University of Amsterdam

Stochastics and Financial Mathematics

Master Thesis

A Tranche Loss Control Variate in a One-Factor Lévy Threshhold Model

Author:
Chris Vermeulen

Supervisor:
prof. dr. B.D. (Drona) Kandhai

Examination date:
March 1, 2024

Daily supervisor:
dr. I. Anagnostou

Korteweg-de Vries Institute for Mathematics

Abstract

Collaterilized Debt Obligations (CDOs) are complex financial instruments widely employed in the financial industry. The valuation of a CDO tranche involves the estimation of expected tranche loss (ETL). This thesis presents a novel control variate variance reduction technique for the estimation of ETL. A tranched version of the Large Homogeneous Portfolio (LHP) approximation is employed as a control variate in a general one-factor Lévy specified threshhold model for correlated defaults. The methodology's efficacy is justified and numerically analysed in the Gaussian- and shifted gamma Copula frameworks. Significant variance reduction for the estimation of ETL can be achieved for a potentially inhomogeneous portfolio of loans, with minimal additional computation time. Furthermore, the control variate technique, as presented in this study, exhibits the potential to enhance more widely adopted risk metrics, such as Value-at-Risk (VaR) and Expected Shortfall (ES).

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Author: Chris Vermeulen, chrisvermeulen@student.uva.nl, 11029072
Supervisor: prof. dr. B.D. (Drona) Kandhai
dr. I. Anagnostou
Daily supervisor: dr. I. Anagnostou
Second Examiner: dr. A. (Asma) Khedher

Examination date: March 1, 2024

Korteweg-de Vries Institute for Mathematics
University of Amsterdam
Science Park 105-107, 1098 XG Amsterdam
`http://kdvi.uva.nl`

Acknowledgments

I would like to express my gratitude towards Ioannis Anagnostou and Christian Lendvai, who were there at the very beginning of this project. Furthermore I would like to thank Sven Karbach for his valuable feedback.

List of Abbreviations and Acronyms

CDO	Collaterized Debt Obligation
CDS	Credit Default Swap
ABS	Asset-Backed-Security
VaR	Value-at-Risk
ES	Expected Shortfall
LHP	Large Homogeneous Portfolio
MSE	Mean Square Error
NV	Notional Value
PD	Probability of Default
LGD	Loss Given Default
EL	Expected Loss
UL	Unexpected Loss
CDF	Cumulative Distribution Function
PDF	Probability Density Function
SLLN	Strong Law of Large Numbers
ETL	Expected Tranche Loss
CMC	Crude Monte Carlo
CV	Control Variate
CVMC	Control Variate Monte Carlo
STD	Standard Deviation

List of Symbols

n	Portfolio Size
N_i	NV of Loan i
l_i	LGD of Loan i
w_i	Exposure Weight of Loan i
p_i	PD of Loan i
$D = (D_1, \ldots, D_n)$	Default Indicator Vector
F_D, F_{ij}, F_i	Multivariate, Bivariate and Marginal CDF of D
L	Portfolio Loss Random Variable
F_L	Portfolio Loss CDF
$\mathbb{E}[L]$	EL
σ_L	UL
VaR_α	VaR Risk Metric
$\rho_{ij,default}$	Default Correlation between D_i and D_j
ρ_{asset}	Asset Correlation
$A = (A_1, \ldots, A_n)$	Critical Variables or Asset Values
$K = (K_1, \ldots, K_n)$	Critical Thresholds
G_A, G_i	Multivariate and Marginal CDF of A
C_A	Copula induced by $G_A, G_1, \ldots, G_n$
C_{ij}	Bivariate Copula induced by G_{ij}, G_i, G_j
H	Infinitely Divisible Distribution
H_1	CDF of A_i under One-Factor Lévy Threshold Model
$X_\rho, X_{1-\rho}^{(i)}$	Systematic and Idiosyncratic Factor
H_ρ, $H_{1-\rho}$	CDFs of Systematic and Idiosyncratic Factors
$h_\rho, h_{1-\rho}$	PDFs of Systematic and Idiosyncratic Factors
$\pi_i(x)$	Conditional PD of Loan i (conditioned on $X_\rho = x$)
$\pi_K(x)$	Conditional PD/EL of a Homogeneous Portfolio
L_∞	Large Homogeneous Portfolio Approximation or Conditional EL
F_∞	CDF of L_∞
c	Attachment Point
d	Detachment point
$L_n^{(c,d)}$	Portfolio Tranche Loss with n Loans
$L_\infty^{(c,d)}$	Tranched LHP Loss or Tranched Conditional EL
s	CDO Fair Premium Spread
N	Number of Monte Carlo Iterations
μ	true ETL
$\hat{\mu}_{crude}$	CMC Estimate of ETL
$\hat{\mu}_{CV}$	CVMC Estimate of ETL
$\sigma_{tranche}$	STD of Tranche Loss $L_n^{(c,d)}$
$\sigma_{tranche,CV}$	STD of CV Adjusted Tranche Loss
σ_{crude}	STD of CMC Estimate
σ_{CV}	STD of CVMC Estimate
η_{crude}	Relative STD of CMC Estimate

η_{CV}	Relative STD of CVMC Estimate
R	STD Ratio CMC / CVMC
r	STD Ratio CVMC / CMC

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Introduction

Over the past two decades, the introduction of Collateralized Debt Obligations (CDOs) has drastically transformed the field of credit risk modeling. CDOs are complex financial instruments that package a variety of debt securities, including bonds and loans, into a single structured product. The success and challenges of CDOs have underscored the importance of effective credit risk modeling in assessing and managing the risks associated with these instruments. Such models are faced with the task of accurately forecasting extreme losses arising from intricate underlying market dynamics, all the while maintaining numerical stability and efficiency.

In the early 2000s, groundbreaking innovations introduced a standardized tool that revolutionized the way investors simulated potential losses on a pool of Asset-Backed Securities (ABS). This marked the beginning of a widespread adoption within the financial industry. For several years, this innovative approach to risk assessment became the standard methodology for valuation of risky CDO tranches. However, eight years later when the 2008 financial crisis unfolded, it exposed that extreme joint default behaviour was not properly modelled. While various alternative solutions have been suggested to address this issue, there is still ample room for improvement in the credit risk models currently employed in the industry.

The aftermath of the crisis prompted a critical reevaluation of risk management practices, particularly in the realm of CDOs. Despite the profound impact and scrutiny, the CDO market demonstrated resilience, gradually recovering over time. Today, CDOs continue to play a vital role in the financial landscape, albeit with a heightened awareness of the need for more sophisticated and robust risk models. The 2008 financial crisis serves as a constant reminder of the importance of adapting and evolving risk management strategies to navigate the complexities of the financial markets.

One of the main challenges that CDO pricing models face is the curse of dimensionality, since they need to account for a multitude of factors and variables that influence the distribution of losses. Monte Carlo simulation is a powerful computational technique that provides a solution to this issue. By simulating a large number of potential future scenarios and outcomes it captures the intricate dynamics of dependent defaults and possible complex portfolio characteristics. Despite its general applicability, one of the method's major downsides lies in the slow rate of convergence towards its true value. In order to double the precision, the number of necesarry iteration must be quadrupled.

Variance reduction techniques are crucial for reducing the computational burden presented by high-dimensional models, making them especially important in Monte Carlo simulations within credit risk models. There exists a diverse array of options aimed at lowering the variance of the output estimator. Among the well-established candidates are control variates, importance sampling, antithetic sampling, stratified sampling, quasi Monte Carlo, common random numbers, and various others. The art of variance reduction lies in selecting a technique that optimally leverages the inherent structure of the problem. Different techniques exhibit varying degrees of suitability for specific problems, emphasizing the importance of discerning choices based on the nature of the modeling challenge at hand. The purpose of this thesis is to present, explore and improve the current Monte Carlo methodology aimed at establishing the fair premium to be paid over a risky CDO tranche.

While the primary focus of this technique's development pertains to CDOs, its applicability extends to enhancing various other stochastics within the same analytical framework. Other risk metrics, such as Value-at-Risk (VaR) and Expected Shortfall (ES), exhibit a potential for substantial enhancement with minimal additional computational complexity. The specific transformations applied to the random variable representing losses, as proposed in this study, possess adaptability to cater to the refinement of these aforementioned risk measures. This observation suggests the potential for broader future development, transcending the immediate confines of this thesis. The principal contributions of this thesis can be decomposed into three main parts.

In the first part, we present a general one-factor Lévy specified framework for correlated defaults. We introduce important credit risk variables and underscore the intricacies of the unconditional loss distribution. Operating within this broad framework, we develop a control variate technique. This technique leverages a tranched version of the Large Homogeneous Portfolio (LHP) approximation, which represents the loss random variable of an infinitely sized homogeneous portfolio, which is essentially a tranched contional expected loss (EL) of the portfolio. Unlike the complex structure of the unconditional loss distribution, the distribution function of the LHP random variable can be explicitly formulated.

Furthermore, we utilize the LHP distribution function to compute the expectation of a tranched transformation of the LHP random variable, serving as a proxy for the expected tranche loss (ETL). Since the expectation of this transformation is known, it serves as a valuable control variate. Additionally, a straightforward argument involving conditional expectations demonstrates the asymptotic optimality of this control variate, hence particularly excelling for portfolios comprising a large number of loans.

Finally, we conduct an extensive numerical study to analyze the method's performance. Sensitivity analysis is performed for homogeneous portfolios, comparing the efficacy of the control variate between the widely used Gaussian Copula framework and the less popular shifted gamma framework. For inhomogeneous portfolios, we compare a cruder but faster variant of the method to the original. The results reveal that the choice of framework significantly influences the control variate efficacy, although improvement is consistently observed.

1 Literature Review

A large body of literature has investigated the theory of credit risk, in particular CDO pricing. Following a brief history on credit risk models, a detailed examination of analytical approximations and variance reduction techniques is presented. Important existing concepts are explored and their potential integration into innovative approaches is provided. Finally, the review narrows its focus towards the use of control variates in a credit risk framework, which serves as the center of study in this thesis.

1.1 History of Credit Risk Models

Fairly pricing a CDO tranche encompasses numerous challenges, but at its essence, it revolves around the intricate task of modeling dependent defaults. Generally speaking almost all models for credit risk can be traced back to (Merton 1974) and (Black and Scholes 1973). Merton adapted Black and Scholes' methodology for pricing options to model the likelihood that a firm defaults. This layed the foundation for *structural models* of default, or *firm-value models* of default. Merton's theory proposes a mechanism to explain a firms default likelihood by examining the connection between its assets and liabilities on a specific time period. Put more generally, default takes place whenever a stochastic variable (or process), typically denoting an assets value, falls below a certain threshhold. For that reason models of this kind are also often referred to as *threshhold models*. Important descendents from the Merton model are for example the KMV model (Crosbie and Bohn 2019) and CreditMetrics (Credit Suisse Financial Products 1997), both widely influential in the financial industry.

On the other hand, *reduced-form models* or *mixture models* aim to explain the dependence between defaults through a set of systematic economic variables or *factors*. CreditRisk+ is a notable example of such a model, which has the structure of a Poisson Mixture model (Credit Suisse Financial Products 1997). In this class of models default times are independent given the realization of some observable economic background process. Hence, the term *factor model* or *conditional independence model* is another popular naming convention. Even though these models differ from structural models by shadowing the underlying mechanics that can lead to default, some structural model specifications can be associated to an equivalent reduced-form model representation, i.e. leading to an identical joint default distribution (McNeil, Frey, and Embrechts 2015).

1.2 Approximations of the Loss Distribution

Vasicek was one of the first to establish a very good analytical approximation for the loss distribution of a homogeneous portfolio, by letting the number of obligors approach infinity (O. A. Vasicek 1987), (O. Vasicek 2002). Gordy extended his work to incorporate for inhomogeneous portfolios as well, along with results regarding the errors induced by asymptotic approximations (Gordy 2003). Another methodology that reshaped the landscape of credit risk, and in particular the CDO market, is presented in (D. X. Li 2000). He suggests to use the Gaussian Copula for modelling dependent default times. The analytical tractability of the Gaussian Copula combined with the power of Monte Carlo simulation handed the market a powerful tool for estimating losses. Many believe that this model was one of the key reasons for the financial crisis of 2008, which is a discussion that exceeds the scope of this thesis. This question highlights the possible limitations of using the Gaussian Copula to model default times, particularly its potential shortcomings in representing tail dependence (Furman et al. 2016). An extensive amount of literature has therefore explored deviations from the Gaussian copula, using for example a fatter tailed Archimedean Copula (Cui 2022) or the Student-t Copula (Daul et al. 2003).

Nevertheless, due to its inherent analytical tractability, the use of the Gaussian Copula remains a very popular modelling choice in credit risk. More general credit risk frameworks have also been developed to accomodate for alternative copula specifications. (Albrecher, Ladoucette, and Schoutens 2006) for example propose a Lévy process specification for a one-factor model. In this thesis, our objective is to take a similar approach to ensure that the new variance reduction techniques can be readily applied in a wide range of settings. The methodology will be assessed in two instances of the Levy-specified one-factor model: firstly, in a Gaussian-, and secondly, in a shifted gamma framework.

1.3 Variance Reduction and Approximations

As previously mentioned Monte-Carlo simulation plays a central role in quantitative finance. In this context, Glasserman argues (Glasserman 2004a)

> "A fundamental implication of asset pricing theory is that under certain circumstances, the price of a derivative security can be usefully represented as an expected value. Valuing derivatives thus reduces to computing expectations. In many cases, if we were to write the relevant expectation as an integral, we would find that its dimension is large or even infinite. This is precisely the sort of setting in which Monte Carlo methods become attractive."

