|_\infty = 1$ 满足等号即可

令有 $\sum_{k=1} |a_{jk}| = \max_i \sum_k |a_{ik}|$ 令 $x_k = \begin{cases} \frac{|a_{jk}|}{a_{jk}}, & a_{jk} \ne 0 \\ 1 \end{cases}$ 因最大1，$\|x\|_\infty = 1$. 且当 $i \ne j$ $\left|\sum_k a_{ik} x_k\right| \le \sum_k |a_{ik}| \le \max_i \sum_k |a_{ik}| = \sum_k |a_{jk}|$ 等号取到

i=j $\left|\sum a_{jk} x_k\right| = \sum_k |a_{jk}|$

例2 $\|\vec{x}\|_{l_1} = \sum |x_i|$ induced matrix norm $\|A\|_1 = \max_k \sum_{i=1} |a_{ik}|$ $\max \sum$ col

□ 同例1. $\|A\| = \sup_{\|x\| \le 1} \|Ax\| = \sup \left\|\sum_k a_k x_k\right\| \le \sup \sum_k |x_k| \|a_k\|$ 因.5 △不等式 $\le (\max_k \|a_k\|) \cdot \sup \sum_k |x_k| \le \max_k \sum |a_{ik}|$ 提出 $\|a_k\|$ 最大 ; 令 j 有 $\sum |a_{ij}| = \max_k \sum |a_{ik}|$ 取 $x_k = \begin{cases} 1 & k=j \\ 0 \end{cases}$ $\|x\|_{l_1} = 1$

例3. 对 $\|A\| = 5$, $\|A^{-1}\| = 2$ 考虑 $Ax = b$ 的计算到 $\hat{x}$，希望 $\|A\hat{x} - b\| = 10^{-6}$，$\|\hat{x} - x\|$ 应如何?

□ $10^{-6} = \|A\hat{x} - Ax\| \le \|A\| \cdot \|\hat{x} - x\|$ ∴ $\|\hat{x} - x\| \ge \frac{10^{-6}}{5}$; 而 $\|\hat{x} - x\| = \|A^{-1}(A\hat{x} - Ax)\| \le \|A^{-1}\| \|A\hat{x} - b\| = 2 \times 10^{-6}$

bound $\frac{10^{-6}}{5} \le \|\hat{x} - x\| \le 2 \times 10^{-6}$

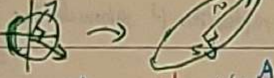
例4 norm equiv 1.46 $T(i)$，换种证法 i) $\|x\|_\infty \le \|x\|_2 \le \sqrt{n} \|x\|_\infty$
ii) $\|x\|_\infty \le \|x\|_1 \le n \|x\|_\infty$

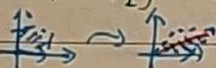
□ i) $\max |x_i| = \sqrt{(\max |x_i|)^2} \le \underbrace{\sqrt{|x_1|^2 + \dots + |x_n|^2}}_{\|x\|_2} \le \sqrt{n \times (\max |x_i|)^2}$

ii) $\max |x_i| \le \underbrace{|x_1| + \dots + |x_n|}_{\|x\|_1} \le n \cdot \max |x_i|$

例4 用 induced norm / operator norm def 估计 $\|A\|_{l_2}$ i) $\begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$ ii) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ 因 pd 又 Ostim. P6 P_k) iii) $\begin{pmatrix} 2\cos\theta & -2\sin\theta \\ 2\sin\theta & 2\cos\theta \end{pmatrix}$

$\boxed{\|A\| = \sup_{x \ne 0} \frac{\|Ax\|}{\|x\|}}$

□ i) eval 5 $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$
10 $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$  最大 stretch 是 10. ∴ $\|A\|_2 = 10$

ii)  考虑 对角线. $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{A} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ∴ 至少 $\|A\|_2 \ge \frac{\left\|\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\|_2}{\left\|\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\|_2} = \frac{\sqrt{5}}{\sqrt{2}} \approx 1.58$ 而 实际为 $1.62 = \sigma_1$

iii) rotation stretch by factor 2 ∴ $\|A\|_2 = 2$

应用1. 用 $\hat{x}$ approx. $Ax = b$ 有 $\frac{\|x - \hat{x}\|}{\|\hat{x}\|} \le \underbrace{\|A^{-1}\| \cdot \|A\|}_{\text{cond number}} \cdot \frac{\|A - \hat{A}\|}{\|A\|}$，where $\hat{A}$ 为 A 的 perturb

□ $Ax + (\hat{A} - A)\hat{x} + A(\hat{x} - x) = b$ 有 $\hat{x} - x = -A^{-1}(\hat{A} - A)\hat{x}$. $\|\hat{x} - x\| \le \|A\| \|A - \hat{A}\| \|\hat{x}\|$ (maximum property)

∴ $\underset{\text{sol relative error}}{\text{LHS}} \le \text{cond. number} \cdot \underset{\text{data relative error}}{\frac{\|A - \hat{A}\|}{\|A\|}}$ cond(A) 过大 有 problem instable

Note $\|A^{-1}\| \cdot \|A\| \ge \|A^{-1} A\| = 1$，且 $\forall M$，$\exists A$ st. cond(A) $\ge M$

□ consider $\varepsilon < 1/M$ 对 $A = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$ 有 $\|A\|_{l_1} = 1$ st. cond(A) $= 1/\varepsilon > M$
$A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\varepsilon} \end{pmatrix}$ $\|A^{-1}\|_{l_1} = 1/\varepsilon$

例5 有 $\|x\|_v = \|\underset{\text{可逆}}{V} x\|$，when $\|\cdot\|$ any norm 易证 $\|\cdot\|_v$ 为 norm. 证 induced $\|A\|_v = \|VAV^{-1}\|$

□ 1° WTS $\forall \|x\|_v = \|Vx\| = 1$ 有 $\|Ax\|_v = \|VAV^{-1}\|$. 但由 $\|VAV^{-1}\|$ def. $\forall \|z\| = 1$ 有

$\|VAV^{-1} z\| \le \|VAV^{-1}\|$ 因此 选 $\underset{\|z\|=1}{z = Vx}$ 有 $\|VAV^{-1} V x\| = \|VAx\| = \|VAV^{-1}\|$

2° 由 $\|VAV^{-1}\|$ def 有 $\|VAV^{-1} z^*\| = \|VAV^{-1}\|$ 令 $x^* = V^{-1} z^*$，有 $\|x^*\|_v = \|z^*\| = 1$ 且 $\|Ax^*\| = \|VAV^{-1} z^*\| = \|VAV^{-1}\|$

证1. $\text{cond}(A) = \max_{\|x\| = \|y\|} \frac{\|Ax\|}{\|Ay\|}$ □ 对 $\|A\| \cdot \|A^{-1}\| = \max_{x \ne 0} \frac{\|Ax\|}{\|x\|} \cdot \max_{z \ne 0} \frac{\|A^{-1} z\|}{\|z\|} = \max_{\|x\|=1} \|Ax\| \cdot \max_{\|y\|=1} \frac{1}{\|Ay\|} = \max_{\|x\|=\|y\|} \frac{\|Ax\|}{\|Ay\|}$
P46.1 def, sup 可取到 (P49.4 NOTE) 取 $x \in$ 同向 $\ne 0$，且 $\|x\| = 1$ $\|x\| = \|y\|$ 的 norm
又令 $y = Az$，因 A 可逆，nullity $= 0$，$y \ne 0$，用 $y \leftarrow \frac{y}{\|y\|}$

2. 对 perturbed $\tilde{x}$ $(A\tilde{x} = \tilde{b})$ 有 $\frac{\|b - \tilde{b}\|}{\|b\|} \frac{1}{\text{cond}(A)} \le \frac{\|x - \tilde{x}\|}{\|x\|} \le \text{cond}(A) \frac{\|b - \tilde{b}\|}{\|b\|}$ bud 非常 不稳定 中

□ 左 $\frac{\|A\| \cdot \|x - \tilde{x}\|}{\|A\| \cdot \|x\|} \ge \frac{\|Ax - A\tilde{x}\|}{\|A\| \cdot \|A^{-1} b\|} \ge \frac{\|b - \tilde{b}\|}{\|A\| \cdot \|A^{-1}\| \cdot \|b\|}$ 右 $\frac{\|A\| \cdot \|x - \tilde{x}\|}{\|A\| \cdot \|x\|} \le \frac{\|A\| \cdot \|A^{-1}\| \|b - \tilde{b}\|}{\|Ax\|}$

3. $\underset{\text{(in 2-norm)}}{\text{cond}(A)} = \begin{cases} \frac{\lambda_{max}}{\lambda_{min}}, & A \text{ pd.} \quad □ \text{ P50 证5} \\ \frac{\sigma_1}{\sigma_n}, & A \text{ 方阵} \\ 1, & A \text{ unitary} \end{cases}$

$A^T A$ pd，eval $\sigma_1^2 \ge \dots \ge \sigma_n^2 > 0$ ∴ $\|A\|_2 = \sigma_1$
$(A^T A)^{-1}$ 即 $1/\sigma_n^2 \ge \dots \ge 1/\sigma_1^2 > 0$ $\|A^{-1}\|_2 = 1/\sigma_n$
P53 T_5 & P_{VI}

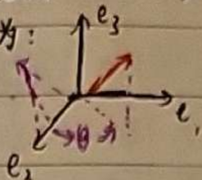
deli得力

Rotation

Date. /

1. 一般只考虑逆时针，如 2D $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ 例2

构建 3D rotate： WLOG 令逆时针为：



rotate around e_1 axis $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

e_2: $\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$

e_3: $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

2. P 对 rotation matrix R

1) (preserve distance) $\|x-y\| = \|Rx-Ry\|$

□ 由 4) & Pss. $P_{(i)}$

2) (preserve angle) $\cos(x,y) = \cos(Rx, Ry)$

□ 后者 $= \frac{x^TR^TRy}{\|Rx\|\cdot\|Ry\|} = \frac{x^Ty}{\|x\|\cdot\|y\|}$ = 前者 $\frac{x^Ty}{\|x\|\cdot\|y\|}$

3) 非 commute ∴ rotate order matter，但 2D Rotation commutative

4) rotation 为 orthogonal

□ 以 2D $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $Q^TQ = I$

5) $|\det R| = 1$ □ 由 4) & Pss. $P_{i)}$ 但对 linear T 后的 area 比上原 area 四.7. ∴ rotate 不 scale area

3. Householder Reflection

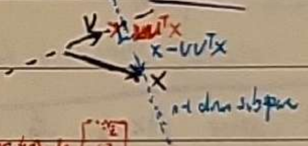
Gram 非 num stable (cond(A) large 时 至少 1 个 col a_j 几乎 $\in \langle a_1 \dots {}_{j-1}\rangle$，∴ Gram 时 $a_j - \sum_i \frac{\langle u_i, a_j\rangle}{\langle u_i, u_i\rangle} u_i$ 会致 close num (numeric P11.3) ∃ cancel error

∴ 提出 stable alg：reflect a_j st. $a_j \in \langle a_1 \dots {}_{j-1}\rangle^{\perp}$

1° def $P_n = I - 2uu^T$ where $\|u\|_2 = 1$，称 reflection P 为 householder Transform

P_2 i) P orthogonal $P^T = I - 2vv^T = P$；$P^TP = P^2 = I - 4vv^T + 4(vv^T)(vv^T) = I$ $\quad 4v\underbrace{(v^Tv)}_{norm=1}v^T$

ii) P reflect x to $n-1$ dim subspace which $\perp v$



□ 因 $v(v^Tx) = proj_v x$ 而 $Px = x - vv^Tx$ 见图

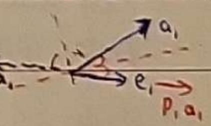
2° (QR decomp) 想找 reflection $\{P_i\}$ st. $P_n \cdots P_1 A = R$ upper triangle $\begin{bmatrix} \ddots \\ 0 \end{bmatrix}$，这样 $Q = P_1^T \cdots P_n^T$，每个 P_i 只处理 R 的 1 col

即 $P_1A = \begin{pmatrix} r_{11} & \vdots & \vdots \\ 0 & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix}$，$P_2P_1A = \begin{pmatrix} r_{11} & r_{12} & \vdots \\ 0 & r_{22} & \vdots \\ \vdots & \vdots & \vdots \end{pmatrix}$ $P_3P_2P_1A = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \vdots \\ \vdots & r_{22} & r_{23} & \vdots \\ & \vdots & r_{33} & \vdots \end{pmatrix}$ Pss. $P_{(iii)}$

2 种 reflect a_1 ① ② 对应 r_{11}

1° P_1: $P_1 \vec{a}_1 = r_{11}\vec{e}_1 = a_1 - 2v_1v_1^Ta_1$，$v_1 = \frac{a_1 - r_{11}e_1}{2v_1^Ta_1}$，而 $\|P_1a_1\|_2 = \|a_1\|_2 = \|r_{11}e_1\| = |r_{11}|$

depend on 1st ele of a_1 近似 因 $\|v_1\|_2 = 1$



∴ pick $r_{11} = -\mathrm{sign}((\vec{a}_1)_1) \cdot \|a_1\|_2$，$v_1 \Leftarrow \frac{a_1 - r_{11}e_1}{\|a_1 - r_{11}e_1\|_2}$ denote $P_1A = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ 0 & & & & \\ \vdots & \vec{a}_2 & A_2 & & \end{pmatrix}$

2° P_2: 因 $\begin{cases} P_2(P_1A)_{*1} = \begin{pmatrix} r_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ P_2(P_1A)_{*2} = \begin{pmatrix} r_{12} \\ r_{22} \\ 0 \\ \vdots \end{pmatrix} \end{cases} = (I - 2v_2v_2^T) \cdot \begin{pmatrix} r_{11} \\ \vdots \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 0 \\ * \\ \vdots \\ * \end{pmatrix}$

$\begin{pmatrix} r_{12} \\ \vec{a}_2 \end{pmatrix} \Rightarrow$ 但 $I - 2v_2v_2^T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & sth & \\ 0 & & & \end{pmatrix}$，乘之 $\Rightarrow \begin{pmatrix} r_{12} \\ sth \cdot \vec{a}_2 \end{pmatrix}$ $\begin{pmatrix} r_{22} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ 类似 1°

~~是 $P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & sth & \\ 0 & & \end{pmatrix}$~~

for sth: → ∴ $r_{22} = -\mathrm{sign}((\vec{a}_2)_1) \cdot \|a_2\|_2$，$v_2 \Leftarrow \frac{a_2 - r_{22}e_1}{\|a_2 - r_{22}e_1\|_2}$ denote $P_2P_1A = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \cdots \\ 0 & r_{22} & r_{23} & \cdots \\ \vdots & \vdots & \vec{a}_3 & A_3 \end{pmatrix}$

3° 同理 $r_{33} = -\mathrm{sign}((\vec{a}_3)_1) \cdot \|a_3\|_2$，$v_3 \Leftarrow \frac{a_3 - r_{33}e_1}{\|a_3 - r_{33}e_1\|_2}$

…… till n°

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例1 $A=\begin{pmatrix}0 & -4\\ 0 & 0\\ -5 & -2\end{pmatrix}$ $b=\begin{pmatrix}-5\\ 6\\ 10\end{pmatrix}$ LS sol $x^* = (A^TA)^{-1}A^Tb = \begin{pmatrix}-5/2\\ 5/4\end{pmatrix}$. Prove by QR using Gram

I) Gram $\left\{\underbrace{\begin{pmatrix}0\\0\\-5\end{pmatrix}}_{a_1}, \underbrace{\begin{pmatrix}-4\\0\\-2\end{pmatrix}}_{a_2}\right\}$: 1° $u_1 = a_1$ $q_1=\begin{pmatrix}0\\0\\-1\end{pmatrix}$ $r_{11}=\|u_1\|=5$

2° $u_2 = a_2 - \mathrm{proj}_{u_1}(a_2) \overset{P33.2有}{=} \begin{pmatrix}-4\\0\\-2\end{pmatrix} - r_{12}\cdot q_1 = \begin{pmatrix}-4\\0\\0\end{pmatrix}$ $r_{12}=q_1^T\cdot a_2 = 2$

$q_2 = \begin{pmatrix}-1\\0\\0\end{pmatrix}$ $r_{22}=\|u_2\|=4$

$\therefore R=\begin{pmatrix}5 & 2\\ & 4\\ \hline 0 & 0\end{pmatrix}$ $Q=\left(\begin{array}{cc|c} | & | & | \\ q_1 & q_2 & q' \\ | & | & | \end{array}\right)^{\text{pick 与 } q_1 q_2 \perp} = \left(\begin{array}{cc|c}0 & -1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{array}\right)$. 由 P46.3 x^* 由 $\hat{R}x=\hat{Q}^Tb$

II) householder 1° $r_{11} = -\mathrm{sign}(0)\|a_1\| = 5$ $v_1 = \dfrac{\begin{pmatrix}0\\0\\-5\end{pmatrix}-5\begin{pmatrix}1\\0\\0\end{pmatrix}}{\|\cdots\|} = \begin{pmatrix}-1/\sqrt2\\ 0\\ -1/\sqrt2\end{pmatrix}$ $P_1 = I-2v_1v_1^T=\begin{pmatrix}0 & 0 & -1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{pmatrix}$, $P_1A=\begin{array}{cc} r_{11} & r_{12} \\ \end{array}\begin{pmatrix}5 & 2\\ 0 & 0\\ 0 & 4\end{pmatrix}$ a_2

2° $r_{22} = -\mathrm{sign}(-4)\cdot\|a_2\| = 4$, $V_2 = \dfrac{\begin{pmatrix}0\\4\end{pmatrix}-4\begin{pmatrix}1\\0\end{pmatrix}}{\|\cdots\|} = \begin{pmatrix}-1/\sqrt2\\ 1/\sqrt2\end{pmatrix}$ $P_2 = I-2V_2V_2^T = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \overset{\text{expand 至 } 3\times3}{\Longrightarrow} \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$

$\therefore R=\begin{pmatrix}5 & 2\\ & 4\\ 0 & 0\end{pmatrix}$ $Q=P_1^TP_2^T=\begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 1\\ -1 & 0 & 0\end{pmatrix}$ ∴同上

Jordan

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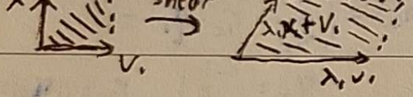
$\begin{pmatrix} -1 & 0 \\ -3 & 1/2 \end{pmatrix}$

1. 对2×2 A smiliar to

(i) $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ L.ind evec 因 . J_{2a}

$\begin{pmatrix} 3 & 1 \\ -1 & 5 \end{pmatrix}$



(ii) $\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$ λ_1 algebraic multi 2, geometric 1 . 取个 $\cancel{x}$, v_1 L.ind , $Ax = \lambda_1 x + v_1$

∴用$\{v_1, x\}$作 basis

(iii) $r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ complex eval, $\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$

2. 3×3 有4个可能 form $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$ $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$ $\begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

Date. /

例1. $\begin{cases} x_1+x_2+2x_3+x_4=1 \\ x_1-2x_2 \quad -x_4=-2 \end{cases}$ 用右.2.

∵ $rk\begin{bmatrix}1&1&2&3\\1&-2&0&-1\end{bmatrix} = rk\begin{bmatrix}1&1&2&3&1\\1&-2&0&-1&-2\end{bmatrix}$，∴ ∃ sol. assign x_3, x_4 random，有 $\begin{bmatrix}1&1\\1&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1-2x_3-x_4\\-2+x_4\end{bmatrix}$

thus $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ 可为 $\begin{bmatrix}1&1\\1&-2\end{bmatrix}^{-1}\begin{bmatrix}1-2x_3-x_4\\-2+x_4\end{bmatrix} = \begin{bmatrix}-\frac{4}{3}x_3-x_4/3\\1-\frac{2}{3}x_3-\frac{2}{3}x_4\end{bmatrix}$ general sol $\begin{bmatrix}x_1\\\vdots\\x_4\end{bmatrix} = \begin{bmatrix}\vdots\\x_3\\x_4\end{bmatrix}$

应用 (minus 1 trick) 解 eq $Ax=0$ 时化为 RREF，并在 missing pivot 处加 $(0\cdots-1\cdots0)$，sol 为右 -1 的 col

如 回. Ex 2 $A=\begin{pmatrix}1&0&9&-8\\0&1&-4&5\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$ 解为 $a\begin{pmatrix}9\\-4\\-1\\0\end{pmatrix}+b\begin{pmatrix}-8\\5\\0\\-1\end{pmatrix}$

1. Gaussian Elimination 不 stable.

eg. $\left(\begin{array}{cc|c}\varepsilon&1&1\\1&1&2\end{array}\right) \Rightarrow \begin{cases}x = \frac{1}{\varepsilon}(1-y) \approx 1 \\ y = \frac{2\varepsilon-1}{\varepsilon-1} \approx 1\end{cases}$ (cancel errors) 问题在于 ε pivot 与其余 entry 相比太小

∴ sol 为 互换 pivot: $\Rightarrow \left(\begin{array}{cc|c}1&1&2\\\varepsilon&1&1\end{array}\right) \Rightarrow \begin{cases}x = 2-y \approx 1\\ y = \frac{1-2\varepsilon}{\varepsilon-1} \approx 1\end{cases}$

∴ 据此为启发，impl Gaussian Elimination w/ pivoting，这样大多数情况 stable

例2 partial pivot $\left(\begin{array}{cccc|c}6&-2&2&4&16\\12&-8&6&10&26\\3&-13&9&3&-19\\-6&4&1&-18&34\end{array}\right)$ $\xRightarrow{\text{max abs = row 2, swap}}$ $\left(\begin{array}{cccc|c}12&-8&6&10&26\\6&-2&2&4&16\\3&\vdots&&\vdots&\vdots\\-6&&&&\end{array}\right) \sim \left(\begin{array}{cccc|c}12&-8&6&10&26\\0&2&&&\vdots\\0&-11&&&\vdots\\0&0&4&&\end{array}\right) \xRightarrow{\text{max abs = row 3, swap}} \begin{pmatrix}&-8&\\&-11&\\&2&\cdots\\&0&\end{pmatrix}$

$\sim \begin{pmatrix}-8&6&\\-11&15/2&\\0&4&\\0&4&\end{pmatrix} \xRightarrow{\text{swap row 3 \& 4}} \cdots \Rightarrow \left(\begin{array}{cccc|c}12&-8&6&10&26\\0&-11&15/2&1/2&25/2\\0&0&4&-13&-21\\0&0&0&3/11&3/11\end{array}\right)$ sol 同 直接高斯消元

Note 此类 partial pivot swap along process，若预先知道那几 row swap，先 swap 再消元 sol 同 边消元边 swap，即先 permute

2. 对 $Ax=b$，$A=\begin{pmatrix}\bar{A}_{n\times n}\\0\end{pmatrix}_{m\times n}$ $(m\geq n)$，$b=\begin{pmatrix}c\\d\end{pmatrix}$ 时 least square sol 同 $\bar{A}x=C$，因：若 $\begin{pmatrix}c_n\\d_{m-n}\end{pmatrix}$

若 A 非此 form，可 change of basis $\underline{P}$ 令 $PA=\begin{pmatrix}\bar{A}\\0\end{pmatrix}$（$P$'s orthogonal，行：change standard basis → col(A) → orthonormal basis (用 Gram) P55 T6），有 $\|b-Ax\|_2 = \|P(b-Ax)\|_2 = \|Pb-\begin{bmatrix}\bar{A}x\\0\end{bmatrix}\|_2$

$= \|\begin{pmatrix}c-\bar{A}x\\d\end{pmatrix}\|_2 = \|c-\bar{A}x\|_2 + \|d\|_2$，min when $\bar{A}x=c$

eg. $A=\begin{pmatrix}1&0&1\\2&3&5\\5&3&-2\\3&5&4\\-1&6&3\end{pmatrix}$ $b=\begin{pmatrix}4\\-2\\5\\-2\\1\end{pmatrix}$，求 $col(A) = \langle \begin{pmatrix}1\\2\\5\\3\\-1\end{pmatrix},\begin{pmatrix}0\\3\\3\\5\\6\end{pmatrix},\begin{pmatrix}1\\5\\-2\\4\\3\end{pmatrix}\rangle \xRightarrow{\text{Gram}} q_1, q_2, q_3$ $\to \perp$ $N(A^T) = \langle \begin{pmatrix}23\\-19\\6\\15\\0\end{pmatrix},\begin{pmatrix}84\\-27\\-3\\0\\15\end{pmatrix}\rangle \xRightarrow{\text{Gram}} p_4, p_5$

P51.3 $\mathbb{R}^5 = R(A) \oplus N(A^T)$，basis $p_1..p_5$

$P = P^T = \begin{bmatrix}-p_1-\\\vdots\\-p_5-\end{bmatrix}$ ($e \to p$, $p \to e$; P orthogonal)

$PA = \begin{pmatrix}6.32 & 4.7 & 1.9\\ & 7.51 & 5.2\\ & & 4.2\\0&0&0\\0&0&0\end{pmatrix}$ $\bar{A}$

$Pb = \begin{pmatrix}2.84\\-1.13\\-3.92\\2.46\\4.58\end{pmatrix}$ (上三行为 c)

NOTE 实不必记算 $N(A^T)$，只需 $\tilde{P} = \begin{pmatrix}-q_1-\\\vdots\\-q_3-\end{pmatrix}$，$P = \begin{bmatrix}\tilde{P}\\ sth\end{bmatrix}$

而 $PA = \begin{bmatrix}\tilde{P}A\\0\end{bmatrix}$ ∴ 只需解 $\tilde{P}Ax = \tilde{P}b$

而 $\bar{A}$ 上△ $\tilde{P}A=\bar{A}$（p_i 是 basis for $N(A^T)$，$A^Tq_4=0$，$0^T=q_4^TA$）

$= \begin{pmatrix}-p_1-\\\vdots\\-p_n-\end{pmatrix}\begin{pmatrix}a_1&\cdots&a_n\end{pmatrix} = \begin{pmatrix}p_1^Ta_1&\cdots&p_1^Ta_n\\ & \ddots & \vdots\\ p_n^Ta_1 & & p_n^Ta_n\end{pmatrix}$（左下 $=0$ by Gram process）

可见 实为 QR decomp: P57.2 $PA=R=\begin{bmatrix}\bar{R}=\bar{A}\\0\end{bmatrix}$，$A=P^{-1}R=P^TR=QR$

3. (LS w/QR)（Q^T 不改变 L_2 norm P55 P117）

$\|Ax-b\| = \|QRx-b\| = \|Rx-Q^Tb\| = \|\begin{bmatrix}\bar{R}\\0\end{bmatrix}x-\begin{bmatrix}\bar{Q}^T\\sth\end{bmatrix}b\| = \|\begin{bmatrix}\bar{R}x-\bar{Q}^Tb\\sth\end{bmatrix}\| = \|\bar{R}x-\bar{Q}^Tb\| + \|sth\|$（前n col；sth 不依赖 x）

min 当 $\bar{R}x=\bar{Q}^Tb$ 取到

Sys of Equations 1

Date. /

1. 对 $A_{m\times n}$，[1].7 $Ax=b$ $\exists$ sol $\Leftrightarrow$ $rkA = rk[A|b]$

□ $\Rightarrow$ b = L.comb of A's col $= \sum_{i=1}^{n} x_i a_i$，thus $b \in \langle a_1 \cdots a_n \rangle$，$rkA = \dim\langle a_1\cdots a_n\rangle = \dim\langle a_1\cdots a_n, b\rangle = rk[A|b]$ [2].J_{1b} (P_2. P_V)

$\Leftarrow$ 令 $rk=r$ 则 A有 r L.ind col $a_1 \cdots a_r$，$[A|b]$ 也有同样 r 个，$\therefore$ b 可由 $a_1\cdots a_n$ L.comb [2] J_{1b}

$\therefore b = \sum_{i=1}^{n} x_i a_i$，weight x_i 即 $\vec{x}$ sol

NOTE, x unique $\Leftrightarrow$ $rkA = rk[A|b] = n$ 可逆

□ $\Rightarrow$ 令 $rk<n$，则有 $y \neq 0$ st $Ay=0$ (col L.dep)，若 x 为 sol，$x+y$ 也 sol，矛盾

$\Leftarrow$ $\exists$ sol (1.)，若非 unique，$Ax=Ay=b$，$A\underbrace{(x-y)}_{\neq 0}=b$ 可 A's col L.ind 矛盾 OR，A可逆，$x=A^{-1}b$ 即 [7].4 [2].1

2. $A_{m\times n}$，$n \geq m$，$rkA=m$，$Ax=b$ 的 sol 为：assign $n-m$ 个 random，剩余 m 个以此 solve

□ 令 $\underbrace{a_1 \cdots a_m}_{m个col}$ L.ind $Ax=b \to x_1a_1+\cdots+x_ma_m = b - x_{m+1}a_{m+1} - \cdots - x_na_n$，assign $\begin{cases} x_{m+1} \leftarrow \varphi_{m+1} \\ \vdots \\ x_n \leftarrow \varphi_n \end{cases}$，$\varphi_i$ random

$\therefore \underbrace{[a_1 \cdots a_m]}_{B，可逆}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \cdots$，$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = B^{-1}[b - \varphi_{m+1}a_{m+1} - \cdots - \varphi_na_n]$

3. (least square) 代码参见 P.20 矩阵例3 / LA 例2.2.2

$\min_x f(x) = \|b - Ax\|_2^2$ 当 A_{mn} 有

- $m=n$ 且 A invertible. uniq sol $x^* = A^{-1}b$，$f(x^*)=0$

- $m>n$，$f(x^*)>0$，有 normal eq $A^TA x^* = A^Tb$ 见 Optim P11.3　　note 当 $m=n$ x^* 也满足 $=A^{-1}b$

  [4].证1　$rkA^TA = rkA = n$　　pm matrix P35. P68. 例3 $= A^+$

  - A^TA 可逆： $x^* = (A^TA)^{-1}A^Tb = \underline{A^+}b$

  - 否： x^* inf many sol，但有同一 $f(x^*)$

□ 除了 Optim P11.3 求 $\nabla=0$；WTS x^* min $f(x)$ $\Leftrightarrow$ $A^TAx^* = A^Tb$ by orthogonal decomposition (P51. J_{2d})

$\Rightarrow$ $b = \underset{\in ColA}{c} + \underset{\in N(A^T)}{d}$ 由 P51. J_{2a}，因 $d \perp Ax-c$ (orthogonal decomp def; $Ax-c \in ColA$)，$f(x) = \|c+d-Ax\|_2^2 \overset{P51.J_0}{=} \|c-Ax\|_2^2 + \|d\|_2^2$，$x^*$ 使 $Ax-c$ 取到

此时 x^* 有 $A^TAx = A^Tc + \underset{d\in N(A^T)，即 A^Td=0}{0} = A^T(c+d) = A^Tb$　　$\Leftarrow$ $A^TAx^* = A^T(c+d) = A^Tc$

$\therefore A^T(Ax^*-c)=0$，$\therefore$ 有 $\underbrace{Ax^*-c}_{\in R(A)} \in N(A^T)$ 因 $\mathbb{R}^m = R(A) \oplus N(A^T)$ 唯一选择为 $Ax^*-c=0$

NOTE 求 pseudo-inverse：令 $B = A^TA$，$y = A^Tb$，即求解 $Bx=y$，when B pd ($m>n$ (当A full-rk) $\in R(A)$) (Optim.P6.P26) $N(A)=\{0\}$

1° cholesky decomp $B = GG^*$ (Optim.P7 1)　2° solve $Gz=y$ by forward sub　3° solve $G^Tx=z$ by backward sub

定理1. (solve normal equation 10 numerical instable)

对 nonsquare A_{mn} ($m>n$) def $\text{cond}_2(A) = \|A\|_2\,\|A^+\|_2$ (对 square 见 P12 定理1)

试证 1° $\text{cond}_2(A) = \frac{\sigma_1}{\sigma_n}$　2° 对 least squares：perturb $\tilde{b}$ 的 sol $\tilde{x}$ ($=A^+\tilde{b}$) 有 $\frac{\|x-\tilde{x}\|_2}{\|x\|_2} \leq \text{cond}_2(A)\frac{\|b-\tilde{b}\|_2}{\|c\|_2}$　rel.error bdd by cond(A)

□ 1° SVD $A = U\underbrace{\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & 0 & \end{pmatrix}}_{记为\begin{bmatrix} S \\ 0 \end{bmatrix}}V^T$，$\therefore \|A\|_2 = \|\Sigma\|_2 = \sigma_1$ (UV orthogonal，由 P55. J_6)，而 $A^+ = (V\Sigma^TU^TU\Sigma V^T)^{-1}V\Sigma^TU^T$

$(S^{-1})^2[S\ 0] = S^{-1}\cdot S^{-1}[S\ 0]$

$= (V[S\ 0]\begin{bmatrix} S \\ 0 \end{bmatrix}V^T)^{-1}V[S\ 0]U^T = V^{T^{-1}}S^{2^{-1}}V^{-1}V[S\ 0]U^T \overset{(3.12)}{=} VS^{2^{-1}}[S\ 0]U^T = V[S^{-1}\ 0]U^T$，$\therefore \|A^+\|_2 = \|[S^{-1}\ 0]\| = \frac{1}{\sigma_n}$

2° 因 $\|x-\tilde{x}\|_2 \leq \|A^+\|\cdot\|b-\tilde{b}\|$ 且对 $b=c+d$ $Ax=c$，$\therefore \|c\| \leq \|A\|\cdot\|x\|$

deli得力

elementary

1. elementary row operation

1) row scaling：某一row 乘非0 scalar α.　$A_{i,:} \to \alpha A_{i,:}$

2) row exchange：互换二行　$A_{i,:} \leftrightarrow A_{j,:}$

3) row adjustment：一row 乘以倍数加于另一row　$A_{j,:} \to A_{j,:} + \alpha A_{i,:}$，$i \neq j$

Gaussian Eliminate $n \times n$ matrix
$O(n^3)$　每对 1st row　$\boxed{n} \to \boxed{\begin{smallmatrix}1\\0\\0\end{smallmatrix}\, n-1}$
scaling, adjust 都为 operation.
1. 1 row 对各行都有 scaling，再对下方的 $n-1$ row
adjust　$n(n-1)$
∴ 对 $k=1 \dots n$，operation 应为 $\sum_{k=1}^{n} k(k-1) = \frac{n^3-n}{3}$

2. matrix R 为 in row echelon form iff

i) 只有 0 的 row 在 R 底部，ii) 非0 row，first nonzero entry 为 $\underline{1}$ (leading entry)，iii) $\forall$ 2 个连续 row，下方的 leading entry 在右

row echelon form 为 reduced row echelon form iff $\forall$ col 有 leading entry 的其他 position 0

例见 四.1.

3. $A, B \in M_{m,n}(F)$，A 与 row equivalent 即 A 可以 elementary row operation 成 B.

易证 $A \sim B$ 为 equivalent relation

且，T_1：$A \sim B$，则 $B = \underset{\text{elementary matrix}}{E}A$；因 row operation 同 column operation，$B = AE'$ ($A \sim B$ by col)

□ row operation：1) $E = \begin{bmatrix} 1 & & \\ & \alpha & \\ & & 1 \end{bmatrix} \to i$ row　2) $E = j\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} i$ 即 $E_{i,:} = e_j^T$，$E_{j,:} = e_i^T$

这样 $(EA)_{i,\alpha} = A_{j,\alpha}$，$(EA)_{j,\alpha} = A_{i,\alpha}$

3) $E = \begin{bmatrix} 1 & & & \\ \alpha & 1 & & \\ & & & 1 \end{bmatrix}$ (i, j) ← j 这样 $(EA)_{j,\alpha} = \alpha A_{i,\alpha} + A_{j,\alpha}$

同3

由 $B_{i,:} = (EA)_{i,:} = E_{i,:}A$ 看出，且 E 由 $A \to B$ 的 op 施加于 I_n 上得到)

对 col operation　1) $E = \begin{bmatrix} 1 & & \\ & \alpha & \\ & & 1 \end{bmatrix}$ ↑ i col　2) $E = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ (j col, i col)　3) $E = \begin{bmatrix} 1 & & \\ & 1 & \alpha \\ & & 1 \end{bmatrix}$ (i, j)

由 $B_{:,i} = AE_{:,i}$ 看出

易知当 A row[col] → B，A^T 对应 col[row] → B^T □ $B = EA$ [$B = AE$] → $B^T = A^T E^T$ [$B^T = E^T A^T$]

T_2：E (elementary matrix) invertible，且 inverse 同 type $\underset{[col]}{row}$ (相反的 $\underset{[col]}{row}$)

□ WLOG，E 由 row op type i 得到，由 T_1 可视为对 I_n 施加同 op. ∴ $I_n \sim E$，∴ $E \sim I_n$ ∴ 有 E'，$I_n = E'E$

注意 prod of elem matrix　elem　✗　$\begin{bmatrix} 5 & \\ & 1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \to \begin{bmatrix} & 5 \\ 1 & \end{bmatrix}$　sum of .. elem ✗ $\begin{bmatrix} 5 & \\ & 1 \end{bmatrix} + \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$

transpose of .. elem ✓ type 1, 2 symmetric，type 3 $\underset{[col]}{row}$ 为 type 3 $\underset{[row]}{col}$ … $A \sim B$ by $\underset{[col]}{row}$ ⇒ $A^T \sim B^T$ by $\underset{[row]}{col}$

$A \sim B$ by $\underset{[col]}{row}$ ⇒ $B \sim A$ by $\underset{[col]}{row}$ ✓ undo op　✗ $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix}$ by row，但不可由col得

T_2 用于求简单的逆矩阵 如 $\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}^{-1} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$，$\begin{bmatrix} 1 & \\ & 3 & \\ & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \\ & 1/3 & \\ & & 1 \end{bmatrix}$，$\begin{bmatrix} 1 & \\ -2 & 1 \\ & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \\ 2 & 1 \\ & & 1 \end{bmatrix}$ (undo 为 2×row 1)

例 $A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 3 & 1 & 2 & 3 \\ 1 & 2 & 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1 & 1 & 4 \\ -1 & -1 & 0 & -5 \\ 1 & 2 & 1 & 0 \end{pmatrix}$ $C = \begin{pmatrix} 3 & 3 & 2 & 4 \\ -1 & -1 & 0 & -5 \\ 1 & 2 & 1 & 0 \end{pmatrix}$ 找到 F st. $A = FC$

□ $A = EB$，note 要找 $B = ?C$ 并 st. $F = E?$，$E = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$，$? = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ∴ $F = E?$

Sys Eq 2

Date. /

1. $x_1 - 2x_2 + x_3 = 0$
$2x_2 - 8x_3 = 8$
$-4x_1 + 5x_2 + 9x_3 = -9$
$\Downarrow$
$A \cdot X = b$

$\Rightarrow$ coefficient matrix $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$

augmented matrix $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$

then, row reduction $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \xrightarrow{\text{elementary op}} \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix}$ echelon 阶梯形 matrix, 此为 reduced echelon

每 row 1st 1 为 leading entry

例1 $f, g \in \mathbb{R}$ $\begin{cases} x_1 + 3x_2 = f \\ cx_1 + dx_2 = g \end{cases}$ 何为 c, d s.t. consistent (无 ~~any~~ sol 即 inconsistent)

解 $\left[\begin{array}{cc|c} 1 & 3 & f \\ c & d & g \end{array}\right] \sim \begin{bmatrix} 1 & 3 & f \\ 0 & d-3c & g-3cf \end{bmatrix}$ $(d-3c)x_2 = g-3cf \in \mathbb{R}$ ∴应令 $d-3c \neq 0$ NOTE: 若 $|A| \neq 0$ 则一定有 uniq. sol

∵ 若有 row $[0 \cdots 0 \mid b]$ inconsistent; 若 consistent $\begin{cases} \#eq = \#var, \text{ uniq. sol 即 } rkA = n \\ \neq \quad , \infty \text{ sol} \end{cases}$ P41.1
即 $rk(A|b) = rkA + 1$, P47.1.

不同的基代表的东西不一样. $[\vec{i}\ \vec{j}\ \vec{k}]$中的$(3,-1,0)$与$[\vec{u}, \vec{v}]$中的$(3,-1)$不一样. 因此坐标中一点应由坐标与基 L. Comb.
对 linear 因为是一个 vector 不变, 互相的结果仍是

2. linear combinations. given $\vec{v}_1, \vec{v}_2, \cdots \vec{v}_p$ in $\mathbb{R}^n$, given scalars $c_1, c_2 \ldots c_p$, 在一条直线上运动

vector $\vec{y}$ defined by $\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p$

$c_1 \cdots c_p$: weight, 可写为 $[\vec{v}_1\ \ \vec{v}_2\ \cdots\ \vec{v}_p\ \ \vec{y}]$

~~Span~~ Span if $\vec{v}_1 \cdots \vec{v}_p$ in $\mathbb{R}^n$, set of all linear combinations of $\vec{v}_1 \cdots \vec{v}_p$ is denoted by $\text{span}\{\vec{v}_1 \ldots \vec{v}_p\} = \{c_1\vec{v}_1 + \ldots + c_p\vec{v}_p \mid \forall c_i \in \mathbb{R}\}$

3. matrix equation $Ax = b$.
$[\vec{a}_1, \vec{a}_2 \ldots \vec{a}_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \ldots + x_n\vec{a}_n$ ($A_{m\times n}$, $X_{n\times 1}$, $\vec{a}_n$ 为列向量)

T ★ $A \in \mathbb{R}^{m\times n}$, the following statements are equivalent:

a. $\forall b \in \mathbb{R}^m$, $Ax = b$ has a solution (不一定 unique)

b. $\forall b \in \mathbb{R}^m$ is a linear combination of A cols

c. the columns (vectors) of A span $\mathbb{R}^m$ (当 $n \geq m$, 见d)

d. $\underline{A}$ (非 aug $[A|b]$) has a pivot position in each row

例1. could a set of n vectors in $\mathbb{R}^m$ span all of $\mathbb{R}^m$ when $n < m$?

解 let $A = [v_1\ v_2\ \ldots\ v_n]$ theorem d. A ~~has~~ cannot have a pivot position in each row

Date. /

Ex $\begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix}$ 1° $\begin{bmatrix} 5\times5-1-24-8 \\ -2\times5+7+9+10 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$

2° $\begin{bmatrix} 5 \\ -2 \end{bmatrix}5 + \begin{bmatrix} 1 \\ -7 \end{bmatrix}\cdot(-1) + \begin{bmatrix} -8 \\ 3 \end{bmatrix}\cdot 3 + \begin{bmatrix} 4 \\ -5 \end{bmatrix}\cdot(-2) = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$，即将 $Ax = [\vec{a}_1 \cdots \vec{a}_4]\cdot x = \sum \vec{a}_i x_i$

例3) $\{\sin(x), \cos(x), 1\} \subseteq \underline{\mathcal{F}(\mathbb{R}, \mathbb{R})}$ L.Ind (P8 2)

□ $\begin{cases} x=0 & a_2+a_3=0 \\ x=\pi/2 & a_1+a_3=0 \\ x=\pi/4 & \end{cases} \Rightarrow a_1=a_2=a_3=0$

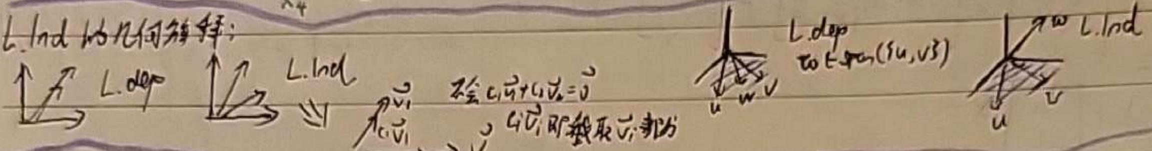
3D/Wronskian

Ex 2 describe all sols of $Ax=0$ in parmetric vector form

□ $A = \begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & -8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix}$，非pivot position $\begin{cases} x_1 = -9x_3+8x_4 \\ x_2 = 4x_3-5x_4 \\ \to x_3, x_4 \text{ free var} \end{cases}$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3+8x_4 \\ 4x_3-5x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$

L.Ind 的几何解释：



Ex3 A is 3×2 with 2 pivot postions. Do $Ax=0$ has nontrival sol? Do $Ax=b$ has at least one sol for every b?

□ $\left[\begin{array}{cc|c} \blacksquare & * & 0 \\ 0 & \blacksquare & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1=0$ $x_2=0$ 1. sol trival

2. no pivot position each row. N.O

Ex4 A 7×5, if colums L.Ind, how many pivot position

□ 5，这样无 free variable，见 F#1

A 5×7, if colums span $\mathbb{R}^5$, how many pivot postions

□ 5，每行都有一个 pivot position - 见 T

Ex5 let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\vec{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\vec{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, $T\ \mathbb{R}^2 \to \mathbb{R}^2$ linear, which $\vec{e}_1 \mapsto \vec{y}_1$ $\vec{e}_2 \mapsto \vec{y}_2$ Find images of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ (即：一旦找到 basis 的 map 地址，可找到 $\forall x \in$ domain 的 map 地址)

□ $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(x_1\vec{e}_1 + x_2\vec{e}_2) = x_1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 6 \end{bmatrix}$

let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\vec{v}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$ $T\ \mathbb{R}^2 \to \mathbb{R}^2$ linear, $\vec{x} \mapsto x_1\vec{v}_1 + x_2\vec{v}_2$

find A s.t $T(\vec{x})$ is $A\vec{x}$ for any $\vec{x}$ (即，找到 $[T]_\beta^\gamma$)

□ $T(\vec{x}) = x_1\vec{v}_1 + x_2\vec{v}_2 = [\vec{v}_1\ \ \vec{v}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$

见 图.1

Ex6 ~~each matrix T is linear.~~ True, $T(\vec{u}+\vec{v}) = \underbrace{A(\vec{u}+\vec{v})}_{\text{matrix 性质}} = A\vec{u}+A\vec{v} = T(\vec{u})+T(\vec{v})$

~~codomain of T $\vec{x} \mapsto A\vec{x}$ is set of all L.Comb of columns of A~~ False

应为 range. codomain 为 $\mathbb{R}^m$, if $A \in \mathbb{R}^{m\times n}$

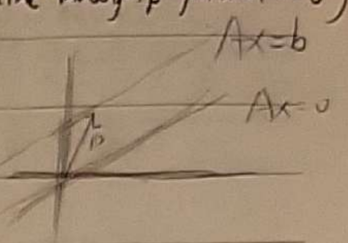
$\{u, v\}$ L.Ind $\Leftrightarrow$ $\{u+v, u-v\}$ L.Ind. ✓ $au+bv=0$ $\frac{a+b}{2}(u+v) + \frac{a-b}{2}(u-v) = 0$

1. solutions of nonhomogeneous systems $Ax=b$.

eg $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-1\\2\\0\end{bmatrix}+x_3\begin{bmatrix}1\\0\\1\end{bmatrix}=\vec{p}+x_3\vec{v}=\vec{p}+t\vec{v}\ (t\in\mathbb{R}^1)$ ← scalar

a point

(line through $\vec{p}$ parallel to $\vec{v}$)



2. Solutions of homogeneous systems $Ax=0$

eg. $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_3\begin{bmatrix}1\\0\\1\end{bmatrix}=x_3\vec{v}=t\vec{v}\ (t\in\mathbb{R}^1)$

3. to $Ax=b$ 对一给定b，一个solution为 $\vec{p}$，则solution set 为 $\vec{w}=\vec{p}+\vec{v}$，$\vec{v}$ 为 $Ax=0$ 的 ∀ solution

(that is, $A\vec{p}=b$)

or: if $Ax=b$ has a solution. solution is unique precisely when $Ax=0$ only has trival solution

对vector space V, $S=\{\vec{v}_1,\dots,\vec{v}_p\}\subseteq V$; $x_1\dots x_p$ scalar (in F)

4. Linear Independent $x_1\vec{v}_1+x_2\vec{v}_2+\dots+x_p\vec{v}_p=\vec{0}$ only trival solution (令 $S=\varnothing$ L.Ind)

Linear dependent $x_1\dots x_p$ not all zero. 无free var = 每row 有 pivot pos.

T #1 (columns of A are L.Ind iff $Ax=0$ has only trival solution)

#2 $S=\{\vec{v}_1\dots\vec{v}_p\}$ L.dep iff at least one of vectors is L.Comb of others

逆否: L.Ind ⟺ 每一个vector不可由其余L.Comb ☐ 对非0 x_i，$\vec{v}_i=\frac{x_1}{x_i}\vec{v}_1+\dots$

可推出: 若 L.Ind $S\subseteq$ V.S. V, let $v\in V-S$, $S\cup\{v\}$ L.dep ⟺ $v\in \mathrm{span}(S)$

☐ ⟹ $a_1s_1+\dots+a_ns_n+av=0$，$a\neq0$ 因 S L.Ind. ∴ v 可被 $\{s_1,\dots,s_n\}$ L.Comb, $v\in\mathrm{span}(S)$

⟸ $v=a_1s_1+\dots+a_ns_n$, note $a_1s_1+\dots+a_ns_n+(-1)v=0$, $-1\neq0$, ∴ $S\cup\{v\}$ L.dep.

一些注意点见图例6-8

注意见 3. 例1

#3 any set $\{\vec{v}_1\dots\vec{v}_p\}$ in $\mathbb{R}^n$ is L.dep if $p>n$ 【eg. 4×5 matrix 的col L.dep】

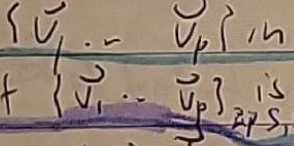
p var 但 n eq must has free var

#5 If $\{\vec{v}_1\dots\vec{v}_p\}$ in $\mathbb{R}^n$ contains $\underbrace{\text{zero vector}}_{}$ (also c=0) then L.dep ☐ 有 $\vec{v}_k=0$ 则 $5\vec{v}_k=0$ (5 ≠ 0)

#6 If $\{\vec{v}_1\dots\vec{v}_p\}$ is L.dep, then $\{\vec{v}_1\dots\vec{v}_p,\vec{v}_{p+1}\}$ also L.dep; (逆否) $\{v_1\dots v_p,v_{p+1}\}$ L.Ind, then $\{v_1\dots v_p\}$ L.Ind

即 $S_1\subseteq S_2\subseteq V$, S_1 L.dep ⟹ S_2 L.dep; S_2 L.Ind ⟹ S_1 L.Ind ☐ 不全0 $a_1\dots a_p$, $a_1\vec{x}_1+\dots+a_p\vec{v}_p+0\vec{v}_{p+1}=0$

5. $A\vec{x}=\vec{b}$ 和 $A\vec{u}=\vec{0}$, $A_{m\times n}$



Transformation/mapping T from $\mathbb{R}^n$ to $\mathbb{R}^m$, $T:\mathbb{R}^n\to\mathbb{R}^m$

each $\vec{x}$ in $\mathbb{R}^n$ to $T(\vec{x})$ in $\mathbb{R}^m$; $\mathbb{R}^n$: domain of T, $\mathbb{R}^m$: codomain of T

$T(\vec{x})$ in $\mathbb{R}^m$ is image of $\vec{x}$, set of all images in range of T

vector space homomorphism, 保证 V, W 有同代数结构 $ax+by\to aT(x)+bT(y)$ 仍线性

T线性可记为 $\vec{u}$ 与 $\vec{v}$ 和 也可是 变换后 不扭曲 同时满足, 泛化 $T:V\to W$

6. T is linear if i) $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$ for all $\vec{u},\vec{v}$ in domain

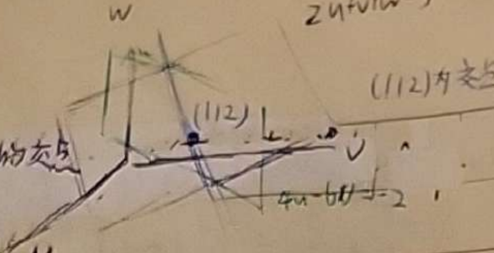
ii) $T(c\vec{u})=cT(\vec{u})$ for all scalars c and for all 定义域中

($T(\vec{0})=\vec{0}$ 有用结论

$T(c\vec{u}+d\vec{v})=cT(\vec{u})+dT(\vec{v})$)

Date. /

例1 对方程组 $\begin{cases} 2u+v+w=5 \\ 4u-6v=-2 \\ -2u+7v+2w=9 \end{cases}$ 以行来看是 plane 的交点



(112)为交点，即sol

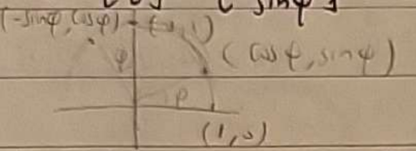
但以列来看，$u\begin{bmatrix}2\\4\\-2\end{bmatrix}+v\begin{bmatrix}1\\-6\\7\end{bmatrix}+w\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}$，comb 系 1 1 5

$\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}1\\3\\5\end{pmatrix}(a\ b)+\begin{pmatrix}2\\4\\6\end{pmatrix}(c\ d)$

证1. $A_{mk}B_{kn}=\sum_{i=1}^{k} \underset{=}{\overset{col}{a_i}} \underset{=}{\overset{row}{b_i}}$ □ $LHS_{ab}=\sum_i a_{ai}b_{ib}$ $RHS_{ab}=\sum_{i=1}^{k} a_{ai}b_{ib}$ 而 $v\otimes u=vu^T$

例2. $T:\mathbb{R}^2\to\mathbb{R}^2$，rotate each point in $\mathbb{R}^2$ about origin through an angle φ. 逆时针 见图例2

□ $\begin{bmatrix}1\\0\end{bmatrix}\to\begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix}$ $\begin{bmatrix}0\\1\end{bmatrix}\to\begin{bmatrix}-\sin\varphi\\ \cos\varphi\end{bmatrix}$ $A=\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}$



例3. Linear T $\mathbb{R}^n\to\mathbb{R}^m$. i) if $\mathbb{R}^n$ onto $\mathbb{R}^m$ ii) if one to one，~~relation~~ relation of m & n
$T: m\times n$

□ i) 即 each row 都为 pivot position，$n\geq m$；ii) col L.Ind，$m\geq n$（事实上，这个结果见 ~~xxx~~ 书 P_1）

例4. $\left[\begin{array}{cc|c}1 & h & -3\\ -2 & 4 & 6\end{array}\right]$ h 为何 consistent

□ $\left[\begin{array}{cc|c}1 & h & -3\\ 0 & 4+2h & 0\end{array}\right]$ $h\ \forall$.（如 $h\neq-2$，x 可为 0）

$\left[\begin{array}{cc|c}1 & h & 2\\ 4 & 8 & k\end{array}\right]$ h k 是 & i) no sol ii) unique iii) infinite many

□ $\left[\begin{array}{cc|c}1 & h & 2\\ 0 & 8-4h & k-8\end{array}\right]$ i) $8-4h=0$, $k-8\neq0$ ii) $8-4h\neq0$ (与k无关) iii) $8-4h=0$, $k-8=0$

例9. 对 $T_1(a_1, a_2, a_3)=(a_1+a_2+1,\ a_2-a_3,\ 2a_3-3)$

□ $T_1(0)=(1,0,-3)\neq0$，不为 L.T

例5. $A_{2\times3}$, one free variable $A\vec{x}=\vec{0}$ is a line in $\mathbb{R}^3$，
two free variable a plane in $\mathbb{R}^3$ 相当于自由点沿伸出线 ($v_3=v_1+t(v_2-v_1)$)

例6. if $\vec{w}$ is L.Comb of $\vec{u}$ & $\vec{v}$ in $\mathbb{R}^n$, then $\vec{u}$ is L.Comb of $\vec{v}$ and $\vec{w}$. X
$\vec{w}=0\vec{u}+1\vec{v}$

例7. if none of vectors in $\{\vec{v}_1\vec{v}_2\vec{v}_3\}$ in $\mathbb{R}^3$ is a multiple of other vector, $\{\vec{v}_1\vec{v}_2\vec{v}_3\}$ L.Ind X
{(100)(010)(110)}，判判三个的不可用互相 multiple

例8. $\vec{v}_1\ \vec{v}_2$ in $\mathbb{R}^4$, $\vec{v}_2$ not a scalar multiple of $\vec{v}_1$, then $\{\vec{v}_1, \vec{v}_2\}$ L.Ind. X
$\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\}$ 有 zero vec. L.dep
需两者都不是对方 scalar multiple

即, linear T $\mathbb{R}^n \to \mathbb{R}^m$ determined by $n\times m$ identity matrix　　linear Map

即, $\forall$ linear T is matrix T

1. $\mathcal{P}_1$ T $\mathbb{R}^n \to \mathbb{R}^m$ linear iff $\exists$ unique matrix $\underline{A}$ s.t ~~standard matrix~~

$$T(\vec{x}) = A\vec{x}, \quad \forall \vec{x} \in \mathbb{R}^n$$

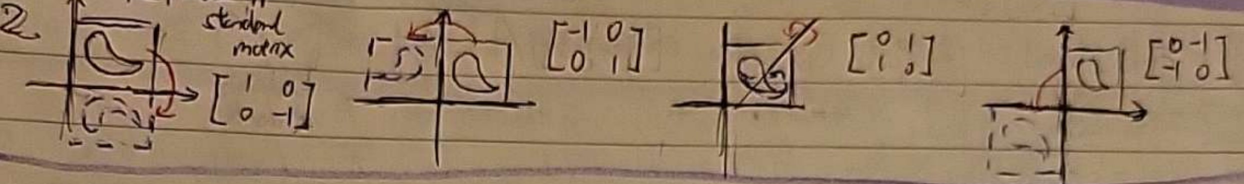
　　(e_j col vector, row $j = 1$)

In fact, $A = [T(\vec{e}_1) \cdots T(\vec{e}_n)]$ (A $m\times n$)

$\square$ $\Rightarrow$) $\vec{x} = I_n\vec{x} = [\vec{e}_1, \dots, \vec{e}_n]\vec{x}$, $T(\vec{x}) = T([\vec{e}_1, \dots, \vec{e}_n]\vec{x}) = x_1T(\vec{e}_1) + \dots + x_nT(\vec{e}_n) = [\underbrace{T(\vec{e}_1), \dots, T(\vec{e}_n)}_{\in\mathbb{R}^m \ A}]\underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}}$

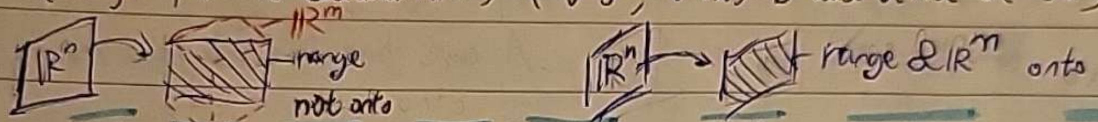
$\Leftarrow$) 回例6

2. 特殊的应用: standard matrix



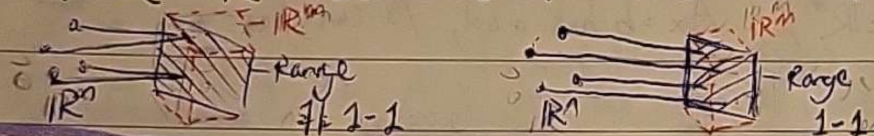
3. $T: \mathbb{R}^n \to \mathbb{R}^m$ onto $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is image of at least one $\vec{x}$ in $\mathbb{R}^n$

(range of T is codomain $\mathbb{R}^m$) ($\forall \vec{b}$, $T(\vec{x}) = \vec{b}$ exist at least one sol)



$T: \mathbb{R}^n \to \mathbb{R}^m$ one-to-one if each $\vec{b}$ in $\mathbb{R}^m$ is image of at most one $\vec{x}$ in $\mathbb{R}^n$

($\forall \vec{b}$, $T(\vec{x}) = \vec{b}$ 无解/唯一解)



即 nullity = 0. 回2.8

推论: ~~$T: V \to W$ linear, T 1-1 $\Leftrightarrow$ Null$(T) = \{0\}$~~

4. $\mathcal{T}_1$ $T: \mathbb{R}^n \to \mathbb{R}^m$ linear, T one to one iff $\underline{T(\vec{x}) = \vec{0} \text{ trival sol only}}$.

$\square$ $\Rightarrow$) linear, $T(\vec{0}) = \vec{0}$ (回.6有用结论) $\overset{对\ T(x)=T(0),\ x=0\ (1\text{-}1)}{\Leftarrow}$) let T not 1-1, let $T(\vec{u}) = T(\vec{v}) = \vec{b}$,

$T(\vec{0}) \neq T(\vec{u} - \vec{v}) \overset{linear}{=} T(\vec{u}) - T(\vec{v}) = \vec{0}$, 矛盾.

$\mathcal{T}_2$ T $\mathbb{R}^n \to \mathbb{R}^m$ linear, A 为 standard matrix for T. 以rank/nullity 表示见回.4

i) T onto $\mathbb{R}^m$ iff columns of A span $\mathbb{R}^m$ $\square$ 由定义, 回$\mathcal{T}$

ii) T one to one iff columns of A L. Ind. $\square$ 由定义, L. dep 有 free variable, 无数解

即 $AB = [A\vec{b}_1 \cdots \underset{col}{\underline{A\vec{b}_n}}]$ 或 $= \begin{bmatrix} \vec{a}_1^T B \\ \vdots \\ \vec{a}_n^T B \end{bmatrix}$, 也 $= \sum_{i=1}^{k} \underset{col}{\underline{a_i}}\ \underset{=row}{\vec{b}_i^T}$ (定理1) sum over outer product　　而 $A\vec{x} = \sum \vec{a}_i x_i$

5. $\mathcal{T}_3$ each column of $A_{mk}B_{kn}$ is a linear combination of columns of A using weights from corresponding column of B. (矩阵证明见回.3)

$\square$ $B = [\vec{b}_1, \dots, \vec{b}_n]$, $A\cdot B = [A\vec{b}_1, \dots, A\vec{b}_n]$ (拿出 B 的 col vector 使用. $A\vec{b}_i = [\vec{a}_1, \dots, \vec{a}_n]\vec{b}_i = \sum_{j=1} a_j b_j$)

6. $AB \neq BA$ in general; $\underline{AB = AC}$ 不一定 $B = C$; $AB = 0$ 不一定 $A = 0$ or $B = 0$

除非 A 可逆, $A^{-1}AB = A^{-1}AC$

$\underline{\underline{I_m}} A = A = AI_n$, $\underline{A(BC) = (AB)C}$　　$A(B+C) = AB + AC$　证明见回.2.

单位矩阵

从 transformation 角度看, 都是先 C 再 B 再 A, 所以不须证明, ABC 都是 standard matrix　　$(B+C)A = BA + CA$

Date. /

例1. $(AB)^T = (B^T A^T)$

□ $(AB)^T_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki}$，而 $(B^T A^T)_{ij} = \sum_k B^T_{ik} A^T_{kj} = \sum_k A_{jk} B_{ki}$

例2. AB $n\times n$，AB invertible，则 A, B invertible.

□ let $C = (AB)^{-1}$，consider BC，$A(BC) = (AB)C = I$　　同理 CA 为 B^{-1}　　若 AB 非 square 不对

而令 $(BC)A = X$，$CAB = B^{-1}XB$，$I = B^{-1}XB$，$X = BIB^{-1} = I$　　$[\ ][\ ] = I$ 可逆

例3 AB 的 last col 全为 0，B 无 zero column，则 A cols L.Ind. ✗

□ $AB = [A\vec{b}_1 \ldots A\vec{b}_n]$，即 $A\vec{b}_n = \vec{0}$，$\vec{b}_n \neq \vec{0}$，∴ 有 nontrivial sol，A col L.dep

OR: $(AB)_{:,-1} = 0 = A(B)_{:,-1}$ 即 $\sum_{i=1} t_i A_{:,i}$ where $B_{:,-1} = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$ 而 $t_i \neq 0$，∴ $\{A_{:,i}\}$ L.dep

B 的 cols L.dep 则 AB 的 col 也 L.dep ✓

□ 各 $(AB)_{:,i} = A(B)_{:,i}$ 而 $B_{:,i}$ 已 L.dep. 可对各 $\{(AB)_{:,i}\}$ 也 $= 0$

例6. i) $CA = I_n$，then $Ax = 0$ only trivial sol. A can't have more col than row.

□ ~~$Ax = 0$，$CAx = 0 = x$，∴ trivial　　可 < row 否则会有 free var~~

nullity = 0

(ii) $AD = I_m$，then $\forall b \in \mathbb{R}^m$，$Ax = b$ has a sol，A can't have more row than col

□ $A\underset{x}{\underline{Db}} = b$　　~~可多行 row 否则不能在各 row 都有 pivot pos.~~

A col span $\mathbb{R}^m$

(iii) A as $m\times n$，∃ $n\times m$ C and D s.t. $CA = I_n$ and $AD = I_m$，prove $m = n$，$C = D$

□ 可逆即同时满足 i) & ii) $-(C_{nm}(A_{mn})D_{nm}) = D = C$　∴ 由 i) & ii) $m = n$

~~者用于 A 可逆，$Ax = 0$ only trivial，A span $\mathbb{R}^n$，满足 四.3.~~

nullity = 0　　rk = n

例7. $A + \varepsilon B$，ε small 证 $A^{-1} - \varepsilon A^{-1}BA^{-1}$ □ $A^{-1} = (A+\varepsilon B)^{-1} = [A(I+\varepsilon A^{-1}B)]^{-1} = (I+\varepsilon A^{-1}B)^{-1}A^{-1}$

Optim P10.3 $= (I - \varepsilon A^{-1}B + \varepsilon^2(A^{-1}B)^2 - \ldots)A^{-1} \approx A^{-1} - \varepsilon A^{-1}BA^{-1}$，即 ignore negligible $\varepsilon^2(A^{-1}B)^2 - \ldots$ term

例4 A invertible $n\times n$，$B_{n\times p}$，证 $[A\ B] \sim [I\ X]$，$X = A^{-1}B$

□ $\underbrace{(E_p \ldots E_1)}_{A^{-1}}A = I$，∴ $(E_p \ldots E_1)B = X = A^{-1}B$

例6. $A = -A^T$，则 diag = 0，且 $\forall x, x^TAx = 0$ □ 1° $A + A^T = 0$

2° $(x^TAx)^T = x^TAx = -x^TAx$，∴ $2x^TAx = 0$

例5 A B X $n\times n$，AX，A−AX 可逆，$(A-AX)^{-1} = X^{-1}B$，证 B 可逆，并 solve X

NOTE 除了证 $B^{-1}B = BB^{-1} = I$ 以外(这里此法较难)，用 $B = X(A-AX)^{-1}$ 可逆 乘积仍可逆 (各 P.2)

□ $X = B(A-AX)$，$X = (I+BA)^{-1}BA$

证1. AB−I 可逆 ⇒ BA−I 可逆 □ WTS BA−I 可逆 ⇒ AB−I 可逆　　四.4.2 ∴ BA−I evl = 0

四.P6 ∴ AB 有 evl = 1　AB−I evl = 0 不可逆　　∴ BA evl = 1 四.P6

↓证 I−AB 可逆 ⇒ I−BA 可逆　法1 $-(I-AB) = AB-I$，apply 1°　　法2 若 $(AB)^k \to 0$ 则

$(I-AB)^{-1}$ ∃，$(I-AB)^{-1} = I + AB + ABAB + ABABAB + \ldots = I + A(I + BA + BABA + \ldots)B = I + A(I-BA)^{-1}B$

optim P10.3

symmetric: $A = A^T$ (对 $\forall$ square A, $A+A^T$ symmetric)
skew-symmetric: $A^T = -A$ ($A - A^T$ skew-symmetric)

Invertible & Transpose

$W = [X \ \vec{x}_0]$ $\quad W^T = \begin{bmatrix} X^T \\ \vec{x}_0^T \end{bmatrix}$ $\quad (rA+sB)_{ij} = rA_{ij} + sB_{ij}$, $(rA^T+sB^T)_{ij} = rA_{ji} + sB_{ji}$

$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^T = \begin{bmatrix} A^T & 0 \\ B^T & D^T \end{bmatrix}$

1. Transpose $A_{m\times n} \to A^T_{n\times m}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T$ $\begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}^T$ $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}^T$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ $\begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}$

matrix ... undo!

求逆可想成线性变换后再变回去(练习 $[\,]$ $[\,]$)

$(A^T)^T = A$ (idempotent)
$(rA+sB)^T = rA^T + sB^T$, $r,s \in F$ (linearity)
$\det(A)^T = \det A$ (4 图.1)
$(AB)^T = B^T A^T$ (见例1)

2. $A_{n\times n}$ invertible if $\exists ! C_{n\times n}$ s.t. $CA = I_{n\times n} = AC$, C 即 A^{-1}

如 C 不唯一, $B = BI = B(AC) = (BA)C = IC = C \quad \therefore B = C$

定义上因两边都要满足, A 必为方阵, 否则 $m\times n$ 无法有设

3. 对于 2×2 的 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if $|A| = ad-bc \neq 0$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

用五种解方程

if $|A| = 0$, not invertible ($A \sim \begin{bmatrix} a & b \\ 0 & d-\frac{bc}{a} \end{bmatrix}$ 不可逆)

4. 若 $A_{n\times n}$ invertible $\Leftrightarrow \forall \vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has unique sol $\vec{x} = A^{-1}\vec{b}$

见 例7.1

□ 若不 uniq, 设 $\vec{u} \neq \vec{x}$ 有 $A\vec{u} = \vec{b}$, $A^{-1}A\vec{u} = \vec{u} = A^{-1}\vec{b} = \vec{x}$

即是说, $A_{n\times n}$ 可逆 $\Leftrightarrow$ col of A L.Ind, span $\mathbb{R}^n$

$[A \ b] \sim [I \ x]$ 可求逆, 见 第4

5. 1. $(A^{-1})^{-1} = A$, i.e. A is invertible, then A^{-1} invertible □ $AA^{-1} = I = A^{-1}A$

2. $(AB)^{-1} = B^{-1}A^{-1}$ □ $AB\cdot(B^{-1}A^{-1}) = AIA^{-1} = I$, 同理 $(B^{-1}A^{-1})AB = I$ 可逆矩阵乘积可逆

回例1

3. $(A^T)^{-1} = (A^{-1})^T$ □ $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$ 4. $r \neq 0$, $(rA)^{-1} = r^{-1}A^{-1}$

6. 对 $A_{m\times n}$, $B_{m\times p}$, 考虑 Augment matrix $m\times(n+p)$ $(A|B)$; note: 对 $\forall M_{m\times n}$, $M(A|B) = (MA|MB)$

□ 对 $M(A|B)$ 的前 n 列, $M(A|B)_{ij}$, $j = 1\cdots n$, 为 $\sum_k M_{ik}A_{kj} = MA_{ij}$, 同理 $j = n+1\cdots n+p$, $M(A|B)_{ij} = MB_{ij}$

以此用于找 inverse. $A_{n\times n}$ 可逆 $\Leftrightarrow (A|I_n) \xrightarrow[\text{row operation}]{\text{elementary}} (I_n|A^{-1})$ (未经左例4)

只对 row, row 为左乘

□: $\Rightarrow$ 对 $A^{-1}(A|I_n) = (I_n|A^{-1})$, 因 $\underbrace{A^{-1} = E_p \cdots E_1}_{M}$, $\therefore E_p \cdots E_1 (A|I_n) = (I_n|A^{-1})$ 每 elementary matrix 代表 row operation

据6. 见 $f(ii)$

$\Leftarrow$ 令 M = row op 的总和, 即 $M = E_p \cdots E_1$, $E_p \cdots E_1(A|I_n) = (I_n|B) = (MA|MI_n)$ $\therefore \begin{cases} MA = I_n \\ M = B \end{cases}$

WTS $B = A^{-1}$. 但由 $MA = I_n$, $M = A^{-1} = B$

7. Linear transform 为 special case of affine transform 线性 : $f(x) = Ax + b$ (linear + translation)

$\boxed{\text{Def: } T \text{ affine iff } \forall \text{scalar } \lambda \quad f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y)}$

7. 図

$X = A_{11}^{-1}A_{21} \quad Y = A_{11}^{-1}A_{12}$

应用.

① If A_{11} 可逆, $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$ 中 $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$

同理 A_{22} 可逆, 舒尔逆为 $A_{11} - A_{12}A_{22}^{-1}A_{21}$, S 称为 <u>schur complement</u>

to $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ 与 A_{11} 都可逆, S 也可逆.

□ lemma $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ 可逆 iff B, C 都可逆, (B C, square)

□ ($\Rightarrow$) $\begin{bmatrix} X & Y \\ Z & W \end{bmatrix}\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ 得 $\begin{cases} BX = I \\ WC = I \end{cases}$; ($\Leftarrow$) $\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$

$\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} = I \begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix} I$, 左边为三个可逆矩阵, 相乘必可逆

(见上面), 因此 $\begin{bmatrix} A_{11} & 0 \\ 0 & S \end{bmatrix}$ 可逆, 由 lemma, S 可逆.