However, a widely established downside of Monte-Carlo methods is its slow $O(\sqrt{n})$ rate of convergence. Variance reduction techniques play a key role in alleviating this issue. By exploiting certain problem or model structures, these techniques can significantly boost the convergence speed towards the true value. In this regard, the works of Glasserman are an important contribution to approximations and variance reduction techniques. Besides establishing analytical approximations of the loss distribution (Glasserman 2004b), he also proposes highly effective importance sampling techniques for computing tail probabilities of the loss distributions (Glasserman and J. Li 2004). A potential downside of this algorithm is its complexity and limitation. Firstly, it requires non-trivial calculation of optimal shift parameters that might be cumbersome for quick and easy implementations. Secondly, while this method has been well-established within the Gaussian copula framework, it is not immediately apparent how this technique can be extrapolated towards other copulas.

Rather than focussing on tail probabilities, Joshi develops a similar importance sampling technique for pricing single tranche CDOs (Joshi 2004). Huang's work provides a solid summary of these methods and their various improvements (Huang, Kwok, and Xu 2023). Quasi-Monte Carlo is another methodology worth mentioning, as it has been an effective variance reduction technique, whose strength lies in its easy implementation and general applicability (Avramidis and L'Ecuyer 2006).

1.4 Control Variates

Control variates are one of the most popular variance reduction techniques, due to their intuitively appealing properties and widespread applications. However, their mentioning in the field of credit risk literature seems to be rather scarce. The use of control variates dates back to a time when computer programming was less straightforward. Interestingly, back then the relevance of variance reduction did not only depend on the extra computational effort necessary to execute the technique, but also on the complexity of implementation. An interesting quote shows the drastically different mindset at the time (Atkinson and Pearce 1976)

> "Additional increases in the efficiency of simulation might be expected in cases which yield sufficient statistics with tractable distributions. But, with a few exceptions, [...], the subject of variance reduction in simulation has had little coverage in the statistical literature. One reason for this may be that, except in simple or artificial examples, it is difficult to obtain sufficient increase in precision to justify the additional complexities in programming."

Known for their straightforward yet highly effective implementation, control variates represent one of the earliest variance reduction techniques. Boyle demonstrated one of the earliest uses of control variates in finance (Boyle 1977). A more comprehensive theoretical framework for control variates was subsequently laid out by (Nelson 1990). The practical application of control variates to quantile estimation was further refined by Hesterberg (Hesterberg and Nelson 1998), thereby expanding their potential use for

Value-at-Risk (VaR) estimation.

Control variates also have been explored for pricing credit derivates, for example in the valuation of exotic options (Zhang et al. 2018), or in particular for Asian options (Lai, Z.-F. Li, and Zeng 2013), (Du, Liu, and Gu 2013). Within the context of CDO pricing frameworks, the idea of utilizing control variates has been advanced by Glasserman and Cheng (Chen and Glasserman 2007).They introduce three distinct approximations for the tranche loss distribution. Firstly, they provide an efficient method for computing the distribution of the number of defaults through a recursive procedure. Utilizing this distribution, they put forth three unique approximations for the conditional tranche loss distribution, based on a homogeneous portfolio, a triangular distribution, and a binomial distribution. They further illustrate the effectiveness of their approximations when used as control variates, emphasizing their significant potential in reducing the variance of the expected tranche loss estimator.

Approximations and control variates are intricately linked. A control variate serves as a stochastic approximation of another random variable, with a primary focus on amplifying covariate fluctuations rather than minimizing mean square error (MSE). In the realm of VaR estimation, a control variate was constructed to estimate tail probabilities by utilizing Vasicek's LHP approximation (Tchistiakov, Smet, and Hoogbruin 2004). Their principles closely resonate with the methodology developed in this thesis. One of the challenges they addressed was the necessity of a strongly correlated control variate, which required suitable parameter adjustments in the LHP. They proposed a moment matching approach between the loss distribution and LHP, which exhibited effective variance reduction through numerical analysis. However, within our framework of CDO pricing, it has been observed that moment matching towards the loss distribution may not yield optimal results. Thus alternative methodologies to maximize the performance of an LHP as control variate will be explored.

2 Credit Risk Management

2.1 Portfolio Credit Risk Variables

A *credit risk model* serves as a crucial tool for quantitative risk managers, whose goal is to evaluate potential losses of a *portfolio* comprised of n risky assets, such as bonds or loans. Usually on a time horizon of one year. For simplicity, our focus will be on modeling the accumulated loss as a singular event in the future. This implies treating the loss as a random variable L rather than a random process evolving over time. Each asset $i \in \{1, \ldots, n\}$ in the portfolio is characterized by its distinct *notional-* or *face value* (NV) $N_i \in \mathbb{R}_{\geq 0}$, *loss given default*(LGD) $l_i \in [0, 1]$ and *probability of default* (PD) $p_i \in [0, 1]$. The default behaviour of all loans $i = 1, \ldots, n$ is modeled by a vector of default indicators $D = (D_1, \ldots, D_n)$, under a multivariate Bernoulli distribution function denoted as F_D

$$F_D(x_1, \ldots, x_n) := \mathbb{P}(D_1 = x_1, \ldots, D_n = x_n) \quad x_i \in \{0, 1\}. \tag{2.1}$$

The marginals of F_D are characterized by individual Bernoulli distribution functions, denoted by F_i

$$F_i(x) := \mathbb{P}(D_i = x) \quad x \in \{0, 1\}, \tag{2.2}$$

each possessing a known expectation p_i and a variance of $p_i(1 - p_i)$. Additionally we denote the bivariate distribution function of pairs of default indicators D_i and D_j by

$$F_{ij}(x, y) := \mathbb{P}(D_i = x, D_j = y) \quad x, y \in \{0, 1\}. \tag{2.3}$$

Furthermore NV and LGD can be combined into a single *exposure* quantity $e_i = N_i l_i$, which quantifies how much is lost by loan i in the case of default. The maximum incurrable loss is thus equal to $\sum_{i=1}^{n} e_i$. Let $w_i = \frac{e_i}{\sum_{i=1}^{n} e_i}$ be the exposure weight associated with loan i, then the pro-rata loss random variable describing the loss of this portfolio over a single time period is given by

$$L := \sum_{i=1}^{n} w_i D_i, \tag{2.4}$$

which attains its values between 0 and 1. More precisely its domain is given by the set

$$\mathcal{L} := \{w \in \mathbb{R} : \exists I \subseteq \{1, \ldots, n\} : \sum_{i \in I} w_i = w\}. \tag{2.5}$$

Furthermore we let F_L denote the distribution function of L,

$$F_L(l) := \mathbb{P}(L = l) \quad l \in \mathcal{L}, \tag{2.6}$$

which is determined by both the distribution function F_D and exposure weights w_i. The central problem for a credit risk officer is accurate modeling of F_L, which in turn involves appropriate modelling of the multivariate Bernouilli distribution F_D.

Risk measures are crucial summarizing quantities of the loss distribution. They offer a mechanism to consolidate information pertaining to the risks associated with a portfolio. This enables the transformation of dense and complex loss information into a singular quantity, serving a dual purpose of facilitating effective communication, but also providing the possibility towards development of techniques dedicated to the estimation of these risk measures. Hence they play a significant role in capital requirement regulation under Basel IV. Some of those risk measures will be listed here. *Expected Loss* (EL) is the expectation of the loss random variable $\mathbb{E}[L]$. *Unexpected loss* is defined as the standard deviation $\sigma_L := \sqrt{\mathrm{Var}[L]}$, indicating the likeliness of large losses. Though being of less consideration in the study presented here, but important for potential future directions, the *value-at-risk* measure (VaR_α) is defined as the α-quantile of the loss distribution, i.e. $\mathrm{VaR}_\alpha := \inf\{x | F_L(x) \geq \alpha\}$. Both EL and UL can readily be computed, the EL is given by

$$\mathbb{E}[L] = \mathbb{E}\Big[\sum_{i=1}^{n} w_i D_i\Big] = \sum_{i=1}^{n} w_i \mathbb{E}[D_i] = \sum_{i=1}^{n} w_i p_i. \tag{2.7}$$

Furthermore the variance of the loss is computed as

$$\begin{aligned}
\mathrm{Var}[L] &= \mathrm{Var}\Big[\sum_{i=1}^{m} w_i D_i] \\
&= \sum_{i=1}^{n} w_i^2 \mathrm{Var}[D_i] + \sum_{i \neq j} w_i w_j \mathrm{Cov}(D_i, D_j) \\
&= \sum_{i=1}^{n} w_i^2 p_i (1 - p_i) + \sum_{i \neq j} w_i w_j (F_{ij}(1,1) - p_i p_j).
\end{aligned}$$

In conclusion, the UL is given by

$$\sigma_L = \sqrt{\sum_{i=1}^{n} w_i^2 p_i (1 - p_i) + \sum_{i \neq j} w_i w_j (F_{ij}(1,1) - p_i p_j)}. \tag{2.8}$$

Figure 2.1 illustrates the typical shape of a loss distribution along with its associated risk measures.

One of the most challenging aspects is modelling the multivariate nature of the default

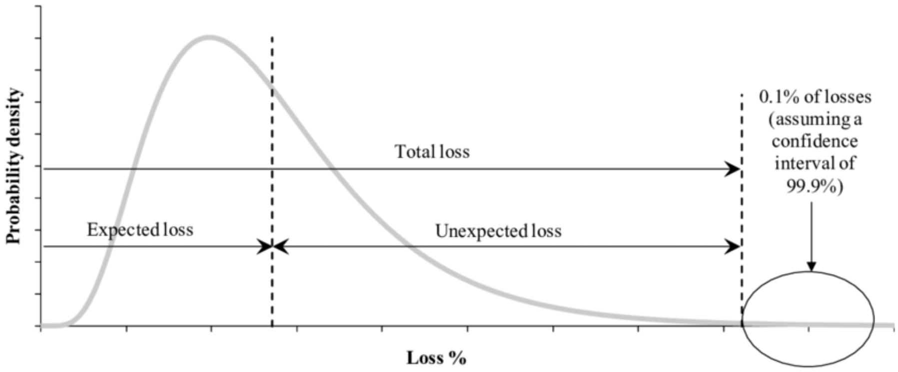

Figure 2.1: Loss distribution with important risk measures.

indicators $D = (D_1, \ldots, D_n)$. Their pairwise dependence can be quantified by the *default correlation*, utilizing Pearson's linear correlation coefficient

$$\rho_{ij,default} := \text{Corr}(D_i, D_j) = \frac{\mathbb{E}[D_i D_j] - \mathbb{E}[D_i]\mathbb{E}[D_j]}{\sqrt{\text{Var}[D_i]\text{Var}[D_j]}} = \frac{F_{ij}(1,1) - p_i p_j}{\sqrt{p_i(1-p_i)p_j(1-p_j)}}. \quad (2.9)$$

This quantity is closely linked to the skew and curtosis of the loss distribution F_L. The subsequent section introduces the overarching class of models that governs both the marginal distributions and the dependency structure of the default indicators, enabling to tune the default correlation to varying degrees.

2.2 Threshhold Model for Correlated Defaults

Various modelling choices exist for constructing the loss distribution. *Threshold models* are an intuitively appealing choice for modelling dependent default indicators $D = (D_1, \ldots, D_n)$. For a detailed discussion regarding such modelling choices, we refer to (McNeil, Frey, and Embrechts 2015) for a more detailed discussion. They propose the following definition for a threshold model:

Definition 2.2.1 (Threshhold Model for Correlated Defaults)**.** *Let $A = (A_1, \ldots, A_n)$ be an n-dimensional random vector and $K = (K_1, \ldots, K_n) \in \mathbb{R}^n$ be a vector of thresholds. The default indicators are defined as*

$$D_i := \mathbb{1}[A_i \leq K_i]. \quad (2.10)$$

(A, K) is said to define a threshold model for default indicators $D = (D_1, \ldots, D_n)$.

Two models (A, K) and $(\tilde{A}, \tilde{K})$ are called *equivalent* if the resultant distribution of D is equal to the distribution of $\tilde{D}$. The random variables $A_1, \ldots, A_n$ in the above definition are often referred to as *critical variables*, which in practice reflect *asset values* associated to the obligors in the portfolio. Let G_A denote the multivariate distribution function of the vector of asset values A, where it is assumed that each asset value attains its values in some domain $\mathcal{A}$,

$$G_A(x_1, \ldots, x_n) := \mathbb{P}(A_1 \leq x_1, \ldots, A_n \leq x_n) \quad x_i \in \mathcal{A} \tag{2.11}$$

We will assume that G_A is continuous in all variables. Similar to the definition of F_i, the marginal distribution function of A_i is denoted by G_i,

$$G_i(x) := \mathbb{P}(A_i \leq x) \quad x \in \mathcal{A}. \tag{2.12}$$

The thresholds $K_1, \ldots, K_n$ are often referred to as *critical thresholds*. They are chosen to let the default probabilities $F_1(1), \ldots, F_n(1)$ match the portfolio PD parameters $p_1, \ldots, p_n$, which is achieved by setting

$$K_i := G_i^{-1}(p_i). \tag{2.13}$$

Then indeed, if the default indicators are specified under a threshold model (A, K) with thresholds (2.13), for all $i \in \{1, \ldots, n\}$ it holds that

$$\begin{aligned} F_i(1) &= \mathbb{P}(D_i = 1) \\ &= \mathbb{P}(A_i \leq K_i) \\ &= \mathbb{P}(A_i \leq G_i^{-1}(p_i)) \\ &= \mathbb{P}(G_i(A_i) \leq p_i) \\ &= p_i, \end{aligned}$$

as $G_i(A_i)$ is uniformly distributed. Through the construction of a threshold model, dependency between default indicators D is established through the dependency between their corresponding asset values A. Hence it is important to distinguish here between the earlier defined default correlation and *asset value correlation*

$$\rho_{ij,asset} := \mathrm{Corr}(A_i, A_j). \tag{2.14}$$

Through the above definitions it becomes clear that there are two main factors that influence the loss distribution, namely the multivariate distribution function of A and the threshholds K.

The specification of G_A dictates the dependency between asset values and subsequently defaults, while the thresholds K influences both the default correlation directly, but also the marginal behaviour of defaults. Copulas serve as a powerful tool to seperate the marginal and joint behaviour of random variables, which are particularly useful in the establishment of a threshold model.

Definition 2.2.2 (Copula). *A copula C is a multivariate distribution function with uniform marginals.*

A central result in the theory of Copula's is Sklar theorem, which states that *any* multivariate distribution function can be decomposed into marginals and a copula. However if the multivariate CDF is assumed to be continuous, as is the case in our discussion, the proof of Sklar's theorem is tremendously simplified.

Theorem 2.2.1 (Sklar's Theorem (Continuous Version)). *Let $H(x_1, x_2, \dots, x_n)$ be a joint distribution function with continuous marginal distribution functions $F_1(x_1), F_2(x_2), \dots, F_n(x_n)$, where each x_i attains values in the domain $\mathcal{X}$. Then there exists a function*

$$C\colon [0,1]^n \to [0,1]$$

such that:

$$H(x_1, x_2, \dots, x_n) = C(F_1(x_1), F_2(x_2), \dots, F_n(x_n)) \quad x_i \in \mathcal{X}. \tag{2.15}$$

C is called the copula induced by $H, F_1, \dots, F_n$.

Proof. See Appendix 7.1.

Thus within this context, the copula associated to $G_A, G_1, \dots, G_n$ is unique and given by the function

$$C_A(u_1, u_2, \dots, u_n) := G_A(A_1 \leq G_1^{-1}(u_1), \dots, A_n \leq G_n^{-1}(u_n)) \quad u_i \in [0,1]. \tag{2.16}$$

Hence the copula C_A evaluated at PD parameter points $p_1, \dots, p_n$ is equal to the probability that the complete portfolio defaults

$$C_A(p_1, \dots, p_n) = G_A(A_1 \leq K_1, \dots, A_n \leq K_n) = F_D(1, \dots, 1).$$

Theorem 2.2.1 shows that the CDF G_A can be written in terms of its marginals G_i and a copula C. We can utilize this fact to establish that the dependency between default indicators is dictated only by the underlying copula C_A of G_A and threshold choices $K_1, \dots, K_n$, but not by the marginals $G_1, \dots, G_n$. Reason being that the thresholds K can always be rechosen to rematch the marginals of D_i. The following Lemma makes this more precise.

Lemma 2.2.2 (Equivalence of Threshold Models). *Let (A, K) and $(\tilde{A}, \tilde{K})$ be two threshold models with default vectors D and $\tilde{D}$, defined as in (2.10). Then the models are equivalent if both*

(i) The marginal distributions of D and $\tilde{D}$ coincide.

(ii) A and $\tilde{A}$ admit the same copula C.

Proof. See Appendix 7.2

Lemma 2.2.2 states that a threshold model (A, K) is in essence uniquely defined by a copula C and thresholds K. Let C_{ij} denote the copula associated with the bivariate CDF G_{ij}, i.e.

$$C_{ij}(u,v) = G_A(G_i^{-1}(u), G_j^{-1}(v)) \quad u,v \in [0,1]. \tag{2.17}$$

The bivariate CDF of default indicators D_i, D_j, defined under a threshold model (A, K), can now also be written in terms of the associated copula function evaluated at the PD parameter points p_i, p_j,

$$F_{ij}(1,1) = \mathbb{P}(D_i = 1, D_j = 1) = G_A(G_i^{-1}(p_i), G_j^{-1}(p_j)) = C_{ij}(p_i, p_j). \tag{2.18}$$

As a result, the expression for default correlation can now be updated to depent only upon the copula C_A of G_A and the PD parameters of the model,

$$\rho_{ij,default} = \frac{C_{ij}(p_i,p_j) - p_i p_j}{\sqrt{p_i(1-p_i)p_j(1-p_j)}}. \tag{2.19}$$

A simple upper bound on the default correlation follows directly from the Fréchet-Hoeffding bounds that hold for any copula.

Lemma 2.2.3 (Fréchet-Hoeffding Bounds)**.** *Let $C(u_1, u_2, \ldots, u_n)$ be a copula function. Then, for any $u_1, \ldots, u_n \in [0,1]$, the following inequalities hold:*

$$\max\left(1 - n + \sum_{i=1}^{n} u_i, 0\right) \leq C(u_1, u_2, \ldots, u_n) \leq \min(u_1, u_2, \ldots, u_n). \tag{2.20}$$

Proof. See Appendix 7.3

The lower Fréchet-Hoeffding bound is also a copula if $n = 2$, corresponding to perfect negative dependence. The upper bound is a copula for any $n \in \mathbb{N}$, corresponding to perfect dependence, also referred to as the *co-monotonicity* copula. Another noteworthy copula is the *independence* copula, defined as

$$C(u_1, \ldots, u_n) = \prod_{i=1}^{n} u_i, \tag{2.21}$$

This copula captures the dependency (or lack thereof) among a set of independent random variables.
If $p_i \leq p_j$, then the upper Fréchet-Hoeffding bound implies that

$$\rho_{ij,default} = \frac{C_{ij}(p_i,p_j) - p_i p_j}{\sqrt{p_i(1-p_i)p_j(1-p_j)}} \leq \frac{p_i(1-p_j)}{\sqrt{p_i(1-p_i)p_j(1-p_j)}} = \sqrt{\frac{p_i(1-p_j)}{(1-p_i)p_j}}. \tag{2.22}$$

Notice that moving p_i and p_j further apart, lowers the upper bound for default correlation. Conversely if $p_i = p_j$, then the upper bound is equal to one. What this shows is that, even though the asset correlation $\rho_{ij,asset}$ might be equal to one, default correlation can still be set arbitrarily low by letting PDs $p_i \to 0$ and $p_j \to 1$.

2.3 One-Factor Levy Threshold Model

The previous section has shown that a threshold model for correlated defaults is specified via a multivariate CDF G_A and threshold vector K. Theorem 2.2.1 provided a means to decompose G_A uniquely into a copula C_A and its marginals $G_1, \ldots, G_n$. This Section provides an appealing approach towards sampling the asset values according to G_A, which will be seen to have a realistic interpretation as well. We start with a definition of the Lévy process

Definition 2.3.1. *A process $X = \{X_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if it possesses the following properties:*

(i) The paths of X are $\mathbb{P}$ a.s right continuous with left limits.

(ii) $P(X_0 = 0) = 1 \quad \mathbb{P}$ a.s.

(iii) For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.

(iv) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

The distribution of a Lévy process possesses the characteristic of infinite divisibility: for any integer n, the distribution of a Lévy process at time t can be expressed as the distribution of the sum of n independent random variables. These random variables precisely represent the increments of the Lévy process over time intervals of length t/n, which are assumed to be independent and identically distributed according to assumptions (iii) and (iv) of Definition 2.3.1. To ensure thoroughness, the definition of infinite divisibility is provided below:

Definition 2.3.2. *Let X be a random variable with CDF $F_X(x)$. The distribution of X is said to be infinitely divisible if, for every positive integer n, there exist n independent and identically distributed random variables $X_1, X_2, \ldots, X_n$ such that their sum $S_n = X_1 + X_2 + \ldots + X_n$ has the same distribution as X.*

The One-Factor Lévy Threshold Model is defined as follows. Consider an infinitely divisible distribution H. Let $X = \{X_s : s \in [0, 1]\}$ be a Lévy process based on H, meaning that $X_1 \sim H$. Denote the distribution function of X_s by H_s and assume that it is continuous. Further assumptions put on X is that X_1 is standardized, meaning $\mathbb{E}[X_1] = 0$ and $\mathrm{Var}(X_1) = 1$. From this immediately also follows the more general fact that $\mathbb{E}[X_s^2] = s$. Besides X, consider also n independent copies of X, notated as $X^{(1)}, \ldots, X^{(n)}$. For a given $\rho \in [0, 1]$, let asset value A_i be given by

$$A_i = X_\rho + X_{1-\rho}^{(i)}. \tag{2.23}$$

A_i is affected by a systematic influence X_ρ (on the portfolio level) and an idiosyncratic influence $X_{1-\rho}^{(i)}$ (on the loan level). Figure 2.2 illustrates how that decomposition affects the joint behaviour of asset values $A_1, \ldots, A_n$. Since X_ρ and $X_{1-\rho}^{(i)}$ are independent and identically distributed, the marginal A_i is standardized and distributed to the law of H_1, as a result of (iii) in definition 2.3.1. Furthermore the pairwise asset value correlation $\rho_{ij,asset}$ is accordingly given by the parameter ρ, since

$$\begin{aligned}\rho_{asset} &= \mathrm{Corr}(A_i, A_j)\\ &= \frac{\mathbb{E}[A_i A_j]}{\sigma_{A_i}\sigma_{A_j}}\\ &= E[X_\rho^2] - \mathbb{E}[X_\rho X_{1-\rho}^{(i)}] - \mathbb{E}[X_\rho X_{1-\rho}^{(j)}] - \mathbb{E}[X_{1-\rho}^{(i)} X_{1-\rho}^{(j)}]\\ &= \mathbb{E}[X_\rho^2] \qquad \text{by independence and } \mathbb{E}[X_s], \mathbb{E}[X_s^{(i)}] = 0, \forall s\\ &= \rho.\end{aligned}$$

In summary, the multivariate distribution G_A induced by the factor model (2.23) presented has identical marginals equal to H_1. The dependency between asset values is established through the systematic factor X_ρ, distributed according to the CDF H_ρ. For any $\rho \in [0, 1]$, we let the induced copula of the model be denoted by C_ρ, i.e.

$$C_\rho(u_1, \ldots, u_n) = \mathbb{P}(A_1 \leq H_1^{-1}(u_1), \ldots, A_n \leq H_1^{-1}(u_n)). \tag{2.24}$$

Notice that C_0 is the independence copula, since

$$\begin{aligned}C_0(u_1, \ldots, u_1) &= \mathbb{P}(X_1^{(1)} \leq H_1^{-1}(u_1), \ldots, X_1^{(n)} \leq H_1^{-1}(u_n))\\ &= \mathbb{P}(H_1(X_1^{(1)}) \leq u_1, \ldots, H_1(X_1^{(n)}) \leq u_n)\\ &= \prod_{i=1}^{n} u_i \quad \text{by independence of } X_1^{(1)}, \ldots, X_1^{(n)}.\end{aligned}$$

Furthermore C_1 is the co-monotonicity copula, i.e. the upper Fréchet-Hoeffding bound, since

$$\begin{aligned}C_1(u_1, \ldots, u_1) &= \mathbb{P}(X_1 \leq H_1^{-1}(u_1), \ldots, X_1 \leq H_1^{-1}(u_n))\\ &= \mathbb{P}(H_1(X_1) \leq u_1, \ldots, H_1(X_1) \leq u_n)\\ &= \mathbb{P}(H_1(X_1) \leq \min(u_1, \ldots, u_n))\\ &= \min(u_1, \ldots, u_n)\end{aligned}$$

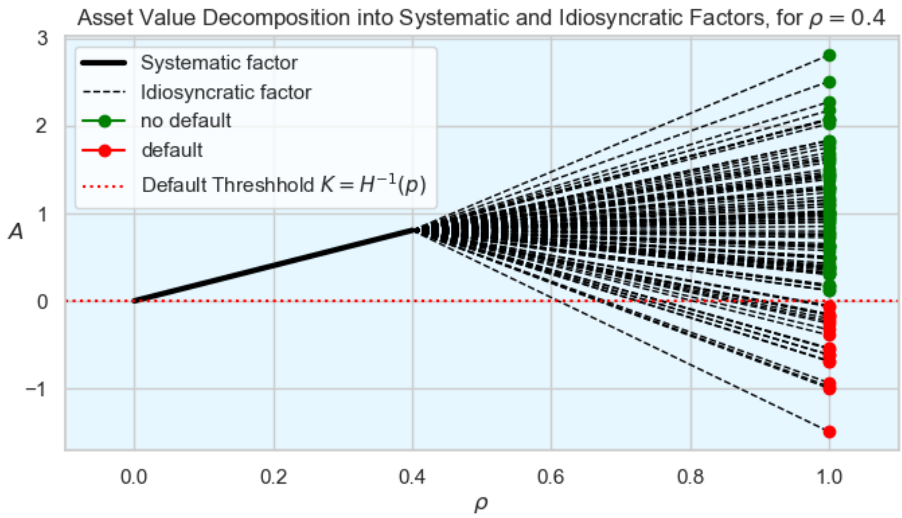

Figure 2.2: An illustration how the default behaviour is influenced by the asset values' decomposition into systematic and idiosyncratic components. The portfolio size is $n = 100$, PD set to $p = 0.5$ and asset correlation set to $\rho = 0.3$.