应用. a standard set of differential equations is transformed by Laplace transforms into $\begin{bmatrix} A - sI_n & B \\ C & I_m \end{bmatrix}\begin{bmatrix} \vec{x} \\ \vec{u} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{y} \end{bmatrix}$, where $A_{n\times n}$, $B_{n\times m}$, $C_{m\times n}$, s a variable, $\vec{u}$, $\vec{x}$, $\vec{y}$ function of s, $\vec{u}$ in $\mathbb{R}_m$ ($\vec{u}_{m\times 1}$) ↑input, $\vec{x}$ in $\mathbb{R}^n$ (state vector), $\vec{y}$ in $\mathbb{R}^m$ (output)

有 transfer function $W(s)$ s.t. $W(s)\vec{u} = \vec{y}$ (input $\mapsto$ output). 若 $A - sI_n$ 可逆, $W(s)$ 为舒尔逆

□ $\begin{cases} (A - sI_n)\vec{x} + B\vec{u} = \vec{0} \\ C\vec{x} + I_m\vec{u} = \vec{y} \end{cases} \Rightarrow \vec{x} = (A - sI_n)^{-1}(-B\vec{u})$, 代入 $\vec{y} = (I_m - C(A - sI_n)^{-1}B)\vec{u} \Rightarrow$ 即 $W(s)$ 为 $\begin{bmatrix} A - sI_n & B \\ C & I_m \end{bmatrix}$ 中 $(A - sI_n)$ 的 schur complement

由上得 舒尔逆有可逆情况, 则可找出 $W(s)^{-1}$, output $\mapsto$ input

例2. 对 $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 逆时针 θ_1 rotate $\rightarrow v_1$, 顺时针 $\theta_2 \rightarrow v_2$ 证 $\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

□ $v_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}$, $v_2 = \begin{pmatrix} \cos -\theta_2 & -\sin -\theta_2 \\ \sin -\theta_2 & \cos -\theta_2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta_2 \\ -\sin\theta_2 \end{pmatrix}$

$\cos(\theta_1 + \theta_2) = \cos(\text{deg between } v_1\, v_2) = \dfrac{\langle v_1\, v_2 \rangle}{\|v_1\| \cdot \|v_2\|}$

例1: $A_{m\times n}$ 的 col form a basis of $\mathbb{R}^m$, 则 $m = n$

□ i) col span $\mathbb{R}^m$ 四.结论6 ii), $n \geq m$ ii) col L.ind 四.丁#3 $n \leq m$ $\Rightarrow n = m$

if $A_{m\times m}$, Col $A \neq \mathbb{R}^m$, 则 Nul $A \neq \{0\}$, A 不可逆

□ 由四.3, $rkA \neq n$ ∴ nullity $\neq 0$, 否则 非满秩, 不可逆 P4.3. OR 即非 ∀ row 有 pivot pos, ∴ $Ax = 0$ 有 nontrivial sol

$A_{3\times 5}$ has 3 pivot col, Is Col $A = \mathbb{R}^3$? Is Nul $A = \mathbb{R}^2$?

□ Col $A = \mathbb{R}^3$, 但 nullspace $\leq \mathbb{R}^5$ ($\dim N(A) \neq N(A)$)

$A_{4\times 7}$ has 3 pivot col, Is Col $A = \mathbb{R}^3$? What is dim Nul A?

□ A 4 row Col $A \leq \mathbb{R}^4$ nullity $= 7 - rkA = 7 - 3 = 4$

Each line in $\mathbb{R}^n$ is one-dimensional subspace of $\mathbb{R}^n$. ✗

□ ~~line 不经 origin 非 subspace, dim only def for subspace~~

四 $A_{6\times 4}$, $B_{4\times 6}$, AB 不可逆 ✓ □ B col > row, col B L.dep 四丁#3 ∴ $Bx = 0$ 有 nontrivial sol, 两边同乘 A $ABx = A0 = 0$, ∴ AB nullity $\neq 0$, 不 full rk

Invertible 2
VS 5

Date. /

1. $A_{n\times n}$. 以下同真同假

即不需同时满足 $CA=AC=I$，因 square

a. A is invertible; A^T is invertible; there is $n\times n$ C s.t $CA=I$; there is $n\times n$ D s.t. $AD=I$

b. A row equivalent to $n\times n$ Identity matrix

c. A has n pivot position; cols of A span $\mathbb{R}^n$; $\forall \vec{b}\in\mathbb{R}^n$, $A\vec{x}=\vec{b}$ has one unique sol

d. $A\vec{x}=\vec{0}$ has only trivial sol; cols of A L. Ind.

只能是 unique sol. 因 $\forall b$, 不会出现 [0 ... 1]

f. linear T $\vec{x}\mapsto A\vec{x}$ is one-to-one; maps $\mathbb{R}^n$ onto $\mathbb{R}^n$; T bijective; T invertible

导出结果 III.4

2. let $T:\mathbb{R}^n\to\mathbb{R}^n$ be L.T. and let A be standard matrix for T. 即 relate L_A & A，见 17

T is invertible iff A is invertible. A^{-1} would be standard matrix for T^{-1}

3. If $A_{m\times n}$, $B_{n\times o}$, A partition into x rols and y cols, B partition into y rows and z rows

then they are comfortable for block multiplication

eg. $AB=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}B_1\\ B_2\end{bmatrix}=\begin{bmatrix}A_{11}B_1+A_{12}B_2\\ A_{21}B_1+A_{22}B_2\end{bmatrix}$

(i,j) in $\text{col}_k(A)\cdot\text{row}_k(B)$ 为 $a_{ik}b_{kj}$

thus $\Sigma(i,j)$ 为 (i,j) in AB

In general $AB=[\text{col}_1\ \text{col}_2 \dots \text{col}_n]\begin{bmatrix}\text{row}_1\\ \text{row}_2\\ \text{row}_n\end{bmatrix}=\text{col}_1\text{row}_1+\dots+\text{col}_n\text{row}_n$

4. block upper triangular $A=\begin{bmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix}$, assume $A_{11\,p\times p}$, $A_{22\,q\times q}$, A invertible

证 $A^{-1}=\begin{bmatrix}A_{11}^{-1} & A_{11}^{-1}A_{12}A_{22}^{-1}\\ 0 & A_{22}^{-1}\end{bmatrix}$

即 $\boxed{\ }$ 令 $AB=I_{p+q}$ $\begin{bmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix}\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}=\begin{bmatrix}I_p & 0\\ 0 & I_q\end{bmatrix}$

$$\begin{cases}A_{11}B_{11}+A_{12}B_{21}=I_p\\ A_{11}B_{12}+A_{12}B_{22}=0\\ A_{22}B_{21}=0\\ A_{22}B_{22}=I_q\end{cases}\Rightarrow$$

- A_{22} 可逆，$B_{22}=A_{22}^{-1}$
- $B_{21}=A_{22}^{-1}\cdot 0=0$
- $A_{11}B_{11}=I_p \therefore B_{11}=A_{11}^{-1}$
- $B_{12}=-A_{11}^{-1}A_{12}A_{22}^{-1}$

见 P10.1

vector space 用这三点可推出 subspace 为 vector space，H 只是 $\mathbb{R}^n$ 的一部分，但保留了代数结构

5. Subspace of $\mathbb{R}^n$ is any set H, a) zero vector in H b) each $\vec{u},\vec{v}$ in H, $\vec{u}+\vec{v}$ in H

可被 c) 推出：$k=0$, $k\vec{u}=\vec{0}\in H$ 了 但仍需要列出，见 P10.1 NOTE

c) each $\vec{u}$ in H, k scalar, $k\vec{u}$ in H. 一个不过原点的线不是 subspace

$Ax=b$ 无解即是说 b 不在 A 的列空间(一个子空间)

$\vec{v}_1\dots\vec{v}_p$ in $\mathbb{R}^n$, All L. Combs of $\vec{v}_1\dots\vec{v}_p$ is subspace in $\mathbb{R}^n$; column space of $A_{m\times n}$ is subspace in $\mathbb{R}^m$

(The null space of $A_{m\times n}$ is set of solutions of homogeneous equation $A\vec{x}=\vec{0}$, a subspace in $\mathbb{R}^n$

注意 col space 为 $\mathbb{R}^m$，为 $\vec{b}$ 的 dim. ∴ col space 也同时为 output $\vec{b}$ 的集合 (又称 $\mathbb{R}^n$)

$\vec{b}$ 一直在 col space，一方面因 $\vec{b}$ 在 subspace. 因 A 的 col 即是接合基 basis 所在的加者 张成的空间即 col space 9

一方面 $\vec{b}$ 永远可被线性连接得到. 无被压缩的情况下，null space 即被可连接到 $\vec{b}$ 的向量张成.

$H_1 = \{(a-3b, b-a, a, b): a, b \in \mathbb{R}\}$ H_1 is subspace of $\mathbb{R}^4$ $\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{v_1}$ $\vec{v_2}$

$H_2 = \{(a,b,c,d): a-2b+5c=d \text{ and } c-a=b\}$, H_2 is subspace of $\mathbb{R}^4$ (定理4.2)

Date. / $\begin{cases} a-2b+5c-d=0 \\ -a-b+c=0 \end{cases}$ set of homogeneous equations, H_2 subspace

例1 $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$ find basis for null space of A.

□ 考虑 $Ax=0$, $[A|0] = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = x_2\underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u} + x_4\underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{v} + x_5\underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{w}$ $\text{span}\{u,v,w\} = \text{Nul}(A)$ 且 u,v,w L.ind ← basis $\{u,v,w\}$

上述 $=0$ 只有 x_2, x_4, x_5 都为0, Note,

证1. $\text{Col}A \cap \text{Nul}A^T = \{\vec{0}\}$ □ 对任意 $Au \in \text{Col}A$, 若 $Au \in \text{Nul}A^T$, $A^TAu = 0$ 即 $u \in N(A^TA) = N(A)$

$\therefore Au = 0$ 即 intersection只有0 四.5

例2 find basis for col space of $A = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (pivot column 为 basis)

□ $\forall \vec{v} \in \mathbb{R}^4$ $\vec{v} = \sum^5 c_i \vec{a_i}$, 但 $\vec{a_3}, \vec{a_4}$ 可被 l. comb. $\therefore$ L.ind basis $\{a_1\ a_2\ a_5\}$

例3 $\vec{v_1} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, $\beta = \{\vec{v_1}, \vec{v_2}\}$, then β is basis for $H = \text{Span}\{\vec{v_1}, \vec{v_2}\}$ since $\vec{v_1}\vec{v_2}$ L.ind

If $\vec{x}$ in H, find coordinate vector □ 因 $\vec{x} \in H$, $\vec{x} = \sum^2 x_i \vec{v_i}$ $\left[\begin{matrix} 3 & -1 \\ 6 & 0 \\ 2 & 1 \end{matrix}\middle|\begin{matrix} 3 \\ 12 \\ 7 \end{matrix}\right] \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{cases} x_1 = 2 \\ x_2 = 3 \end{cases}$

对此们有 $[x]_{\text{natural}} = Q_{\beta \to \text{natural}} [x]_\beta$

$\begin{bmatrix} 3 & -1 \\ 6 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$

例4 $\begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}$ 若可化为 $\begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 可知 第3、4列为free variable, 第1、2、5列为

basis, 但只有 $\begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 3 \\ 4 \end{bmatrix}\begin{bmatrix} -9 \\ 2 \\ 1 \\ -8 \end{bmatrix}$ 是而不是 $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ 和 $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, 即化简后的列不是basis

pivot col 表示线性相关性, 各pivot col 都是 L.ind, 有free variable的即可被他们表示

例5 $\alpha_1 \dots \alpha_r$ L.ind. $u \in \text{span}A$, $v \notin \text{span}A$, 证 $\forall t \in \mathbb{F}$, $\{tu+v, \alpha_1 \dots \alpha_r\}$ L.ind

□ $0 = \sum_{i=1}^r b_i\alpha_i + b_{r+1}(tu+v)$, 若 $b_{r+1} \neq 0$, $tu+v = t\sum_{i=1}^r a_i\alpha_i + v = \sum_{i=1}^r \frac{-b_i}{b_{r+1}}\alpha_i$, 则 $v \in \text{span}A$

矛盾; $b_{r+1} = 0$, 而 α_i L.ind, b_i $i=1,\dots,r$ $=0$ $\therefore$ L.ind

若 $k \geq 2+n$ $a_i \in \mathbb{R}^n$, 则 $\{a_1 \dots a_k\}$ L.dep. 证 $\exists$ scalar $b_1 \dots b_k$ (其中至少1个非0) st. $\sum^k b_i = 0$, $\sum^k b_i a_i = 0$

□ def $\overline{a_i} = \begin{bmatrix} 1 \\ a_i \end{bmatrix} \in \mathbb{R}^{n+1}$, $\overline{a_i}$ 仍L.dep, $\therefore$ 有非全0 b_i st. $\sum b_i \overline{a_i} = \vec{0} = \begin{bmatrix} \sum b_i \\ \vdots \end{bmatrix}$

例6 AB, A diagonal matrix, 即 scalar times rows of B ✓ A row i is $[0\,0 \dots d_i \dots 0]$, AB row i is $[0\,0 \dots d_iB \dots 0]$

If $AC = 0$, then $A = 0$ or $C = 0$ ✗ $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$(A+B)(A-B) = A^2 - AB + BA - B^2$, only when A, B commute 才为 $A^2 - B^2$ ✓

每个 square matrix is a product of elementary matrixs. ✗ 即写 $A = E_p \cdots E_1 I$, 但 A 不一定可逆 四 11)

(非 $A \sim I$)

A 可逆, $r \neq 0$, $(rA)^{-1} = r^{-1}A^{-1}$ ✓

$A_{n\times n}$ $\det(CA) = C^n \det(A)$ ✓ P_{3-2} P_{viii}

[10] 即使 $AB \neq BA$, $\det BA = \det AB$ ✓ $P_7\ 22$ $\det(A+B) = \det A + \det B$

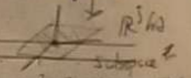
$\begin{bmatrix}3\\8\\-1\end{bmatrix}\begin{bmatrix}6\\2\\5\end{bmatrix}$ 是 L.Ind，但不 span $\mathbb{R}^3$，因此不是 basis

$\boxed{\dim 2}$ & $\boxed{\det 3}$ & $\boxed{\text{span } 2}$

$\mathbb{R}^2$ 不是 $\mathbb{R}^3$ 的 subspace，$\mathbb{R}^2$ 的是 2 个值，不是 $\mathbb{R}^3$ 向量的 subset.

$H = \{[\,\vdots\,]\}$ 才是 $\mathbb{R}^3$ 的 subspace

pivot col of matrix A form a basis for col space of A 见左！(P2.3)



$\mathbb{R}^3$ 的 subspace，一定是三维的，平面也不是，任意一个三维空间中

1. Basis for vector space V (P8) is L.I. set in V that spans V.

eg1. $\{\vec{e}_1 \dots \vec{e}_n\}$ $\boxed{\text{standard basis}}$ for $\mathbb{R}^n$, thus 可逆 $n\times n$ 矩阵的 cols 即 basis for $\mathbb{R}^n$ ($C(A)=\mathbb{R}^n$)

eg2. 对 $V=\{0\}$，因 $\text{span}(\phi)=\{0\}$，ϕ 为 V 的 basis (P2.定义)

2. if set $\beta = \{\vec{b}_1 \dots \vec{b}_p\}$ (ordered) is basis for subspace H, for each $\vec{x}$ in H, coordinate of $\vec{x}$ relative to basis β (unique, 可证) is $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \dots + c_p\vec{b}_p$, and $[\vec{x}]_\beta = \begin{bmatrix}c_1\\ \vdots \\ c_p\end{bmatrix}$ is coordinate vector relative to β. ($\vec{x} = [\vec{b}_1 \dots \vec{b}_p][\vec{x}]_\beta$ 线性组合基；即 $[\vec{b}_1 \dots \vec{b}_p]^{-1}\vec{x} = [\vec{x}]_\beta$)

The dimension of a nonzero subspace H is number of vectors in any basis for H!

The rank of A is the dimension of col space of A. } dim of subspace $\{0\}$ is defined as 0

如 $\vec{x}$ 左 A 变换后 $\vec{b}$ 为直线，rank = 1. 后为平面，rank = 2 …，即 number of dim in output $\vec{b}$.

3. (Rank Theorem) If A has n col, $\overset{\text{pivot col}}{\text{rank}}\, A + \overset{\text{free variable}}{\dim}\, \text{Nul}(A) = n$: 证明见图形，直线的

泛化的 proof: $T: V \to W$ linear, $\underset{\dim(\text{Null}(T))}{\text{nullity}(T)} + \underset{\dim(\text{Range}(T))}{\text{rank}(T)} = \dim(V)$ (***3 c)

□ let $\dim(V)=n$, $\text{nullity}(T)=k$, $\{v_1,\dots,v_k\}$ basis for $\text{Null}(T)$，可 extend 为 $\{v_1,\dots,v_n\}$ 为 V 的 basis，令 $S=\{T(v_{k+1}) \dots T(v_n)\}$

WTS S 为 $\text{Range}(T)$ 的 basis 即 i) S span: $\text{Range}(T) \overset{***10\, J_2}{=} \text{span}(T(v_1),\dots,T(v_n)) = \text{span}(T(v_{k+1}),\dots,T(v_n)) = \text{span}(S)$ (前 k 个为 0)

ii) S L.Ind. let $\sum_{k+1} a_i T(v_i) = \vec{0}$，$T$ linear，$T(\sum_{k+1} a_i v_i) = 0$，$\sum_{k+1} a_i v_i \in \text{Null}(T)$，∴ 可被 $\{v_1,\dots,v_k\}$ 表示 (**3. J_1)：$\sum_{k+1} a_i v_i = \sum_{1}^{k} b_j v_j$

由 $\{v_1,\dots,v_n\}$ V basis，$\sum^{k} b_i v_i - \sum_{k+1} a_i v_i = 0$，$a_i = b_i = 0$ ∴ $\{T(v_{k+1}),\dots,T(v_n)\}$ basis，rank $k = n-k$

4. $A_{n\times n}$，A 可逆除了前面上页的定理外：

a). Col of A form basis of $\mathbb{R}^n$

b) Col space $A = \mathbb{R}^n$; $\dim \text{Col} A = n$; $\text{rank}\, A = n$

□ 以泛化表示：($A_{n\times n}$ 可视为 linear T.(同理)) linear $T: V \to W$, T bijection $\Leftrightarrow$ $\text{rank}(T) = \dim(T)$; T 1-1 ∴ nullity = 0 (因 J_2)

~~只需 V W dim 同，1-1 即 onto！ rank T = dim(Range(T)) = dim V = dim W ∴ range(T) = W (***3 J_3), by def onto~~ rank T = dim V

5. Determinant $\det A_{n\times n} = a_{11}\det A_{11} - a_{12}\det A_{12} + \dots + (-1)^{n+1} a_{1n} A_{1n} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$ 作为 (1,j) cofactor C_{1j}

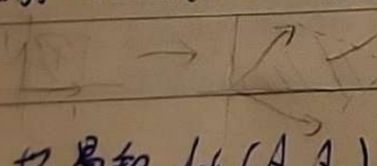
(对于三阶矩阵，det 的证明 $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}$ $\Delta = a_{11}a_{22}a_{33} + \dots$ ~~即 det A. 如 A 可逆，$\Delta \neq 0$. 即 Δ~~ 有重要意义，称为行列式)

cofactor 不只是 C_{1j}，可对 ∀ row/col cofactor expansion.

6. Row operations: 1° 两行互换，det 改变号 2° 一行加到另一行，det 不变 3° 一行 × k, det × k

推论 1° 两行相同，det = 0. 2° 两行成比例，det = 0 3° 一行都 0，det = 0 ~~从中知 A 不可逆~~ $\Leftrightarrow$ det = 0 (无 pivot position)

7. 行列式可表示线性变换后 (1,0)(0,1) (basis) 围成的区域扩大了多少倍，因此只要 det ≠ 0，即化为不能用当前维度表示 (不一定 0 !!!)，A 就可用相反的变换变回来，A^{-1} 存在



det (二维 det 即比例面积)，

即！一般 linear T invertible 除非 map 成更小维度空间，∴ 扩大 0 倍 (压没了)

也易知 $\det(A_1A_2) = \det(A_1)\det(A_2)$ 拉伸两次

应用 $A=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ $B=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ Pauli $\underbrace{\text{spin matrices}}_{\text{量子力学 electron spin}}$, $AB=-BA$, anticommute

应用，寻找 $(-2,0)\ (0,3)\ (1,3)\ (-1,0)$ 围成平行四边形面积 起点移到 $\overset{A}{(0,0)}, \overset{B}{(2,3)}\ \overset{C}{(3,3)}\ \overset{D}{(1,0)}$

即 $D(2,3)$ 和 $(1,0)$ $\left|\begin{vmatrix}2&1\\3&0\end{vmatrix}\right|=|3|=3$

例 to $A^n=0$ for some $n>1$, $(I-A)^{-1}$?

□ $(I-A)\underbrace{(I+A+A^2+\dots+A^{n-2}+A^{n-1})}_{(I-A)^{-1}}=I-A^n=I$

应用. let a,b positive, find area of $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1$

□ 从单位圆映射到 $\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1$ 全方位

$A\vec{u}=\vec{x}$, $\vec{u}=\begin{bmatrix}u_1\\u_2\end{bmatrix}$ $\vec{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}au_1\\bu_2\end{bmatrix}$

得到 $A=\begin{bmatrix}a&0\\0&b\end{bmatrix}$

$\text{area} = |\det A|\cdot \underset{\text{原先面积}}{A_{\text{circle}}} = ab\pi(1^2)=\pi ab$

例 Vandermonde $V=\begin{bmatrix}1&x_1&x_1^2&\cdots&x_1^{n-1}\\1&x_2&x_2^2&\cdots&x_2^{n-1}\\\vdots&&&&\vdots\\1&x_n&x_n^2&\cdots&x_n^{n-1}\end{bmatrix}$. $\vec{y}=(y_1,\dots,y_n)$ in $\mathbb{R}^n$, $\vec{c}=(c_0,\dots,c_{n-1})$ in $\mathbb{R}^n$

$V\vec{c}=\vec{y}$. $p(t)=c_0+c_1t+\dots+c_{n-1}t^{n-1}$

a) show $p(x_1)=y_1$, $p(x_2)=y_2$ … $p(x_n)=y_n$. $p(t)$ interpolating polynomial for $(x_1,y_1)\dots(x_n,y_n)$. 因 graph pass through points

b) 若 $x_1,\dots,x_n$ distinct. show col of V L.Ind.

□ a) $\begin{bmatrix}1&x_1&x_1^2&\cdots&x_1^{n-1}\\1&x_2&x_2^2&\cdots&x_2^{n-1}\\\vdots&&&&\vdots\\1&x_n&x_n^2&\cdots&x_n^{n-1}\end{bmatrix}\begin{bmatrix}c_0\\c_1\\\vdots\\c_{n-1}\end{bmatrix}=\begin{bmatrix}c_0+c_1x_1+\dots+c_{n-1}x_1^{n-1}\\\vdots\\c_0+c_1x_n+\dots+c_{n-1}x_n^{n-1}\end{bmatrix}=\begin{bmatrix}p(x_1)\\p(x_2)\\\vdots\\p(x_n)\end{bmatrix}=\begin{bmatrix}y_1\\\vdots\\y_n\end{bmatrix}$

b) $n-1$ 次多项式只有 $n-1$ 个解. 若 $\vec{y}=\vec{0}$, $p(x_1)\dots p(x_n)$ n 个多项式都为0. 若 L.dep. 则 $(c_0\dots c_{n-1})$ 有 nontrivial 解.

因 $x_1\dots x_n$ 不同，即 $n-1$ 次多项式有 n 个根，矛盾. ∴ 只有 $c_0=c_1=\dots=c_{n-1}=0$ 才使 n 个多项式为0. $\vec{c}$ 只有 trivial sol. 则此 c L.Ind

应用: 证 $\det\begin{bmatrix}A&0\\C&D\end{bmatrix}=\det A\cdot\det D=\det\begin{bmatrix}A&B\\0&D\end{bmatrix}$ (用于求 eigenvalues (三角块矩阵))

$\begin{bmatrix}A&0\\C&D\end{bmatrix}=\begin{bmatrix}A&0\\0&I\end{bmatrix}\begin{bmatrix}I&0\\C&D\end{bmatrix}$ ∴ $\det\begin{bmatrix}A&0\\C&D\end{bmatrix}=\det A\cdot\det D$

$\det\begin{bmatrix}A&B\\0&D\end{bmatrix}=\det\begin{bmatrix}A&B\\0&D\end{bmatrix}^T=\det\begin{bmatrix}A^T&0\\B^T&D^T\end{bmatrix}=\det A^T\det D^T=\det A\det D$

应用. 用代数推导证明 $\vec{u}\times\vec{w}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\u_1&u_2&u_3\\w_1&w_2&w_3\end{vmatrix}$

□ 考虑 $f\begin{bmatrix}x\\y\\z\end{bmatrix}=\det\begin{bmatrix}x&u_1&w_1\\y&u_2&w_2\\z&u_3&w_3\end{bmatrix}$ 可知这是线性的. 因 input $\mathbb{R}^3$ output $\mathbb{R}^1$, 则 $f(x)$ 可用 $A\vec{x}$

表示 $A_{1\times3}$ (其实就是个1×3矩阵的变换) $[A_1\ A_2\ A_3]\begin{bmatrix}x\\y\\z\end{bmatrix}$ 可看作点乘，几何可定义为 (A_1,A_2,A_3)

在 $(x\ y\ z)$ 上的投影乘 (x,y,z) 长度. 拉普拉斯展开 $\begin{cases}A_1x=(u_2w_3-u_3w_2)x\\A_2y=(u_3w_1-u_1w_3)y\\A_3z=(u_1w_2-u_2w_1)z\end{cases}$ 可知 (x,y,z) 换为 $(\vec{i},\vec{j},\vec{k})$ 可成为向量.

因此 $A\begin{bmatrix}\vec{i}\\\vec{j}\\\vec{k}\end{bmatrix}$ 为一个向量 定义为 $\vec{u}\times\vec{w}$ 就是在行列式写出

例 于 A $A^{adj}=\begin{bmatrix}3&5&4\\1&0&1\\2&1&1\end{bmatrix}$ find A^{-1}

□ $A^{-1}=\frac{\operatorname{adj}A}{\det A}=\frac{1}{6}\begin{bmatrix}-1&-1&5\\1&-5&1\\1&7&-5\end{bmatrix}$

$|A|=0$. 即线性 tranform 后面积变为0，被压缩，维度减少，但这样就有一个 input 对多个 output，不是一个合格可做的，如压缩后b在这压缩后的面/线上，有 sol，否则 $A\vec{x}=\vec{b}$ 无解 (压缩)

Invertible 3 & eigen 1

1. $A_{n\times n}$, $\det A^T = \det A$. 即 col operation 与 row operation 效果相同 P7 T3

2. Cramer's Rule. $A_{n\times n}$ 可逆. For any $\vec{b}$ in $\mathbb{R}^n$, unique $\vec{x}$ is $x_i = \dfrac{\det A_i(\vec{b})}{\det A}$, $i=1,2..n$. where $A_i(\vec{b})$ is A replacing ith col by $\vec{b}$

$A \cdot I_i(\vec{x}) = A[\vec{e}_1 \cdots \vec{x} \cdots \vec{e}_n] = [A\vec{e}_1 \cdots A\vec{x} \cdots A\vec{e}_n] = [\vec{a}_1 \cdots \vec{b} \cdots \vec{a}_n] = A_i(\vec{b})$

$\det A \cdot \det(I_i(\vec{x})) = \det A_i(\vec{b})$, $\det(I_i(\vec{x})) = \vec{x}$ 即出之，只对唯一解适用，无穷解求不出

3. A 可逆 (non-singular), $A^{-1} = \dfrac{\operatorname{adj} A}{\det A}$

jth col of A^{-1} is $\vec{x}$ that satisfies $A\vec{x} = \vec{e}_j$，而 A^{-1} 的 (i,j) entry 为 $x_i = \dfrac{\det A_i(\vec{e}_j)}{\det A}$

note $\det A_i(\vec{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}$. 因此 A^{-1} 的整体为 $A^{-1} = \dfrac{1}{\det A}\underbrace{\begin{bmatrix} C_{11} & C_{21} & C_{n1} \\ C_{12} & C_{22} & C_{n2} \\ C_{1n} & C_{2n} & C_{nn} \end{bmatrix}}_{\operatorname{adj} A}$

注意 $\operatorname{adj} A \cdot A = A \cdot \operatorname{adj} A = \det A \cdot I$ ($A^{-1}\det A = \operatorname{adj} A$, 同乘A)

4. eigenvector of $A_{n\times n}$ is $\boxed{\text{non zero}}$ vector $\vec{x}$ s.t. $A\vec{x} = \lambda\vec{x}$ for some scalar λ 参见 [24]

λ eigenvalue of $A_{n\times n}$ if $\exists$ nontrivial sol $\vec{x}$ for $A\vec{x} = \lambda\vec{x}$.

亦即, λ is eigenvalue iff $(A-\lambda I)\vec{x} = \vec{0}$ has nontrivial sol. The set of all sols is $\operatorname{Null}(A-\lambda I)$

This set a subspace of $\mathbb{R}^n$, eigenspace of A corresponding to λ

定理 1. eigenvalue of triangular matrix are entries on main diagonal 如 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ λ 只有1，只有一个

□ 因 triangular det 为 diagonal 相乘

定理 2. 0 is eigenvalue iff A not invertible. [5] T1

□ $A\vec{v} = 0\vec{v} = 0$ $\therefore A\vec{v}=0$ nontrivial sol $(\vec{v}\neq 0)$ $\therefore$ A 非 1-1, nullity $\neq 0$ 则非 full-rank

定理 3. DEF polynomial $f(t) \in P[\mathbb{F}]$ split over F iff $\exists\, a_1 .. a_n, c \in \mathbb{F}$ (不一定 distinct), $f(t) = c(t-a_1)..(t-a_n)$

eg. $\boxed{(t^2+1)}(t-2)$ 非 split over $\mathbb{R}$, 但 split over $\mathbb{C}$ $(t-i)(t+i)$

diagonalizable T characteristic polynomial split over $\mathbb{F}$ 即: 若有非 distinct 特征值，有 repeated. algebra multiplicity > 1

□ [25] T1a show $f(\lambda)$ 为 poly，且讨论和 ind. of basis. ∴对某 β, $[T]_\beta = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$，对其 $f(\lambda) = (\lambda_1 - t)..(\lambda_n - t) = (-1)^n (t-\lambda_1)..(t-\lambda_n)$

*即使 split 不一定 diagonalizable.

定理 4. (Cayley-Hamilton) $\forall$ square A satisfy characteristic func. 即若 $f(\lambda) = (-1)^n\lambda^n + k_{n-1}\lambda^{n-1} + .. + k_1\lambda + k_0$, 有 $(-1)^n A^n + k_{n-1}A^{n-1} + .. + k_1 A + k_0 I = 0$

deli得力

Date.

例1 $A=\begin{bmatrix}4&-1&6\\2&1&6\\2&-1&8\end{bmatrix}$ eigenvalue = 2, find basis for eigenspace for $\lambda=2$

解 $A-2I=\begin{pmatrix}2&-1&6\\2&-1&6\\2&-1&6\end{pmatrix}$, ∴ 对 $(A-2I)x=0$ $\left[\begin{array}{ccc|c}2&-1&6&0\\&&&0\\&&&0\end{array}\right]$ 故 $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=x_2\underbrace{\begin{bmatrix}1/2\\1\\0\end{bmatrix}}_{u}+x_3\underbrace{\begin{bmatrix}-3\\0\\1\end{bmatrix}}_{v}$ → basis

例2 $A_{n\times n}$ 的每一行 sum = s, 证特征值为 s; $B_{n\times n}$, 如每一列 sum = s, 特征值也为 s

解 1° $A\begin{bmatrix}1\\\vdots\\1\end{bmatrix}=\begin{bmatrix}a_{11}+\dots+a_{1n}\\\vdots\\a_{n1}+\dots+a_{nn}\end{bmatrix}=s\begin{bmatrix}1\\\vdots\\1\end{bmatrix}$ 2° A^T 与 A 同 eigval

应用. dynamic system. $A=\begin{bmatrix}.95&.03\\.05&.97\end{bmatrix}$, dynamic system defined by $\vec{x}_{k+1}=A\vec{x}_k$, $\vec{x}_0=\begin{bmatrix}.6\\.4\end{bmatrix}$

解 $\det\begin{bmatrix}.95-\lambda&.03\\.05&.97-\lambda\end{bmatrix}\Rightarrow\lambda=1/0.92$ 此时 $\vec{v}_1\begin{bmatrix}3\\5\end{bmatrix}$ $\vec{v}_2\begin{bmatrix}1\\-1\end{bmatrix}$ 为特征向量,

$\vec{x}_0=c_1\vec{v}_1+c_2\vec{v}_2$ ($\vec{v}_1,\vec{v}_2$ basis for $\mathbb{R}^2$) $=[\vec{v}_1\ \vec{v}_2]\begin{bmatrix}c_1\\c_2\end{bmatrix}$

整理 $[c_1\ c_2]=[\vec{v}_1\ \vec{v}_2]^{-1}\vec{x}_0=\begin{bmatrix}3&1\\5&-1\end{bmatrix}^{-1}\begin{bmatrix}.6\\.4\end{bmatrix}=\begin{bmatrix}.125\\.225\end{bmatrix}$

$\vec{x}_1=A\vec{x}_0=c_1A\vec{v}_1+c_2A\vec{v}_2=c_1\vec{v}_1+c_2(.92)\vec{v}_2$; $\vec{x}_2=A\vec{x}_1=c_1\vec{v}_1+c_2(.92)^2\vec{v}_2$

∴ $\vec{x}_k=c_1\vec{v}_1+c_2(.92)^k\vec{v}_2=(.125)\begin{bmatrix}3\\5\end{bmatrix}+(.225)(.92)^k\begin{bmatrix}1\\-1\end{bmatrix}$, 当 $k\to\infty$, $\vec{x}_k=.125\vec{v}_1$

A 可为 Markov chain 的 migration matrix, $\vec{x}_0$ 初始人口, $\vec{x}_k$ 为k年后人口, 可知会稳定下来

例3 $A=\begin{bmatrix}1&3&3\\-3&-5&-3\\3&3&1\end{bmatrix}$ 求 A^k; $B=\begin{bmatrix}2&4&3\\-4&-6&-3\\3&3&1\end{bmatrix}$; $C=\begin{bmatrix}5&-8&1\\0&0&7\\0&0&-2\end{bmatrix}$ 习4 $P=\begin{pmatrix}1&-1&-1\\-1&1&0\\1&0&1\end{pmatrix}$

解 1° $\det(A-\lambda I)=0$ $\begin{cases}\lambda_1=1 & v_1=[1\ -1\ 1]^T\\ 2 & v_2=[-1\ 1\ 0]^T\\ 2 & v_3=[-1\ 0\ 1]^T\end{cases}$ $\{v_{1,2,3}\}$ L.Ind. ∴ $A=PDP^{-1}$ ∴ $A^k=PD^kP^{-1}$, $D^k=\begin{pmatrix}1&&\\&(-2)^k&\\&&(-2)^k\end{pmatrix}$

2° B 有 $\lambda_1=1$, $v_1=\begin{pmatrix}1\\-1\\1\end{pmatrix}$ $\lambda_2=\lambda_3=-2$, $v_2=\begin{pmatrix}-1\\1\\0\end{pmatrix}$ 非 $\mathbb{R}^3$ basis, 不可 diagonalizable.