2.4 A Note on Exposure

It is generally not possible to express the multivariate Bernouilli distribution F_D, resultant from threshold model (A, K) with A generated through (2.23), in simple terms. Hence also the one-dimensional loss random variable inherits this complexity. The choice of NVs N_i and LGDs l_i, combining into exposures e_i (or exposure weight w_i), determine the space in which the discrete random variable L attains its values. Its realized outcome is affected purely by the realized outcome of the default indicators $D_1, \dots, D_n$, which are mapped into a loss through the deterministic map $\gamma\colon \{0,1\}^n \to \mathcal{L}$. Its image $\mathcal{L}$ is the collection of all values that that L can attain, see (2.5). By definition the map γ is surjective, however it is not injective per sé. If we have a portfolio of two loans with exposures weights $w_1 = \frac{1}{3}, w_2 = \frac{2}{3}$, then $\mathcal{L} = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, every outcome of the default indicator vector D has its individual loss coupled to it. On the other hand a portfolio with weights $w_1 = w_2 = \frac{1}{2}$, the possible values the loss can attain is given by $\mathcal{L} = \{0, \frac{1}{2}, 1\}$. Its (degree of) injectivity directly affects the probabilistic nature of L, since in the absence of injectivity two different default realizations could lead to the same loss. What this means is that exposure weights do not merely specify a "space" in which L attains its values. The number of ways in which distinct subsums can result in the same value also impacts the probability distribution assigned to a specific value.

That latter insight underscores the complexity of the loss random variable under the model described above. Several Levy Processes $X_s, X_s^{(1)}, \dots, X_s^{(n)}$, governed by the law of an infinitely divisible distribution H, are combined through (2.23) to generate asset values $A_1, \dots, A_n$. These undergo an additional transformation into default indicators $D_1, \dots, D_n$, determined by specific threshold choices $K_1, \dots, K_n$. Subsequently, these indicators are further aggregated into a sum weighted by exposure, culminating into the one-dimensional loss random variable L. The following section disects the CDF of L. And shows that, under some strong assumptions on the portfolio characteristics, the distribution of L could be dramatically simplified.

2.5 Loss Distribution

Consider a threshhold model (A, K), with A generated via (2.23), generating default indicator vector $D = (D_1, \dots, D_n)$ with arbitrary PDs $p_1, \dots, p_n \in [0, 1]$. Further assume that all NVs and LGDs are equal to unity, i.e. $N_i = l_i = 1$, resulting in homogeneous exposure weights $w_i = \frac{1}{n}$. Then the loss random variable L simplifies to

$$L = \frac{1}{n} \sum_{i=1}^{n} D_i, \tag{2.25}$$

which is a finite sum of dependent and non-identical Bernouilli random variables D_i. However notice that, conditional on the systematic factor X_ρ, the default indicators D_i are in fact independent. In eye of this it is useful to define the conditional PD of loan i by the function

$$\begin{aligned} \pi_i(x) &:= \mathbb{P}(A_i \leq K_i | X_\rho = x) \\ &= \mathbb{P}(X_\rho + X_{1-\rho}^{(i)} \leq K_i | X_\rho = x) \\ &= \mathbb{P}(X_{1-\rho}^{(i)} \leq K_i - X_\rho | X_\rho = x) \\ &= H_{1-\rho}(K_i - x). \end{aligned} \tag{2.26}$$

Utilizing this function it holds that the distribution of nL given $X_\rho = x$ follows a Poisson-Binomial distribution with parameter n and success probabilities $\pi_1(x), \dots, \pi_n(x)$. Let $\mathcal{I}_k$ denote the set of all subsets $I \subset \{1, \dots, n\}$ that are of size k. The distribution function of nL conditional on $X_\rho = x$ is thus given by

$$\mathbb{P}(nL = k | X_\rho = x) = \sum_{I \in \mathcal{I}_k} \prod_{i \in I} \pi_i(x) \prod_{i \in I^c} (1 - \pi_i(x)). \tag{2.27}$$

Integrating over X_ρ yields the unconditional loss distribution

$$\mathbb{P}(nL = k) = \int_{\mathrm{ran}(X_\rho)} \sum_{I \in \mathcal{I}_k} \prod_{i \in I} \pi_i(x) \prod_{i \in I^c} (1 - \pi_i(x)) dH_\rho(x). \tag{2.28}$$

It is evident that even after the first measures in homogenizing the portfolio, which was setting the exposures and notionals equal to one, evaluation of the conditional and unconditional loss distributions (2.27) and (2.28) remains an intensive computation. Especially for larger portfolios, as the number of evaluations for each point of integration within (2.28) grows with the cardinality of the set $\mathcal{I}_k$, given by $\binom{n}{k}$.

An important concept to aid in the simplification of this setting is *exchangebility.* Analysis of the loss distribution can be tremendously simplified if exchangebility on the underlying threshold model (A, K) is assumed.

Definition 2.5.1 (Exchangebility). *An n-dimensional random vector $X = (X_1, \dots, X_n)$ is called exchangeable if*

$$(X_1, \dots, X_n) \stackrel{d}{=} (X_{\Pi(1)}, \dots, X_{\Pi(n)}) \quad \textit{for any permutation } \Pi. \tag{2.29}$$

Furthermore (A, K) is called an exchangeable threshold model for default indicators $D = (D_1, \dots, D_n)$ if D is exchangeble.

For an exchangeable threshold model the bivariate default distributions F_{ij} are identical for every pair of loans (i, j). As a result, a single asset correlation and default correlation describe the overall dependency between defaults within the portfolio. Furthermore the defaults are now governed by a single threshold parameter $K = H_1^{-1}(p)$. Thus for a homogeneous portfolio it makes sense to consider the conditional PD on portfolio level, which we will denote by

$$\pi_K(x) := H_{1-\rho}(K - x), \tag{2.30}$$

which is essentially the same as the conditional EL of the portfolio loss. In the exchangeable case, the random variable nL conditional on $X_\rho = x$ is distributed according to a binomial distribution with parameters n and p, simplifying expression (2.28) into

$$\mathbb{P}(nL = k) = \int_{\operatorname{ran}(X_\rho)} \binom{n}{k} (H_{1-\rho}(K - x))^k (1 - H_{1-\rho}(K - x))^{n-k} dH_\rho(x). \tag{2.31}$$

Even though the exchangebility assumption provides a simpler way of expressing the unconditional loss distribution, obtaining it would still require computationally intensive numerical integration over X_ρ. This problem can be alleviated by letting the number of obligors n approach infinity, which is for large portfolios a very good approximation of the unconditional loss distribution.

2.6 Large Homogeneous Portfolio Approximation

A well known approximation for the unconditional loss distribution of a homogeneous portfolio is given by the Large Homogeneous Portfolio (LHP) approximation, whose origin can be traced back to the works of Vasicek (O. Vasicek 2002), (O. A. Vasicek 1987). This approximation is the limit of the discrete distribution function $\mathbb{P}(L = \frac{k}{n})$ as $n \to \infty$ for a homogeneous portfolio. In other words, it is the distribution function of

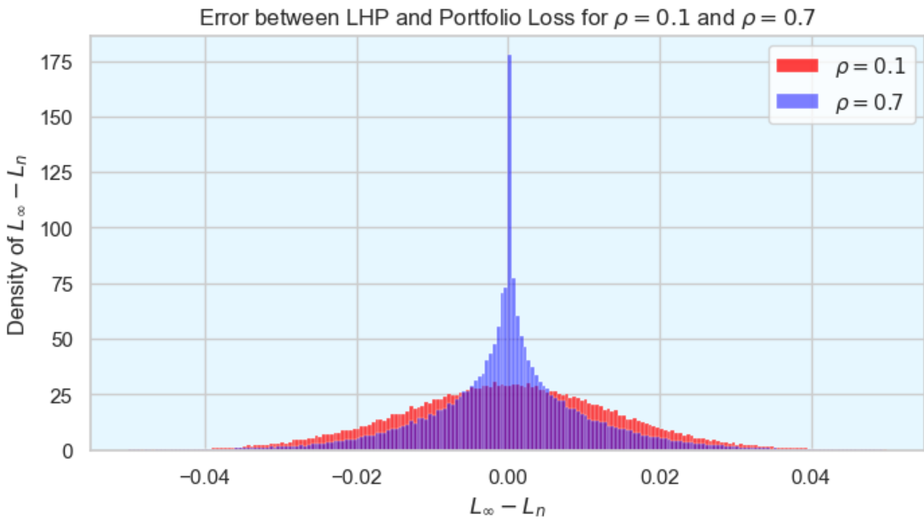

Figure 2.3: Simulation of the density of the error $L_\infty - L_n$ for two correlation parameters and a portfolio size of 1000.

the limiting random variable

$$L_\infty := \lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} D_i. \tag{2.32}$$

The distribution of L_∞ is found as followed. First observe that conditional on $X_\rho = x$, the loss random variable $L_n = \frac{1}{n}\sum_{i=1}^{n} D_i$ converges a.s. to $\mathbb{E}[D_i|X_\rho = x] = \pi_K(x)$ by the Strong Law of Large Numbers (SLLN). Note the addition of the subscript n to L_n for specifying the portfolio size. The unconditional random variable L_n thus converges almost surely to the random variable $\pi_K(X_\rho)$, which is called the LHP approximation. Its CDF, which we will generally denote by F_∞, is given by

$$\begin{aligned} F_\infty(x) := \mathbb{P}(L_\infty \le x) &= \mathbb{P}(H_{1-\rho}(K - X_\rho) \le x) \\ &= \mathbb{P}(K - X_\rho \le H_{1-\rho}^{-1}(x)) \\ &= 1 - H_\rho(K - H_{1-\rho}^{-1}(x)). \end{aligned} \tag{2.33}$$

The distribution function in (2.33) is different from the CDF of L_n, but gets closer as $n \to \infty$. Figure 2.3 shows two overlapping histograms of the difference $L_\infty - L_n$ for two different correlations ρ. From these histograms of the difference $L_\infty - L_n$ it seems that the first moments align for all portfolio sizes, which is indeed the case since

$$\mathbb{E}[L_\infty] = \mathbb{E}[\mathbb{E}[\mathbb{1}[A_i \le K]|X_\rho]] = p = \mathbb{E}[L_n] \quad \forall n.$$

However higher moments do not coincide. Let $C_\rho(u,v) = \mathbb{P}(A_i \le H_1^{-1}(u), A_j \le H_1^{-1}(v))$ be the copula function for the bivariate CDF of any two asset values. By the exchangability assumption, this copula describes all dependencies between asset values completely. The second moment of L_n is then given by

$$\begin{aligned}
\mathbb{E}[L_n^2] &= \mathbb{E}\Big[\Big(\frac{1}{n}\sum_{i=1}^{n} D_i\Big)^2\Big] \\
&= \frac{1}{n^2}\Big((n^2-n)\mathbb{E}[D_iD_j] + n\mathbb{E}[D_i^2]\Big) \\
&= \Big(1-\frac{1}{n}\Big)C_\rho(p,p) + \frac{p}{n} \\
&= C_\rho(p,p) + \frac{1}{n}\Big(p - C_\rho(p,p)\Big).
\end{aligned}$$

As a reminder $\mathbb{E}[D_iD_j]$ is indeed equal to the bivariate copula of the multivariate CDF G_A, since

$$\mathbb{E}[D_iD_j] = \mathbb{P}(A_i \le H_1^{-1}(p), A_j \le H_1^{-1}(p)) = C_\rho(p,p).$$

Since $L_n \to L_\infty$ almost surely, the second moment of L_∞ is equal to $C_\rho(p,p)$. Again by the upper Fréchet-Hoeffding bound of the bivariate copula we have $p \ge C_\rho(p,p)$, hence the second moment of L_∞ is a lower bound for the second moment of L_n. Its difference decreases as n grows.

It is also important to elaborate on the situation if the portfolio is not homogeneous, but has distinct thresholds $K_1, \ldots, K_n$. What can we then say about the limit of L_n as $n \to \infty$? The problem is that, unlike the homogeneous case, we must be provided with an infinite sequence of thresholds $K_1, K_2, \ldots$ and exposure weights $w_1, w_2, \ldots$, such that we can evaluate the limit $\sum_{i=1} w_i D_i$. There are certain conditions that must be placed on those sequences of values in order for the limit to exist and can be found in (Gordy 2003). We will, however, proceed differently.

Recall that for a homogeneous portfolio, with PD parameter $p = H_1(K)$ and homogeneous exposures set to unity, the LHP approximation is given by the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} D_i = \pi_K(X_\rho),$$

in other words the conditional PD of the portfolio. This is the same as the conditional EL $\mathbb{E}[L_n|X_\rho]$, as a consequence of the homogenuity assumption. Consider now an inhomogeneous portfolio with PD parameters $p_1 = H_1(K_1), \ldots, p_n = H_n(K_n)$ and exposure weights $w_1, \ldots, w_n$. Define the uniformly distributed random variable $(\hat{w}, \hat{K})$ on the domain $\{(w_1, K_1), \ldots, (w_n, K_n)\}$, and define the limiting random variable of the loss L_n

of an inhomogeneous portfolio as

$$L_\infty = \lim_{n\to\infty} \sum_{i=1}^{n} \hat{w}_i \mathbb{1}[A_i \le \hat{K}_i] \tag{2.34}$$

where the pairs $(\hat{w}_1 \hat{K}_1), (\hat{w}_2, \hat{K}_2), \ldots$ are iid draws from $(\hat{w}, \hat{K})$. It is not hard to see that under this construction, the limiting random variable L_∞ is again given by the conditional EL of the portfolio. Since all A_i are also identically distributed and conditionally independent on the systematic factor X_ρ, we can apply the SLLN again as follows:

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \hat{w}_i \mathbb{1}[A_i \le \hat{K}_i] &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} n\hat{w}_i \mathbb{1}[A_i \le \hat{K}_i] \\
&\stackrel{d}{=} \mathbb{E}[n\hat{w}\mathbb{1}[A_1 \le \hat{K}]|X_\rho] \\
&= \sum_{i=1}^{n} \mathbb{E}[nw_i \mathbb{1}[A_1 \le \hat{K}_i]|X_\rho]\mathbb{P}((\hat{w}, \hat{K}) = (w_i, K_i)) \\
&= \frac{1}{n} n \sum_{i=1}^{n} w_i \mathbb{E}[\mathbb{1}[A_1 \le \hat{K}_i]|X_\rho] \\
&= \sum_{i=1}^{n} w_i \pi_i(X_\rho) \\
&= \mathbb{E}[L_n|X_\rho],
\end{aligned}$$

Recall that for a homogeneous portfolio, the distribution function of the LHP random variable $\pi_K(X_\rho)$ is given by (2.33). In contrast, the distribution function of $\sum_{i=1}^{n} w_i \pi_i(X_\rho)$ is not as readily computed, since the inverse of the weighted sum is harder to find. It will be seen, however, that for the purpose of the methodology developed here, knowing the full distribution of L_∞ for inhomogeneous portfolios is not necesarry.

We end this chapter with two Lemmas (Graybill 1967) that tell an important story regarding the relationship between the random variables L_∞ and L_n. They state that essentially L_∞ is the optimal transformation of the systematic factor X_ρ for the purpose of approximating L_n.

Lemma 2.6.1. *Let (X, Y) be a random vector. Then for any X measurable function f we have*

$$|\mathrm{Corr}(Y, f(X)| \le \mathrm{Corr}(Y, \mathbb{E}[Y|X])$$

Proof. See Appendix 7.4

This Lemma tells us is that L_∞ is the maximally correlated random variable with L_n if we only use the information of X_ρ. Furthermore L_∞ minimizes the MSE with L_n, as a consequence of the following Lemma

Lemma 2.6.2. *Let (X, Y) be a random vector. For any X measurable function f we have*

$$\mathbb{E}[(Y - f(X))^2] \geq \mathbb{E}[(Y - \mathbb{E}[Y|X])^2]$$

Proof. See Appendix 7.5

In conclusion, for both homogeneous and inhomogeneous portfolios, the LHP random variable is well defined and is essentially another naming for the conditional EL. Lemma 2.6.1 and 2.6.2 highlight the potential of the role L_∞ can play in estimating complicated stochastics that might be functions of L_n. The Lemmas essentially state that there is no better way of utilizing the information provided by the systematic factor X_ρ for the purpose of estimating L_n. However the question remains whether the advantages of L_∞ apply as well to a complicated transformation of L_n. The following chapters aim to answer that question in the context finding the fair premium of a CDO tranche.

3 CDO Pricing Models

3.1 Collateralized Debt Obligations

A Collateralized Debt Obligation (CDO) is a financial instrument designed to reallocate the credit risk associated with a specific set of assets. If the underlying set of assets is a portfolio of *Credit Default Swaps* (CDS), then the term *synthetic* CDO is often used. The redistributing of risk is achieved by partitioning the potential loss associated with the reference portfolio among *tranches*. The CDO *issuer*, also called the *protection buyer*, typically structures the tranches in such a way that it attains a desired credit rating, provided by rating agencies such as Moore's and S&P. Arranged in order of seniority, common names for tranches are *equity*-, *mezzanine*-, *senior*-, and *super-senior* tranche. The *investor*, also called the *protection seller*, engaged in a CDO transaction selects a specific tranche in which to invest, that exposes them to the credit risk inherent to the chosen tranche. In this process, investors essentially engage in an arrangement through which they offer credit protection to the CDO issuer for that particular tranche, accepting responsibility for any potential losses. In return for bearing this credit risk, investors are entitled to receive compensation in the form of premiums, commonly calculated as a *spread* over the tranche notional. Junior tranches, such as equity and mezzanine face early losses, while senior tranches are impacted only when the lower tranches have already fully absorbed their maximum possible loss. Tranching results in higher risk for junior tranches, leading to their lower market value compared to the less risky senior tranches. This risk diversification serves as the primary objective of a CDO, designed to attract a broad spectrum of investors. Risk-averse institutions like banks or pension funds are typically inclined to invest in senior CDO tranches, while smaller, more risk-tolerant firms may choose mezzanine or even equity tranches for their investments. A typical CDO tranche structure is depicted in Figure 3.1.

Even though this thesis does not focus on all details that are involved in pricing a CDO, for completeness we shall provide a full specification of the framework here. CDO contracts are generally specified for a given time horizon $[0, T]$, with T being the maturity date. Losses of the underlying portfolio are assumed to follow a stochastic process $L(t)$ with respect to a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, where $\mathbb{Q}$ is the risk-neutral measure, and adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let $\{r_t\}_{t\geq 0}$ denote the interest rate process, which is assumed to be known. A tranche is defined by its *attachment* and *detachment* points $0 \leq c < d \leq 1$. Assume that upon pre-defined moments in time $0 = t_0 < t_1, \ldots, t_m < T$ an exchange of payments is done between the protection seller and -buyer. The protection seller receives premium at times t_i for the period $[t_{i-1}, t_i]$,

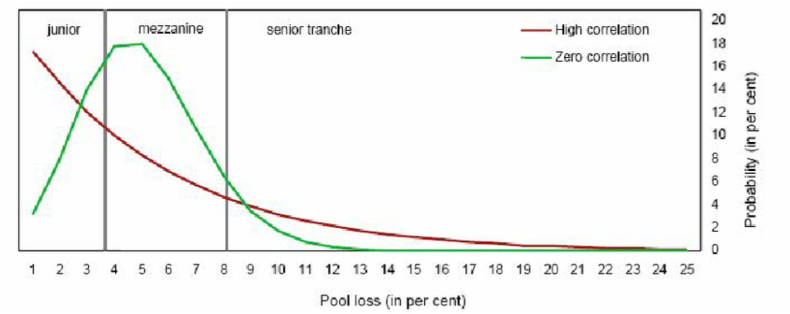

Figure 3.1: Plot of a common CDO tranche structure with two loss distributions of varying correlation. The higher the correlation, the more probabilistic weight of the loss distribution is put into senior tranches.

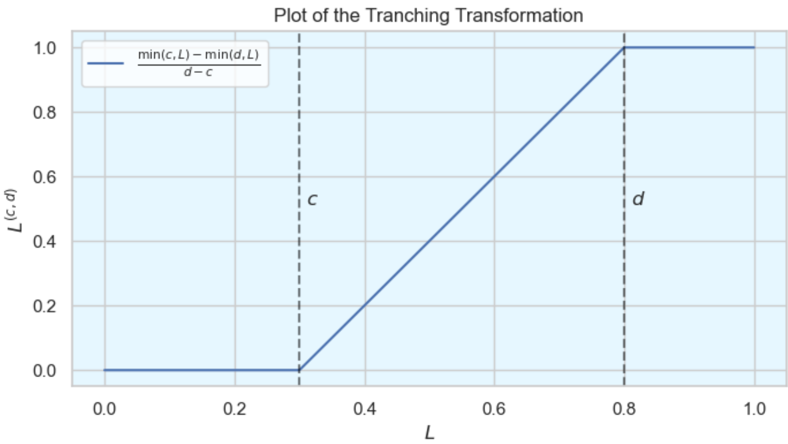

Figure 3.2: A plot of the tranching transformation.

while the protection buyer compensates the protection seller for losses that have occurred in that same period. The premium spread that is to be payed over the tranche notional is denoted by s. Further we denote $L^{(c,d)}(t)$ to be the pro-rata loss of the tranche at time t, which is fully specified in terms of the loss process $L(t)$ via

$$L^{(c,d)}(t) = \frac{\min(L(t), d) - \min(L(t), c)}{d - c} \tag{3.1}$$

Figure 3.2 visualizes this transformation through a plot. Under the risk neutral measure $\mathbb{Q}$ let the discount factor $\mathbb{E}\left[e^{-\int_{t_0}^{t} r(u)du}\right]$ be denoted by $D(t)$. The value of the *premium leg* is then computed as the present value of all expected spread payments

$$\text{premium leg} = s \sum_{j=1}^{m} \Delta t_j (1 - \mathbb{E}[L^{(c,d)}(t)] D(t_j) \tag{3.2}$$

where $\Delta t_j = t_j - t_{j-1}$. The expected payment to be made by the protection seller to the protection buyer at time t_j is equal to $\mathbb{E}[L^{(c,d)}(t_j)] - \mathbb{E}[L^{(c,d)}(t_{j-1})]$. The value of the *protection leg* is computed as the sum of the present value of expected default payments,

$$\text{protection leg} = \sum_{j=1}^{m} (\mathbb{E}[L^{(c,d)}(t_j)] - \mathbb{E}[L^{(c,d)}(t_{j-1})])D(t_j). \tag{3.3}$$

Since at issuance the premium spread s is determined, a CDO is fairly priced if and only if the total exchange of payments is equal, i.e.

$$s = \frac{\sum_{j=1}^{m} (\mathbb{E}[L^{(c,d)}(t_j)] - \mathbb{E}[L^{(c,d)}(t_{j-1})])D(t_j)}{\sum_{j=1}^{m} \Delta t_j (1 - \mathbb{E}[L^{(c,d)}(t)])D(t_j)}. \tag{3.4}$$

The only unknown quantities here are the *expected tranche losses* (ETL) $\mathbb{E}[L^{(c,d)}(t_j)]$, whose values are governed by the underlying loss distribution. Thus the central problem of establishing the fair premium spread of a CDO tranche, is finding a convenient methodology for calculating the ETL.

3.2 Expected Tranche Loss

Rather than focussing on a multi-time step horizon as previously presented, we continue with a simplified single-period setting. We denote the loss incurred on a tranche with attachment and detachment points $0 \leq c < d \leq 1$ as the random variable

$$L_n^{(c,d)} = \frac{\min(d, L_n) - \min(c, L_n)}{d - c}, \tag{3.5}$$

which is normalized by $d - c$ such that $L_n^{(c,d)}$ attains its maximum and minimum value at 1 and 0 respectively. We again assume that the default indicator vector D is implied by a threshold model (A, K), where A is generated using (2.23). In order to arrive at the fair spread given in equation (3.4), we must evaluate the expected tranche loss $\mathbb{E}[L_n^{(c,d)}]$. Let us write out this quantity in full form:

$$\mathbb{E}[L_n^{(c,d)}] = \mathbb{E}\Big[\frac{\min\left(d, \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}[X_\rho + X_{1-\rho}^{(i)} \leq K_i]\right) - \min\left(c, \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}[X_\rho + X_{1-\rho}^{(i)} \leq K_i]\right)}{d - c}\Big].$$

Recall that the unconditional loss distribution (2.28) is of complicated form, so attempts to calculate $\mathbb{E}[L_n^{(c,d)}]$ exactly can be discarded. We proceed our study into $\mathbb{E}[L_n^{(c,d)}]$ by first looking at the optimal approximation of $L_n^{(c,d)}$ given the systematic factor X_ρ, which from Section 2.6 is seen to be given by the conditional expectation $\mathbb{E}[L_n^{(c,d)}|X_\rho]$. This random variable is the optimal proxy of the true tranche loss $L_n^{(c,d)}$ if we are only allowed to incorporate the information of X_ρ, as it is maximally correlated and has minimal MSE by Lemmas 2.6.1 and 2.6.2 respectively. However, unfortunately the expectation of $\mathbb{E}[L_n^{(c,d)}|X_\rho]$ is still cumbersome to compute. Instead we are looking for a proxy of $L_n^{(c,d)}$ of which we *do* know the exact expectation. Similar to the procedure in Section 2.6, we take the limit as $n \to \infty$ and consider the random variable $\lim_{n\to\infty} \mathbb{E}[L_n^{(c,d)}|X_\rho]$ as a proxy for $\mathbb{E}[L_n^{(c,d)}|X_\rho]$.

Assume that the portfolio is homogeneous, i.e. the PD parameter is described a single quantity K. The limiting random variable of the portfolio loss L_n is thus given by the LHP random variable $L_\infty = \pi_K(X_\rho)$. Application of the continuous mapping theorem yields that the limit of the tranche loss random variable $L_n^{(c,d)}$ is given by

$$\lim_{n\to\infty} L_n^{(c,d)} = \lim_{n\to\infty} \frac{\min(d, L_n) - \min(c, L_n)}{d-c} = \frac{\min(d, L_\infty) - \min(c, L_\infty)}{d-c} := L_\infty^{(c,d)} \quad (3.6)$$

As a result of the following Lemma, the boundedness of $L_n^{(c,d)}$ together with the a.s. convergence of $L_n^{(c,d)}$ to $L_\infty^{(c,d)}$ implies that also the conditional expectation $\mathbb{E}[L_n^{(c,d)}|X_\rho]$ converges a.s. to $L_\infty^{(c,d)}$.