3° C 有 $\lambda=5,0,-2$ distinct, 可对角化

例4 $AP=PD$, D diagonal, then nonzero col of P must be eigenvector of A. ✓ 见前的证明

A has n L.Ind eigenvectors, 则 A^T 也是 ✓ 即A可对角化, $A^T=(PDP^{-1})^T=(P^T)^{-1}D^T(P)^T$, 可对角化, 有n个L.Ind.特征向

可逆与可对角化无太大关系. $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ 可对角化但不可逆; $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ 可逆但不可对角化

① A^{-1} 的特征值为 λ^{-1}

□ 令 $A^{-1}x = tx$, $t \neq 0$ (A 可逆)

则 因 $Ax = \lambda x$, $A^{-1}A^{-1}Ax = A^{-1}A^{-1}\lambda x$

$tx = A^{-1}x = \lambda A^{-1}tx = \lambda t^2 x$, $t = \lambda t^2$, $t = \frac{1}{\lambda}$

P

② 若 $A^2 = 0$, then 唯一的特征值 $= 0$

□ 令 $Ax = \lambda x$, $AAx = 0 = \lambda^2 x$, $\therefore \lambda = 0$

eigen 2

③ A^T 与 A 特征值相同 □ $f(\lambda) = \det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I)$

④. T^m 特征值 λ^m □ $T^m(v) = \lambda \cdots \lambda v$

⑤ $I_n - A$ eigval 为 $1-\lambda_1, \dots, 1-\lambda_n$

⑥ AB, BA eval 相同 …

若 AB eval $= \lambda$ $\lambda Bv = B \cdot (ABv) = (BA)(Bv)$ $\therefore$ BA eval $= \lambda$

2. λ is eigenvalue iff $\det(A - \lambda I) = 0$ (需有 free variable, i.e. $\det = 0$) (characteristic polynomial)

即 非 full-rank

推论见图.2

(如为 $(\lambda - 2)^2 = 0$ $\lambda_1 = \lambda_2 = 2$, 称为 multiplicity $= 2$)

3. $A, B_{n \times n}$. A is similar to B iff 可找可逆 P, s.t $P^{-1}AP = B$ or $A = PBP^{-1}$

change of basis 后 2 种 …

changing $A \to P^{-1}AP$ is similarity transformation

If A, B similar, they have same eigenvalue

同 algebraic/geometric multiplicity, 但 evec diff

*同 eigval 不一定 similar,

* share trace: $tr(B) = tr(P^{-1}AP) = tr(PP^{-1}A) = tr A$

□ 因 $B = P^{-1}AP$ $\det(B - \lambda I) = \det(P^{-1}AP - \lambda P^{-1}IP) = \det(P^{-1}(A - \lambda I)P) = \det P^{-1} \det(A - \lambda I) \det P = \det(A - \lambda I)$

4. 令 $E_\lambda = \{x \in V \mid T(x) = \lambda x\}$ eigenspace of λ, 可知 $E_\lambda \leq V$

即 $Nul(T - \lambda I)$!! 为 eigenvec 和 zero

非 eigenvec

T_1 对 $T: V \to V$, λ algebra multiplicity m, 有 $1 \leq$ geometric multiplicity $= \dim(E_\lambda) \leq m$

即 $f(t)$ characteristic poly 中的 $(t-\lambda)^m$

□ 令 E_λ basis $\{v_1 \cdots v_p\}$. 因 $E_\lambda \leq V$, extend 为 $\{v_1 \cdots v_p, \cdots v_n\}$ basis for V. 考虑 $[T]_\beta = A$. 由 3. T_3 应用

note 对前 p: $[T(v_i)]_\beta = \begin{bmatrix} 0 \\ \vdots \\ \lambda \\ \vdots \\ 0 \end{bmatrix}$ $\therefore A = \begin{bmatrix} \lambda I_p & B \\ 0 & C \end{bmatrix}$, 而 $f(t) = \det\begin{pmatrix} (\lambda - t)I_p & B \\ 0 & C - tI_{n-p} \end{pmatrix} = \det(\lambda - t)I_n \cdot \det(C - tI_{n-p})$

$= \lambda v_i$

$= (\lambda - t)^p \cdot \det$, 因 $(\lambda - t)^p$ 为 $f(t)$ 一个 factor, $\therefore m \geq p = \dim(E_\lambda)$

与 t 无关的 poly

T_2 令 $\lambda_1 \cdots \lambda_k$ distinct 特征值 for T, 令 S_i E_{λ_i} 的 L. ind. subset. $S = \bigcup_{i=1}^{k} S_i$ L. ind. subset of V

□ 令 $S_i = \{v_{i1} \cdots v_{in_i}\}$, 考虑 $a_{11}v_{11} + \dots + a_{1n_1}v_{1n_1} + \dots + a_{k1}v_{k1} + \dots + a_{kn_k}v_{kn_k} = 0 = \sum_{i=1}^{k}\sum_{j=1}^{n_i} a_{ij}v_{ij}$

但对各 i, 令 $w_i = \sum_{j=1}^{n_i} a_{ij}v_{ij} \in E_{\lambda_i}$. 这样 $w_1 + \dots + w_k = 0$, 表明 $w_i = 0$ (若不, 有 $w_{n_1} \dots w_{n_m}$ 非 0, 则对这 m 个 eig. vec. 因 T_2a, L. ind. $\therefore w_{n_i} = 0$ 矛盾). 而 S_i L. ind. $\therefore a_{ij} = 0$ for all j

对 $A_{n \times n}$, 如 distinct eigenvalue 为 $\lambda_1 \dots \lambda_p$: ($p < n$)

a. $1 \leq k \leq p$, $1 \leq \dim$ Eigenspace λ_k ≤ multiplicity of λ_k

b. A 可对角化 iff sum of dim of eigenspace $= n$, 这只有 i) 特征多项式 ~~…~~ 可被 split over ii) 每个 λ_k 的 dim ES $= \lambda_k$ 的 multiplicity 这二者同时满足才可能

c. 如 A 可对角化 且 β_k 是 dim ES λ_k 的 basis, 则 $\beta_1 \dots \beta_p$ forms an eigenvector basis for $\mathbb{R}^n$

构成所有向量

(split over: $t^2 - 1$ ✓ 因 $= (t+1)(t-1)$, 但 $t^2 + 1$ 不可, 可对角化 A 的特征多项式 splits, 但反之不对, 除非 b. ii))

***1

vs 6

Date. /

1. Subspace 可视为 set of all sols to a system of homogeneous linear equation → Null space ①
   也可视为 set of all L. Comb. of certain specified vectors → Col space ②

①: $A\vec{x}=\vec{0}$ $\vec{x}=0$ 肯定在; $\vec{u},\vec{v}$ in Null, $A(\vec{u}+\vec{v})=A\vec{u}+A\vec{v}=\vec{0}$; $cA\vec{u}=Ac\vec{u}=\vec{0}$

②: 见前几页, 令 $\vec{v}_1,\dots,\vec{v}_p$ span Col, weight 全为0则为$\vec{0}$; $\vec{u},\vec{v}$ 相加也可写为 L. Comb; $c\vec{u}$ 只是乘在 weight 上
(Col space 被 span 出)

从求 Nul A 的过程中, 得知 ① 亦转化为 ② (将 ① 中 sol један 写为 sol 表一些向量 L. Comb)

①② 从代性变换来看, ① 为 Kernal, ② 为 range

例 $\left\{\begin{bmatrix}a\\b\\c\end{bmatrix}: a+b+c=2\right\}$ 不为 subspace 无0 / 不为 homogeneous equation

$\left\{\begin{bmatrix}b-2d\\5+d\\b+3d\\d\end{bmatrix}: b,d\in\mathbb{R}\right\}$ × 无0 / 有5这个常数, 不能写为 L. Comb

$\left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix}: \begin{matrix}a-2b=4c\\2a=c+3d\end{matrix}\right\}$ ✓ 记为 Nul A, $A=\begin{bmatrix}1&-2&-4&0\\2&0&-1&-3\end{bmatrix}$

$\left\{\begin{bmatrix}c-6d\\d\\c\end{bmatrix}: c,d\in\mathbb{R}\right\}$ ✓ $\begin{bmatrix}1\\0\\1\end{bmatrix}c+\begin{bmatrix}-6\\1\\0\end{bmatrix}d$ 为 Col A, $A=\begin{bmatrix}1&-6\\0&1\\1&0\end{bmatrix}$

例 用 subspace 解释 为何 $\begin{cases}x_1-3x_2-3x_3=0\\-2x_1+4x_2+2x_3=0\\-x_1+5x_2+7x_3=0\end{cases}$ 已知解为 $x_1=3, x_2=2, x_3=-1$ 时, 解的倍数也为解

□ 可记为 Nul A, $A=\begin{bmatrix}1&-3&-3\\-2&4&2\\-1&5&7\end{bmatrix}$, $\vec{x}=\begin{bmatrix}3\\2\\-1\end{bmatrix}$ in Nul A, $c\vec{x}$ 也在 (Nul A 为 subspace)

同理, 用 Col Space 可知 $\begin{matrix}5x_1+x_2-3x_3=0\\-9x_1+2x_2+5x_3=1\\4x_1+x_2-6x_3=9\end{matrix}$ 有解时, $\begin{matrix}5x_1+x_2-3x_3=0\\-9x_1+2x_2+5x_3=5\\4x_1+x_2-6x_3=45\end{matrix}$ 也有解. ($5\begin{bmatrix}0\\1\\9\end{bmatrix}=\begin{bmatrix}0\\5\\45\end{bmatrix}$, 也在 Col A)

见P63

应用 $\mathbb{P}_n$ 为 polynominal space, $\vec{p}_n(t)=a_0+a_1t+\dots+a_nt^n$ 可证为 Vector Space, 在 statistical trend analysis 中用到

(lagrange Interpolation) 对 $c_0\dots c_n\in\mathbb{F}$ 且 distinct, DEF $f_i(x)=\dfrac{(x-c_0)\dots(x-c_{i-1})(x-c_{i+1})\dots(x-c_n)}{(c_i-c_0)\dots(c_i-c_{i-1})(c_i-c_{i+1})\dots(c_i-c_n)}$
$=\prod\limits_{k=0,k\neq i}^{n}\dfrac{x-c_k}{c_i-c_k}$, 证 $\{f_0\dots f_n\}$ 为 $\mathbb{P}_n(\mathbb{F})$ 的 basis

□ $\sum\limits_{i=0}^{n}a_if_i=$ zero poly, 即 $\forall x$ poly $=0$, $\therefore \sum\limits_{i=0}^{n}a_if_i(c_j)=0$ 对 $j=0\dots n$

但 $f_i(c_j)=\begin{cases}0 & i\neq j\\1 & i=j\end{cases}$, $\sum a_if_i(c_j)=a_j$ (只有当 $i=j$ 才为1) $\therefore$ 对各 $a_j=0$ 而 $\dim(\mathbb{P}_n)=n+1$, $\therefore \{f_0\dots f_n\}$ 这 n+1 L.Ind 为 basis

因此对 poly $p(x)\in\mathbb{P}_n(\mathbb{F})$, $p(x)=\sum a_if_i$, 而对 $j=0\dots n$, $p(c_j)=\sum a_if_i(c_j)=a_j$

$\therefore p=\sum\limits_{j=0}^{n}p(c_j)f_j$

可知 对 poly 若其对已知的 n+1 个 distinct c_i, $p(c_i)$ 都 $=0$.
则 $p=\sum 0f_i=0$, zero func!!

eg1 已知 poly 经过 (1 8) (2 5) (3 -4), 找出 poly 的拟合.

□ 对 poly p $\begin{cases}p(1)=8\\p(2)=5\\p(3)=-4\end{cases}$ 选 $\begin{matrix}c_0=1\\c_1=2\\c_2=3\end{matrix}$ 则 $f_0=\dfrac{(x-2)(x-3)}{(1-2)(1-3)}$ $p=8\cdot f_0+5f_1+4f_2=-3x^2+6x+5$

eg2. $c_0\dots c_n$ distinct, def $T:\mathbb{P}_n(\mathbb{F})\to\mathbb{F}^{n+1}$, $f\to\begin{pmatrix}f(c_0)\\\vdots\\f(c_n)\end{pmatrix}$, 证 T isomorphism

□ linerity 易证, 且 dim 相同 只经 show 1-1 令 $T(f)=0$, $\therefore$ 对各 c_i, $f(c_i)=0$ $\therefore f$ 为 zero vec.
四.4

应用 Verify $S=\{1,t,t^2,\dots,t^n\}$ is a basis for $\mathbb{P}_n$. (S 为 standard basis)

□ S span $\mathbb{P}_n$; $c_0\cdot1+c_1\cdot t+\dots+c_n\cdot t^n=\vec{0}(t)=0$ 代数上 多于 n个0点的多项式为0多项式
因此 $c_0=0=c_1=c_2\dots$, S L.Ind

***2

span 3

set 内 $\forall$ vector 可由 basis 唯一表示

1-(The spanning set theorem) 若 finite set $S=\{\vec{v}_1,\dots\vec{v}_p\}$, span(S) = Vec. Space V, 则 S 的某 subset 为 V 的 basis

□ 若 $S=\phi$ or $\{\vec{0}\}$, span(S) 和 $V=\{0\}$, S 视为 basis. 若 S 有非 0 项, 构造 basis by 步骤 ⓐ:

ⓐ: pick 非0 $\vec{v}_i$, note $\{\vec{v}_i\}$ L.Ind, 若可能, 找剩余的加上可构成 L.Ind. 的 $\vec{v}_i$ … 直到找到 $\beta=\{u_i\dots u_k\}$ L.Ind.

WTS β 为 basis, 即 β spans V: i) span(β) $\subseteq$ V 因 $\beta\subseteq S$, span(β) $\subseteq$ span(S) = V

ii) V $\subseteq$ span(β) 若 $v\in V$ 也在 β 内, 自然 $v=1\cdot v\in$ span(β); $v\in V-\beta$, 若 $v\notin S$, v 为 S 的 L.Comb 表示, 同 $v\in S$,

但: $\beta\cup\{v\}$ 为 L.dep (否则 ⓐ 会包进), 由 回 4 #2 推得 $v\in$ span(β)

eg: 若 $S=\{(\dots)\}$ span $\mathbb{R}^3$. pick $\begin{pmatrix}2\\-3\\5\end{pmatrix}$, $\begin{pmatrix}-4\\6\\-10\end{pmatrix}=2\begin{pmatrix}2\\-3\\5\end{pmatrix}$, 不 include, $\begin{pmatrix}1\\2\\1\end{pmatrix}$ 不是 include, … exclude, ∴ basis = $\{(\dots),(\dots),(\dots)\}$

or: 即找出 free variable, $[\dots]\to[\dots]$ 去 1st, 3rd, 4th 10

例 find basis for $x+2y+z=0$ in $\mathbb{R}^3$.

□ 可视为 sol for homogenous equation. Null A, $A=[1\ 2\ 1]$, $\{[\dots],[\dots]\}$

let $\vec{v}_1=\begin{bmatrix}0\\1\\1\end{bmatrix}$ $\vec{v}_2=\begin{bmatrix}0\\1\\1\end{bmatrix}$ $\vec{v}_3=\begin{bmatrix}0\\1\\0\end{bmatrix}$, let H be set of vectors in $\mathbb{R}^3$ whose 2nd 3rd entry are equal

Then $\forall\vec{x}$ in H is L.Comb of $\vec{v}_1\vec{v}_2\vec{v}_3$, $\begin{bmatrix}s\\t\\t\end{bmatrix}=s\begin{bmatrix}1\\0\\1\end{bmatrix}+(t-s)\begin{bmatrix}0\\1\\1\end{bmatrix}+s\begin{bmatrix}0\\1\\0\end{bmatrix}$

但 $\{\vec{v}_1\ \vec{v}_2\ \vec{v}_3\}$ 不是 basis for H, 因 $\vec{v}_1\ \vec{v}_2\ \vec{v}_3$ 甚至不在 H 里, 基是在 H 里的向量, 且 L.Ind 且 span

2、Coordinate (线性关系也保留) 回 2 例题见回 Ex5

比较 10 页, 对一个 given T, $[T]_\beta^\gamma$ 为 unique, 见左手, 而 10 的 Thm Show 对 given output, 有且只有一个 T st. basis → output

$\beta=\{\vec{b}_1,\dots\vec{b}_n\}$ 为 V 的 basis, 对 $x\in V$, 若 $x=\sum a_i\vec{b}_i$, 令 coord $[x]_\beta$ 为 $\begin{bmatrix}a_1\\\vdots\\a_n\end{bmatrix}$, 事实上,

对 $T: V\to\mathbb{F}^n$, $x\to[x]_\beta$ 不仅 linear 且 bijective

□ 对 $x=\sum\alpha_i b_i$, $y=\sum\beta_i b_i$, $T(x+ky)=\begin{bmatrix}\alpha_1+k\beta_1\\\vdots\\\alpha_n+k\beta_n\end{bmatrix}=T(x)+kT(y)$ ∴ linear; 1-1, 令 $T(x)=0=\begin{bmatrix}0\\\vdots\\0\end{bmatrix}$ ∴ $a_i=0$ ∴ $x=\sum 0b_i=0$ ∴ nullity = 0 1-1

onto: note dim V = n = dim F^n, 由回 dim 相同时 1-1 即⇒ onto

推广, 已用 col vector 表示 Vector space 的一 ele, 用 matrix 表示 其 L.T. v_n ($T(v_j)\in W$, 用 γ 表示) (于是) basis 的映射为像, 即可推得 …

对 $T: V\to W$, $\beta=\{v_1,\dots,v_n\}$, $\gamma=\{w_1,\dots,w_m\}$, 对 v_j (j: $1\to n$) → $\sum_{i=1}^m a_{ij}w_i$ (a_{ij} i: 1→m unique, 由 … T_1)

记 a_{ij} 组成的 matrix ($m\times n$) 为 $[T]_\beta^\gamma$, 对 $[x]_\beta$, $T([x]_\beta)$ 为 $[x]_\gamma$, $[T]_\beta^\gamma=[[T(v_1)]_\gamma,\dots,[T(v_n)]_\gamma]$ 合并 a b c 3

对 $x=\sum x_iv_i$, $T(x)=\sum x_iT(v_i)$, 同 $[T]_\beta^\gamma[x]_\beta=[[T(v_1)]_\gamma\dots]\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}=x_1[T(v_1)]_\gamma+\dots$ check L.Ind 时可用 $af_1(x)+bf_2(x)+cf_3(x)=0$ 取3个 x 值

另一证明见回 4: 总可用 matrix 研究 $\forall$ vector space, 这样证明如 $\mathbb{R}^n$

回 这种 coordinate mapping 是一种 isomorphism from V to $\mathbb{R}^n$. 虽然可以维度不同 但 "act alike"

例 对 $T:\mathbb{R}^3\to\mathbb{R}^2$, $(a_1,a_2,a_3)\to(2a_1+3a_2-a_3,\ a_1+a_3)$, 找 $[T]_\beta^\gamma$, ii) $T:\mathbb{R}^2\to\mathbb{R}^3$, $\begin{bmatrix}a_1\\a_2\end{bmatrix}\to\begin{bmatrix}a_1-a_2\\a_1\\2a_1+a_2\end{bmatrix}$, $\beta=\{[\dots],[\dots]\}$, $\gamma=\{[\dots],[\dots],[\dots]\}$

□ 由 $[[\dots]_\gamma,[\dots]_\gamma,[\dots]_\gamma]=\begin{bmatrix}2&3&-1\\1&0&1\end{bmatrix}$ ii) $[[\dots]_\gamma[\dots]_\gamma]=\begin{bmatrix}-1/3&-1/3\\2&3\\2/3&4/3\end{bmatrix}$

3-DEF $\mathcal{L}(V,W)$ 为 all L.T. from V to W, 可证其为 V.S. (addition, scalar multiply 为 of f+g 之类…)

Thm (矩阵表示法也 linear) 对 $T,U\in\mathcal{L}(V,W)$, $[T+U]_\beta^\gamma=[T]_\beta^\gamma+[U]_\beta^\gamma$, $[kT]_\beta^\gamma=k[T]_\beta^\gamma$

□ i) 有 a_{ij}, b_{ij} st. $T(v_j)=\sum_i a_{ij}w_i$, $U(v_j)=\sum_i b_{ij}w_i$, $(T+U)(v_j)=\sum_i(a_{ij}+b_{ij})w_i$ ∴ 左 $=a_{ij}+b_{ij}=$ 右

ii) $kT(v_j)=k\sum a_{ij}w_i$ ∴ 左 $=ka_{ij}=$ 右

***3

VS 7 & dim 3

Date. /

PEF见四 1

T_1 $\beta=\{s_1,\dots,s_n\}$ 为 V.S. 的 subset. β 为 basis $\Leftrightarrow$ $\forall v\in V$, v 为 β 的 unique L.comb..

□ $\Rightarrow$ 见 P2 P_{iv} $\Leftarrow$ i) β L.Ind. 对 $\vec{0}\in V$, 唯一表示即trivial表示, ∴ coeff $=0$

ii) span, trivial since $\forall v$ 有 L.comb

T_2 (Replacement) 对 V.S. V, 有 n-ele G, $\mathrm{span}(G)=V$, 有 m-ele L.Ind. L 且 $L\subseteq V$. 证 $\exists H\subseteq G$, $m\le n$ 且 n-m ele 且 $L\cup H$ span V 回 #6

□ induction on m: $m=0$, $L=\phi$, take $H=G$; 对 $m+1$, $L=\{v_1,\dots,v_{m+1}\}$, 因 $\{v_1,\dots,v_m\}$ L.Ind 由 inductive hypothesis, 有 n-m ele $\{u_1,\dots,u_{n-m}\}\subseteq G$ 且 $\{v_1,\dots,v_m\}\cup\{u_1,\dots,u_{n-m}\}$ ② span V, 即: 对 $v_{m+1}\in V$, $a_1v_1+\dots+a_mv_m+b_1u_1+\dots+b_{n-m}u_{n-m}=v_{m+1}$ ①

$n-m>0$ (若 $n-m=0$, $v_{m+1}\in\mathrm{span}(\{v_1,\dots,v_m\})$, $L=\{v_{m+1}\}\cup\{v_1,\dots,v_m\}$ L.dep, 矛盾) 回 4 #2, ∴ 同理 b_i 不全为 0, 令 $b_1\neq0$, $u_1=(-b_1^{-1}a_1)v_1+\dots$

即 n-m 至少 $\ge$ 1 对 n-m-1 ele $H=\{u_2,\dots,u_{n-m}\}$, WTS $\mathrm{span}(L\cup H)=V$. i) ($\subseteq$): $L\cup H\subseteq V$, span $\subseteq V$ P2 T_1

ii) ($\supseteq$): 由 ① $u_1\in$ span, ② $\subseteq$ span, ∴ span(②) $=V\subseteq$ LHS

P Replacement 的推论

a.) If vector space V basis $\beta=\{\vec{b}_1,\dots,\vec{b}_n\}$, then any set in V with more than n vectors is L.dep 回 #6

□ 法1. 若 L.Ind, 对大于 n 的 set 有 subset S, 有 n+1 ele. 则 S 也 L.Ind. 由 T_2, $n+1\le n$, 矛盾

法2. col > row, 有 free variable, L.dep 回 4 T#3

b) V 的 basis n vector, then $\forall$ basis has n vectors

□ 若 $\beta_1\beta_2$ 为 basis, β_1 n vector, β_2 m vectors, 若 $m>n$, 由 a) β_2 L.dep ∴ $m\le n$. 对 β_1 若 $n>m$, 同理 $n\le m$ ∴ $m=n$

c.) $\forall$ L.Ind. subset L 可被 expand 为 V 的 basis

□ 若 $\dim(V)=n$, $|L|>n$, 由 a) L dep; 若 $|L|\le n$, 有 $H\subseteq\beta$ st $L\cup H$ span V (T_2), $|L\cup H|$ at most n

d) (basis theorem) $\dim(V)=n$, any V 的 spanning set at least n ele, 若 n ele 则为 V basis;

any L.Ind L 有 n ele 为 V 的 basis b)

* n个 vector in $\mathbb{R}^n$ span $\mathbb{R}^n$ 也为 basis

若不, 即有 v_k L.dep, ∴ $\mathrm{span}\{v_i\}=\mathrm{span}\{v_i\}/v_k$ 但 n-1 vec 不可能 span $\mathbb{R}^n$

□ i) let G spanning set. $H\subseteq G$ 为 basis (***2.1 subset), $|H|=n$. 由 $|G|\ge|H|$, $|G|\ge n$. 若 $|G|=n$, $G=H$

ii) 回 c) 中 β 换为 spanning set, 有 $n-n=0$ subset H. st. $L\cup H=L$ span V. ∴ L span 且 L.Ind. L 为 basis

T_3 $W\le V\Rightarrow\dim(W)\le\dim(V)$, 且 basis for W 可 extend 为 basis for V 也即: 等号当 $W=V$ 成立

□ i) let $\dim(V)=n$. if $W=\{0\}$, $\dim(W)=0\le n$; 若 if $\exists x_1\in W$ 且 $x_1\neq0$, 由 ***2.1 步骤 ④ 构建

L.Ind. $\{x_1,\dots,x_k\}$, 由 $P_{a)}$ $k\le n$ 否则 L.Dep. 余下的 $x_i\cup\{x_1,\dots,x_k\}$ L.dep, ∴ $x_i\in\mathrm{span}(x_1,\dots,x_k)$, ∴ $\mathrm{span}(x_1\dots x_k)=W$, basis

ii) 由 $P_{b)}$

例 1 对 $\{u_1\dots u_s\}$, $\{v_1\dots v_t\}$ $s>t$, u_i 为 V 的 L.comb. 证 U L.dep

□ $\mathrm{span}\,U\subseteq\mathrm{span}\,V$, 且 span U, span V 为 vec. space (P2 T_1) ∴ $\mathrm{span}\,U\le\mathrm{span}\,V$, 据 T_3 $\dim(\langle U\rangle)\le\dim(\langle V\rangle)\le t<s$ 据 $P_{d)}$

~~回 T#3 的结论, 但因 V 不一定 L.Ind, 不可写为 $[[u_1]_V\dots]$ 然后说 col # > row #~~ $\dim(\langle U\rangle)<s$, 必有 ele 可被 L.comb. ∴ L.dep

例 2 $\{\vec{v}_1\dots\vec{v}_p\}$ span V, $\dim V\le p$ ✓; $\{\vec{v}_1\dots\vec{v}_p\}$ L.Ind in V, $\dim V\ge p$ ✓

$\{\vec{v}_1\dots\vec{v}_p\}$ L.dep in V, $\dim V\le p$ ✗; If any set of p element in V fail to span V, $\dim V>p$ ✓

与 L.d 无关. 如全是 0

例 3 $\alpha_1\dots\alpha_3$ L.dep, $\alpha_2\dots\alpha_4$ L.Ind; 证 $\alpha_1\in\langle\alpha_2\,\alpha_3\rangle$, $\alpha_4\notin\langle\alpha_1\,\alpha_2\,\alpha_3\rangle$

□ 有不全为 0 c_i st. $c_1\alpha_1+\dots+c_3\alpha_3=0$, $c_1\neq0$, 否则 $\alpha_2\,\alpha_3$ L.dep ∴ $\alpha_1=\frac{-c_2}{c_1}\alpha_2+\frac{-c_3}{c_1}\alpha_3$ 方法类似 回 T#2

若 $\alpha_4\in\mathrm{span}(\alpha_1\dots\alpha_3)=\mathrm{span}(\alpha_2\,\alpha_3)$ (P2 $P_{v)}$), 不 L.Ind.

***4

Rank & basis

Rank 1

T_3 A_{nm} $rk = 1 \Leftrightarrow \exists\ u \in \mathbb{R}^n\ v \in \mathbb{R}^m,\ uv^T = A$

□ $\Leftarrow$ 对 $x \in \mathbb{R}^m$, $Ax = uv^Tx = (v^Tx)u$, 即 map $\forall x$ to span$\{u\}$

$\Rightarrow$ $\because \forall x \in \mathbb{R}^m$ 有 $Ax = k\vec{u}$, for some $\vec{u} \in \mathbb{R}^m$ $\therefore$ 对 x = natural basis e_i 成立, $\therefore A = (k_1u, k_2u \ldots k_mu) = u(k_1 \ldots k_m) = uv^T$

应用 40 equation 42 variable 的 homogeneous system 有 2 个解 (不是 multiply), 其他均可用此 2 解表达, 证 对 nonhomogeneous system with same coefficient 有解.

□ $A_{40 \times 42}$ 因 $Ax = 0$ 有 2 Lind sol, nullity = 2, $rk\,A = 40$. $\therefore Col\,A = \mathbb{R}^{40}$. $\forall b$, $Ax = b$ 有 sol

T_4 A $rk = k$, 则 $A = \sum^k$ rk-1-matrix □ $A = U\Sigma V^T$, 令 $\Sigma = \sum \Sigma_i$, $\Sigma_i = \mathrm{diag}(0 \ldots \sigma_i\ 0 \ldots 0)$ $\therefore A = \sum^k U\Sigma_i V^T$

T_5 (Eckart-Young) 对 $rk = r$ A_{mn} 令 $rk = k$ B_{mn} 对 $k < r$, A 的 rk-k approxi. $\hat{A}(k) = \sum_{i=1}^{k} \sigma_i \vec{u}_i \vec{v}_i^T$, 满足

1) $\hat{A}(k) = \underset{rk\,B = k}{\mathrm{argmin}}\ \|A - B\|_2$ 即 rk A approxi 在 L_2 matrix norm 下最优

ii) $\|A - \hat{A}(k)\|_2 = \sigma_{k+1}$

应用: Rank 1 矩阵在算法/singular value decomposition 有用

Let $A_{2\times3}$, rank $A = 1$ $\vec{u}_1$ first col of A, $\vec{u}_1 \neq 0$. 证 必有 $\vec{v}$ in $\mathbb{R}^3$ s.t. $A = \vec{u}\vec{v}^T$

□ rank $A = 1$, $\vec{u}_1$ 为 basis for Col A, $\vec{u}_2 = x\vec{u}_1$, $\vec{u}_3 = y\vec{u}_2$, $\vec{v} = \begin{bmatrix}1\\x\\y\end{bmatrix}$

(若 $\vec{u}_1 = \vec{0}$, 而 $\vec{u}_2$ 为 basis, 令 $\vec{u}_3 = x\vec{u}_2$, $\vec{v} = \begin{bmatrix}0\\1\\x\end{bmatrix}$; $\vec{u}_1, \vec{u}_2 = \vec{0}$, $\vec{u}_3$ 为 basis, $\vec{v} = \begin{bmatrix}0\\0\\1\end{bmatrix}$)

PP $A_{m\times n}$ 有 rank 1 iff outer product 存在, i.e. $A = \vec{u}\vec{v}^T$ for $\vec{u}$ in $\mathbb{R}^m$, $\vec{v}$ in $\mathbb{R}^n$

$x = a\beta_1 + b\beta_2 + c\beta_3$ 新的 x 坐标为 $ar_1 + br_2 + cr_3$

新 basis $r_1\ r_2\ r_3$ $(r_1, r_2, r_3) = Q\begin{pmatrix}a\\b\\c\end{pmatrix}$

矩阵本身也只是 identity map ← 矩阵不会对 value 做什么, 而是只是转换为另一 basis 的表示

3. Basis change 若 given V, 有 $\beta = \{x_1 \ldots x_n\}$, $\beta' = \{x_1' \ldots x_n'\}$. 为了找出互换 coord 的 matrix, def $T: x_i' \to \sum_{i=1}^n Q_{ij} x_i$

T_1 let β, β' two basis for V, let $Q = [I_V]_{\beta'}^{\beta}$, 则 Q invertible 且 $\forall v \in V$, $[v]_\beta = \underset{\beta' \to \beta}{Q}[v]_{\beta'}$

□ note I_V bijection, thus $\exists$ inverse. and $[v]_\beta = [I_V(v)]_\beta = [I_V]_{\beta'}^{\beta}[v]_{\beta'}$. 因.1 充分, Q invertible 因 4

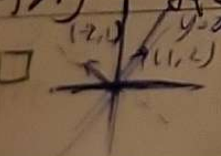
之所以直接 def Q 为 I_V 的 matrix 形式, 因 想实现 对于 v, $[v]_{\beta'} \to [v]_\beta$, 而 $[v]_\beta = [T(v)]_\beta = Q[v]_{\beta'}$ 只有 identity 才能做到. 因.4

若已有 $T: V \to V$ 在 β 中, 想知道 T 在另一 basis 的表达式 expression: 因.1 末

T_{2a} $[T]_{\beta'} = \underset{\beta' \to \beta}{Q^{-1}}[T]_\beta \underset{\beta' \to \beta}{Q}$ 所以 右→左读, β' 表示 $\to \beta \to \beta'$ □ $Q[T]_{\beta'} = [I_V]_{\beta'}^{\beta}[T]_{\beta'} = [I_V T]_{\beta'}^{\beta} = [T \cdot I_V]_{\beta'}^{\beta} = [T]_\beta Q$

两者 similar

例) 找出 沿 $y = 2x$ 对称 相对的 linear T 矩阵



□ 找 $(2,1)$ 去 (1,2) 正交, note $(1,2)$ $(-2,1)$ linear Indpdnt. 且 $T(1\,2) \to (1\,2)$ $T(-2,1) \to (2,-1)$ 它们为 β' $[T]_{\beta'} = \underset{\beta \to \beta}{Q^{-1}}[T]_\beta \underset{\beta' \to \beta}{Q}$

$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = Q^{-1}?\begin{bmatrix}1 & -2\\2 & 1\end{bmatrix}$

$? = \begin{bmatrix}-0.6 & 0.8\\0.8 & 0.6\end{bmatrix}$ $\therefore T(a\,b) = ?\begin{pmatrix}a\\b\end{pmatrix}$

***5 basis 2 & Decomposition — 即有 $\sum c_i v_i' = v_i$ 之可解 $[v_1' \cdots v_n'][c] = v_i$, $[v_i]_{\beta'}$ 回.1

若 β, β' 已经给出, 可用 ***9 2.(3)1.

$\beta = \{v_1 \cdots v_n\}$ $\beta' = \{v_1' \cdots v_n'\}$ OR $Q_{\beta\to\beta'} = [[v_1]_{\beta'}, \dots] = [v_1' \cdots v_n']^{-1}[v_1 \cdots v_n]$ 已知 $P = [I_W]_{r}^{r'}$, $P^{-1} = [I_W^{-1}]_{r'}^{r}$

T_{2d} $[x]_{\beta'} = [v_1' \cdots v_n']^{-1}[v_1 \cdots v_n][x]_\beta$ □ $x = \sum x_i v_i = [v_1 \cdots v_n][x]_\beta = \sum x_i' v_i' = [v_1' \cdots v_n'][x]_{\beta'}$

T_{2b} $T: V \to W$, 则 $[T]_{\beta'}^{r'} = \underset{r\to r'}{P^{-1}}[T]_\beta^r \underset{\beta'\to\beta}{Q}$ □ $[T]_{\beta'}^{r'} = [I_W T I_V]_{\beta'}^{r'} = [I_W][T][I_V]_{\beta'}^{\beta} = P^{-1}[T]_\beta^r Q$

note 对 $T: V \to V$ 有 $[T]_{\beta'}^{r} = \underset{\beta\to r}{P}[T]_\beta$

T_{2c} $\underset{\beta\to r}{R}\,\underset{\alpha\to\beta}{Q} = \underset{\alpha\to r}{S}$ □ $[I_V]_\beta^r [I_V]_\alpha^\beta = [I_V]_\alpha^r = \underset{\alpha\to r}{S}$

应用 LU Factorization (pd 则为 Cholesky, Option $P_7.1$)

1. If $A_{m\times n}$ 可在无 row interchange 时化为 echelon form, 则

$\underset{mn}{A} = \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \\ & & 1 \end{pmatrix} \begin{pmatrix} \square & & \\ 0 & \ddots & \\ & & \square \end{pmatrix}_A$ 可不为 U 若 diagonal = zero, A 非可逆

$L_{m\times m}$ lower △ $U_{m\times n}$ echelon form

□ 即 A 可由 elementary row operation reduce 为 U, $E_p E_{p-1} \cdots E_1 A = U$, $\therefore A = \underbrace{(E_p \cdots E_1)^{-1}}_{L} U$ inverse 仍为 上△ 回.P_1 b) product 仍为 上△ 仍为上△

P i) LU not unique, 但若 require $\mathrm{diag}(L) = (1 \dots 1)$, unique □ 若 $D \neq I_n$, $A = LU = \underline{LD}\,\underline{D^{-1}U}$ 新 U, 上△; 新 L, lower △

若 LD lower △, D lower △; 可 $D^{-1}U$ 仍是上△ $\therefore D$ 只能 diag matrix 由 $D \neq I_n$, 可 $D_{ii} \neq 1$, 但 $(LD)_{ii} = \sum L_{ij} D_{ji} = D_{ii} \neq 1$ (diag L 为 (1…1)) 可新 L 未 diag = (1…1), ∴不存在 D

ii) $\det \underset{mn}{A} = \prod_{i=1}^{n} U_{ii}$ □ $\det L \det U = 1 \cdot \prod U_{ii}$ 对于 △ 等于 diag 回.1

iii) solve Sys Eq $Ax = b$ 方法: i) solve $Ly = b$ 用 forward/backward sub 因 L△, ii) solve $Ux = y$

iv) 非 ∀ matrix 有 LU $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. 若有 LU 可能 at least 1 个 L, U 非可逆

v) (PLU) 若要调换 / pivot 有 $PA = LU$ P46.1

例1. $A = \begin{pmatrix} 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \end{pmatrix}$ max|row| 13, 18, 6, 12 scale factor; 1 ÷ scale vec $3/13, 6/18, 6/6, 12/12$; $P_1 = \begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix}$ $\Rightarrow \begin{pmatrix} 6 & -2 & 2 & 4 \\ -6 & 4 & 1 & -18 \\ 3 & -13 & 9 & 3 \\ 12 & -8 & 6 & 10 \end{pmatrix}$ $E_1 = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ -1/2 & & 1 & \\ -2 & & & 1 \end{pmatrix}$ $\begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & 2 & 3 & -14 \\ 0 & -12 & 8 & 1 \\ 0 & -4 & 2 & 2 \end{pmatrix}$ 18, 13, 12 scale factor 保持 不是 updated

右 $2/18, 12/13, 4/12$ 3rd max

$P_2 = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 6 & -2 & 2 & 4 \\ & -12 & 8 & 1 \\ & 2 & 3 & -14 \\ & 4 & 2 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & 1/6 & 1 & \\ & -1/3 & & 1 \end{pmatrix}$ $\begin{pmatrix} 6 & -2 & 2 & 4 \\ & -12 & 8 & 1 \\ & 0 & 13/3 & -83/6 \\ & 0 & -2/3 & 5/3 \end{pmatrix}$ 18, 12 3rd max, 故 $P_3 = I$

$E_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 2/13 & 1 \end{pmatrix}$ $E_3 P_3 E_2 P_2 E_1 P_1 A = U = \begin{pmatrix} 6 & -2 & 2 & 4 \\ & -12 & 8 & 1 \\ & & 13/3 & -83/6 \\ & & & -6/13 \end{pmatrix}$

NOTE

swap along 消元 1 不 swap first (permute) 再消元. 可 find new elementary op $\tilde{E}$ st. $\tilde{E}_3 \tilde{E}_2 \tilde{E}_1 P_3 P_2 P_1 A = U$, 有 $\begin{cases} \tilde{E}_1 = P_3 P_2 \underbrace{E_1 P_2^{-1}}P_3^{-1} = P_3 P_2 E_1 P_2^T P_3^T \\ \tilde{E}_2 = P_3 E_2 P_3^T \\ \tilde{E}_3 = E_3 \end{cases}$ undo $P_2 P_3$ (先剥 $P_1 A$) 再用 E_1, 再 redo P_{23} 回用 P_{23} Pset P(ii)

$\therefore$ PLU 有 $\begin{cases} P = P_3 P_2 P_1 \\ L = \tilde{E}_1^{-1} \tilde{E}_2^{-1} \tilde{E}_3^{-1} \\ U = \tilde{E}_3 \tilde{E}_2 \tilde{E}_1 P A \end{cases}$

例2 LU on $\begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & & & \\ & 3 & & \cdots & \\ & -9 & & & \\ & 12 & & & \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ & 3 & 1 & 2 & -3 \\ & & & 2 & 1 \\ & & & 4 & 7 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & -1 & 5 & -2 \\ & 3 & 1 & 2 & -3 \\ & & & 2 & 1 \\ & & & & 5 \end{pmatrix} = U$

化为对角 = 1 $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{pmatrix}$

例3 LU 快速解 A^{-1}; find $Ax = e_1 \dots Ax = e_n$ 即化为解 sys eq

* SGD - UV decompose, 令 $A = UV$, SGD $\min \|UV - A\|_2^2$ where U, V learned param

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Date. /

应用 Leontief Input-Output Model

1. econ divid in n sectors that produce goods & services $\vec{x}$ in $\mathbb{R}^n$ production vector
→ production level (output)

open sector consume goods & services, $\vec{d}$ final demand vector, goods & services demands by open sector

producer 为了生产,需要 input, intermediate demand

Leontief 想找到 $\vec{x}$ s.t. intermediate demand $+ \vec{d} = \vec{x}$, 完全平衡

为此,假设 for each sector, ∃ unit consumption vector $(\vec{c})$ in $\mathbb{R}^n$, input needed per unit of output

可知 intermediate demand $= x_1\vec{c}_1 + x_2\vec{c}_2 + \dots = C\vec{x}$, $C = [\vec{c}_1\ \vec{c}_2\ \dots]$, $\vec{c}_1 x_1$ 为 sector 1 为了生产 x_1 unit 单位需要消耗的 input

这样 $\vec{x} = C\vec{x} + \vec{d} \Rightarrow (I - C)\vec{x} = \vec{d}$

2. 例如 econ 有 manufacturing, Agriculture, Services 三个 sectors.