Lemma 3.2.1. *Let $X_1, X_2, \dots$ be a sequence of bounded non-negative random variables that converges a.s. to X. Then $\mathbb{E}[X_i|Y]$ converges a.s. to $\mathbb{E}[X|Y]$ for any random variable Y.*

Proof. See Appendix 7.6

Thus, even though $\mathbb{E}[L_n^{(c,d)}|X_\rho]$ might in this case not be equal in distribution to $L_\infty^{(c,d)}$ for any finite n, its limit is, since clearly we have that

$$\mathbb{E}[L_\infty^{(c,d)}|X_\rho] \stackrel{d}{=} L_\infty^{(c,d)}.$$

Figures 3.3 and 3.4 confirm the behaviour of Lemmas 2.6.1 and 3.2.1. Figure 3.3 shows that the MSE between $\mathbb{E}[L_n^{(c,d)}|X_\rho]$ and $L_\infty^{(c,d)}$ vanishes, confirming its convergence. Furthermore Figure 3.4 shows that the correlation $\mathrm{Corr}(L_n^{(c,d)}, \mathbb{E}[L_n^{(c,d)}|X_\rho])$ acts as an upper bound for the correlation $\mathrm{Corr}(L_n^{(c,d)}, L_\infty^{(c,d)})$, however by a small amount. This is very good news, as it means that $L_\infty^{(c,d)}$ is extremely close to the optimal proxy $\mathbb{E}[L_n^{(c,d)}|X_\rho]$, especially for larger portfolios. $L_\infty^{(c,d)}$ serves as the asymptotically maximally correlated random variable for assessing portfolio loss within a (c, d) tranche. Hence, especially for large portfolios, it can serve as an exceptionally effective control variate as will be seen in Section 4.2. We will refer to $L_\infty^{(c,d)}$ as the *tranched* LHP random variable, which is essentially a tranched conditional EL of the portfolio loss. The following section shows that its expectation can be computed in a fast way, independent from portfolio size.

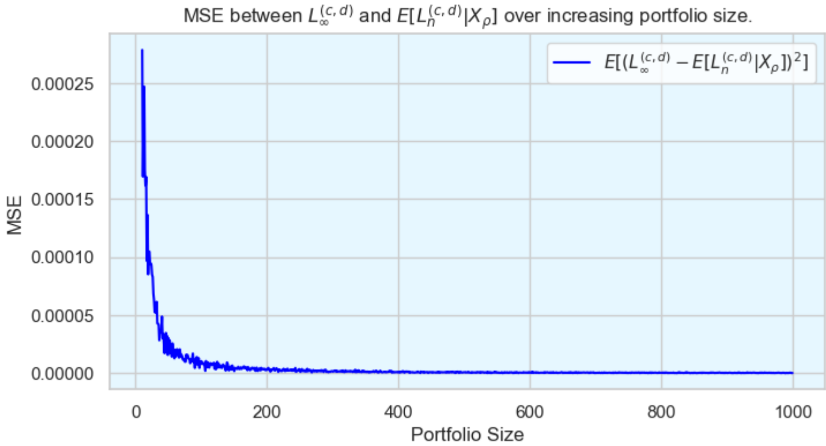

Figure 3.3: A plot of the MSE $\mathbb{E}[(L_\infty^{(c,d)} - \mathbb{E}[L_n^{(c,d)}|X_\rho])^2]$ under increasing portfolio size. It is approaching zero as a result of Lemma 3.2.1

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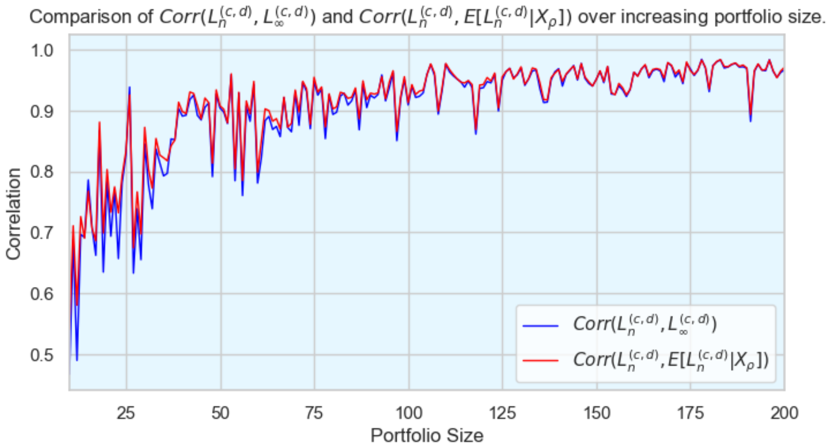

Figure 3.4: A comparison of the correlations $\mathrm{Corr}(L_n^{(c,d)}, L_\infty^{(c,d)})$ and $\mathrm{Corr}(L_n^{(c,d)}, \mathbb{E}[L_n^{(c,d)}|X_\rho])$ under increasing portfolio size. The correlations are almost identical, but $\mathrm{Corr}(L_n^{(c,d)}, L_\infty^{(c,d)})$ is dominated by $\mathrm{Corr}(L_n^{(c,d)}, \mathbb{E}[L_n^{(c,d)}|X_\rho])$ by a very small margin.

3.3 Expectation of tranched LHP

$L_\infty^{(c,d)}$ proves to be an effective control variate due to its ability to allow for analytical computation of its exact expectation in both homogeneous and inhomogeneous portfolios. In the case of homogeneous portfolios, the LHP approximation manifests as a single LHP random variable. The distribution function of its tranched transformation can be explicitly derived in terms of the original distribution function, facilitating the computation of the expectation.

However, for inhomogeneous portfolios, the scenario is somewhat more intricate. The sum of single LHP random variables, weighted by exposure, lacks a straightforward, explicitly defined distribution function. Despite this complexity, the property of the sum being a decreasing function of the systematic factor enables a simple substitution, leading to the derivation of an integral expression for the expectation of the tranched LHP.

3.3.1 Homogeneous Portfolio

If the portfolio is homogeneous with PD parameter $p = H(K)$, then the LHP random variable has the simple form

$$L_\infty = \pi_K(X_\rho) = H_{1-\rho}(K - X_\rho),$$

which has expectation equal to p. Recall that the tranched LHP random variable is given by

$$L_\infty^{(c,d)} = \frac{\min(d, L_\infty) - \min(c, L_\infty)}{d - c}.$$

Since this is a non-negative random variable, its expectation is equal to

$$\mathbb{E}[L_\infty^{(c,d)}] = \int_0^1 \mathbb{P}(L_\infty^{(c,d)} \geq l)dl.$$

For any $l \in [0,1]$, the probability $\mathbb{P}(L_\infty^{(c,d)} \geq l)$ can be decomposed into

$$
\begin{aligned}
\mathbb{P}(L_\infty^{(c,d)} \geq l) &= \mathbb{P}(L_\infty < c)\mathbb{P}(0 \geq l | L_\infty < c) \\
&+ \mathbb{P}(c \leq L_\infty < d)\mathbb{P}(L_\infty \geq l(d-c)+c | c \leq L_\infty < d) \\
&+ \mathbb{P}(L_\infty \geq d)\mathbb{P}(1 \geq l | L_\infty \geq d) \\
&= \mathbb{P}(L_\infty < c)\mathbb{1}[l=0] + \mathbb{P}(c \leq L_\infty < d)\mathbb{P}(L_\infty \geq l(d-c)+c | c \leq L_\infty < d) \\
&+ \mathbb{P}(L_\infty \geq d) \\
&= \mathbb{P}(L_\infty < c)\mathbb{1}[l=0] + \mathbb{P}(c \leq L_\infty < d)\frac{\mathbb{P}(L_\infty \geq l(d-c)+c, c \leq L_\infty < d)}{\mathbb{P}(c \leq L_\infty < d)} \\
&+ \mathbb{P}(L_\infty \geq d) \\
&= \mathbb{P}(L_\infty < c)\mathbb{1}[l=0] + \mathbb{P}(l(d-c)+c \leq L_\infty < d) + \mathbb{P}(L_\infty \geq d) \\
&= \mathbb{P}(L_\infty < c)\mathbb{1}[l=0] + \mathbb{P}(L_\infty < d) - \mathbb{P}(L_\infty \leq l(d-c)+c) + \mathbb{P}(L_\infty \geq d) \\
&= \mathbb{P}(L_\infty < c)\mathbb{1}[l=0] + 1 - \mathbb{P}(L_\infty \leq l(d-c)+c).
\end{aligned} \tag{3.7}
$$

Using this expression, notice that then indeed we get

$$
\begin{aligned}
\mathbb{P}(L_\infty^{(c,d)} = 0) &= \mathbb{P}(L_\infty \leq c) \\
\mathbb{P}(L_\infty^{(c,d)} = 1) &= \mathbb{P}(L_\infty \geq d).
\end{aligned}
$$

We have written the distribution function of the tranched LHP random variable $L_\infty^{(c,d)}$ in terms of the distribution function of L_∞, that can be evaluated using (2.33), we write

$$
\mathbb{P}(L_\infty^{(c,d)} \leq l) = \mathbb{P}(L_\infty \leq l(d-c)+c) - \mathbb{P}(L_\infty \leq c)\mathbb{1}[l=0]. \tag{3.8}
$$

Thus the expectation of $L_\infty^{(c,d)}$ can be numerically computed through the following integral:

$$
\begin{aligned}
\mathbb{E}[L_\infty^{(c,d)}] &= \int_0^1 \mathbb{P}(L_\infty^{(c,d)} \geq l)dl \\
&= \int_0^1 \mathbb{P}(L_\infty < c)\mathbb{1}[l=0] + 1 - \mathbb{P}(L_\infty \leq l(d-c)+c)dl \\
&= 1 - \int_0^1 \mathbb{P}(L_\infty \leq l(d-c)+c)dl \\
&= 1 - \frac{\int_c^d \mathbb{P}(L_\infty \leq u)du}{d-c} \\
&= 1 - \frac{\int_c^d 1 - H_\rho(K - H_{1-\rho}^{-1}(u))du}{d-c}
\end{aligned}
$$

Substituting $H_{1-\rho}(K-v) = u$ yields

$$\mathbb{E}[L_\infty^{(c,d)}] = 1 - \frac{\int_c^d 1 - H_\rho(K - H_{1-\rho}^{-1}(m))dm}{d-c} = 1 - \frac{\int_{x_d}^{x_c} h_\rho(K-v)(1-H_\rho(v))dv}{d-c} \tag{3.9}$$

where $x_c = \pi_K^{-1}(c)$ and $x_d = \pi_K^{-1}(d)$

3.3.2 inhomogeneous Portfolio

Calculating the expectation of a tranched LHP for an inhomogeneous portfolio is also possible. The LHP random variable for an inhomogeneous portfolio, with exposures e_i and PD parameters $p_i = H(K_i)$ is defined as the following transformation f of the systematic factor X_ρ,

$$L_\infty = \mathbb{E}[L_n|X_\rho] = \sum_{i=1}^{n} w_i \pi_i(X_\rho) = \sum_{i=1}^{n} w_i H_{1-\rho}(K_i - X_\rho) := f(X_\rho). \tag{3.10}$$

In the analysis for the homogeneous case, the distribution function of the tranched LHP could be calculated in terms of the distribution function of the LHP. However we are now faced with the problem that it is not clear how to express the distribution function of the inhomogeneous LHP $f(X_\rho)$, which is in fact a weighted sum of LHP approximations. The problem lies in the fact that there is no simple expression for the inverse of f, in contrast to the homogeneous case where $f(\cdot) = H_{1-\rho}(K-\cdot)$ is invertible. Neverthless we can still exploit that f is, for most choices of H, strictly decreasing. Thus an inverse f^{-1} does exist, which allows us to rewrite the following integral

$$\mathbb{E}[L_\infty^{(c,d)}] = \int_0^1 \mathbb{P}(L_\infty^{(c,d)} \geq l)dl = 1 - \int_0^1 \mathbb{P}(L_\infty \leq l(d-c)+c)dl$$

By letting $l(d-c)+c = u$, this integral becomes

$$1 - \frac{\int_c^d \mathbb{P}(L_\infty \leq u)du}{d-c}$$

Now we further substitute $u = f(v)$, which transforms the integral again into

$$\mathbb{E}[L_\infty^{(c,d)}] = 1 - \frac{1}{(d-c)} \sum_{i=1}^{n} \int_{x_d}^{x_c} w_i h_{1-\rho}(K_i - x)(1-H_\rho(v))dv \tag{3.11}$$

where $x_c = f^{-1}(c)$ and $x_d = f^{-1}(d)$. Notice that if $K_i = K$ and $e_i = e$ for all i then indeed (3.11) reduces to (3.9).

4 Monte Carlo for Expected Tranche Loss estimation

4.1 Crude Monte Carlo

Crude Monte Carlo (CMC) is a powerful computational method for the estimation of integrals. Conceptually it approaches estimating the volume of a set by treating it as a probability. By taking random samples from all possible outcomes and finding the fraction that falls within a specific set, it produces a volume estimate. As a simple example, suppose we wish to estimate the following unknown integral,

$$\mu = \int_0^1 f(x)dx \tag{4.1}$$

From a probability viewpoint this integral is equal to the expected value $\mathbb{E}[f(U)]$ for a uniform random variable U. Hence if we were able to access the outcomes of independent uniform random variables $U_1, \ldots, U_n$ and are able to produce pointwise evaluations $f(U_i)$, we could approximate μ via the CMC estimate

$$\mu \approx \hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i) \tag{4.2}$$

If f is integrable on $[0, 1]$, then the SLLN ensures that $\hat{\mu}_n$ is a consistent estimate of μ, i.e.

$$\hat{\mu}_n \xrightarrow{n\to\infty} \mu \quad \text{a.s.} \tag{4.3}$$

furthermore if f is square integrable on the unit interval, the variance of $\sigma_f^2 = \mathrm{Var}[f(U)]$ exists. The Central Limit Theorem tells us then that the error $\hat{\mu}_n - \mu$ is approximately normal with mean 0 and variance $\frac{\sigma_f^2}{n}$. Thus not only do we obtain an estimate for μ, but also information regarding its potential deviation from the truth. The shape of the standard error $\frac{\sigma_f}{\sqrt{n}}$ is a critical aspect of Monte Carlo. To reduce this error by half, you need to quadruple the number of points. If you aim to attain an additional decimal place of precision, it necessitates increasing the number of points by a factor of 100. For one-dimensional problems such as this, the $\sqrt{n}$ convergence rate is quite slow, much better convergence rates can be achieved. However its power lies in the fact that the relatively slow convergence rate is independent of the dimension of the integral. The procedure described above remains applicable when dealing with integrals over hypercubes $[0, 1]^d$ in any dimension, no matter how large. Hence, Monte Carlo methods are particularly appealing when addressing high-dimensional problems.