Purchased from	M	A	S
	Inputs Consumed per Unit of Output		
M	.50	.40	.20
A	.20	.30	.10
S	.10	.10	.30

如 M 要生产 100 units, 需要

$100\vec{c}_1 = 100\begin{bmatrix} .50 \\ .20 \\ .10 \end{bmatrix} = \begin{bmatrix} 50 \\ 20 \\ 10 \end{bmatrix}$

而 $C = \begin{bmatrix} .50 & .40 & .20 \\ .20 & .30 & .10 \\ .10 & .10 & .30 \end{bmatrix}$

如 $\vec{d}$ 为 50 M, 30 A, 20 S, $\vec{x}$ 可用以下找到

$I - C = \begin{bmatrix} .5 & -.4 & -.2 \\ -.2 & .7 & -.1 \\ -.1 & -.1 & .7 \end{bmatrix}$ $\quad [I-C\ \ \vec{d}] \sim [I\ \ \vec{x}]$ $\quad \vec{x} = \begin{bmatrix} 226 \\ 119 \\ 78 \end{bmatrix}$, 生产水平

3. 实际上 $\vec{x}$ 必须 economically feasible, 即无负数 entries; col sums of $C < 1$, 因 sector 应 input < output

则如 $C, \vec{d}$ 无 negative entries 且 col sums of $C < 1$, $I - C$ 有逆.

比如 $\vec{d}$ 确定, industries 使 $\vec{x}$ 生产水平正好为 $\vec{d}$, 即生产出的正好被全消耗, 则

intermediate demand 为 $C\vec{d}$, 而为了 $C\vec{d}$, 需要 intermediate demand $C(C\vec{d})$ ...

$\vec{x} = \vec{d} + C\vec{d} + C^2\vec{d} \dots = (I + C + C^2 + \dots)\vec{d}$, 根据 $(I-C)(I + C + C^2 + \dots + C^m) = I - C^{m+1}$ ①

If col sum of $C < 1$, $C^m \to 0$ $\therefore I - C^{m+1} \to I$ ...,

因此 由① $(I - C)^{-1} = I + C + C^2 + \dots + C^m$

4. $\vec{d} \to \vec{d} + \Delta\vec{d}$, $\vec{x} \to \vec{x} + \Delta\vec{x}$, 即 production level change with demand $(\vec{d} + \Delta\vec{d} = (I-C)\vec{x} + (I-C)\Delta\vec{x})$

而如 $\Delta\vec{d}$ 只有一个 sector 提高需求, $[I-C]^{-1}\begin{bmatrix} 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} (I-C)^{-1} \text{的} \\ \text{第一列} \end{bmatrix} = \vec{x}$, 即 哪怕只一个 sector 提高, 整个 sector 都会提高生产力

deli得力

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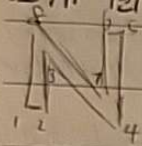
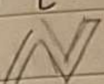
note 此为 $T: \mathbb{R}^3 \to \mathbb{R}^3$，但有很好的性质，对满足(单)的 $\mathbb{R}^3$ ele，有单纯的 map，且我们也只关心其余的 $\mathbb{R}^2$

Date. /

应用 图像学

1.  $D = \left[\begin{array}{cccc|cccc} 0 & .5 & .5 & 6 & 6 & 5.5 & 5.5 & 0 \\ 0 & 0 & 6.42 & 0 & 8 & 8 & 1.58 & 8 \end{array}\right]$ 被 $A = \begin{bmatrix} 1 & .25 \\ 0 & 1 \end{bmatrix}$ shear transform 后为 

(各个点的坐标)

2. 但是 matrix multiplication 也即 linear Transformation 不能直接平移，因此引入 homogeneous coordinate，(x,y) in $\mathbb{R}^2$ 可视为 $(x,y,1)$ in $\mathbb{R}^3$，这样 $(x,y) \mapsto (x+h, y+k)$ 可用 $(x,y,1) \to (x+h, y+k, 1)$

standard matrix for L.T. 为 $\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$ $\left(\begin{bmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+h \\ y+k \\ 1 \end{bmatrix} \right)$

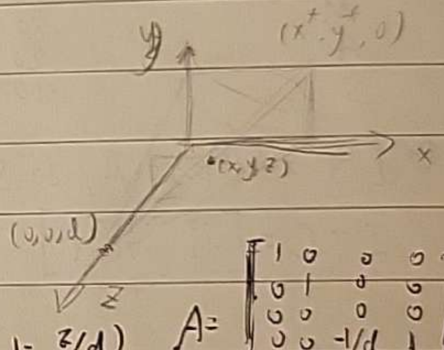
而非平移，即 L.T in $\mathbb{R}^2$ 在 $\mathbb{R}^3$ 中记为 $\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$，A 为 $\mathbb{R}^2$ 的 L.T. 的 standard matrix

如 $\begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ counterclockwise rotation $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ reflexion $y=x$ $\begin{bmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Scale x by s, y 为 t

3. 3D 同理，可写为 (X, Y, Z, H) if $H \neq 0$, and $x = \frac{X}{H}$, $y = \frac{Y}{H}$, $z = \frac{Z}{H}$ 为 homogeneous for (x,y,z)

eg. $(10,-6,14,2)$ 与 $(-15,9,-21,-3)$ 都为 $(5,-3,7)$ 的 homogeneous

4. 3D 的投影到 2D 平面



从 $(0,0,d)$ 看，将 (x,y,z) 投到 viewing plane $(x^*, y^*, 0)$

可知 $\frac{x^*}{d} = \frac{x}{d-z}$，$x^* = \frac{x}{1-\frac{z}{d}}$，同理 $y^* = \frac{y}{1-\frac{z}{d}}$

即 $(x,y,z,1) \mapsto \left(\frac{x}{1-z/d}, \frac{y}{1-z/d}, 0, 1\right) = (x, y, 0, 1-z/d)$ $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix}$ 己

即 $\{(x,y,0,t)\} \oplus \{(0,0,z,0)\} = \mathbb{R}^4$，$T: \mathbb{R}^4 \to \mathbb{R}^4$ 将其 $(x,y,z,1) \to (x,y,0,1-z/d)$

而此应用 … $(x\,y\,z\,1)$ …

应用 difference equation

1. vector ~~sp~~ space $\mathbb{S}$ of discrete-time signal 中有 signal，function sampled at discrete time intervals

有 form $(y_0, y_1, \dots)$ 每个 y_i 为采样，记为 $\{y_k\}$ $y_k = (-1)^k$ $\{y_k\} = (1,-1,1,-1 \dots)$

L.ind for 3 signals 为 $c_1 u_k + c_2 v_k + c_3 w_k = 0$ for all k 有 $c_1 = c_2 = c_3$，k 可为 $\forall \geq 0$

无论是否 L.ind，上式 k 也可写为 $k+1$, $k+2$，则 $\begin{bmatrix} u_k & v_k & w_k \\ u_{k+1} & v_{k+1} & w_{k+1} \\ u_{k+2} & v_{k+2} & w_{k+2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

左 matrix 为 Casorati matrix of signal，det 为 Casoratian

If Casorati matrix 可逆 for at least one value of k，则 $c_1 = c_2 = c_3 = 0$，$\{u_k\}, \{v_k\}, \{w_k\}$ L.ind

2. given signal $\{z_k\}$，$a_0 y_{k+n} + a_1 y_{k+n-1} + \dots + a_{n-1} y_{k+1} + a_n y_k = z_k$ for all k

为 linear difference / linear recurrence equation of order n. If $\{z_k\}$ zero sequence, homogeneous

在 digital signal processing，linear difference equation 为 linear filter，$a_0, \dots, a_n$ 为 filter coefficients

$\{y_k\}$ input，$\{z_k\}$ output，$\{z_k\}$ 为 0 则称被 filtered out，$\{y_k\}$ 即为 sol

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Date. /

3. Sol for homogeneous difference equation has form $y_k = r^k$ for some r.

eg. Find $y_{k+3} - 2y_{k+2} - 5y_{k+1} + 6y_k = 0 \quad \forall k$.

代入 r^k $\quad r^k(r-1)(r+2)(r-3)=0$。 $\quad 1^k, (-2)^k, 3^k$ 都为 sol. 比如 3^k,

$3^{k+3} - 2\cdot 3^{k+2} - 5\cdot 3^{k+1} + 6\cdot 3^k$

综上，r^k 为 sol iff r is root of auxiliary equation $r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n = 0$ $\quad = 3^k(27-18-15+6)=0$

4. given $a_1 \dots a_n$, mapping $T: \mathbb{S} \to \mathbb{S}$, $\{y_k\} \mapsto \{w_k\}$ by $w_k = y_{k+n} + a_1 y_{k+n-1} + \dots + a_{n-1} y_{k+1} + a_n y_k$

可证 T linear, 则 sol for homogeneous ($\{w_k\} = \{0\}$) 即 T 的 kernal

1° If $a_n \neq 0$, and $\{z_k\}$ given, $y_{k+n} + a_1 y_{k+n-1} + \dots + a_{n-1} y_{k+1} + a_n y_k = z_k$ for all k has unique sol if $y_0 \dots y_{n-1}$ specified.

□ define $y_n = z_0 - [a_1 y_{n-1} + \dots + a_{n-1} y_1 + a_n y_0]$, 并且逐步定义 define $y_{n+1}, y_{n+2} \dots$

2° set H of all sols of $y_{k+n} + a_1 y_{k+n-1} + \dots + a_{n-1} y_{k+1} + a_n y_k = 0$ for all k is n-dim Vector

□ H 为 $\mathbb{S}$ subspace 因可视为 T 的 kernal. $\forall \{y_k\}$ in H, $F: H \to \mathbb{R}^n$, $\{y_k\} \to (y_0, y_1, \dots, y_{n-1})$

可证 F linear. T. 由 1°, ∃ unique $\{y_k\}$ in H st. $F(\{y_k\}) = (y_0, y_1, \dots y_{n-1})$, F one to one, onto

F isomorphism, 因 $\dim H = \dim \mathbb{R}^n = n$ 见上上页黑示例例)

(3°) find basis for set of all sols for $y_{k+3} - 2y_{k+2} - 5y_{k+1} + 6y_k = 0 \quad \forall k$

由本页 eg 知 $1^k, (-2)^k, 3^k$ 为解，且由上页 Cartasian, $k=0$ $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ 可逆, L.ind.

any L.ind of 3 vectors in 3-dim space is basis.

~~***~~ 9.2
basis 3

Date. /

例1. 算 $Q_{\beta\to\beta'}$ i) $v_1' = v_1 + 3v_2 - 4v_3$, $v_2' = 2v_1 - v_2 + 5v_3$, $v_3' = 4v_1 + 5v_2 + 3v_3$ ii) $v_1 = v_1' + v_2' + 3v_3'$, $v_2 = 2v_1' - v_2' + 4v_3'$, $v_3 = 3v_1' + 5v_3'$

法1
□ i) $Q = [I_V]_\beta^{\beta'} = (R_{\beta'\to\beta}^{-1}) = \begin{pmatrix} 1 & 2 & 4 \\ 3 & -1 & 5 \\ -4 & 5 & 3 \end{pmatrix}^{-1}$ ii) $Q = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 3 & 4 & 5 \end{pmatrix}$

法2 i) $Q = [v_1' \cdots v_n']^{-1}[v_1 \cdots v_n]$, 而 $\begin{bmatrix} 1 & 3 & -4 \\ 2 & -1 & 5 \\ 4 & 5 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1' \\ \vdots \\ v_n' \end{bmatrix} \xRightarrow{\text{转置}} [v_1 \cdots v_n]\begin{bmatrix} 1 & 3 & -4 \\ & \vdots & \end{bmatrix}^T = [v_1' \cdots v_n']$

ii) $\begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 4 \\ 3 & 0 & 5 \end{bmatrix}\begin{bmatrix} v_1' \\ \vdots \\ v_n' \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

证1. β, r orthonormal basis for $\mathbb{R}^n$, 证 $Q_{\beta\to r}$ orthogonal
basis 即 T_{2c} B 的 β

□ 右 $Q_{\beta\to r} \; Q_{e\to\beta} = Q_{e\to r}$ T   ∴ $Q = TB^{-1}$ 由 P33. βv_i)

但 一般 change basis matrix 不 orthogonal

应用3. 令 A_{mn} 秩 r, β $\mathbb{R}^n$ basis $= \{\underbrace{x_1 \cdots}_{\leftarrow R(A^T)\text{ basis}} {}_r \; \underbrace{x_{r+1} \cdots n}_{\leftarrow N(A)\text{ basis}}\}$ ← 旧 P31. T_{2a} 3.1, r $\mathbb{R}^m$ basis $\{\underbrace{y_1 \cdots {}_r}_{\leftarrow R(A)\text{ basis}} \; \underbrace{y_{r+1} \cdots m}_{\leftarrow N(A^T)\text{ basis}}\}$

$i \le r$ $i > r$

則 $[L_A]_\beta^r = \begin{bmatrix} I_{rr} & 0 \\ 0 & 0 \end{bmatrix}$ □ $A_i x_i \in R(A)$ $L_A(x_i) = 0$ 旧 .7 → L_i (in) of $y_{1\sim r}$ 則 $A = \underset{\underset{r\to e}{Q}}{T}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\underset{\underset{e\to\beta}{Q}}{B^{-1}}$ $[L_A]_e^e$

若 $rk = r = \min(n, m)$ (Full-Rank) 則 $\begin{cases} m < n & [I \;\; 0] \\ m > n & \begin{bmatrix} I \\ 0 \end{bmatrix} \end{cases}$

eg1. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $\beta = \{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \vdots \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\}$ $r = \{\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \vdots \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\}$

$[L_A]_\beta^r = [\begin{pmatrix} 14 \\ 32 \\ 50 \end{pmatrix}_r \begin{pmatrix} 32 \\ 77 \\ 122 \end{pmatrix}_r \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_r] = \begin{pmatrix} \boxed{\begin{matrix} -2 & -2 \\ 8 & 17 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix}$ C

eg2. $A = \begin{pmatrix} -1 & 1 \\ 2 & -1 \\ 0 & 1 \end{pmatrix}$ $\beta = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ $r = \{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \vdots \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\}$

$[L_A]_\beta^r = [\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}_r \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}_r] = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 3 & 1 \end{matrix}} \\ \begin{matrix} 0 & 0 \end{matrix} \end{pmatrix}$ C

证2. 对 orthonormal basis (i.e. $\langle e_i e_j \rangle = \delta_{ij} = \begin{cases} 0 & i \ne j \\ 1 & i = j \end{cases}$), 对 $x = \sum^n \alpha_i e_i$ ie. 由 basis 表示.
右 $\alpha_i = \langle x, e_i \rangle$ □ 則 $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$, Ax 对 e_1: $\langle x e_1 \rangle = \alpha_1 \langle e_1 e_1 \rangle = \alpha_1$
只比 走班

***10. linear Map 2

1 $\mathcal{T}_1$ $T: V \to W$, V, W 为 vector space, T linear, 则 $Null(T) \leq V$, $Range(T) \leq W$

□ $Null(T)$: i) $T(0_V) = 0_W$, $\therefore 0_V \in Null(T)$ iii) $T(cx) = cT(x) = c \cdot 0_W = 0_W$

ii) $T(x+y) = T(x)+T(y) = 0_W + 0_W = 0_W$

$Range(T)$: i) $T(0_V) = 0_W$, $\therefore 0_W \in Range(T)$ iii) let $T(\alpha) = x$

ii) let $T(\alpha) = x$, $T(\beta) = y$, note $T(\alpha+\beta) = x+y$. $\therefore x+y \in Range(T)$ note $T(c\alpha) = cT(\alpha) = cx \in Range(T)$

$\mathcal{T}_2$ linear $T: V \to W$, $\beta = \{v_1, \dots, v_n\}$ 为 V basis, 则 $Range(T) = span(T(\beta)) = span(T(v_1), \dots, T(v_n))$

□ 对②: note $T(\beta)$ 即 map 为 $\{T(v_i)\}$. 对① $span(T(\beta)) \subseteq Range(T)$, 因 $T(\beta) \in W$, 且 $Range(T) \leq W$, 由 $\mathcal{T}_1$

$Range(T) \subseteq span(T(\beta))$, let $\alpha \in V$, $T(\alpha) = x \in Range(T)$, $\alpha = \sum \beta_i v_i$, T linear, $\therefore x = \sum \beta_i T(v_i)$

例1 对 $T: \mathbb{P}_2[\mathbb{R}] \to M_{2\times2}(\mathbb{R})$, $f(x) \to \begin{bmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{bmatrix}$, 求 $Range(T)$ basis

□ $\mathbb{P}_2[\mathbb{R}]$ 的 basis $\beta = \{1, x, x^2\}$, $Range(T) = span\{T(1), T(x), T(x^2)\} \Rightarrow \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & \\ & 0 \end{bmatrix}, \begin{bmatrix} -3 & \\ & 0 \end{bmatrix} \right\} \Rightarrow$ 1st & 2nd

dim = 2

例2 对 $T: \mathbb{P}_2[\mathbb{R}] \to \mathbb{P}_3[\mathbb{R}]$, $f(x) \to 2f'(x) + \int_0^x 3f(t)dt$, 是否 bijection 回 $\mathcal{T}_1$ 皿.3

□ 对 $\beta = \{1, x, x^2\}$, $Range(T) = span(\{3x, 2+\frac{3}{2}x^2, 4x+x^3\})$, dim = 3, nullity = 0, $\therefore$ 1-1, 而 $3 < dim(\mathbb{P}_3[\mathbb{R}]) = 4$, not onto

$\mathcal{T}_3$ 对 $\beta_V = \{v_1, \dots, v_n\}$, 对 $\forall w_1, \dots, w_n \in W$, $\exists$ unique linear $T: V \to W$, such that $\forall v_i \to w_i$

选的不同 w_i 找出不同 T, 这样找到的有 T (即是说, T, U 只需证有 $U(v_i) = T(v_i)$ 则 $U = T$)

□ 对 $x \in V$, $x = \sum^n a_i v_i$, DEF $T: V \to W$, $x \to \sum^n a_i w_i$.

T linear: $T(k\alpha+\beta) = T(\sum(kc_{1_i} + c_{2_i})v_i) = \sum^n (kc_{1_i}+c_{2_i})w_i$ 而 $T(k\alpha) + T(\beta) = \sum^n kc_{1_i} w_i + \sum^n c_{2_i} w_i$

~~且 对 $v_i = \sum_{j\neq i}^n 0 \cdot v_j + v_i$. $T(v_i) = w_i$ 且 unique: 若 $\exists U: V \to W$, 对 $x \in V$, $U(x) = T(x)$ $\therefore U = T$~~

但可有 bijection T (不一定是 $\mathbb{R}^2$, 见 回.2)

但随意不行, 如 $T: \mathbb{R}^2 \to \mathbb{R}^2$, $x \to 0$)

因例 3 1 的论证

P i) 对 linear $T: V \to W$, $dim V < dim W$, T 不可能 onto (这即说: T bijection $\Rightarrow dim V = dim W$)

$dim V > dim W$, T 不可能 1-1

□ 由 basis Thm, $rank T \leq dim V < dim W$; $nullity(T) = dim V - rank T \geq dim V - dim W > 0$

ii) T 1-1 $\iff$ T 将 V 的 L.Ind subset map $\to$ W 的 L.Ind subset 1-1, $Null(T) = \{0\}$

~~□ $\Rightarrow$ 对 V 的 $\beta = \{v_1, \dots, v_n\}$, 考虑 $T(v_i)$, $\sum a_i T(v_i) = 0$, $T(\sum a_i v_i) = 0$, $\sum a_i v_i = 0$, $a_i = 0$~~

$\Leftarrow$ 法1: 假设不1-1, 则 $\exists x \neq y$ 而 $T(x) = T(y)$. 对 $z = x - y \neq 0$, $\{z\}$ L.Ind. 但 $T(z) = T(x) - T(y) = 0$, $\{0\}$ L.dep. 矛盾

法2, 考虑 $x \in V$ 使 $T(x) = 0$, $\because \{0\}$ L.dep, $\{x\}$ L.dep, 但对单元素的 set L.dep 只有 $x = 0$, $\therefore Null(T) = \{0\}$ 1-1

iii) T 1-1, 且 $S \subseteq V$. S L.Ind $\iff$ $T(S)$ L.Ind.

□ $\Rightarrow$ 由 $P_{ii)}$ 知 T 1-1 $\Leftarrow$ 设 $S = \{s_1, \dots, s_n\}$, 令 $\sum a_i s_i = 0$, $\underset{T\ \text{linear}}{T(\sum a_i s_i) = 0}$, $\sum a_i T(s_i) = 0$, $\underset{T(S)\ \text{L.Ind}}{a_i = 0}$, S L.Ind.

iv) T 1-1 且 onto, 证 T 将 $\beta_V \to \beta_W$ (反过来也可以, 因这代表 $dim V = dim W$, $V \cong W$ (回.2) $\therefore$ 有 bijection T, 而因不一定此 bijection unique)

□ 由 $P_{iii)}$ $\{T(v_i)\}$ L.Ind, 且 $Range(T) = span(\{T(v_i)\})$. 而 T onto, $\therefore Range(T) = W$, $\therefore \{T(v_i)\}$ span. $\therefore \{T(v_i)\} = \beta_W$

~~v) 若 $S \subseteq V$ L.Ind, 而 $T(S)$ L.dep 则 T 不 1-1~~

□ 考虑若 T 1-1, 则由 $P_{iii)}$ 知 $T(S)$ L.Ind (或: 对 $\sum a_i T(s_i) = 0$, $\exists a_i \neq 0$, 而由 linearity $T(\sum a_i s_i) = 0$

(当然若 contrapositive 再 apply iii) 也行) 因 $\exists a_i \neq 0$, $\therefore \sum a_i s_i \neq 0$, $\therefore Null(T) \neq 0$, not 1-1

vi) 对 已知的 β basis for V. given 任意可逆 $n\times n$ 矩阵 Q, 可找出另一 basis γ

即: 令 $\gamma_j = \sum_{i=1}^n Q_{ij}\beta_i$, $\gamma = \{\gamma_j\}$ 为 V basis, 且 Q 为 change basis matrix

~~□ $\sum_j a_j \gamma_j = 0 = \sum_j a_j \sum_i Q_{ij}\beta_i = \sum_i (\sum_j a_j Q_{ij})\beta_i = 0$ 但 β basis, $\therefore \forall i$, $\sum_j a_j Q_{ij} = 0$~~

~~即 $(a_1 \cdots a_n)\begin{pmatrix} Q_{11} & \cdots & Q_{1n} \\ \vdots & \ddots & \vdots \\ Q_{n1} & \cdots & Q_{nn} \end{pmatrix} = (a_1 \cdots a_n)Q = (0 \cdots 0) = 0$. Q 可逆, $(a_1 \cdots a_n)QQ^{-1} = 0Q^{-1} = 0$~~

$\therefore$ 各 $a_j = 0$ $\therefore \gamma$ basis

法3. 假设不1-1, $\exists x \neq 0$ s.t. $T(x) = 0$. 考虑 L.Ind $\beta = \{v_1 \cdots v_n\}$, $x = \sum a_i v_i$, $T(x) = 0 = \sum a_i T(v_i)$, 但对 β 的 $T(\beta)$ 也是 L.Ind.

$\therefore a_i = 0$, 但 $x \neq 0$ 矛盾

*** 11 linear Maps

例1 $T: M_{2\times3}(F) \to M_2(F)$, $\begin{pmatrix} a_{11} & & \\ & & a_{23} \end{pmatrix} \to \begin{pmatrix} 2a_{11}-a_{12} & a_{13}+2a_{12} \\ 0 & 0 \end{pmatrix}$, T bijection?

□ Nullity: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + a_{22}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + a_{23}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

nullity = 4, rank = 2，不1-1 也不 onto

应用1 DEF: 对 $W_1 \oplus W_2 = V$, $T: V \to V$ 为 projection W_1 along W_2 if. $\forall x_1 \in W_1, x_2 \in W_2$，对 $x = x_1 + x_2$, $T(x) = x_1$

i) 对 $T: \mathbb{R}^2 \to \mathbb{R}^2$，若T为 project on y-axis along x-axis，$T(a,b) = (0,b)$

project on y-axis along line $L = \{(s,s)\}$，$T(a,b) = T((0,b-a)+(a,a)) = (0,b-a)$

ii) 对 $T: \mathbb{R}^3 \to \mathbb{R}^3$，若T为 project on xy-plane along line $L = \{(a,0,a)\}$，$T(a,b,c) = T((a-c,b,0)+(c,0,c)) = (a-c,b,0)$

iii) T linear 且 $W_1 = \{x \in V \mid T(x) = x\}$，$W_1 = Range(T)$，$W_2 = Null(T)$

□ 对 $x = x_1 + x_2$, $y = y_1 + y_2$，$T(\alpha x + y) = \alpha x_1 + y_1 = T(\alpha x) + T(y)$；$W_1 \subseteq \{x \in V \mid T(x) = x\}$ trivial；$x \in RHS$, $x = x_1 + x_2$, $T(x) = x = x_1 \in W_1$；$x \in W_1$, $T(x) = x$, $x \in Range(T)$；$x \in Range(T)$, $\exists y = y_1 + y_2 \in V$ s.t. $T(y) = x = y_1 \in W_1$

$x \in W_2$, $T(x) = T(0+x) = 0$, $x \in Null(T)$

$x \in Null(T)$, $T(x_1 + x_2) = T(x) = x_1 = 0$，∴ $x = x_2 \in W_2$

iv) 证 ∃ subspace W' and $T: V \to V$ s.t. T project on W along W' 习题 P11 T_3

□ 令 $\beta_W = \{v_1, \dots, v_k\}$ 为 W basis，可 extend 为 $\beta_V = \{v_1, \dots, v_n\}$（***3. β），也构造 $W' = \{v_{k+1}, \dots, v_n\}$，$V = W_1 \oplus W_2$

应用2 DEF: $T: V \to V$ linear，对 $W \leq V$，W T-invariant if $T(W) \subseteq W$

eg. $\{0\}$, V, Range(T), Null(T) 都 T-invariant

DEF $T_W: W \to W$ 为 restriction of T on W, $x \to T(x)$

i) 若 W T-invariant，T_W linear □ $T_W(x+ky) = T(x+ky) = T(x) + kT(y) = T_W(x) + kT_W(y)$

ii) 若 T project on W along W'，则 W T-invariant，$T_W = I_W$

□ $T(x+x') = x \in W$，∴ $T(W) \subseteq W$，而 $\forall x \in W$, $T_W(x) = T(x) = T(x+0) = x$, $I_W(x) = x$，∴ $I_W = T_W$

iii) $V = Range(T) \oplus W$ 且 W T-invariant，则当 V finite-dim，$W = Null(T)$

□ $W \subseteq Null(T)$: 对 $w \in W$, $T(w) \in W$，但也 $\in Range(T)$，而 $Range(T) \cap W = \{0\}$（⊕ def），∴ $T(w) = 0$，∴ $w \in Null(T)$

$Null(T) \subseteq W$: $\dim V = rank\,T + nullity\,T \leq rank\,T + \dim W$（P17 T_{26}）∴ 而 $W \subseteq Null(T)$（***3. T_3），∴ $W = Null(T)$

iv) W T-invariant，$Null(T_W) = Null(T) \cap W$ 且 $Range(T_W) = T(W)$

□ LHS ⊆ RHS, $x \in W$, $T_W(x) = T(x) = 0$，∴ $x \in RHS$；$x \in Range(T_W)$, $T_W(x') = T(x') = x$，∴ $x \in T(W)$

RHS ⊆ LHS, $x \in W$ 且 $x \in Null(T)$，则 $T(x) = 0 = T_W(x)$；$x \in T(W)$, $T(x') = x$. 而 $T_W(x') = T(x')$

NOTE iii) V 非 finite 不成立. eg. $T: P(\mathbb{R}) \to P(\mathbb{R})$, $f \to f'$，令 $W = \{0\}$, $Range(T) = P(\mathbb{R})$, $Null(T) = \{f' 为 0, f 为 constant\} = \mathbb{R} \neq W$

例4. V finite dim, $T: V \to V$ linear. if $rank(T) = rank(T^2)$，则 $Range(T) \cap Null(T) = \{0\}$，且 $V = Range(T) \oplus Null(T)$

□ 对 $x \in Range(T^2)$，$\exists w \in V$ s.t. $T^2(w) = x$，∴ 有 $T(w)$ st. $T(T(w)) = x$，$x \in Range(T)$，∴ $Range(T^2) \subseteq Range(T)$，而 rank 相同，

∴ $Range(T^2) = Range(T)$（***3. T_3），由 rank Thm，$nullity(T) = nullity(T^2)$ 若 $x \in R(T) \cap N(T)$，$T(x) = 0$ 且 $\exists w$ st. $T(w) = x$，

则 $T(T(w)) = T(x) = 0$，∴ $w \in Null(T^2) = Null(T)$，∴ $T(w) = 0 = x$ ∴ x 只可为 0. 而 Range(T), Null(T) ⊆ V, $V = Range(T) + Null(T)$

例5 对 linear $T: V \to V$，且 $T = T^2$，证 $V = \{y : T(y) = y\} \oplus Null(T)$

□ $V = \{y \mid T(y) = y\} + Null(T)$: 对 $\forall x \in V$, $x = T(x) + x - T(x)$，$T(x - T(x)) = T(x) - T(x) = 0$，∴ $\in Null(T)$；对 $T(x)$, $TT(x) = T^2(x) = T(x)$

而 intersection = $\{0\}$: 若 $x \neq 0$, $T(x) = x = 0$，矛盾

例6. T, U nonzero L.T: $V \to W$, if $R(T) \cap R(U) = \{0\}$, $\{T, U\}$ L.ind subset of $\mathcal{L}(V, W)$

□ $aT + bU = \underline{0}$（此为 0 的 func.）即：对 $x \neq 0 \in V$, $aT(x) + bU(x) = 0$，$T(ax) = U(-bx)$ ∴ $ax = -bx = 0$，$a = b = 0$

例7 若 $k \geq n+1$ 则 $\{a_1 \dots a_k\} \in \mathbb{R}^n$ L.dep（1.3. T.13），而若 k ≥ n+2，∃ scalar $\alpha_1 \dots \alpha_k$ st. 至少 1 个 $\alpha_i \neq 0$，$\sum_{i=1}^{k} \alpha_i a_i = 0$ 且 $\sum^{k} \alpha_i = 0$

□ 令 $\bar{a}_i = \begin{bmatrix} 1 \\ a_i \end{bmatrix} \in \mathbb{R}^{n+1}$，$\{\bar{a}_i\}_i$ L.dep ∴ 有非全 0 α_i st. $0 = \sum^{k} \alpha_i \bar{a}_i = \begin{bmatrix} \sum \alpha_i \\ \sum \alpha_i a_i \end{bmatrix}$

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Date. ¹这即 回.Ex1的ℓ2, 四的 col 组合 $\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\3\end{pmatrix}+b\begin{pmatrix}2\\4\end{pmatrix}$

1. 17 的了, 当 B col vector, $B=(b_1,\dots,b_n)^T$, $AB=\sum_{i=1}^{n} b_i A_{:,i}$ 四 P_1 i)&ii) 四.3.ii)

□ ∵ $\{e_1\dots e_n\}$ 为 $\mathbb{F}^n$ 的 basis, $B=\sum_1^n b_i e_i$, ∴ $A(b_1e_1+\dots+b_ne_n) = \underline{b_1Ae_1+\dots+b_nAe_n} = \underline{\sum b_i A_{:,i}}$

类似的, 当 A row vector, $A=(a_1\dots a_n)$ $AB=\sum_{i=1}^{n} a_i B_{i,:}$ □ $AB=(B^TA^T)^T$ → 转上行情况

$(a\ b)\begin{pmatrix}1&2\\3&4\end{pmatrix} = a(1\,2)+b(3\,4)$

2. $T: V\to W$ (dim n, m, $A=[T]_\beta^\gamma$), 证 $\mathrm{rank}(T)=\mathrm{rank}(L_A)$ $\mathrm{nullity}(T)=\mathrm{nullity}(L_A)$

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow \phi_\beta & & \downarrow \phi_\gamma \\ \mathbb{F}^n & \xrightarrow{L_A} & \mathbb{F}^m \end{array}$$

□ $\mathrm{rank}\,T = \dim(\underline{R(T)})$ ($\le W$) $\mathrm{rank}\,L_A = \dim \underline{R(L_A)}$ ($\le \mathbb{F}^m$)

~~由回例1想到 $\phi_\gamma: W\to \mathbb{F}^m$, WTS: $\phi_\gamma: R(T)\to$~~ $\boxed{\phi_\gamma R(T) = R(L_A)}$

~~⊆): $x''\in\phi_\gamma R(T)$, 有 $x'\in R(T)$ st. $\phi_\gamma(x')=x''$, 有 $x\in V$ st. $\phi_\gamma(T(x))=x''$~~ ∴ $\underline{L_A\phi_\beta(x)=x''}$ 12.2

∴ 有 $\phi_\beta(x)$ st. $L_A(\phi_\beta(x))=x''$ ⊇) 类似, ∴ $\phi_\gamma R(T) = R(L_A)$

∴ 由回例1. $\mathrm{rank}\,L_A = \underline{\dim R(L_A) = \dim(\phi_\gamma R(T))} = \underline{\dim R(T)} = \mathrm{rank}\,T$

nullity 由 rank Thm 知

3. ~~对AB m rows, $N(A^T)\subseteq N(B^T) \Rightarrow R(B)\subseteq R(A)$~~ 相同, 因若 $u\in N(A^T)$, 由⊆ $u\in N(B^T)$

□ 考虑 $[A\ v]$ (多1 col), 由四.3, $\begin{cases} \dim R(A) + \dim N(A^T) = m \\ \dim R([A\ v]) + \dim N([A\ v]^T) = m \end{cases}$ $u^Tv = u^TBx = (B^Tu)^Tx = 0$ 而 $[A\ v]^Tu = \begin{bmatrix}A^Tu\\v^Tu\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

令 $v\in R(B)$, 有 x st. $Bx=v$

$\Rightarrow \dim R(A) = \dim R([A\ v])$ ∴ v 可写作 A 的 col 线性comb. ∴ $v\in R(A)$ 引 ~~…~~

4. ~~对 A_{mn} B_{np} $\dim R(AB) = \dim R(B) - \dim(N(A)\cap R(B))$~~

~~□ 构 map $T: R(B)\to R(AB)$, $y\to Ay$. 因 $T(cy_1+y_2) = T(B(cx_1+x_2)) = AB(cx_1+x_2)$~~ ($Bx_1=y_1$ …)

$N(T)=N(A)\cap R(B)$ 因 $y\in N(T) \Leftrightarrow Ay=0$ 且 $y\in R(B)$

$= \underbrace{cABx_1}_{cT(y_1)} + \underbrace{ABx_2}_{T(y_2)}$, linear. ∴ 由四.3, $\dim R(B) = \underline{\mathrm{nullity}\,T} + \underline{\mathrm{rk}\,T}$ ($\dim R(AB)$)

5. ~~$N(A)=N(A^TA)$ □ ⊆ 因 $Ax=0$ A^TAx 也 $=0$ ⊇ 有 $A^TAx=0$ $x^TA^TAx = 0 = \|Ax\|^2$, 由 ≥ 0~~ $Ax=0$

6. ~~$V\le\mathbb{R}^n$, 证 ∃ matrix $U\ V$ st. $V=R(V)=N(U)$~~ PS1. $V\oplus V^\perp=\mathbb{R}^n$

~~□ 令 $\{v_1\dots v_r\}$ basis of $\dim = r$ V, $V=[v_1\dots v_r]$, $R(V)=V$~~ 又令 $\{u_1\dots u_{n-r}\}$ basis of $V^\perp$

$U = \begin{bmatrix} -u_1- \\ \vdots \\ -u_{n-r}- \end{bmatrix}$, $V^\perp = R(U^T) \xRightarrow{\text{取}\perp} \underline{V = R(U^T)^\perp = N(U)}$ PS1 ?

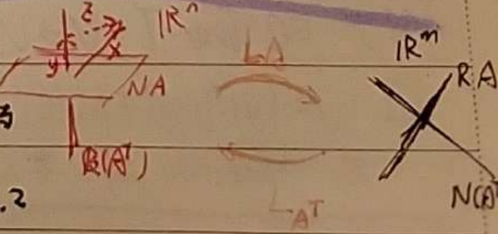
7. ~~证 L_A (A_{mn}) bijection $R(A^T)\to R(A)$; L_{A^T} bijection $R(A)\to R(A^T)$~~

PS1. $\mathbb{R}^n = R(A^T)\oplus N(A)$

□ 1° onto) $b\in R(A)$, 由 $R(A)$ 定义 有 $Ax=b$, $x\in\mathbb{R}^n$, 但 $x = \underset{R(A^T)}{y} + \underset{NA}{z}$ ∴ $Ax=Ay=b$.

1-1) $Ax=0$ 只有trivial sol, 因若有 $x\ne 0\in R(A^T)$ 即有 $A^Ty=x$, $AA^Ty=0$ 只有 $y=0$

2° 由1°, L_{A^T} 为 $R(A)\to R(A^T)$ bijection 见定理1. ***7.2



Liner Map 4

Date. /

1. 由 ~~xxx~~ 2.2. 推出矩阵乘法:

对 $T: V\to W$, $U: W\to Z$, let $A=[U]_\beta^\gamma$, $B=[T]_\alpha^\beta$

$\alpha=\{v_1 \dots v_n\}$ $\beta=\{w_1 \dots w_m\}$ $\gamma=\{z_1 \dots z_p\}$, consider $U\circ T: V\to Z$, 易证其 linear,

DEF $AB=[UT]_\alpha^\gamma$, 对 v_j: $UT(v_j)=U(\sum_{k=1}^m B_{kj}w_k) \overset{\text{定义}[T]_\alpha^\beta}{} = \underline{\sum_k^m B_{kj}U(w_k)}$ (linearity)

$=\sum_{k=1}^m B_{kj}(\sum_{i=1}^p A_{ik}z_i) = \sum_{i=1}^p(\sum_{k=1}^m A_{ik}B_{kj})z_i$, ∴ 令 $C_{ij}=\sum_{k=1}^m A_{ik}B_{kj}$, 故 $(AB)_{ij}=C_{ij}$

NOTE, 因 $f\circ g$ 一般 $\neq g\circ f$, 很容易知 $AB\neq BA$, $\begin{pmatrix}1&1\\0&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\begin{pmatrix}1&1\\0&0\end{pmatrix}$

易证 $[UT]_\alpha^\gamma=[U]_\beta^\gamma[T]_\alpha^\beta$, 几乎是定义　　$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\1&1\end{pmatrix}$

2. P_1: 正如 对 $U_1,U_2\in\mathcal{L}(V,W)$, $T\in\mathcal{L}(W,Z)$, $T(U_1+U_2)=T(U_1)+T(U_2)$ 等, matrix representation 有 (一些注意是定义的例子)

let A $m\times n$, B, C $n\times p$, D, E $q\times n$

i) $A(B+C)=AB+AC$; $(D+E)A=DA+EA$ □ $[A(B+C)]_{ij}=\sum_{k=1}^n A_{ik}(B+C)_{kj}=\sum A_{ik}B_{kj}+\sum A_{ik}C_{kj}$

ii) 对 $k\in\mathbb{F}$, $k(AB)=(kA)B=A(kB)$

Kronecker delta, $\delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$, 之后 I_n 可由 δ_{ij} def: $(I_n)_{ij}=\delta_{ij}$

iii) $I_mA=A=AI_n$ □ $(I_mA)_{ij}=\sum_{k=1}^m(I_m)_{ik}A_{kj}=\sum_{k=1}^m\delta_{ik}A_{kj}=\underline{A_{ij}}$ 只有当 $k=i$ $\delta_{ik}=1$

Identity transform: $V\to V$, $x\to x$

iv) 若 $\dim V=n$, $[I_V]_\beta=I_n$ □ $[I_V]_\beta=[[I_V(\beta_1)]_\beta,\dots]=[[\beta_1]_\beta,\dots]=[e_1,\dots,e_n]=I_n$

NOTE cancel law 不成立, eg $A=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $A^2=O=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, 若 $A\cdot A=O=A\cdot O$, $A=O$, 矛盾

3. 见 T_3 的证明 let A $m\times n$, B $n\times p$, i) $\underbrace{AB_{:,j}}_{}$ $= A\cdot B_{:,j}$ (AB 的 col j) ii) $B_{:,j}=Be_j$

□ i) $AB_{:,j}=\begin{pmatrix}(AB)_{1j}\\ \vdots\\ (AB)_{mj}\end{pmatrix}=\begin{pmatrix}\sum A_{1k}B_{kj}\\ \vdots\\ \sum^n A_{mk}B_{kj}\end{pmatrix}=A\begin{pmatrix}B_{1j}\\ \vdots\\ B_{nj}\end{pmatrix}=AB_{:,j}$ ii) $(Be_j)_i=\sum_k^p B_{i,k}e_k=B_{i,j}$ 只有当 $k=j$ $e_k=1$

OR: 显然 $(BI)_{:,j}=B\cdot I_{:,j}=B\cdot e_j$

当然 $AB_{i,:}=A_{i,:}B$, 但用的少

□ $(AB)_{i,:}=((AB)_{i,1},\dots,(AB)_{i,p})=(\sum_{k=1}^n A_{ik}B_{k,1},\dots,\sum_{k=1}^n A_{ik}B_{kp})=[A_{i1},\dots,A_{in}]B=A_{i,:}B$

4. ~~xxx~~ 2. 红字的另一证明: $[T(u)]_\gamma=[T]_\beta^\gamma[u]_\beta$

□ fix $u\in V$, def $f: F\to V$, $a\to au$; $g: F\to W$, $a\to aT(u)$, let $\alpha=\{1\}$ basis for $\mathbb{F}$. note $g=Tf$

把1的像g"记"

$[T(u)]_\gamma=\underset{\text{ele 以 }\gamma\text{ 表示}}{\underline{[g(1)]_\gamma}}=[g]_\alpha^\gamma=[Tf]_\alpha^\gamma=[T]_\beta^\gamma[f]_\alpha^\beta=[T]_\beta^\gamma\cdot[f(1)]_\beta=[T]_\beta^\gamma[u]_\beta$

由定义 $[g]_\alpha^\gamma=[[g(\text{basis})]_\gamma]=[[g(1)]_\gamma]$ 　F 的 basis 只有 $\{1\}$

5. DEF $L_A: \mathbb{F}^n\to\mathbb{F}^m$, $x\to Ax$, A $m\times n$. L_A linear (由性质 P_1 i)&ii)) 性质3.ii)

P_2 i) $[L_A]_\beta^\gamma=A$ □ $[[L_A(e_1)]_\gamma,\dots,[L_A(e_n)]_\gamma]=[[Ae_i]_\gamma]_{i=1}^n=\underline{[A_{:,i}]_{i=1}^n}=A$

ii) $L_A=L_B\iff A=B$ □ $A=[L_A]_\beta^\gamma=[L_B]_\beta^\gamma=B$, ∴ $A=B$

iii) $L_{A+B}=L_A+L_B$; $L_{kA}=kL_A$ □ P_1 i)&ii)

iv) (回 P_1) $T: \mathbb{F}^n\to\mathbb{F}^m$ liner, ∃ unique $m\times n$ C st. $T=L_C$. □ $C=[T]_\beta^\gamma$, $[T(x)]_\gamma=[T]_\beta^\gamma[x]_\beta=Cx$, unique 见 P_2 ii)

v) let E $n\times p$, $L_{AE}=L_AL_E$ (这证明矩阵乘 associative, $L_{A(BC)}=L_AL_{BC}=(L_AL_B)L_C$)

□ $L_{AE}(e_j)=(AE)(e_j)=\underset{\text{性质3.ii)}}{\underline{AE_{:,j}}}=\underset{\text{性质3 i)}}{A\cdot E_{:,j}}=A\underset{\text{性质3.ii)}}{(E\cdot e_j)}=L_AL_E(e_j)$, ∴ 两者相同 (~~xxx~~10, T_3 证了)

Date. /

例1 $T: V \to W$ isomorphism, $V_0 \le V$, i) 证 $T(V_0) \le W$, ii) $\dim V_0 = \dim T(V_0)$

□ i) 对 $x', y' \in T(V_0)$ $\exists x, y \in V$, $T(x)=x'$, $T(y)=y'$ 对 $x'+y' = T(x+y) \in T(V_0)$ $kx' = T(kx) \in T(V_0)$

ii) T 1-1, ∴ 对 β_{V_0}, $T(\beta_{V_0})$ L.Ind ($\cancel{\text{xxx}}$ 10, P_{11})

且 $\forall x' \in T(V_0)$, 有 $T(x)=x'$, 而 $x \in \langle \beta_{V_0} \rangle$

∴ $x' = \sum a_i T(\beta_{V_0, i})$ ∴ span

∴ $\{T(\beta_{V_0})\}$ 为 basis, 与 β_{V_0} 同元素

例2. V basis $\beta = \{v_1 \cdots v_n\}$ W basis $\gamma = \{w_1 \cdots w_m\}$, $\exists!\ T_{ij}: V \to W$, $v_k \to \begin{cases} w_i & k=j \\ 0 & k \ne j \end{cases}$ ($\cancel{\text{xxx}}$ 10, T_3)

证 $\{T_{ij}: i=1 \cdots m, j=1 \cdots n\}$ basis for $\mathcal{L}(V,W)$, M^{ij} 为 $m \times n$, 全0除了row i col j 为1, 证

$M^{ij} = [T_{ij}]_\beta^\gamma$, 且 $\exists!\ \Phi: \mathcal{L}(V,W) \to M_{m\times n}(\mathbb{F})$, $T_{ij} \to M^{ij}$, 证 Φ isomorphism ($\cancel{\text{xxx}}$ 10, T_3)

□ i) $\sum\sum a_{ij} T_{ij} = 0$, $\mathcal{L}(V,W)$ 的0为 $f(x)=0, \forall x \in V$. ∴ $\underbrace{\sum_i\sum_j a_{ij} T_{ij}(v_k)}_{\text{特别地}} = 0$, ∴ $\sum_i a_{ik} T_{ik}(v_k) = 0$

对 $k=1 \cdots n$, $\sum_i a_{ik} w_i = 0$ ∴ $a_{ik}=0$ 而 k 又 arbitrary ∴ L.Ind.

ii) $[T_{ij}]_\beta^\gamma = \left[[T_{ij}(v_k)]_\gamma\right]_{k=1}^n = M^{ij}$ (只有 $k=j$ 才有 w_i, 且只有 i row 有1 $[w_i]_\gamma = e^i$)

iii) note 由i) T_{ij} basis $\Phi(T_{ij}) = M^{ij}$ 又为 $M_{m\times n}(\mathbb{F})$ 的 basis, ∴ 由 ($\cancel{\text{xxx}}$ 10, P_{11}) isomorphism

例3 $S: P_3[\mathbb{R}] \to M_2(\mathbb{R})$, $f \to \begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix}$ is S invertible?

□ 法1 $S(ax^2+bx+c) = \begin{pmatrix} f(1) & f(2) \\ f(3) & f(4) \end{pmatrix} = 0$ ∴ $x=1,2,3,4$ 时 $f=0$, ∴ $a=b=c=0$, $\mathrm{Null}(S)=\{0\}$, 1-1, $\boxed{\text{III}}$ 4.

且 $\dim P_3[\mathbb{R}] = \dim M_2(\mathbb{R})$, bijection invertible

法2 $[S]_\beta^\gamma = \cdots$ 证 $[S]_\beta^\gamma$ 之矩阵可逆

法3 直接找出 $S^{-1}: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to p$ $\begin{cases} p(1)=a \\ p(2)=b \\ p(3)=c \\ p(4)=d \end{cases} \xrightarrow{\text{lagrange}} p(x) = \sum_{i=0}^{3} p(i+1) f_i$, 证 $S(S^{-1}(x)) = x$

例4. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ let $\beta = \left\{\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix}\right\}$ $\gamma = \left\{\begin{pmatrix}5\\6\end{pmatrix}\begin{pmatrix}7\\8\end{pmatrix}\right\}$ 找出 $[L_A]_\gamma^\beta = P^{-1} A Q$ 的 $P\ Q$

□ 法1 $[L_A]_\gamma^\beta = \underbrace{[I]^\beta_\alpha [L_A]_\alpha^\alpha [I]_\gamma^\alpha}_{\text{标准是}\alpha} = ([I]_\beta^\alpha)^{-1} A \underbrace{([I]_\gamma^\alpha)}_{P^{-1}} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} A \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}$

法2 $[L_A]_\gamma^\beta = [I]_\gamma^\beta [L_A]_\gamma^\gamma = \left[\begin{pmatrix}5\\6\end{pmatrix}_\beta \begin{pmatrix}7\\8\end{pmatrix}_\beta\right] \underbrace{Q^{-1}}_{\gamma \to \alpha} A \underbrace{Q}_{\gamma \to \alpha}$, 即 $P = Q([I]_\gamma^\beta)^{-1} = Q[I]_\beta^\gamma$

In fact, T invertible $\iff [T]_\beta^\gamma$ invertible

□ $\Rightarrow$ 根1. $\Leftarrow$ let B $n\times n$ s.t. $B[T]_\beta^\gamma = I_n$，由根2.基本，由 $\mathcal{L}(W,V)\cong M_{n\times n}(\mathbb{F})$，和 **10.$T_3$，$\exists! U \in \mathcal{L}(W,V)$, s.t. $U: w_j \to \sum_{i=1}^{n} B_{ij} v_i$，$\therefore B=[U]_\gamma^\beta$，WTS $U=T^{-1}$：$[UT]_\beta = [U]_\gamma^\beta [T]_\beta^\gamma = BA = I_n = [I_V]_\beta$

(note $i=j$, dim 相同) $\in \gamma$ $\in \beta$ 即 [1].P_2 iv) $\therefore UT = I_V$ [1].P_1 iv)

1，对 bijection liner $T: V\to W$，考虑 $T^{-1}: W\to V$ 也 linear，由此引入矩阵 invertible

根据 T invertible，$\underline{T^{-1}\text{ 也 invertible}}$ (**10.P_i)，$[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$ (identity map，记 A) [1].P_1 iv) [1].1 notation

□ 因 T bijection，$\underline{\dim V = \dim W = n}$，$[T]_\beta^\gamma$ $n\times n$，而 $T^{-1}T = I_V$，$\therefore I_n = [I_V]_\beta = [T^{-1}T]_\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$

同理 $I_n = [I_W]_\beta = [TT^{-1}]_\beta = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta$ $\therefore$ invertible

2. 事实上，只要 $\dim V = \dim W \iff V \cong W$（即 $\exists$ bijection liner $T: V\to W$）

□ $\Rightarrow$ 令 $\beta = \{v_1 \cdots v_n\}$，$\gamma = \{w_1 \cdots w_n\}$ 由 **10.T_3 $\exists$ unique $T: V\to W$, $v_i \to w_i$，$\therefore$ **10.P_{ii} $\underline{T \text{ 1-1}}$，而 $\underline{\mathrm{Rge}(T) = \mathrm{span}(T(v_i)) = \mathrm{span}(w_i) = W}$ (**10.T_2) $\therefore$ onto $\therefore$ bijection

$\Leftarrow$ **10.P_i)

NOTE，由此，若 $\dim V = n$，$V \cong \mathbb{F}^n$，另一种视角看 **2.2 $\nearrow$ **2.2

1° $\Phi: \mathcal{L}(V,W) \to M_{\dim(W)\times\dim(V)}(\mathbb{F})$, $T \to [T]_\beta^\gamma$ 2° $\phi_\beta: V \to \mathbb{F}^{\dim(V)}$: $x \to [x]_\beta$

两者皆 isomorphism

liner 见 **2.T

bijection：$\forall A \in M$，$\exists$ 且 $\exists!$ T s.t. $\Phi(T) = [T]_\beta^\gamma = A$，由 $[T]_\beta^\gamma$ 的性质和 unique

进一步说明 $\dim(\mathcal{L}(V,W)) = \dim(W)\times\dim(V)$ （$M_{m\times n}(\mathbb{F})$ dim mn）

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\ \mathbb{F}^n \arrow[r, "L_A"'] & \mathbb{F}^m \end{tikzcd}$$

对 $x \in V$，$L_A\phi_\beta(x) = [x]_\gamma$，即：$[T]_\beta^\gamma [x]_\beta = [T(x)]_\gamma$（见 [1].4）

对 $x \in V$，$\phi_\gamma T(x) = [x]_\gamma$

$\therefore L_A \phi_\beta = \phi_\gamma T$

~~NOTE 即使 $W \cong V$，不代表只有一个 isomorphism，考虑 **4.2 basis change~~

即左相似性是 A 和 A ($M_n(\mathbb{F})$)

3. 由 ***与 T_{2b} 当 $T: L_A: \mathbb{F}^n \to \mathbb{F}^n$，$[L_A]_\gamma = Q^{-1}_{\gamma\to\beta}[L_A]_\beta Q_{\gamma\to\beta} = Q^{-1}AQ$ $\to$ j col 即 r_j

eg. $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{pmatrix}$ $\gamma = \left\{\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$，$Q = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$，$\therefore [L_A]_\gamma = Q^{-1}AQ = \begin{pmatrix} 0 & 2 & 8 \\ -1 & 4 & 6 \\ 0 & -1 & -1 \end{pmatrix}$

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Date. /

(β, γ standard)

例1 1° let $\beta = \{\binom{2}{1}, \binom{3}{1}\}$ basis for $\mathbb{R}^2$，求 β^*，2° $T: P_1[\mathbb{R}] \to \mathbb{R}^2$，$p(x) \mapsto (p(0), p(2))$，求 $[T^t]_{\gamma^*}^{\beta^*}$

□ 1° $\beta^* = \{f_1, f_2\}$ 对 f_1 $f_1\binom{2}{1} = 1 = f_1(2 * \binom{1}{0} + \binom{0}{1}) = 2f_1\binom{1}{0} + f\binom{0}{1}$

$f_1\binom{3}{1} = 0 = 3f_1(e_1) + f_1(e_2)$ ∴ $f_1(e_1) = -1$，$f_1(e_2) = 3$

$f_1(x, y) = -x + 3y$

$f_2(x, y) = x - 2y$

2° $[[g_1 T]_{\beta^*}, [g_2 T]_{\beta^*}] = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

这样 $g_1 T = af_1 + bf_2$，$g_1 T(1) = g_1\binom{1}{1} = g_1(e_1) + \underset{=0}{g_1(e_2)} = 1$；$g_1 T(1) = af_1(1) + bf_2(1) = a$ ∴ $a = 1$

$g_1 T(x) = g_1\binom{0}{2} = 2g_1(e_2) = 0$；$g_1 T(x) = af_1(x) + bf_2(x) = b$ ∴ $b = 0$ ∴ $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

例2. $V = \mathbb{R}^3$，$f_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x - 2y$，$f_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + y + z$，$f_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y - 3z$，证 $\{f_1 \cdots f_3\}$ basis of V^* 且找出 V 对应的 basis

□ i) 设 $\sum a_i f_i\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underset{=}{0}$ (zero func) ∴ 对 $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$，$\sum a_i f_i\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underset{=}{0}$ (1F 为 0) ∴ $\begin{cases} a + b = 0 \\ -2a + b + c = 0 \\ b - 3c = 0 \end{cases} \Rightarrow a = b = c = 0$

ii) $f_1(v_1) = 1$，$f_2(v_1) = 0$，$f_3(v_1) = 0$ $\Rightarrow \begin{cases} x_1 - 2y_1 = 1 \\ x_1 + y_1 + z_1 = 0 \\ y_1 - 3z_1 = 0 \end{cases}$ ∴ $v_1 = (\frac{2}{5}, \frac{-3}{10}, \frac{-1}{10})$ 同理 $v_2 = (\frac{3}{5}, \frac{3}{10}, \frac{1}{10})$ $v_3 = (\frac{1}{5}, \frac{1}{10}, \frac{-3}{10})$

例3. $\forall$ basis in V^* 为 V 一些 basis 的 dual basis，即对 $x = \sum a_i \underset{V的某basis}{?_i}$，$f_j(\sum a_i ?_i) = a_j$

□ let $\{f_1 \cdots f_n\}$ basis for V^*，将 V^* 视为右边 1 的 V 中的 PEF，总 ∃ 了由 V^* basis 做出的 V^{**} basis $\{\hat{x}_1 \cdots \hat{x}_n\}$

由 def 得出，对于 $t \in V^*$，$t = \sum a_i f_i$，$\hat{x}_j(t) = \hat{x}_j(\sum a_i f_i) = a_j = t(x_j)$，特别地，对于 f_j，

$\{x_i\}$ 为 V basis，因 $\hat{x}$ isomorphism，且有一 $\hat{x}_i \to \hat{x}_i$，且个数一样，∴ basis

$f_j(\underset{\in V}{x}) = f_j(\sum a_i x_i) = a_j$，∴ $\{f_1 \cdots f_n\}$ 是 $\{x_1 \ldots x_n\}$ 的 dual basis

线性 linear functional　SQL (aggregate func 也是其中一种)
↑ vec. sp. over field

$V \xrightarrow{T} W \rightarrow F$, $g \in W^*$, $gT \in V^*$

1 对 L $T: V \to F$, 令 $V^* = \mathcal{L}(V,F)$ dual space　正 ***2. 2 的 DEF, $\mathcal{L}(V,F)$ 为 V.S. 回.2

由 回 2 结论, $\dim(V^*) = \dim V \cdot \dim F = \dim V$, $\therefore V^* \cong V$

DEF: 对 $x \in V$, $x = \sum a_i v_i$, $f_i(x) = a_i$ 为 coord function

T_{1a} 令 $\beta^* = \{f_1 \dots f_n\}$, β^* basis for V^*, 即: 对 $\forall f \in V^*$, $f \in \mathrm{span}\{f_1 \dots f_n\}$

g-f 对 basis map 相同 ***10. T_5　回. β, $f_i(v_j)$ 为 kronecker delta

□ 令 $g = \sum f(v_i) f_i$, WTS $g = f$. ∵ span: $g(v_j) = \sum f(v_i) f_i(v_j) = \sum f(v_i) \delta_{ij} = f(v_j)$　***3. P.d) span. ∴ basis

T_{1b} 对 $T: V \to W$, DEF $T^T: W^* \to V^*$, $g \to gT$, $[T^T]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^T$ 见上图

□ i) 对 $g \in W^*$, $T^T(g) = gT \in \mathcal{L}(V,F)$, $\therefore T^T$ 映射 map 到 V^*　令 $A = [T]_\beta^\gamma$

ii) $T^T(f+kg) = fT + kgT$, linearity　iii) 对 $\beta = \{v_1 \dots v_n\}$ $\gamma = \{w_1 \dots w_m\}$, $\beta^* = \{f_1 \dots f_n\}$ $\gamma^* = \{g_1 \dots g_m\}$ 考虑 $([T^T]_{\gamma^*}^{\beta^*})_{ij} =$

即 $[T^T(g_j)]_{\beta^*}$ 的 i-row, 由 T_{1a}, $T^T(g_j) = g_j T \in V^*$, $\therefore = \sum_i g_j T(v_i) f_i$
***2 的构造

$\therefore$ i-row $= g_j T(v_i) = g_j(\sum_k A_{ki} w_k) = \sum_k A_{ki} g_j(w_k) = \sum_k A_{ki} \delta_{jk} = A_{ji}$　(***2 构造)

2. 考虑 V^{**}, Def $\hat{x}: V^* \to F$, $f \to f(x)$ $(x \in V)$ 易证 $\hat{x}$ L.T., 证 $\psi: V \to V^{**}$, $x \to \hat{x}$ 为 isomorphism

□ i) L.T. 考虑 $\widehat{x+ky}$ 与 $\hat{x} + k\hat{y}$. 对各 $f \in V^*$, $\widehat{x+ky}(f) = f(x+ky) = f(x) + kf(y) = (\hat{x} + k\hat{y})(f)$ (V^* 为 set of L.T.)

ii) 1-1. 若 $\psi(x) = 0$, 即 $\hat{x} = 0$ (zero func), 即 对 $f \in V^*$, $\hat{x}(f) = 0$ (scalar), $f(x) = 0$. 而 $x \in V$, $x = \sum a_i v_i$, $\sum a_i f(v_i) = 0$

因 f 任意, 对各 f_j, $\sum_i a_i f_j(v_i) = 0$, $\sum_i a_i \delta_{ji} = 0$, 各 $a_j = 0$ $\therefore x = 0$ (回. T_1)

iii) onto:　回.4

$\because \dim V^{**} = \dim \mathcal{L}(V^*, F) = \dim V^* \cdot \dim F = 1 \cdot \dim V$ ∴ 同 dim 又 1-1. ∴ onto

3. T_{2a} 令 $W < V$ (proper), ∃ nonzero linear functional $f \in V^*$ st $f(x) = 0$, $\forall x \in W$

***3. T_5　$\hat{x}$ ∃, 由 T1. 的 Def

□ 对 W basis $\{v_1 \dots v_m\}$ 延伸至 V basis $\{v_1 \dots v_n\}$, $W < V$ ∴ $m < n$. 取 $f_{m+1} \in V^*$ basis, $\forall x \in W$, $f_{m+1}(\sum_1^m a_i v_i) = 0$

而对 $v_{m+1} \in V$, $f_{m+1}(v_{m+1}) = 1$, $\therefore f_{m+1}$ nonzero

T_{2b} 令 $T: V \to W$.　i) T onto $\iff$ T^T 1-1　ii) T 1-1 $\iff$ T^T onto

□ i) ($\Rightarrow$) let $g \in \mathrm{Nul}(T^T)$, $g \in W^*$, $\forall w \in W$, consider $g(w) = g(T(w')) = T^T(g)(w') = 0$　(onto; zero func, 因 $g \in \mathrm{Nul}$; $\in F$)

$\therefore$ g zero func. $\therefore \mathrm{Nul}(T^T) = \{0\}$ ∴ 1-1. ($\Leftarrow$) assume $R(T) \neq W$, 而 $R(T) < W$, 由 T_{2a},　回 T. 7.

有 nonzero $g \in W^*$, 且 $g(w) = 0$, $\forall w \in R(T)$. 但 因 nonzero, $T^T(g) \neq 0$ (zero func). 但 $\forall x \in V$, $gT(x) = g(\underset{\in R(T)}{?}) = 0$ 矛盾　***10. T.

ii) $\Rightarrow$ 因 $R(T) \leq W$, 且 $\{T(v_i)\}$ 为 $R(T)$ basis (v_i V basis, ***10 P.i)), 对 $\forall f \in V^*$, Def $g \in W^*$ as: $gT(v_i) = f(v_i)$
$g(w_j) = 0$　W basis 中非 $T(v_i)$ 的

note 这样 $T^T(g)(v_i) = gT(v_i) = f(v_i)$ $\therefore T^T(g) = f$ ***10. T_5 basis map 相同 ∴ onto

$\Leftarrow$ assume 非 1-1, $\exists x \neq 0 \in V$, st. $T(x) = 0$, def $f \in V^*$ as $f(x) \neq 0$, 而 T^T onto ∴ 有 $g \in W^*$ st. $gT = f$

consider $gT(x) = g(0) = 0$　但 $f(x) \neq 0$ ∴ 矛盾 ($g \in \mathcal{L}(W,F)$, g L.T. send $0_W \to 0_F$)

Date: /

例1. product of 2 matrix, rank 为两 matrix 之和 rank ☐ 可以令 $AB=0$, $A,B\neq 0$ ∴ ×

例2 将 $A=\begin{pmatrix}1&2&1\\1&0&1\\1&1&2\end{pmatrix}$ 写为 elementary 的乘积

☐ A 可逆 由定理 P_{iii} $A\sim\begin{pmatrix}1&2&1\\0&-2&0\\1&1&2\end{pmatrix}\sim\begin{pmatrix}1&2&1\\0&-2&0\\0&-1&1\end{pmatrix}\sim\begin{pmatrix}1&2&1\\0&1&0\\0&-1&1\end{pmatrix}\sim\begin{pmatrix}1&0&1\\0&1&0\\0&-1&1\end{pmatrix}\sim\begin{pmatrix}1&&1\\&1&\\&&1\end{pmatrix}\sim\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$ $(I_n=E_6\cdots E_1A)$

$E_1=\begin{pmatrix}1&&\\-1&1&\\&&1\end{pmatrix}$ $E_2=\begin{pmatrix}1&&\\&1&\\-1&&1\end{pmatrix}$ $E_3=\begin{pmatrix}1&&\\&-1/2&\\&&1\end{pmatrix}$ $E_4=\begin{pmatrix}1&-2&\\&1&\\&&1\end{pmatrix}$ $E_5=\begin{pmatrix}1&&\\&1&\\&1&1\end{pmatrix}$ $E_6=\begin{pmatrix}1&&-1\\&1&\\&&1\end{pmatrix}$ $\underline{A=E_1^{-1}\cdots E_6^{-1}}$

例3 对 $T,U: V\to W$, a) $R_{ange}(T+U)\subseteq R(T)+R(U)$ b) $\text{rank}(T+U)\leq \text{rank}\,T+\text{rank}\,U$

☐ a) $y=T(x)+U(x)$ for some x b) $\underbrace{\text{rank}\,T+U\leq}_{}\overset{3.\ T_3}{\dim(R(T)+R(U))}\overset{P_{12}.\ T_{20}}{=}\dim R(T)+\dim R(U)-\dim(R(T)\cap R(U))$ $\leq \text{rank}\,T+\text{rank}\,U$

NOTE 这说明 $\text{rank}(A+B)=\text{rank}(L_{A+B})\overset{\boxed{\text{III}}.P_2\,iii)}{=}\text{rank}(L_A+L_B)\leq\text{rank}\,A+\text{rank}\,B$

例4. $A_{m\times n}$, $B_{n\times p}$, $\text{rank}\,A=m$, $\text{rank}\,B=n$, 求 $\text{rank}(AB)$

☐ $L_A(\mathbb{F}^n)=\mathbb{F}^m$, $L_B(\mathbb{F}^p)=\mathbb{F}^n$, 对 $L_{AB}(\mathbb{F}^p)=L_AL_B(\mathbb{F}^p)=L_A(\mathbb{F}^n)=\mathbb{F}^m$, ∴ rank $=m$

例5. 若 A $m\times n$ rank m, $\exists B_{n\times m}$ st. $AB=I_m$; 相对, 若 $B_{n\times m}$ rank m, $\exists A_{m\times n}$ st. $AB=I_m$

☐ i) 即 $R(L_A)=\mathbb{F}^m$, onto. 对 $\mathbb{F}^m$ basis $\{e_1\cdots e_m\}$, 对 e_i, $\exists L_A(\underset{\in\mathbb{F}^n}{v_i})=e_i$, 令 $B=(v_1\cdots v_m)$

$AB=(Av_1\cdots Av_m)\overset{\boxed{\text{II}}\,3}{=}(L_A(v_1)\cdots L_A(v_m))=(e_1\cdots e_m)=I_m$

ii) 考虑 $\text{rank}(B^T)=m$, ∴ B^T 为 $m\times n$, $\text{rank}=m$, 由 i), $\exists C$ st. $B^TC=I_m=(C^TB)^T$, ∴ 令 $C^T=A$ (右定 $P_{i)}$)

例6. i) A 3×1, B 1×3, 则 AB at most rank 1

ii) C 3×3 rank 1, 则 $\exists A$ 3×1, B 1×3, st. $C=AB$

☐ i) 令 $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\cdot(b_1\ b_2\ b_3)=\begin{pmatrix}a_1b_1&a_1b_2&a_1b_3\\a_2b_1&a_2b_2&a_2b_3\\a_3b_1&a_3b_2&a_3b_3\end{pmatrix}$, 可知各 col 比例关系 (eg. 2r 为 1r $\times\frac{a_2}{a_1}$) ∴ rank ≤ 1

ii) 令 C 1 row L.I.d. 令 $C=\begin{pmatrix}-c_1-\\-ac_1-\\-bc_1-\end{pmatrix}$, ∴ $A=\begin{pmatrix}1\\a\\b\end{pmatrix}$, $B=c_1$

例7. 对 $A=\begin{pmatrix}1&0&-1&2&6\\-1&1&3&-1&0\\-2&1&4&-1&3\\3&-1&-5&1&-6\end{pmatrix}$, i) M 5×5 rank 2, $AM=0$

ii) B 5×5, $AB=0$, 证 B rank ≤2

☐ i) 令 $M=(m_1\ m_2\ 0\ 0\ 0)$, $AM=0$ 即 $Am_1=0\in\mathbb{F}^4$, $Am_2=0\in\mathbb{F}^4$, 快方法为: 求 $(A|0)$ 来求 $Ax=0$

null space 为 $\left\{\begin{pmatrix}x_3+3x_5\\-2x_3+x_5\\x_3\\-2x_5\\x_5\end{pmatrix}\right\}$, basis 为 $\left\{\begin{pmatrix}1\\-2\\1\\0\\0\end{pmatrix},\begin{pmatrix}3\\1\\0\\-2\\1\end{pmatrix}\right\}$. ∴ 可令 $M=\begin{pmatrix}1&3&1&1&1\\-2&1&1&1&1\\1&0&0&0&0\\0&-2&1&1&1\\0&0&1&1&1\end{pmatrix}$

ii) 令 $B=(b_1\ b_2\ b_3\ b_4\ b_5)$ 即各 b_i 在 $Ax=0$ 的解空间, 但对 L_A: $\underset{5}{\dim(\mathbb{F}^5)}=\overset{3}{\text{rank}(L_A)}+\text{nullity}(L_A)$

∴ $\text{nullity}(L_A)=2$, 而 $b_i\in N(L_A)$ ∴ rank ≤2

证1. $rkA^TA=rkAA^T=rkA=rkA^T$ (Gram matrix rk eq)

☐ 由 $\boxed{\text{II}}.5$ & $\boxed{\text{III}}.3$

(Linear Map 7)

回.3

研究对 matrix representation 的 L.T. 的性质. def $rank(A) = rank(L_A)$

Date. /

1. T_{1a} A $m\times n$, 有 P $m\times m$, Q $n\times n$ 可逆, i) $rank(AQ) = rank A$ ii) $rank(PA) = rank A$ iii) $rank(PAQ) = rank A$

回.P_2 ii) def of range; L_Q onto, $Q = [L_Q]$ 可逆, L_Q 可逆

□ i) 由 $Range(L_{AQ}) = Range(L_A L_Q) = L_A L_Q(F^n) = L_A(L_Q(F^n)) = L_A(F^n) = R(L_A)$ (回.1)

$\therefore rank(AQ) = dim(Range(L_{AQ})) = dim(Range(L_A)) = rank A$ (按定义) ii) $R(L_P L_A) = L_P L_A(F^n) = L_P(Range(L_A))$

$\therefore rank(PA) = dim L_P(R(L_A)) = dim(R(L_A)) = rank(A)$

☆ 这即说 elementary row/col operation 不改变 rank (回.T_2, E 可逆) (L_P isomorphism, 回.1, T为 L_P)

T_{1b} rank of matrix 同 max # of L.ind. col (回.3 的内容) 回.3.ii)

□ $rk\,A = rank L_A = dim(R(L_A)) = dim(L_A(\beta)) \overset{10.T_2}{=} dim(span\{A(e_i)\}_i) = dim(span\{A_{:,i}\}_i)$ (按定义)

T_{1c} let A $m\times n$, A rank $= r \Rightarrow \exists$ 可逆 B, C s.t. $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = D = BAC$ 想象成 composite / AB $B^{-1}A^{-1}$ $(AB)^{-1}$..