4.2 Control Variate Theory

Control Variates (CV) represent a versatile and effective variance reduction technique in Monte Carlo simulations, notable for their simplicity and efficiency. In virtually every stochastic simulation scenario, CVs serve as a viable approach to decrease the variance. It achieves this by exploiting a dependency structure between the random variable at study and another random variable whose expectation is assumed to be known. The amount of obtained variance reduction is dictated by the strength of dependency between the two random variables.

Suppose Y is a random variable whose expectation is unknown. Let $Y_1, \dots, Y_n$ be n i.i.d. copies of Y. The CMC estimate of the expectation $\mu_Y = \mathbb{E}[Y]$ is given by

$$\hat{\mu}_Y = \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

We can decrease the variance of $\hat{\mu}_Y$ by observing another random variable X, with known expectation $\mu_X = \mathbb{E}[X]$. Let $\hat{\mu}_X$ denote the CMC estimate of μ_X. If there is a dependency between X and Y, then the error $X - \mathbb{E}[X]$ provides us information regarding error $Y - \mathbb{E}[Y]$. By knowing $X - \mathbb{E}[X]$ we can push the CMC estimate into the right direction. For some appropriately chosen $\beta \in \mathbb{R}$, the CV improved CMC estimate, which we will refer to as the CVMC estimate, is given by

$$\hat{\mu}_Y^{CV} = \hat{\mu}_Y - \beta(\hat{\mu}_X - \mathbb{E}[X]). \tag{4.4}$$

Since we shift the CMC estimate $\hat{\mu}_Y$ linearly in the error, the term *linear* CV is often used. Let us compute which choice of β obtains the minimal variance:

$$\mathrm{Var}[\hat{\mu}_Y^{CV}] = \mathrm{Var}[\hat{\mu}_Y] + \beta^2 \mathrm{Var}[\hat{\mu}_X] - 2\beta \mathrm{Cov}(\hat{\mu}_Y, \hat{\mu}_X)$$

This is a quadratic polynomial in β with a unique minimum located at

$$\beta^* = \frac{\mathrm{Cov}(Y, X)}{\mathrm{Var}[X]} \tag{4.5}$$

for this choice of β the resulting variance is given by

$$\mathrm{Var}[\hat{\mu}_Y^{CV}] = (1 - \mathrm{Corr}(Y, X)^2)\mathrm{Var}[\hat{\mu}_Y] \tag{4.6}$$

The achieved variance reduction is directly tied to the squared correlation between X and Y. As demonstrated in the conclusive findings outlined in Section 2.6, it becomes evident that $\mathbb{E}[Y|X]$ serves as the optimal control variate when restricted to utilizing only the information provided by X.

4.3 Tranched LHP as Control Variate

All the building blocks are now in place to present the methodology aiming to efficiëntly estimate the ETL for a possibly inhomogeneous portfolio. We assume that correlated defaults are generated under some one-factor Lévy threshhold model (A, K). The Crude Monte Carlo estimator for the expected tranche loss $\mathbb{E}[L_n^{(c,d)}]$ is obtained by simulating N i.i.d. outcomes of the tranche loss random variable $L_n^{(c,d)}$ and calculating their sample mean. This entails the following steps:

1. Generate $n \times N$ idiosyncratic factors $X_{1-\rho}^{(i,j)}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, N$
2. Generate N systematic factors $X_\rho^{(j)}$ for $j = 1, \ldots, N$
3. Compute loss $L_{n,j} = \sum_{i=1}^n w_i \mathbb{1}[X_\rho^{(j)} + X_{1-\rho}^{(i,j)} \leq K_i]$
4. Transform into tranche losses $L_{n,j}^{(c,d)} = \frac{\min(d, L_{n,j}) - \min(c, L_{n,j})}{d-c}$
5. Compute the CMC estimate $\hat{\mu}_{crude} = \frac{1}{N} \sum_{j=1}^N L_{n,j}^{(c,d)}$

Our goal is to enhance the precision of the estimate $\hat{\mu}_{crude}$ by identifying a random variable X with a known expected value that demonstrates a strong correlation with $L_n^{(c,d)}$, and subsequently employing it as a linear control variate. Prominent contenders for this purpose are the systematic and idiosyncratic factors. These factors possess known expected values and notably exhibit a significant correlation with the tranche loss.

We are now confronted with a vast array of possibilities in harnessing the information offered by the factors during simulation. It is important to observe that the parameter ρ governs the extent of dependence between the loss random variable L_n and the factors X_ρ, $X_{1-\rho}^{(1)}, \ldots, X_{1-\rho}^{(N)}$. When ρ takes on higher values, the systematic factor X_ρ exerts a substantial influence on the portfolio's overall loss. Conversely, when ρ is small, the portfolio's loss becomes more reliant on the vector of idiosyncratic factors. Similarly, the portfolio size n also determines which factors wield greater influence. In smaller portfolios, the idiosyncratic factors may better capture the behavior of the portfolio loss. On the other hand in larger portfolios, their behavior is smoothed out by the Law of Large Numbers, and the loss behavior is predominantly guided by the systematic factor.

From Section 2.6 we have already seen that for the choice of $c = 0, d = 1$, i.e. full portfolio loss, the unique maximizer of correlation given the information of X_ρ is the LHP L_∞. Recall that L_∞ refers to the same object as the conditional EL $\mathbb{E}[L_n | X_\rho]$. Hence it is also the optimal control variate for estimating the full portfolio loss. For a portfolio with homogeneous PD this random variable is in distribution equal to the LHP random variable $\pi_K(X_\rho)$. Recall that for a portfolio that does not have homogeneous

PDs and exposures, L_∞ is a weighted sum of LHP random variable:

$$L_\infty = \sum_{i=1}^{n} w_i \pi_i(X_\rho). \tag{4.7}$$

It has previously been shown that (4.7) is asymptotically the maximally correlated random variable with the tranche loss given the systematic factor, and hence the optimal control variate. The steps required for obtaining a CMC estimate, presented in the beginning of this Section, can now be updated to incorporate (4.7) as a control variate:

1. Generate $n \times N$ idiosyncratic factors $X_{1-\rho}^{(i,j)}$ for $i = 1, \dots, n$ and $j = 1, \dots, N$
2. Generate N systematic factors $X_\rho^{(j)}$ for $j = 1, \dots, N$
3. Compute loss $L_{n,j} = \sum_{i=1}^{n} w_i \mathbb{1}[X_\rho^{(j)} + X_{1-\rho}^{(i,j)} \leq K_i]$
4. Transform into tranche losses $L_{n,j}^{(c,d)} = \frac{\min(d, L_{n,j}) - \min(c, L_{n,j})}{d-c}$
5. Transform $X_\rho^{(j)}$ into $L_\infty^{(j)} = \mathbb{E}[L_n | X_\rho^{(j)}]$ using (4.7) $\quad j = 1, \dots, N$
6. Transform $L_\infty^{(j)}$ into $L_{\infty,j}^{(c,d)} = \frac{\min(d, L_\infty^{(j)}) - \min(c, L_\infty^{(j)})}{d-c}$
7. Approximate optimal β with sample covariance $\widehat{\mathrm{Cov}}$ and sample variance $\widehat{\mathrm{Var}}$

$$\beta^* = \frac{\widehat{\mathrm{Cov}}(L_{n,j}^{(c,d)}, L_{\infty,j}^{(c,d)})}{\widehat{\mathrm{Var}}[L_{\infty,j}^{(c,d)}]}$$

8. Updated CVMC estimate is computed as

$$\hat{\mu}_{CV} = \frac{1}{N} \sum_{j=1}^{N} L_{n,j}^{(c,d)} - \beta^* \Big(\frac{1}{N} \sum_{j=1}^{N} L_{\infty,j}^{(c,d)} - \mathbb{E}[L_\infty^{(c,d)}] \Big)$$

where $\mathbb{E}[L_\infty^{(c,d)}]$ can be computed using expression (3.11).

5 Numerical Simulation and Analysis

This Section provides an extensive numerical analysis to analyse the effectiveness of the Crude Monte-Carlo methodology with and without the use of control variates. The analysis shall be executed in two frameworks, focussing on the Gaussian and shifted gamma distributions. In the former case, the factors will follow a Gaussian distribution, while in the latter case they are shifted gamma distributed. The following section explains those frameworks in more detail.

5.1 Framework Setups

5.1.1 Gaussian Framework

The adoption of Gaussian-distributed asset values, akin to the Merton Model, represents the most commonly employed model in practical applications. In our Lévy model specification, this simplifies to selecting H as the standard normal distribution, which is infinitely divisible , since any sum of independent normal random variables is also normal. Within this framework, the resultant Lévy processes are also called Wiener processes, leading to $X_\rho \sim \mathcal{N}(0, \rho)$ and $X^{(i)}_{1-\rho} \sim \mathcal{N}(0, 1-\rho)$. The normal distribution also belongs to a location-scale family, allowing us to express the random asset values $A_1, \dots, A_n$ in a more straightforward manner as follows:

$$A_i = \sqrt{\rho} Z + \sqrt{1-\rho}\epsilon_i \quad i = 1, \dots, n \tag{5.1}$$

Here, Z and $\epsilon_1, \dots, \epsilon_n$ are standard normal random variables. The conditional PD can be written in the following simple form:

$$\pi_K(x) = \mathbb{P}(A_i \leq K | X_\rho = x) = \mathbb{P}(\sqrt{\rho}Z + \sqrt{1-\rho}\epsilon \leq K | \sqrt{\rho}Z = x) = \Phi\Big(\frac{K - x}{\sqrt{1-\rho}}\Big), \tag{5.2}$$

which yields the LHP random variable

$$L_\infty = \Phi\Big(\frac{K - \sqrt{\rho}Z}{\sqrt{1-\rho}}\Big). \tag{5.3}$$

Following the lines of calculation (2.33) and the symmetric properties of the standard normal CDF Φ, the CDF of (5.3) is given by

$$F_\infty(x) = \Phi\Big(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - K}{\sqrt{\rho}}\Big). \tag{5.4}$$

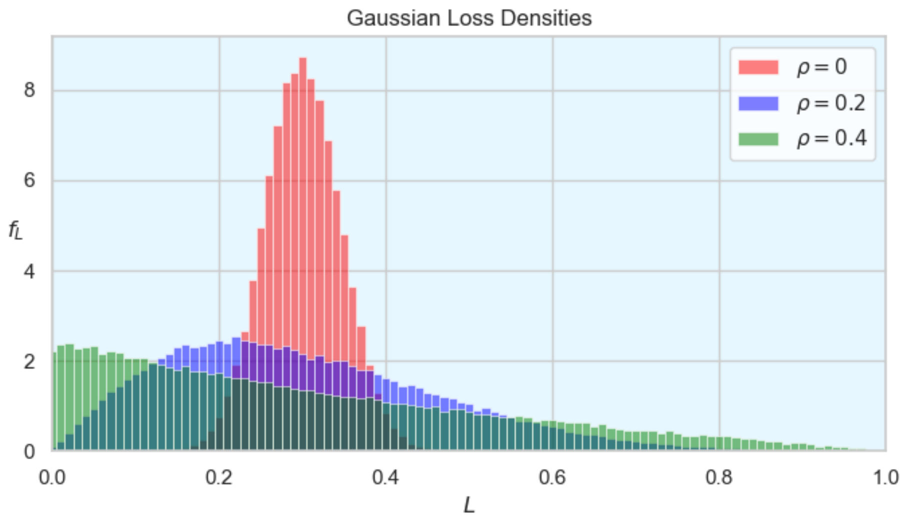

Figure 5.1: Gaussian loss distribution with loan PD equal to 0.3 and asset correlations set to $\rho_{asset} = 0, 0.2, 0.4$.

Recall that the tranched LHP random variable is given by

$$L_\infty^{(c,d)} = \frac{\min(d, L_\infty) - \min(c, L_\infty)}{d - c}. \tag{5.5}$$

In line with equation (3.9), the expectation for a homogeneous tranched LHP random variable can be expressed analytically. For the Gaussian framework it has the following appealing form (Kalemanova, Schmid, and Werner 2007):

$$\mathbb{E}[L_\infty^{(c,d)}] = \frac{\Phi_2(\Phi^{-1}(c), K; 0, \Sigma) - \Phi_2(\Phi^{-1}(d), K; 0, \Sigma)}{d - c}, \tag{5.6}$$

where $\Phi_2(x, y, \Sigma)$ is the bivariate normal distribution function with mean 0 and covariance matrix

$$\Sigma = \begin{pmatrix} 1 & -\sqrt{1-\rho} \\ -\sqrt{1-\rho} & 1 \end{pmatrix}.$$

The expectation of the Gaussian tranched LHP of an inhomogeneous portfolio, which is essentially a tranched conditional expectation, can also be computed analytically. Via computation of the formula presented in (3.11).
Figure 5.1 shows the simulated loss density for Gaussian distributed factors, with asset correlations set to $\rho_{asset} = 0, 0.2$ or 0.4.