□ 经过初等变换 s.t. $A \to \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$ 时, 即 $D = \underbrace{E_p \cdots E_1}_{B} A \underbrace{F_1 \cdots F_q}_{C}$, 而可逆乘积仍可逆

且有 r个1 (operation 不改变 rank, & T_{1b})

P_1 i) $rank A^T = rank A$ 即使由回.3, $Nullity(A) \neq Nullity(A^T)$, 因 A 可能不同 # 行

回.5.3, B,C 都可逆; D的性质 s.t. rk 为 r

□ 由 T_{1c}, 有 $D = BAC$, $\therefore D^T = (BAC)^T = C^T A^T B^T$, 因 C^T, B^T 可逆, 由 T_{1a} iii) $rank(C^T A^T B^T) = rank A^T = rank D^T \overset{T_{1c}}{=} r = rank A$

ii) (T_{1b} 拓展) rank of matrix 同 max # of L.ind. row

□ max # of L.ind row 同 max # of L.ind col of A^T, 而 $rank A^T \overset{P_1 i)}{=} rank A$

iv) 在 T_{1b} 后本性质说明: row/col generate 的 subspace dim 为 rank (col space)

iii) 可逆矩阵为 elementary matrix 乘积 回.T_2 inverse elementary 仍 elementary

□ A $n\times n$ 可逆, 则 rank $= n$ ($Null(L_A) = \{0\}$ (1-1) $\therefore rank(L_A) = n = rank A$ (按定义)) $\therefore$ 由 T_{1c}, $D = I_n$, 且有 $D = I_n = BAC = \underbrace{E_p \cdots E_1}_{B} A \underbrace{F_1 \cdots F_q}_{C}$, $\therefore A = E_1^{-1} \cdots E_p^{-1} F_q^{-1} \cdots F_1^{-1}$

iv) $rank A = rank\, kA$, k nonzero scalar. □ WTS $Range(L_A) = R(L_{kA})$, 而 $R(L_{kA}) = L_{kA}(F^n) = kL_A(F^n) = L_A(F^n) = R(L_A)$ 对 $L_A(x)$ 有 $kL_A(x/k)$ (L_A L.T) OR 用 T_{1a} 也 回.P_2 iii)

T_2 $T: V \to W$, $U: W \to Z$; A B 为矩阵, 证: a) $rank(UT) \leq rank(U)$ b) $rank(UT) \leq rank T$ c) $rank(AB) \leq rank A$ d) $rank AB \leq rank B$

□ a) $R(UT) = UT(V) = U(R(T)) \subseteq U(W) = R(U)$, $rank UT \leq rank U$ (回.3.T_3; $R(T) \subseteq W$)

c) $rank(L_{AB}) = rank(L_A \circ L_B) \leq rank(L_A)$ (由 a)) d) $rank\, AB = rank\, AB^T = rank\, B^T A^T \leq rank\, B^T = rank B$ (P_1 i); 由 c))

b) 令 $A = [U]_\beta^\gamma$, $B = [T]_\alpha^\beta$, 则 $AB = [UT]_\alpha^\gamma$ (回.1), $\therefore rank AB = rank UT \leq rank B = rank T$ (d)) $rank T = rank(L_{[T]_\alpha^\beta}) = rank(L_B) = rank B$ 回.2 按 def

即 $rank(AB) \leq min\{rk(A), rk(B)\}$

T_3 (Frobenius) $rk AB + rk BC \leq rk B + rk ABC$

□ $rk AB = rk B - dim(R(B) \cap N(A))$ (回.四) 又, 令 $B \leftarrow BC$, $rk ABC = rk BC - dim(R(BC) \cap N(A))$

$\therefore rk AB + rk BC = rk B + rk ABC + dim R(BC) \cap N(A) - dim R(B) \cap N(A)$ ≤ 0 因 $R(BC) \subseteq R(B)$, 左 $\leq$ 右

Note: 当 $B \leftarrow I_n$ 为 (Sylvester) $C \leftarrow B$ $rk AB \geq rk A + rk B - n$

Date. /　　　即有无eigval的L.T.

例1　a) T: rotate $\mathbb{R}^2$ by $\pi/2$　b) 令 $C^\infty(\mathbb{R}) = \{$set of func: $\mathbb{R}\to\mathbb{R}$ | f有任意阶导数$\} \leq \mathcal{F}(\mathbb{R},\mathbb{R})$

$T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}),\ f \to f'$　找出eigenvector

□ a) 几何角度，无T(v)为v的伸展(kv) ∴ 无eigenvector，not diagonalizable 无此β

b) 若f eigenvec. 有 $T(f) = f' = \lambda f$　1-order DE: $f(t) = ce^{\lambda t}$ ∴ f在 $c\neq 0$ 时为eigenvector 对应λ为∀ real ($\lambda = 0$, f为zero func)

例2: V vec. space, k scalar. kI_V diagonalizable?

□ note 对∀ basis β, $[kI_V]_\beta = kI$，对角矩阵 ∴ 无论何basis都是 (由f(λ)，仅一eigenval = k)

若A similar to $\underset{\text{scalar matrix}}{\underline{\underline{kI}}}$，则 $A = kI$；且 diagonalizable 只有一eigenval的为 scalar matrix

□ $A = P^{-1}kIP = kP^{-1}IP = kI$；令 $D = \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$ 只有一eigenval, $\lambda_1 = \dots = \lambda_n = \lambda$

由此证 $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ 无法 diagonalizable　□ $f(\lambda) = (1-\lambda)(1-\lambda)$，一eigenval，若可diagonalizable，为scalar matrix，False

例3、证无basis in $\mathbb{R}^2$ st. $[T]_\beta = A = \begin{pmatrix}1&2\\2&6\end{pmatrix}$，$[T]_\gamma = B = \begin{pmatrix}2&1\\1&3\end{pmatrix}$

□　令 $B = \underset{\gamma\to\beta}{Q^{-1}} A \underset{\gamma\to\beta}{Q}$，$\det(C-\lambda I) = \det(Q^{-1}AQ - \lambda Q^{-1}IQ) = \det(Q^{-1}(A-\lambda I)Q)$

$= \det Q^{-1} \det(A-\lambda I) \det Q = \det A-\lambda I$，即 characteristic poly 相同，但 $f_A(t) = \lambda^2 - 7\lambda + 2 \neq f_B(t) = \lambda^2 - 5\lambda + 5$

例4. 若 $T: M_2(\mathbb{R}) \to M_2(\mathbb{R}),\ A \to A^T$. 求T的特征值

□ 令 $T(A) = \lambda A$，$A^T = \lambda A$，而 $T(A^T) = T(\lambda A) = A = \lambda^2 A$ ∴ $\lambda = \pm 1$.
$\lambda = 1$，eigspa = {symmetric}　$\lambda = -1$ eigspa = {skew-symmetric}，dim见P12 例1

例5 证 real symmetric 的eigenval 也real

□ 对 $Ax = \lambda x$，$\langle Ax, x\rangle = \langle \lambda x, x\rangle = \lambda\langle x,x\rangle$
$= \bar{\lambda}\langle x,x\rangle$ P49 P3 ii) ∴ $(\lambda - \bar{\lambda})\langle x,x\rangle = 0$ ∴ $\lambda = \bar{\lambda}$
只有x=0 才为0，且eigenvec ≠ 0

~~例6，real symmetric 的 distinct eigenvec 且 mutually orthogonal~~

□　对 $\lambda_1 \neq \lambda_2$，$Av_1 = \lambda_1 v_1$，$\langle Av_1, v_2\rangle = \langle\lambda_1 v_1, v_2\rangle = \lambda_1\langle v_1 v_2\rangle$
$\lambda_1 \neq \lambda_2$　$Av_2 = \lambda_2 v_2$
∴ $\underline{\lambda_1\langle v_1,v_2\rangle = \lambda_2\langle v_1 v_2\rangle}$ ∴ $\langle v_1, v_2\rangle = 0$　　$= \underset{v_1^T A^T v_2)}{\sqrt{Av_1}^T v_2} = \langle v_1, \underset{\text{symmetric}=A}{A^T} v_2\rangle = \langle v_1, \lambda_2 v_2\rangle$

~~证1. $tr(A) = \lambda_1 + \dots + \lambda_n$~~　法1 ∀ matrix similar to Jordan Form, similar matrix share eigvalue.
∴ $tr(A) = tr(P^{-1}JP) = tr(PP^{-1}J) = tr(J) = \sum\lambda_i(J) = \sum\lambda_i(A)$　　法2 由右 $\mathcal{T}_{1n}$ k_{n-1} 由 $\lambda_1 + \dots + \lambda_n$ 组成

证2 skew-symmetric $A^T = -A$ 有 eval 0 or purely imaginary

□ $x \in \mathbb{C}^n$　$x^T A x \begin{cases} = \lambda x^T x = \lambda\|x\|^2 \\ = (A^T x)^T x = \bar{\lambda}\|x\|^2\end{cases} \Rightarrow \lambda = -\bar{\lambda}$
conjugate $-\bar{\lambda}\bar{x}$

diagonalization & eigen3 — 即对A，先change basis为β st. D为diagonal，再变换回来 找det

回中，令n阶方阵A为$[T]_\alpha$，若T diagonalizable，有$[T]_\beta$ diagonal

即 $A = \underset{\beta\to\alpha}{Q}\, D\, \underset{\beta\to\alpha}{Q^{-1}}$ (standard basis)

1. $T: V\to V$ diagonalizable iff ∃ ordered basis β s.t. $[T]_\beta$ diagonal

令方阵 A diagonalizable iff L_A diagonalizable 即 similar to diagonal

Note, 若 $D=[T]_\beta$ diagonal，对 $\forall v_j\in\beta$，$T(v_j)=\sum_i D_{ij}v_i = D_{jj}v_j = \lambda_j v_j$ } T diagonalizable ⟺ $T(v_j)=\lambda_j v_j$

而若 $T(v_j)=\lambda_j v_j$，则 $[T]_\beta = D = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$ $[[T(v_1)]_\beta\ [T(v_2)]_\beta \dots]$

称 $v\in V$ eigenvector of T ⟺ $\exists\lambda\in F$，$T(v)=\underset{\text{eigenvalue}}{\lambda} v$

对应地，$A_{n\times n}$，$v\in F^n$ eigenvector of A ⟺ v eigenvector of L_A

∴ 以eigenval/eigenvec表述，diagonalizable 指T的basis $\beta=\{v_1\dots v_n\}$为set of eigenvector，且因β basis，v_i distinct nonzero

因β basis，说明$A_{n\times n}$ ($L_A: F^n\to F^n$) 不会超过n个eigenval

2. λ eigenval ⟺ $\det(A-\lambda I_n)=0$ 记为f(λ)，characteristic polynomial 回.2 note: $A-\lambda I$ nullity > 0，$rk(A-\lambda I)<n$ 回.4.T_1 P7.T_1

□ ⟺ 非零 $Av=\lambda v$ ⟺ $(A-\lambda I_n)(v)=0$ ⟺ $A-\lambda I_n$ not invertible ⟸ $\det(A-\lambda I_n)=0$

对应的，characteristic polynomial of T 为 $f(\lambda)=\det([T]_\beta - tI_n)$

且 f(λ) indep. of chosen basis 回回.3 即使不同basis，而matrix representation similar ∴ f(λ)相同

T_{1a} characteristic polynomial 为degree n，且leading coeff 为 $(-1)^n$

□ WTS $f(\lambda)=(-1)^n\lambda^n + k_{n-1}\lambda^{n-1}+\dots+k_1\lambda+k_0$. In fact, by induction $f(\lambda)=(A_{11}-t)\cdots(A_{nn}-t)+q(\lambda)$，

q(λ) degree ≤ n-2：令对n-1成立，对n，$\begin{vmatrix}A_{11}-\lambda & & \\ & \ddots & \\ & & A_{nn}-\lambda\end{vmatrix} \overset{\text{最末行expand}}{=} (A_{nn}-\lambda)\underbrace{\begin{vmatrix}A_{11}-\lambda & & \\ & \ddots & \\ & & A_{n-1,n-1}-\lambda\end{vmatrix}}_{\text{满足归纳条件}} + \sum_{i=1}^{n-1}(-1)^{n+i}A_{ni}\det\tilde{A}_{ni}$ degree ≤ n-3

去掉 n row i col 即无$A_{nn}-\lambda$，$A_{ii}-\lambda$ ∴ degree at most n-2

$=(A_{nn}-\lambda)(A_{11}-\lambda)\cdots(A_{n-1,n-1}-\lambda)+(A_{nn}-\lambda)q(\lambda)+\underbrace{q'(\lambda)}_{\text{degree}\le n-2}$

∴ 展开 $(A_{11}-\lambda)\cdots(A_{nn}-\lambda)$，λ coeff 为 $(-1)^n$ $k_{n-1}=(-1)^{n-1}tr(A)$

NOTE，$f(0)=k_0=\det(A-0\cdot I_n)=\det(A)$. 见P$T_4$

T_{1b} $A_{n\times n}$ 最多n distinct eigenvalues □ 由T_{1a}，eigenval为f(λ)的zero. ∴ 最多n个不同实根

T_{1c} 对 $T:V\to V$，λ为eigenvalue of T. 则 $v\in V$ eigenvector ⟺ $v\neq 0 \wedge v\in Null(T-\lambda I)$

□ ⟹ $v\neq 0$，且有 $T(v)=\lambda v$ ∴ $T(v)-\lambda v = (T-\lambda I)(v)=0$ ⟸ 即$(T-\lambda I)(v)=0$，$T(v)=\lambda v$ 即E_λ (回.4)

3. T_{2a} let $\lambda_1\dots\lambda_k$ distinct eigenval of T，$\{v_1\dots v_k\}$ L.ind

□ induction on k，k-1 T，则对k，let $\sum_{i=1}^k a_iv_i = 0$ ①，两边apply func: $T-\lambda_k I$: $a_1[T(v_1)-\lambda_k I(v_1)]+\dots=0$

$a_1(\lambda_1-\lambda_k)v_1+\dots+a_k 0 = 0$ ∵ $\{v_1\dots v_{k-1}\}$ L.ind. ∴ $a_1(\lambda_1-\lambda_k)=\dots=a_{k-1}(\lambda_1-\lambda_{k-1})=0$ 而λ_i distinct

∴ $\lambda_i-\lambda_k\neq 0$ ∴ $a_1=\dots=a_{k-1}=0$

则①：$a_kv_k=0$ 但v_k eigenvector，∴ $v_k\neq 0$ ∴ $a_k=0$

T_{2b} T有n distinct eigenval ⟹ T diagonalizable □ $\{v_1\dots v_n\}$ L.ind. (T_{2a}) ∴ 由1.故，有basis of eigenvec. ∴ T diagonalizable

Note T_{2b} 逆非true. I 的eigenval只有1，但diagonalizable 例见回.例3.1

4. diagonalization 的好处是简化matrix power，若 $A=PDP^{-1}$ (找D与P)，$A^k = PD^kP^{-1}$，如回.例3

$D=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$ $P=\begin{pmatrix}| & & |\\ v_1 & \cdots & v_n\\ | & & |\end{pmatrix}$ 因P叫 eigendecomposition ⟺ $\{v_i\}$为basis

且 $\det A = \det P \det D \det P^{-1} = \det D = \prod\lambda_i$ 同result as P7.T_4

1. diagonal matrix $\begin{pmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_n \end{pmatrix}$ 记为 $diag(v)$,
易知 对 $x \in \mathbb{F}^n$, $diag(v) \cdot x = \sum x_i v_i = v \odot x$
且 $diag(v)^{-1} = diag([\frac{1}{v_1} \cdots \frac{1}{v_n}]^T)$

triangular matrix $\begin{pmatrix} a & b & \\ 0 & \ddots & \ddots \end{pmatrix}$ 易用cofactor expansion 证 $\det = \prod diag_i$
strict triangular $\begin{pmatrix} 0 & 0 & \\ 0 & \ddots & 0 \end{pmatrix}$ main diagonal $= 0$, $\det = 0$
tri $\underset{\times}{+}$ tri 仍为 tri

2. 用(a) square A nilpotent iff $\exists m \geq 1$ st. $A^m = 0$ □ characteristic $f(\lambda) = (-1)^n \lambda^n$
strict triangular nilpotent : $A^n = 0$ 由 Cayley-Hamilton $f(A) = (-1)^n A^n = 0$

(b) triangular A inverse 仍为 triangular (若可逆) □ 法1 $A^{-1} = \frac{1}{\det A} adj A$ 图.3 若 A upper/lower triangular, $C_{ij} = 0$ $i<j$ / $i>j$
note C_{ij} 顺序 $\therefore adj(A)$ upper/lower
法3 对上△ A 拆为 $\underset{\text{diag}}{D}(I + \underset{\text{对角线上0}}{B})$, $A^{-1} = (I+B)^{-1} D^{-1}$, 因 $(I+B)^{-1} = I - B + B^2 - \cdots + (-1)^{n-1} B^{n-1}$ (之后为0 因 $B^n = 0$)
法2 令 $A^{-1} = [a_1' \cdots a_n']$ $AA^{-1} = [Aa_1' \cdots Aa_n'] = I$ $\therefore Aa_k' = e_k = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} k$ 若 A lower triangular thus a_k' 在 k row 上 必为 0 $\therefore A^{-1}$ lower triangular
Note 可逆 $\Leftrightarrow$ $\det \neq 0$ $\Leftrightarrow$ $diag_i \neq 0$

measure 一个门将 unit ball vector transform 成 ellipsoid 的 size ($\|x\|\le 1$, stretch)

Date.

因 $\|A\|_{op} \ge \forall x\ne 0\ \frac{\|Ax\|}{\|x\|}$ ∴ $\|A\|_{op}$ 也称 induced norm 或 4.

1. (operator norm) $\|A\|_{op} = \sup_{x\neq 0}\frac{\|Ax\|}{\|x\|} = \inf\{c\ge 0 \mid \|Ax\|\le c\|x\|, \forall x\} = \|A\|_{ab} = \sup_{\|x\|_b\le 1}\|Ax\|_a$ (ab 2 norm)

证明为 norm ☐ 有 property i) $\forall A, x$ 有 $\|Ax\|_a \ge 0$, $\forall A\ \sup_x \|Ax\|_a \ge 0$ ∴ $\forall A\ \sup_{\|x\|_b\le 1}\|Ax\|_a \ge 0$

当 $A=0$, $\|0\|_b = \sup_{\|x\|_b\le 1}\|0x\|_a = 0$, 而若 $\|A\|_{ab}=0$, $\sup_{\|x\|_b\le 1}\|Ax\|_a = 0$ (def), ∴ $\forall \|x\|_b\le 1$ 有 $\|Ax\|_a = 0 \Rightarrow Ax=0 \Rightarrow A=0$

ii) $\sup\|\alpha Ax\|_b = \sup|\alpha|\,\|Ax\|_b = |\alpha|\sup\|Ax\|_b$ iii) 有 RHS $\|A\|_{ab}+\|B\|_{ab} = \sup\|Ax\|_a + \sup\|By\|_a \ge \|Ax\|_a + \|By\|_a$, $\forall \|x\|_b, \|y\|_b\le 1$. 因 x, y 任意, 取 $u=x=y$ 且 $\|u\|_b\le 1$, 有上式 $= \|Au\|_a + \|Bu\|_a \ge \|Au+Bu\|_a$ ∴ by def $\|A\|_{ab}+\|B\|_{ab} \ge \sup_{\|u\|_b\le 1}\|Au+Bu\|_a = \|A+B\|_{ab}$

2. 定理 i) $\forall \|\cdot\|_a$ cont. on V.S. V under topology induced by $\|\cdot\|_1$ (unif. cont.)

☐ WTS $\forall\varepsilon>0\ \exists\delta>0$ st. $\|x-x'\|_1<\delta \Rightarrow |\,\|x\|_a - \|x'\|_a| < \varepsilon$

由有 property iii) on $\|\cdot\|_a$ $\begin{cases}\|x\|_a - \|x'\|_a = \|x'+(x-x')\|_a - \|x'\|_a \le \|x-x'\|_a \\ \|x'\|_a - \|x\|_a = \|x-(x-x')\|_a - \|x\|_a \le \|x-x'\|_a\end{cases}$ ∴ RHS $\le \|x-x'\|_a$

因 x under V 有 basis $\{e_i\}^n$ st. $x=\sum^n \alpha_i e_i$, $x'=\sum^n \alpha_i' e_i$, $\|x-x'\|_a \le \sum^n |\alpha_i-\alpha_i'|\cdot\|e_i\|_a \le \|x-x'\|_1 \cdot \boxed{M}$ ($\max\|e_i\|_a$)

∴ 若令 $\delta = \frac{\varepsilon}{M}$, $\|x-x'\|_1<\delta$ 有 $|\,\|x\|_a - \|x'\|_a| \le \|x-x'\|_a \le \|x-x'\|_1\cdot M < \varepsilon$

ii) (norm equivalent) $\forall$ 2 $\|\cdot\|_a, \|\cdot\|_b$ on V.S. V equiv, 即 $\exists$ real $0<C_1\le C_2$ st. $\forall x\in V$, $C_1\|x\|_b \le \|x\|_a \le C_2\|x\|_b$

step 1) (norm transitivity) 若 $\|\cdot\|_a, \|\cdot\|_b$ equiv. $\|\cdot\|_1$, 则 $\|\cdot\|_a$ equiv. $\|\cdot\|_b$

☐ 有 $C_1\|x\|_1 \le \|x\|_a \le C_2\|x\|_1$, $C_1'\|x\|_1 \le \|x\|_b \le C_2'\|x\|_1$ ∴ $\frac{C_1'}{C_2}\|x\|_a \le \|x\|_b \le \frac{C_2'}{C_1}\|x\|_a$ ∴ equi.

step 2) 只需证 $C_1\|x\|_1 \le \|x\|_a \le C_2\|x\|_1$. 若 $x=0$ trival, 若 $x\ne 0$, 取 $u=\frac{x}{\|x\|_1}$ 则 WTS $C_1 \le \|u\|_a \le C_2$

Note $\|u\|_1 = 1$. 由 Extreme Value Thm, cont. $\|\cdot\|_a$ on compact set $\{u \mid \|u\|_1 = 1\}$ (close & bounded) 必有 max min in set

∴ 令 $C_1 = \min_{\|u\|_1=1}\|u\|_a$, $C_2 = \max_{\|u\|_1=1}\|u\|_a$ 因 $u\ne 0$, $C_2\ge C_1>0$, 且 $C_1\le\|u\|_a\le C_2$

3. 证 $\|A\|$ induced norm 为 matrix norm

☐ i) 对 $A\ne 0$ 有 $\|x\|=1$, st. $Ax\ne 0$, ∴ $\|Ax\|\ne 0$ (vec norm def), 又若 $A=0$ 无论 x 为何 $Ax=0$, ∴ $\|Ax\|=0$

ii) $\|cA\| = \max\|cAx\| = \max|c|\,\|Ax\| = |c|\max\|Ax\|$ 对 induced) $\forall y\ne 0$ 有 $x=\frac{y}{\|y\|}$ 满足 $\|x\|=1$

iii) 可找到 $\|x_0\|=1$ & $\|A+B\| = \|(A+B)x_0\|$, ∴ $\|Ax_0+Bx_0\| \le \|Ax_0\|+\|Bx_0\|$ (iii) of vec. norm) ∴ $\|Ay\| = \|A\|y\|x\| = \|y\|\cdot\|Ax\| \le \|y\|\cdot\|A\|$

$\le \|A\|\|x_0\| + \|B\|\|x_0\|$ (由 induced) of matrix norm)

iv) 可找 $\|x_0\|=1$, 有 $\|ABx_0\| = \|AB\|$

∴ $\|ABx_0\| \le \|A\|\cdot\|Bx_0\| \le \|A\|\,\|B\|\,\|x_0\|$ (induced) of matrix norm)

$A^TA = UDU^T$ spectral decomp $[A^TAx]_x = [A^TA]_x[x]_x = Q^{-1}_{x\leftarrow e}[UDU^T]Q_{x\leftarrow e}[x]_x = U^{-1}UDU^TU[x]_x = D[x]_x = \sum c_i\lambda_i x_i$ (note $U=Q_{x\leftarrow e}$) (recent upper … 2 P25 定理)

4. matrix norm induced by vec norm $\|x\|_2 = \sqrt{\langle x,x\rangle}$, 为 $\|A\| = \sqrt{\lambda_1} = \sigma_1$, λ_1 为 A^TA 的 largest eval ($\sigma_1(A)$ max singular v…)

☐ $\|Ax\|^2 = \langle x, A^TAx\rangle$ 因 A^TA psd Optim. P6. P1b), 有 $\lambda_1\ge\cdots\ge\lambda_n\ge 0$ & orthonormal evec basis $\{x_i\}$ P50. 1

取 $\|x\|=1$. $= \langle c_1x_1+\cdots+c_nx_n,\ c_1\lambda_1x_1+\cdots+c_n\lambda_nx_n\rangle$ 令 x 由 $\langle x_i\rangle$ 表示, 因 $\|x\|=1$ 有 $c_1^2+\cdots+c_n^2=1$

($\{x_i\}$ orthonormal) $= \lambda_1c_1^2+\cdots+\lambda_nc_n^2 \le \lambda_1(c_1^2+\cdots+c_n^2) = \lambda_1$, 等号当 $x\in x_1$ 取到 $\langle x_1, A^TAx_1\rangle = \langle x_1, \lambda_1x_1\rangle = \lambda_1$

$\|A\| = \max\|Ax\| = \sqrt{\lambda_1}$

5. (Rayleigh) 对 P pd 有 $\lambda_{min}\|x\|^2 \le x^TPx \le \lambda_{max}\|x\|^2$ R P63 T1

Inner product & norm

norm 与 metric space (1回) $d(x,y)=\|x-y\|$

0. 复习了 def norm $\|x\|$, $\|\cdot\|: \mathbb{R}^n \to \mathbb{R}$
 i) $\|x\|>0$ if $x\neq\vec{0}$, $\|x\|=0 \Leftrightarrow x=\vec{0}$
 ii) $\|\alpha x\| = |\alpha|\,\|x\|$
 iii) (triangle) $\|x+y\| \le \|x\|+\|y\|$

L_1 $\|x\|_{l_1}=\sum_i |x_i|$
l_2 $\sqrt{\sum_i |x_i|^2}$
l_p $\sqrt[p]{\sum_i |x_i|^p}$
l_∞ $\max_i |x_i|$

1. $\langle\cdot,\cdot\rangle: \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ 为 inner product iff
 a. positivity $\langle x,x\rangle \ge 0$ ($x=0$ 等号成立)
 b. symmetry $\langle x,y\rangle = \langle y,x\rangle$
 c. additivity $\langle x+y,z\rangle = \langle x,z\rangle+\langle y,z\rangle$
 d. homogeneity $\langle rx,y\rangle = r\langle x,y\rangle$, $r\in\mathbb{R}$

P_1 i) $\langle x,y+z\rangle = \langle y+z,x\rangle = \langle y,x\rangle+\langle z,x\rangle = \langle x,y\rangle+\langle x,z\rangle$
 ii) $\langle x,ry\rangle = \langle ry,x\rangle = r\langle y,x\rangle = r\langle x,y\rangle$

2. def Euclidean inner product $\langle x,y\rangle = \sum^n x_iy_i = x^Ty$
 且 norm $\|x\| = \sqrt{\langle x,x\rangle} = \sqrt{x^Tx}$

P_2 i) (Cauchy-Schwarz) $|\langle x,y\rangle| \le \|x\|\,\|y\|$, $x=\alpha y$ 等号成立, $\alpha\in\mathbb{R}$ 即 $(\sum x_iy_i)^2 \le (\sum x_i^2)(\sum y_i^2)$

□ 若 $\|x\|=\|y\|=1$, $0\le \|x-y\|^2 = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2 = 2-2\langle x,y\rangle$ (1.a), $\therefore \langle x,y\rangle \le 1$ 等号 $x=y$ (1.d)
当 x,y 有 $=0$, 显然. 若 $x,y\neq 0$. 令 $x' \leftarrow x/\|x\|$, $y' \leftarrow y/\|y\|$, now $\|x'\|=\|y'\|=1$ $\therefore \langle x',y'\rangle = \frac{\langle x,y\rangle}{\|x\|\cdot\|y\|} \le 1$
且令 $x' \leftarrow -x$, $\langle x',y\rangle = -\langle x,y\rangle \le \|x\|\cdot\|y\|$ 上行未 $\therefore |\langle x,y\rangle| \le \|x\|\cdot\|y\|$ 等号 $\frac{x}{\|x\|} = \frac{\pm y}{\|y\|}$

证明此 $\|\cdot\|$ 确为 norm
 ii) $\|x\| \ge 0$, 等号 $x=0$ 1.a
 iii) (homogeneity) 对 $r\in\mathbb{R}$, $\|rx\| = |r|\,\|x\|$ □ $\sqrt{\langle rx,rx\rangle} = \sqrt{r^2\langle x,x\rangle}$ 1d
 iv) (三角不等式) $\|x+y\| \le \|x\|+\|y\|$
 □ $\|x+y\|^2 = \|x\|^2 + 2\underbrace{\langle x,y\rangle}_{\le \|x\|\|y\|\ \text{i)}} + \|y\|^2 \le (\|x\|+\|y\|)^2$

 v) $\big|\,\|x\|-\|y\|\,\big| \le \|x-y\|$
 □ $\|x\| = \|x-y+y\| \le \|x-y\| + \|y\|$ (iv) $\therefore \|x\|-\|y\| \le \|x-y\|$ 同理 $\|y\|-\|x\| \le \|y-x\| = \|x-y\|$ (iii)

Note $\|\cdot\|$ 为 unif. cont: $\forall\varepsilon>0$, 令 $\delta=\varepsilon$, 当 $\|x-y\|<\delta=\varepsilon$, 有 $\big|\,\|x\|-\|y\|\,\big| \le \|x-y\| < \varepsilon$ 见左 v)

3. 对 $\mathbb{C}^n$, inner product $\langle x,y\rangle := \sum^n x_i\bar{y}_i$

P_3 i) $\langle x,y\rangle = \overline{\langle y,x\rangle}$ □ 右右 $\overline{\sum y_i\bar{x}_i} = \sum \bar{y}_i\bar{\bar{x}}_i = \sum \bar{y}_i x_i$ conj 性质
 ii) $\langle x, r_1y+r_2z\rangle = \bar{r}_1\langle x,y\rangle + \bar{r}_2\langle x,z\rangle$ □ $\overline{\langle r_1y+r_2z,x\rangle} = \overline{r_1\langle y,x\rangle + r_2\langle z,x\rangle}$ (i), 1.d $= \overline{r_1\langle y,x\rangle}+\ldots = \bar{r}_1\,\overline{\langle y,x\rangle}+\ldots$ conj 性质

4. matrix norm 不仅需满足前述 3 property, 应还有 $\|AB\| \le \|A\|\,\|B\|$ eg. Frobenius $\|A\|_F = \sqrt{\sum_i\sum_j A_{ij}^2}$
希望 connect matrix norm & vector norm, $\therefore$ 研究 induced norm (与之1 vp. norm equiv)
令 $\|\ \|_{(n)}$, $\|\ \|_{(m)}$ vec norm on $\mathbb{R}^n$, $\mathbb{R}^m$, matrix norm $\|\cdot\|$ induced by $\|\cdot\|_{(n)}$, $\|\cdot\|_{(m)}$ iff $\forall A\in\mathbb{R}^{m\times n}$, $x\in\mathbb{R}^n$ $\|Ax\|_{(m)} \le \|A\|\,\|x\|_{(n)}$ 证明 $\|I\|=1$
eg. $\|A\| = \max_{\|x\|_{(n)}=1} \|Ax\|_{(m)}$ 为 induced norm (见左了)

NOTE 由左 v) cont + VS compact $\forall A$ $\exists \|x_0\|=1$ st. $\|Ax_0\| = \|A\|$ (即 $\|A\|$ attainable)

5. spectral radius $\rho(A) = \max\{|\lambda(A)|\}$ note $\|A\| \ge \|Ax\| = |\lambda|\cdot\|x\| = \rho(A)$, induce norm 下界 radius. 有时用 $\rho(A)$ 近似 $\|A\|_2$
 令 $\lambda\text{-}x$ 对应 $|\lambda(A)|_{max}$, norm x st. $\|x\|=1$

Date. /

证1. real symmetric A ∀eigenval ∈ℝ

□ 考虑 $\lambda(\underbrace{\bar{x}^T x}_{\text{complex inner prod}}) = \bar{x}^T(\lambda x) = \bar{x}^T A x \overset{A \text{ symmetric}}{=} (A\bar{x})^T x = (\bar{\lambda}\bar{x})^T x = \bar{\lambda}(\bar{x}^T x)$

$\therefore (\lambda - \bar{\lambda})(\bar{x}^T x) = 0$ ... $\therefore \lambda = \bar{\lambda}$

$\langle x, x\rangle_{\mathbb{C}} \geq 0$, 若 $=0$ 则 $x=0$, 但 x 为 evec

---

1. (Spectral Thm for symmetric A)

i) ∀eigenval $\lambda \in \mathbb{R}$ 且对应 eigvec real (证1)　总有n个eigvec

ii) distinct eval 的 evec orthogonal (与 P_{in})

iii) ∃ diag $D \in \mathbb{R}^{n\times n}$ & orthogonal $U \in \mathbb{R}^{n\times n}$ s.t. $A = UDU^T$, $D = \text{diag}(\lambda_{1\ldots n})$ U orthogonal

左 compact 定 / 就 svd decomp → 称为 spectral decomposition □ 由 级四.4 ⇒ diagonalizable, 因 $U = [v_1 \ldots v_n]$, $U^T U = I$. $\therefore A = UDU^T = UDU^{-1}$　定2　U unitary

iv) $\{v_1 .. v_n\}$ eigvec 为 orthogonal basis (即 $v_i^T v_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$)

即使不一定有n distinct eigval　□ U可逆, ∴ full-rank, ∴ L.Ind. $\{v_i\}$, 可由 Gram Schmidt 生成 orthonormal

---

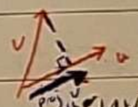
证2 diagonalizable A 有 orthogonal evec ⇒ A symmetric

□ A 可 [25].4 $A = PD\underbrace{P^{-1}}_{=P^T \text{ 因 evec orthogonal}}$　$A^T = P^{-1\,T} D^T P^T = P^{-1\,T} D P^{-1} = A$, $P^{-1\,T} = P$

num stable 见 [36].3　例见 [37]. 例1

2. (Gram-Schmidt Process)

对 L.Ind $\{v_{1..k}\}$ → orthonormal L.Ind $\{e_{1..k}\}$. Def $\text{proj}_u(v) = \frac{\langle u, v\rangle}{\langle u, u\rangle} u$, $\text{proj}_0(v) \overset{\Delta}{=} 0$



易知 $\text{proj}_u v$ 与 u 同方向 且 $v - \text{proj}_u v \perp u$

□ $u_1 = v_1$　$e_1 = \text{norm}(u_1) = \frac{u_1}{\|u_1\|}$

$u_2 = v_2 - \text{proj}_{u_1}(v_2)$　$e_2 = \text{norm}(u_2)$

⋮

$u_k = v_k - \sum_{i=1}^{k-1} \text{proj}_{u_i}(v_k)$　$e_k = \text{norm}(u_k)$

Induction 证各自 orthogonal　eg. $u_1 u_2 = v_1^T v_2 - v_1^T \text{proj} = 0$

即 each new $u_i \perp \forall u_j, j = 1 \ldots i-1$

若 v_k 为 1st L.dep vector 则 $u_j = 0$, break Alg

即 $u_k = v_k - \text{proj}_{\langle u_1..k-1\rangle}(v_k)$ where proj 为 projection construct by [52].2

→ $u_k = v_k - \sum \frac{\langle u_i, v_k\rangle}{\langle u_i, u_i\rangle} u_i = v_k - \sum (e_i^T v_k) \cdot e_i$

证3 A symmetric A^{-1} 也 symmetric　□ $(A^{-1})^T = (A^T)^{-1} = A^{-1}$

---

2. 即使 $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ symmetric 也有 spectral decomp 的 U (eg. natural basis I), 因 构建 by process on A

总可找到 L.Ind $u_{2..n}$ wrt v_1, 再 Gram Schmidt

- 令 $\lambda_1 v_1$ epair, find orthonormal $\{v_1, u_{2..n}\}$ basis, 令 $U_1 = (v_1\ u_2 \ldots u_n)$, note $U_1^T A U_1$ symmetric
- 1st col of $U_1^T A U_1$ 为 $(U_1^T A) v_1 = U_1^T \lambda_1 v_1 = \lambda_1 \begin{pmatrix} -v_1- \\ -u_2- \\ \vdots \\ -u_n- \end{pmatrix} \begin{pmatrix} v_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\therefore U_1^T A U_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & B_{(n-1)\times(n-1)} \end{pmatrix}$, 因 B symmetric
- apply process on B .. until find orthonormal basis

证4. $\{x_{1..n}\}$ mutual orthogonal, $x_i \neq 0$. 证 $\{x_{1..n}\}$ L.Ind. □ 对 $\sum^n \alpha_i x_i = \vec{0} \overset{\text{乘 } x_j,\ j=1..n}{\Longrightarrow} (\sum \alpha_i x_i)^T x_j = \vec{0}^T x_j = 0$

$\alpha_j x_j^T x_j$

$\alpha_j = 0$

证5 对 symmetric A　$\lambda_{min}(A^{-1}) = \frac{1}{\lambda_{max}(A)}$; $\lambda_{max}(A^{-1}) = \frac{1}{\lambda_{min}(A)}$

□ A eval ℝ 见 [15].①

orthogonal & eigen 4

T_0 $x \perp y$ 为 $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ 展开 $\|x+y\|^2$

1、 x, y orthogonal iff $\langle x, y\rangle = 0$ ‖ A orthogonal matrix iff $A^T = A^{-1}$ (即 $A^TA = I$)

Generalization of rotation P31 & reflexion & permutation P34

讲 col of A 皆 orthogonal 且 norm = 1 (orthonormal)

□ $\forall i,j$ $(A^TA)_{ij} = I_{ij} = 0 = \sum_k A_{ki}A_{kj}$ col dot product

$i=j$ 时 $=1$ 验证行 row 也 orthonormal

2. Prop a) 对 symmetric A，其 eigenval $\in \mathbb{R}$ 且 对 eigvec $v, \hat v$ associate with eval $\lambda \neq \hat\lambda$，$v, \hat v$ orthogonal

A 有 $v_i = \lambda_i$ … (基础 1)

□ $\hat v^T(Av) = \lambda \hat v^T v$，$= (A^T\hat v)^T v = \hat\lambda \hat v^T v$ ∴ 只当 $\hat v^T v = 0$ 时 $\lambda \hat v^T v = \hat\lambda \hat v^T v$

1b) $A = \sum_{i=1}^{n} \lambda_i u_i u_i^T$，当 $\{(\lambda_i, v_i)\}_{i=1}^{n}$ 为 eigen pairs，v_i normalized

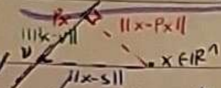
□ $AI = A = A\sum v_i v_i^T$，考虑 $Av_i = \lambda_i v_i$，同乘 v_i^T 并 sum over i，$\sum_i A v_i v_i^T = \sum_i \lambda_i v_i v_i^T = AI = A$

1c) $I = \sum^n v_i v_i^T$ $\{v_i\}^n$ eigen vec

□ $\{v_i\}^n$ orthogonal basis，∴ 对 $\forall x = \sum_k^n \alpha_k v_k$，考虑 $\boxed{\sum_i^n v_i v_i^T} x = \sum_i^n \sum_k^n v_i v_i^T \alpha_k v_k = \sum_k^n \alpha_k v_k = x$

$= \delta_{ik}$ Kronecker delta，orthogonal …

因 x 任意 ∴ □ 必为 I



同证为 subspace，即使 V 非 subspace

3、 令 VS 的 orthogonal complement $V^\perp = \{x \mid v^Tx = 0, \forall v \in V\}$ (有 $V \oplus V^\perp = \mathbb{R}^n$，当 V subspace)

称 linear transform T orthogonal projection onto V，iff $\forall x \in \mathbb{R}^n$，$T(x) \in V$ 且 $x - T(x) \in V^\perp$ ($T: VS\,\mathbb{R}^n \to V$)

Thm P orthogonal projection onto subspace $V = R(P)$ $\Leftrightarrow$ $P^2 = P = P^T$ (symmetric)

def, $x - Px \in R(P)^\perp$ $N(P^T)$

□ $\Rightarrow$ $R(I-P) \subseteq R(P)^\perp$，∴ $\forall x \in V$ 有 $P^T(I-P)x = \vec 0$，∴ $P^T(I-P) = 0$ zero matrix ∴ $P^T = P^TP$

∴ $P = P^{T^T} = (P^TP)^T = P^TP = P^T$ $\Leftarrow$ 考虑 $\forall y \in V$，$(Py)^T(I-P)x = y^TP(I-P)x = 0$

∴ $(I-P)x \in R(P)^\perp$

2a) $R(A)^\perp = N(A^T)$ 且 $N(A)^\perp = R(A^T)$

□ 1° $\subseteq$ 令 $x \in R(A)^\perp$，$\forall v \in R(A)$，则 $\exists y \in V$ $v = Ay$，由 $(Ay)^Tx = 0$，$y^TA^Tx = 0$，即 $A^Tx = 0$ $x \in N(A^T)$

$\supseteq$ 令 $x \in N(A^T)$ $\forall v \in R(A)$ ——，考虑 $v^Tx = y^TA^Tx = 0$

$L_A: \mathbb{R}^n \to \mathbb{R}^m$

$\mathbb{R}^n = N(A) \oplus R(A^T)$

$\mathbb{R}^m = N(A^T) \oplus R(A)$

2° apply $\perp$ on both side 且 由 2c) 即 $N(A)$ vec $\perp$ A^T col span vector

2b) $V^{\perp\perp} = V$ $\subseteq$ 令 $\{a_1, \dots a_n\}$ V，$\{b_1, \dots b_n\}$ $V^\perp$ basis，by def $V = R(A = [\vec a_1 \dots \vec a_n])$，$V^\perp = R(B = [\vec b_1 \dots \vec b_n])$ ∴ WTS $R(B) \subseteq R(A)$

令 $x \in N(A^T)$ $R(A)^\perp = V^\perp$ ∴ $\forall v \in V$ 有 $v^Tx = 0$，因 $Bv \in V$，$(Bv)^Tx = 0 = v^T(B^Tx)$，因 v arbitrary，$B^Tx = 0$，$x \in N(B^T)$

$\supseteq$ 易证 by def ∴ 由 図 3

2c) $V \subseteq W \Rightarrow W^\perp \subseteq V^\perp$ □ $\forall w' \in W^\perp$ ∴ $\forall w \in W$ 有 $w^Tw' = 0$ 而 $\forall v \in V$ 也 $\in W$

2d) P 可以 orthogonal decomposition VS. $\forall x \in VS$，$\underbrace{Px}_{\in V} + (I-P)x \in V^\perp$ 由 def

unique

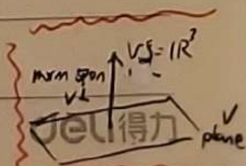
2e) P orthogonal on V，则 $Px = x, \forall x \in V$ 且 $R(P) = V$ 且 $N(P) = V^\perp$

□ 1° x orthogonal decomp 为 $Px + (I-P)x$，但 $x = x + 0$ 也为 因 $x \in V$，∴ $Px = x$ (orthogonal decomp unique)

2° by def $R(P) \subseteq V$ 但 $Px = x$ ∴ $V \subseteq R(P)$

3° 同取 $\perp$ $N(P)^\perp = V^\perp$ 2 $V = R(P^T) = R(P)$

2f) $\forall x \in VS$. $\|Px\| \le \|x\|$ □ $\|x\|^2 \overset{2d)}{=} \|Px + (I-P)x\|^2 \overset{T_0}{=} \|Px\| + \dots \le \|Px\|$



orthocell 2.

Date. /

1. matrix A 为 projection matrix iff $A^2 = A$ (idempotent)

即 $P^T = P = P^2$, 见上之 3

若 $\langle Px, y\rangle = \langle x, Py\rangle$ 则为 orthogonal projection, 若不 orthogonal 则 oblique

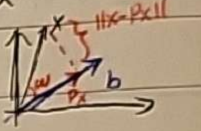
$\begin{pmatrix}x\\y\\z\end{pmatrix}$ 与 $\begin{pmatrix}x\\y\\0\end{pmatrix}$ 连5线 xy-plane 因为 $P=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$ 有 $P^2\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\0\end{pmatrix} = P\begin{pmatrix}x\\y\\z\end{pmatrix}$

P: i) projection A evol = 0/1 ☐ $A^2 v = \lambda^2 v = Av = \lambda v$, $\therefore (\lambda^2-\lambda)v = 0$ (非0 vec)

ii) proj A B 乘不一定 proj 例 $\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1/2&1/2\\1/2&1/2\end{pmatrix} = \begin{pmatrix}1/2&1/2\\0&0\end{pmatrix}$

除非若 $AB = BA$ $(AB)^2 = ABAB = AABB = A^2B^2 = AB$

2. construct projection on 1° 1-D line span$\{\vec{b}\}$ of vector $\vec{x}$



- project $\vec{x}$ on line, $\therefore Px = \lambda\vec{b}$
- projected Px 与 x closest, 即 $\min\|x - Px\|$

只当垂直, $\therefore \langle Px - x, b\rangle = 0 = \langle \lambda b - x, b\rangle \iff \lambda = \frac{b^T x}{\|b\|^2}$

- $\therefore$ projected $Px = \frac{b^T x}{\|b\|^2} b$ Note 由此可知 $\|Px\| = \|\lambda b\| = |\lambda|\cdot\|b\| = \frac{|b^T x|}{\|b\|^2}\|b\| = |\cos w|\|x\|\cdot\|b\|\cdot\frac{\|b\|}{\|b\|^2}$ $= |\cos w|\cdot\|x\|$

且 $= b\lambda = \frac{bb^T}{\|b\|^2}x$ $\therefore P = \frac{bb^T}{\|b\|^2}$ P i); Px 为 P evec, eval = 1

2° Project on $V \subseteq \mathbb{R}^n$, $\dim V = m \geq 1$ 有 basis $\{\vec{b}_{1\ldots m}\}$

- project x on V, $Px = \sum \lambda_i b_i = [\vec{b}_1 \ldots \vec{b}_m]\begin{bmatrix}\lambda_1\\ \vdots \\ \lambda_m\end{bmatrix} = B\lambda$
- 垂直/orthogonal to all V's basis $\{b_{1\ldots m}\}$, $\therefore \langle Px - x, b_1\rangle = 0 = b_1^T(x - B\lambda)$

$\vdots$

$\langle Px - x, b_m\rangle = 0 = b_m^T(x - B\lambda)$

m,n $n,1$

$\iff \begin{pmatrix}-b_1-\\ \vdots \\ -b_m-\end{pmatrix}(x - B\lambda) = 0$

$\iff B^T(x - B\lambda) = 0$

$\iff B^T B\lambda = B^T x$

$\therefore \lambda = (B^T B)^{-1}B^T x = B^{+}x$ (pseudo-inverse)

- $\therefore$ projected $Px = B\lambda$, $P = B(B^T B)^{-1}B^T$ (当 m=1, $B^T B$ scalar, reduce 为 1°)

例 1. 1° 对 line $\left\langle\begin{pmatrix}1\\2\\2\end{pmatrix}\right\rangle$, projection $P = \frac{1}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}(1\ 2\ 2) = \frac{1}{9}\begin{pmatrix}1&2&2\\2&4&4\\2&4&4\end{pmatrix}$, note 对任意 x 如 $\begin{pmatrix}1\\1\\1\end{pmatrix}$ $P(x) = \frac{1}{9}\begin{pmatrix}5\\10\\10\end{pmatrix}$

且 $P^2(x) = P(x)$

2° 对 plane $V = \left\langle\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix}\right\rangle$, $P = B(B^T B)^{-1}B^T$ = $\begin{pmatrix}1&0\\1&1\\1&2\end{pmatrix}\ldots = \frac{1}{6}\begin{pmatrix}5&2&-1\\2&2&2\\-1&2&5\end{pmatrix}$, $P(x) = B\lambda = B\begin{pmatrix}5\\-3\end{pmatrix}$

$B^T B\lambda = B^T x$

应用 least square 即 find $Ax = b$ 无 sol 时的近似 sol : find $\vec{x}^* \in Col(A)$ 使与 b closest

(orthogonal projection of b onto R(A)

quadratic form & norm 2 → P34.

1. quadratic form is $f: \mathbb{R}^n \to \mathbb{R}$, $x \to x^T Q x$, $Q \in \mathbb{R}^{n\times n}$ $\sum_{i,j} Q_{ij} x_i x_j$ 是齐2次 ∴ quadratic

NOTE Q 应 symmetric, 若非, 可用 $Q \leftarrow \frac{1}{2}(Q+Q^T)$ 代替, Q_{new} symmetric

of quadratic form $x^T Q x$ $\begin{cases} \text{positive definite} & >0 \text{ for } x\neq 0 \\ \text{positive semi definite} & \geq 0 \ \forall x \\ \text{negative definite} & <0 \ \forall x\neq 0 \\ \text{negative semi definite} & \leq 0 \ \forall x \end{cases}$

2. (Sylvester Criterion) $x^T Q x$ p.d. definite iff all leading principal minor of $Q > 0$ (det of Q successively rm last row & col; P49.3)

证 1° 令 Δ_i ith leading prin. minor, $\Delta_0 = 1$. 令 $x = \sum^n x_i \vec{e}_i = \sum^n \tilde{x}_i \vec{v}_i$ (natural basis of $\mathbb{R}^n$) ∴ $x = [v_1 \dots v_n]\tilde{x}$, ∴ $x^T Q x =$

$\tilde{x}^T V^T Q V \tilde{x} = \tilde{x}^T \tilde{Q} \tilde{x}$ when $\tilde{Q} = V^T Q V$, note $\tilde{q}_{ij} = \langle v_i, Q v_j \rangle$ 希望 $\tilde{q}_{ij} = 0, i\neq j$

考虑 build basis v_i as $\begin{cases} v_1 = \alpha_1 e_1 \\ v_2 = \alpha_{21} e_1 + \alpha_{22} e_2 \\ \vdots \\ v_n = \alpha_{n1} e_1 + \dots + \alpha_{nn} e_n \end{cases}$ 希望 pick $\{\alpha_{ij}\}$ s.t. 对 v_i 有以下: ⇑ $\forall 1 \leq j \leq i-1$, $\langle v_i, Q e_j \rangle = 0$ ⟹ $\langle v_i, Q v_j \rangle = 0$), 但 $\langle e_i, Q v_i \rangle = 1$

∴ sys on $\begin{cases} \alpha_{i1} q_{11} + \alpha_{i2} q_{12} + \dots + \alpha_{ii} q_{1i} = 0 \\ \vdots \\ \alpha_{i1} q_{i-1,1} + \alpha_{i2} q_{i-1,2} + \dots + \alpha_{ii} q_{i-1,i} = 0 \\ \alpha_{i1} q_{i1} + \alpha_{i2} q_{i2} + \dots + \alpha_{ii} q_{ii} = 1 \end{cases}$

$\Rightarrow \begin{bmatrix} q_{11} & \cdots & q_{1i} \\ \vdots & & \vdots \\ q_{i1} & \cdots & q_{ii} \end{bmatrix} \begin{bmatrix} \alpha_{i1} \\ \vdots \\ \alpha_{ii} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$ 由 Cramer, $\alpha_{ii} = \frac{1}{\Delta_i} \det \begin{pmatrix} q_{11} & \cdots & q_{1,i-1} & 0 \\ \vdots & & & \vdots \\ q_{i1} & \cdots & q_{i,i-1} & 1 \end{pmatrix} = \frac{\Delta_{i-1}}{\Delta_i}$

此时 $\tilde{Q} = \begin{bmatrix} \alpha_{11} & & 0 \\ & \ddots & \\ 0 & & \alpha_{nn} \end{bmatrix} = \begin{bmatrix} 1/\Delta_1 & & \\ & \ddots & \\ & & \Delta_{n-1}/\Delta_n \end{bmatrix}$ ∴ 在 $\{v_i\}$ basis 化 quadratic $x^T Q x$ 为 $\frac{1}{\Delta_1}\tilde{x}_1^2 + \dots + \frac{\Delta_{n-1}}{\Delta_n}\tilde{x}_n^2$

2° $\Delta_i > 0 \Leftrightarrow$ p.d. ⟹ sum square, >0 ⟸ 若 ∃k s.t. $\Delta_k = 0 = \det Q_k = \det \begin{pmatrix} q_{11} & \cdots & q_{1k} \\ \vdots & & \vdots \\ & & q_{kk} \end{pmatrix}$

∴ $N(Q_k) \neq \{0\}$, 有 $u \neq 0$ s.t. $Q_k u = 0$. ∴ 对 $x = \begin{bmatrix} u \\ 0 \end{bmatrix}$ $x^T Q x = u^T Q u = 0$, 但因 $x \neq 0$, p.d. 应 >0, ∴ 矛盾 ∴ $\Delta_i > 0$

Note 对 psd, psd ⟺ all principal minor ≥ 0, eg. $\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix}$ $\Delta_1 = 2$, $\Delta_2 = 0$, $\Delta_3 = 0$ 但 eval $0, -1.4, 1.4$ indefinite (Optim. P6 P20)

3. Rayleigh quotient $R_A(x) = \frac{x^T A x}{x^T x}$, A symmetric. note i) $R_A(x) = R_A(\alpha x)$, $\alpha \neq 0$, $\vec{x} \neq \vec{0}$ ii) 若 x 为 A eigvec with eval λ, $R_A(x) = \lambda$

Th, $\|x\|_2 = 1$ 有 $\lambda_{min} \leq x^T A x \leq \lambda_{max}$ λ wrt A

证 证 $\leq \lambda_{max}$ $A = UDU^T$ (P50 1.), 令 $y = U^T x$, $\max_{\|x\|=1} x^T A x = \max_{\|y\|=1} y^T D y = \max_{y_1^2 + \dots + y_n^2 = 1} \sum \lambda_i y_i^2$ ($y \leftrightarrow x$ 1-1, 因 U orthogonal)

max 当 $i \in I = \{\text{indice of } \lambda_{max}\}$ (可能多个) $\sum_{i \in I} y_i^2 = 1$, $i \notin I$ $y_i = 0$, 此时 val $= \sum_{i \in I} \lambda_i y_i^2 = \lambda_{max} \sum_{i \in I} y_i^2 = \lambda_{max}$

由 i) 得 (min-max) $\forall x \neq 0$ $\lambda_{min} \leq R_A(x) \leq \lambda_{max}$

4. 对 matrix induced norm, 可逆 A 有 $\|A^{-1}\| \geq (\|A\|)^{-1}$ (in general $\|A^{-1}\| \gg$ RHS) 证 $\underbrace{\|AA^{-1}\|}_{1,\ P49.4\text{证}} \leq \|A\| \cdot \|A^{-1}\|$ (P49.4 P.2)

Perturbutation Matrix
(col representation)

Date. /

1. 对 permutation $\pi : \{1 \cdots m\} \to \{1 \cdots m\}$: $\begin{pmatrix} 1 & 2 & \cdots & m \\ \pi(1) & \pi(2) & \cdots & \pi(m) \end{pmatrix}$ 有 permutation $P_{m \times m}$, $P_{ij} = \begin{cases} 1 & \pi(i) = j \\ 0 & \neq \end{cases}$

$$P = \begin{bmatrix} - e_{\pi(1)} - \\ \vdots \\ - e_{\pi(m)} - \end{bmatrix}$$

e.g. 对 $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 5 & 3 \end{pmatrix}$ $P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

果 i) Pv permutates 各 v_i, $Pv = \begin{bmatrix} v_{\pi(1)} \\ \vdots \\ v_{\pi(n)} \end{bmatrix}$ 因 $v_i = \sum_k^n p_{ik} v_k = v_{\pi(i)}$

但 orthogonal 非 perm
因 orth 可有 entry $\notin \{0,1\}$

ii) P orthogonal, $\therefore P_\pi^{-1} = P_\pi^T = P_{\pi^{-1}}$ 因 $P_\pi P_\pi^T(i,j) = \sum_k P_{ik} P^T_{kj} = \sum_k P_{ik} P_{jk} = \delta_{ij}$ $\therefore P_\pi P_\pi^T = I_n$

iii) $P_\sigma P_\pi v = P_{\pi \circ \sigma} v$. $(\pi \circ \sigma)(i) = \pi(\sigma(i))$; $v^T P_\sigma P_\pi = v^T P_{\pi \circ \sigma}$

iv) $v^T P$ permutate 各 v_i, $[v_1 \cdots v_n] P = [v_{\pi^{-1}(1)} \cdots v_{\pi^{-1}(n)}]$

v) $\exists k$ s.t. $P_\pi^k = I$ 因 $\pi \circ \pi \circ \cdots$ 总会轮回 $\forall$ perm, k 称为 ~~perm 的 order~~

vi) eval 为 unity $z^n = 1$ 的 $n \leq s$ 个 (in unit complex circle) 因 P orthogonal, $\therefore$ preserve norm 右 P (ii)

$\|Pv\| = \|v\| = \|\lambda v\| \Rightarrow |\lambda| = 1$, 又 有 $P^k = 1$ $\therefore \lambda^k = 1$ $\therefore \lambda = e^{\frac{i 2\pi m}{k}}$, $m = 1\,2\,3 \ldots$ Algebra P1

Unitary / orthogonal 3

P51.1

若 real 称为 orthogonal　　Hermitian / conjugate transpose

1. 对 matrix U, 称为 Unitary iff $U^H U = I_n = UU^H$

P i) $|\det U| = 1$ 由 $T_1 . J_2$　　ii) $\langle Ux, Uy\rangle = \langle x, y\rangle$ (preserve complex inner product) $(Ux)^H Uy = x^H U^H U y$

iii) preserve L_2 vector norm, $\forall x$, $\|Ux\|_2 = \|x\|_2$　$x^T U^T U x = x^T x$　x 可为 complex ∴ conjugate　　由此 angle 不变 $\cos\theta = \frac{x^T y}{\|x\|\|y\|}$ 在 $x \leftarrow Ux$, $y \leftarrow Uy$ 下不变

iv) (Takagi Factorization) 对 symmetric A, $\exists$ unitary U, real ≥ 0 diagonal Σ s.t. $A = U\Sigma U^T$

U cols 为 orthonormal eigvec for $A\bar{A}$, Σ diagonal entry 为 square root of $A\bar{A}$ eigval

v) U unitary $\Leftrightarrow$ col orthonormal set □ $U^H U = [U^H u_1 \dots U^H u_n] = I_n$ 即 $(U^H u_j)_i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$, $u_i^T u_j$

NOTE 由 vi) U unitary $\Leftrightarrow$ row orthonormal

vi) AB unitary 则 A^T, A^{-1}, AB unitary □ $A^T{}^T A^T = AA^T = 1$; $B^T A^T A B = 1$

T_1 U_{nn} unitary, 若 U triangular 且 diag entry > 0 (multiple), $U = I$

假定上三角 (设 U)

□ u_1 为 norm 1 又 u_1 为 $\begin{pmatrix} \square \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ∴ $u_1 = e_1$; $u_2^T u_1 = 0$ 且 $\|u_2\| = 1$ 只有 $u_2 = e_2$ ...

2. (non-square orthonormal set) 若 A_{mn} has col orthonormal, 则 $A^H A = I_n$

□ $(A^H A)_{ij} = \sum_k A^H_{ik} A_{kj} = \sum A_{ki} A_{kj} = \langle a_i, a_j\rangle = \delta_{ij}$

3. eval of unitary U in complex unit circle. □ P iii) $\|Uv\| = |\lambda|\,\|v\|$, $= \|v\|$ thus $|\lambda| = 1$

4. Q_{mn} col orthonormal note 只有 $m \geq n$, 此时 SVD $V = I_{nn}$, $\Sigma = I_{mn}$, $U = (Q | \dots)$

□ $m < n$ 时不可能有此 Q 因 col of Q in $\mathbb{R}^m$ 上不会有比 m 多的 orthonormal

∴ 因 T_2 $Q^T Q = I_n$ $\sigma_i = 1$, ∴ $\Sigma = I_{mn}$, $V = I_n$　则 $Q_{mn} = U_{mn} I_{mn} I_n = (Q_{mn} | u_{n+1} \dots u_m)$

where u_i orthonormal. note 此 u_i 必可 用可找到 $\{u_{n+1 \dots m}\}$ L.ind 再 Gram Schmidt

5. $\|U\|_2 = 1$ □ $\|U\|_2 = \sup_{x \neq 0} \frac{\|Ux\|_2}{\|x\|_2} = 1$ (P iii))

6. (preserve matrix L_2 norm) $\|AU\|_2 = \sup_{\|x\|=1} \|AUx\| \overset{P\,iii)}{=} \sup_{\|Ux\|=1} \|AUx\| \overset{y \leftarrow Ux}{=} \sup_{\|y\|=1} \|Ay\| = \|A\|_2$

Date. /

例1 Rank 分解 $A=\begin{pmatrix}1&3&1&4\\2&7&3&9\\1&5&3&1\\1&2&0&8\end{pmatrix} \sim \overset{\text{RREF}}{\begin{pmatrix}1&0&-2&0\\0&1&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}}$ ⇒ $C=\begin{pmatrix}1&3&4\\2&7&9\\1&5&1\\1&2&8\end{pmatrix}$, $F=\begin{pmatrix}1&0&-2&0\\0&1&1&0\\0&0&0&1\end{pmatrix}$

$a_3=-2a_1+a_2$

例2 QR 分解 $A=\begin{pmatrix}12&-51&4\\6&167&-68\\-4&24&-41\end{pmatrix}$ ⇒ col. l.i.d. 用 P50.2 Gram Process → $\begin{bmatrix}12&-69&-58/5\\6&158&6/5\\-4&30&-33\end{bmatrix}$ → $\begin{bmatrix}-6/7&-69/175&-58/175\\3/7&158/175&6/175\\-2/7&6/35&-33/35\end{bmatrix}$

$u_1 \quad u_2 \quad u_3$

Q = $q_1 \quad q_2 \quad q_3$

$Q^TA=Q^TQR=R=\begin{pmatrix}14&21&-14\\0&175&-70\\0&0&35\end{pmatrix}$

例3 用 QR 解 sys eq. $A\vec{x}=\vec{b}$.

同乘 Q^T

$QR\vec{x}=\vec{b} \iff R\vec{x}=Q^T\vec{b}$，因 R upper △，用 backward sub 解

$\begin{pmatrix}r_{11}x_1+\cdots+r_{1n}x_n\\ r_{22}x_2+\cdots+r_{2n}x_n\\ \vdots\\ r_{nn}x_n\end{pmatrix}=Q^T\vec{b}$

Date. /

1 对 A_{mn} $rk=r$ |rank decomposition| 为 $A_{mn} = \underset{\text{full-col-rank}}{C_{mr}} \underset{\text{full-row-rank}}{F_{rn}}$

$rk=r$, $R(A)=span\{a_1 \ldots a_n\}$

令 A 中 c.lnd 的 r 个 col vector $\vec{c}_1 \ldots r$，$C=[\vec{c}_1 \ldots \vec{c}_r]$，而 $\forall j=r+1 \ldots n$ 有 $\vec{a}_j$ 被前 r 个 L.comb，$F=\begin{pmatrix} 1 & & c_{1,r+1} \ldots c_{1n} \\ & \ddots & \vdots \\ & 1 & c_{r,r+1} \ldots c_{rn} \end{pmatrix}$

$(\vec{a}_j = \sum_{i=1}^{r} f_{ij} \vec{c}_i)$

P1 a) not unique. 对 $\forall$ 可逆 R，$C \leftarrow CR$，$F \leftarrow R^{-1}F$ 也为 rk decomposition 见 P1a

b) $\begin{cases} m<n & rkA=m : A = I_m A \\ > & \quad\; n : A = A I_n \end{cases}$

2. $m \geq n$ A_{mn} $rk=n$ (col L.Ind, col full rank) |QR decomposition| 为 $A = \underset{\text{unitary}}{Q_{mm}} R_{mn}$ $\xrightarrow{\text{upper triangular}}$ $\begin{bmatrix} \bar{R}_{nn} \\ 0 \end{bmatrix}$ 且 $= \bar{Q}\bar{R}$ P50. 注

Q 为 A_m L.Ind col $\{a_i\}$ Gram-Schmit Process 生成 的 orthonormal $[q_1 \ldots q_n]$，因 $\forall k=1 \ldots n$ 有 $a_k = r_{k1}q_1 + \ldots + r_{k,k-1}q_{k-1} + r_{kk}q_k$

$q_i^T v_k = q_i^T a_k$ $\|u_k\| = \|a_k\|$

$\therefore \bar{Q}_{mn} = (q_1 \ldots q_n)$ $\bar{R} = \begin{pmatrix} r_{11} & \ldots & r_{1n} \\ 0 & \ddots & r_{nn} \end{pmatrix}$ 易知 $Q^TQ = I_n$ $\bar{R}$ entry ≥ 0

$Q = (\bar{Q} | Q')$ pick st Q orthogonal

P1 T_2 ± 1 P55. P1

P2 a) 若 A square (即 m=n $\Rightarrow$ invertible)，R 也可逆 $\because$ $\det A = \det Q \det R$ $\therefore |\det A| = |\det R| \neq 0$

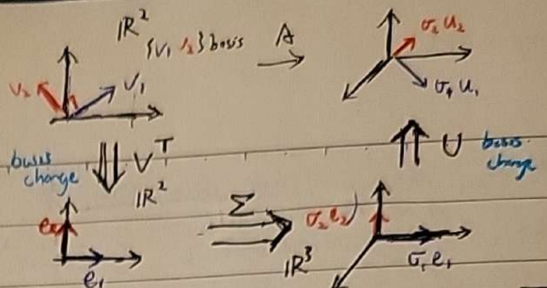
b) QR 不 unique ?3.2

3. LU decomposition ***5.应用 4. UV decomposition ***5.*

5. 对 pd cholesky / LDL^T 见 P1

Date. /

例1. SVD 几何意义 ; 对 $A \in \mathbb{R}^{3\times 2}$



对 linear $T: \mathbb{R}^n \to \mathbb{R}^m$

B C 为 $\mathbb{R}^n$ $\mathbb{R}^m$ standard basis

$\bar{B}$ $\bar{C}$ 〃〃 任意 basis

- V 为 $Q_{\bar{B}\to B}$ ∴ $V^T = V^{-1} = Q_{B\to\bar{B}}$，将 $[x]_B$ 转换为 $\bar{B}$ representation
- Σ scale each dim by σ_i 并 ~~删~~/增 dim 成 $\mathbb{R}^m$ ($\bar{C}$ 表示)
- U 为 $Q_{\bar{C}\to C}$，令 output of $T(x)$ 处于 $[T(x)]_C$

应用1 A $\overset{\text{user}}{\text{movie rating}}$ 若 i) all user rate use linear

ii) rating 无 noise

iii) u_i typical movie

v_i typical viewer

例2. SVD $A = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 4 & & & 0 \\ & 2 & & 0 \\ & & \boxed{0} & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \\ 1/\sqrt{2} & 1/2 & 1/2 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$ 找 basis for $N(A)$, $R(A)$

解 1° U orthogonal ∴ 可逆 ∴ bijection ∴ $Ux = 0 \iff x = 0$ ∴ 只用考虑 x s.t. $\Sigma V^T x = 0$ (因 3.ii)

而显然，考虑 y s.t. $\Sigma y = 0$，y 应 $\begin{pmatrix} 0 \\ 0 \\ * \\ * \end{pmatrix}$ 即应令 x 满足 $V^T x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ 和 $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\xRightarrow{V^T \text{ orthogonal}}$ $x = V\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = V$ 3rd col / 4th col

2° V^T bijection ∴ 只用考虑 $U\Sigma$ 如何 map input $x \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ 因 $\Sigma x = \begin{pmatrix} 4a \\ 2b \\ 0 \end{pmatrix}$ ∴ $U\Sigma x$ 有 $\begin{pmatrix} 4a/\sqrt{2} - 2b/\sqrt{3} \\ 4a/\sqrt{2} + 2b/\sqrt{3} \\ 2b/\sqrt{3} \end{pmatrix}$

$= \begin{pmatrix} 4/\sqrt{2} \\ 4/\sqrt{2} \\ 0 \end{pmatrix} a + \begin{pmatrix} -2/\sqrt{3} \\ 2/\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix} b$, $R = \left\langle \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \right\rangle$

$m\,\boxed{A}^{\,n} = \boxed{U}\;\boxed{\Sigma}\;\boxed{V^T}$ Σ 为 △ $\begin{pmatrix}\sigma_1 & 0 \\ 0 & \ddots\ \sigma_n \\ 0 & \cdots\ 0 \\ 0 & \cdots\ 0\end{pmatrix}$ or $\begin{pmatrix}\sigma_1 & & 0 & 0 \cdots 0 \\ 0 & \ddots & \sigma_n & 0 \cdots 0\end{pmatrix}$

Date.

1. SVD $\boxed{\text{any}}$ $A_{mn} = U_{mm}\,\Sigma_{mn}\,V^T_{nn}$ (orthogonal) 令 rk $A = r$ $\sigma_1 \geq \dots \geq \sigma_r > \sigma_{r+1} = \dots = \sigma_{\min(n,m)} = 0$

$\boxed{\Sigma}$. S.t. diagonal with singular val σ_i, $\sigma_i^2(A) = \lambda_i(A^TA) \geq 0$ 因 A^TA psd option. P_6.

$\therefore A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$ 即 sum of rk-1 matrix 4. T_3

2. 构建 SVD:

1° A^TA symmetric 且 psd (P6. $P_{(2)}$), $\therefore$ eigen diagolize (Psu. 1) 为 $P\Lambda P^{-1} = P\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}P^T$, $\lambda_i \geq 0$ (psd)

假若 SVD $\exists$, $A^TA = V\Sigma^T U^T U \Sigma V^T = V\begin{bmatrix}\sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2\end{bmatrix}V^T \rightarrow \begin{cases} V^T = P^T \\ \sigma_i^2 = \lambda_i \end{cases}$

$\therefore A^TA$ 的 eigenvec (orthonormal by Spectral Thm) 为 V^T row / V col
即 $v_i \perp v_j$

2° $\{v_i\}$ orthonormal, 而 $\{Av_i\}_{i=1}^{r}$ 在 $m \geq r$ 下 basis 且 $(Av_i)^T(Av_j) = v_i^T(A^TA)v_j = v_i^T\lambda_j v_j = 0$ orthogonal basis

3° 因此用 $\{Av_i\}$ 构建 U, 为 orthonormal $u_i = \frac{Av_i}{\|Av_i\|} = \frac{Av_i}{\sqrt{v_i^TA^TAv_i}} = \frac{Av_i}{\sqrt{\lambda_i}\|v_i\|} = \frac{Av_i}{\sigma_i}$, $i = 1 \dots r$

(singular val eq)

u_i 为 AA^T evec: $AA^Tu_i = \frac{A\cdot(A^TAv_i)}{\sigma_i}$

4° 对 remaining $m-r$, find L.I.nd. vec wrt $u_{1 \dots r}$ 并 Gram-Schmidt orthonormalize

3. $P_{(1)}$ orthogonal Q 有