5.1.2 Shifted Gamma Framework

In light of what is mentioned regarding the financial crisis of 2008, in this section we will consider an alternative choice of infinitely divisible distribution H, namely the shifted gamma distribution, which is also infinitely divisible as the Gamma distribution is infinitely divisible. The Gaussian framework faced the criticism that it lacks tail dependence, a concept that quantifies the joint probabilistic nature of extreme events. The *shifted gamma* framework aims to eleviate this issue by incorporating a stronger probability of extreme losses. Interestingly it does not achieve this by heavy tails, as the Gamma distribution is still a light tailed distribution. Instead it introduces a much larger probability of full portfolio loss through its bounded domain.

For some parameter $a > 0$, we define a Gamma process $G = \{G_t : t \in [0,1]\}$ with parameters $\alpha = a$ and $\beta = \sqrt{a}$. Then we have $\mathbb{E}[G_1] = \sqrt{a}$ and $\mathrm{Var}[G_1] = 1$. Let $G_s, G_s^{(1)}, \ldots, G_s^{(n)}$ be $n+1$ i.i.d. copies of G. As the driving Lévy process for the factors we let

$$X_s = \sqrt{a}s - G_s \quad X_s^{(i)} = \sqrt{a}s - G_s^{(i)} \tag{5.7}$$

And similarly as before define the asset values A_i as

$$A_i = X_\rho + X_{1-\rho}^{(i)} \tag{5.8}$$

Then accordingly $\mathbb{E}[A_i] = 0$ and $\mathrm{Var}[A_i] = 1$. Figure 5.2 shows the shifted gamma density of the systematic factors for three different asset value correlations. Furthermore Figure 5.3 shows a histogram of three shifted gamma loss densities for the same varying asset value correlations.

Similar to the Gaussian model, the weight of the distribution is smoothed out towards the left as the asset correlation increases, providing a fatter tail. However one notable difference in comparison to the Gaussian model is the high peak at full portfolio loss $L = 1$, which seems to also increase with correlation. Reason this occurs is due to the upper boundedness of the factors. First note that the domains of X_ρ and $X_{1-\rho}^{(i)}$ are $(-\infty, \sqrt{a}\rho]$ and $(-\infty, \sqrt{a}(1-\rho)]$ respectively. Thus if the systematic factor falls below $K - \sqrt{a}(1-\rho)$, then no idiosyncratic factor can be high enough such that $A_i \geq K$. Hence all the weight from the probability $\mathbb{P}(X_\rho \leq K - \sqrt{a}(1-\rho))$ is put onto $L = 1$.

The independence of the event $X_\rho \leq K - \sqrt{a}(1-\rho)$ from the portfolio size implies a discontinuity in the density of the LHP approximation at $x = 1$. Figure 5.4 illustrates the conditional PD function $\pi_K(x)$ across various correlation parameters. The knack's location for a given K and ρ is situated at $K - \sqrt{a}(1-\rho)$. Furthermore, a lower bound on $\pi_K(x)$ can be established for all $x \in (-\infty, \sqrt{a}\rho]$. As the systematic factor cannot surpass $\sqrt{a}\rho$, for an infinitely large portfolio, there will always be a proportion of loans greater than $\mathbb{P}(X_{1-\rho}^{(i)} \leq K - \sqrt{a}\rho)$ that default. It's worth noting that this lower bound diminishes with increasing asset correlation, a trend evident in Figure 5.4.

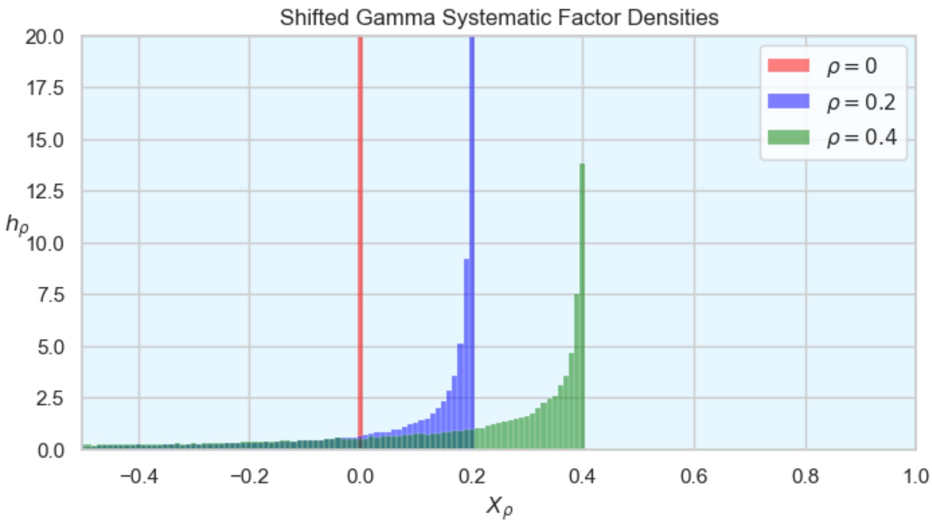

Figure 5.2: Simulated densities of the systematic factors X_ρ for the shifted gamma framework. The correlations are set to $0, 0.2$ and 0.4. Observe the peak at $X_0 = 0$, which indeed should be the case under the model assumptions. The weight of the distribution gets more spread out under increasing ρ, and the right bound of the domain also increases with ρ.

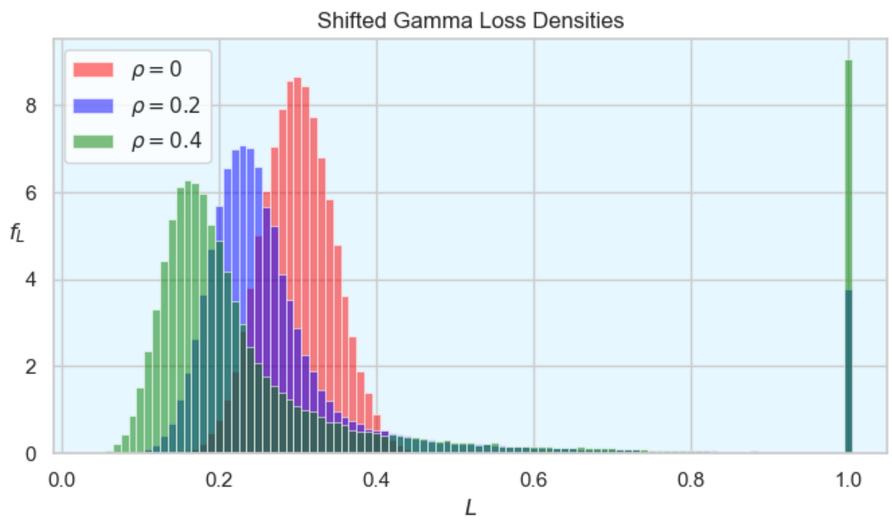

Figure 5.3: Simulated densities of the portfolio loss L for the shifted gamma framework. Correlations are set to 0, 0.2 and 0.4. Observe the sharp peak at $L = 1$, an effect resultant from the boundedness of the factor distribution.

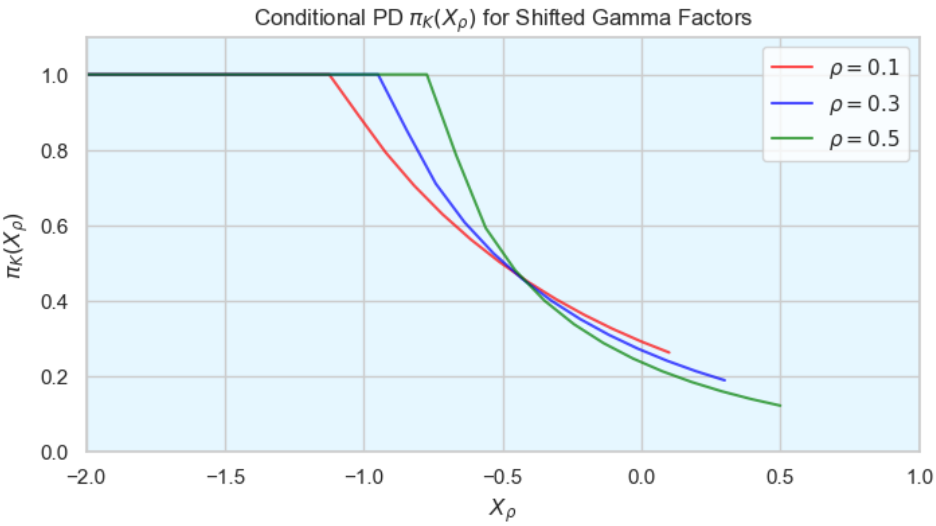

Figure 5.4: Shifted gamma conditional PD function π_K for asset correlations $\rho = 0.1, 0.3, 0.5$ and $a = 1$. The location of the knack is $K - \sqrt{a}\rho$ and the lower bound $\mathbb{P}(X_{1-\rho}^{(i)} \leq K - \sqrt{a}\rho)$ is achieved at $X_\rho = \sqrt{a}\rho$.

5.2 Output Analysis Notation

This section aims to introduce some notation and concepts that we use to evaluate the performance of the algorithm. The Crude Monte-Carlo (CMC) estimate is denoted by $\hat{\mu}_{crude}$, aiming to estimate the true expected tranche loss (ETL) $\mu := \mathbb{E}[L_n^{(c,d)}]$, and computed via the procedure described in Section 4.3. The standard deviation of $\hat{\mu}_{crude}$ is denoted by σ_{crude}. Let $\sigma_{tranche} := \sqrt{\mathrm{Var}[L_n^{(c,d)}]}$ to be the standard deviation of the tranche loss. Then we have by the independence of individual outcomes that $\sigma_{crude} = \frac{\sigma_{tranche}}{\sqrt{N}}$, which states that in order to halve the standard deviation, the number of iterations must be quadrupled. Furthermore the CMC estimate with (3.6) as a linear control variate is denoted by $\hat{\mu}_{CV}$ with standard deviation σ_{CV}.

The standard deviations of both estimates provide a quantitative measure of precision. Clearly, the lower the standard deviation, the better the estimate. However, if the true mean, that which the Monte Carlo algorithm aims to estimate, are very low, consequently, the standard deviations are also low. This does not imply that for parameter inputs resulting in a very low ETL, the precision of their estimates is higher. Instead we should consider the *relative standard deviation* (relative STD), which takes into account the value of the true mean. The relative STDs are denoted by

$$\eta_{crude} := \frac{\sigma_{crude}}{\mu}, \quad \eta_{CV} := \frac{\sigma_{CV}}{\mu} \tag{5.9}$$

Another naming convention for the relative STD is *coefficient of variation*. Furthermore we consider the ratio of the two standard deviations, denoted by

$$R := \frac{\sigma_{crude}}{\sigma_{CV}}. \tag{5.10}$$

This quantity, which we would like to maximize, captures how many Monte-Carlo iterations are spared by the control variate implementation. Let $\sigma_{tranche,CV}$ be the standard deviation of the control variate adjusted tranche loss, i.e.

$$\sigma_{tranche,CV} = \sqrt{\mathrm{Var}\left[L_n^{(c,d)} - \beta^*(L_\infty^{(c,d)} - \mathbb{E}[L_\infty^{(c,d)}])\right]}$$

Then also the relation $\sigma_{tranche,CV} = \frac{\sigma_{tranche}}{\sqrt{N}}$ holds. The effectiveness of the control variate technique can be interpreted by asking how many CMC iterations would have to be added to achieve the same level of precision as if we included the control variate methodology. To explain this in more detail, let N_{crude} and N_{CV} be the number of iterations to achieve a certain level of precision with and without control variates. If we wish to ask how many CMC iterations are needed to achieve the same accuracy, the following equation should hold

$$\sigma_{CV} = \frac{\sigma_{tranche,CV}}{\sqrt{N_{CV}}} = \frac{\sigma_{tranche}}{\sqrt{N_{crude}}} = \sigma_{crude} \implies \sqrt{\frac{N_{CV}}{N_{crude}}} = \frac{\sigma_{tranche,CV}}{\sigma_{tranche}} = \frac{1}{R}$$

Hence, $N_{crude} = R^2 N_{CV}$. Thus, the number of necessary samples to achieve the same accuracy as CMC is decreased by a factor of R^2.

The theoretical nature of linear control variates allows us to write R in terms of the correlation between $L_n^{(c,d)}$ and $L_\infty^{(c,d)}$. Recall that (4.6) tells us that

$$\sigma_{CV} = \sqrt{1 - \mathrm{Corr}(L_n^{(c,d)}, L_\infty^{(c,d)})^2}\sigma_{crude},$$

which results in the following convenient expression for R

$$R = \frac{1}{\sqrt{1 - \mathrm{Corr}(L_n^{(c,d)}, L_\infty^{(c,d)})^2}} := \frac{1}{r}. \tag{5.11}$$

Given the potential for R to be arbitrarily high, our subsequent analysis concentrates on visualizing estimates of the quantity

$$r = \sqrt{1 - \mathrm{Corr}(L_n^{(c,d)}, L_\infty^{(c,d)})^2},$$

as r remains bounded between 0 and 1. In the upcoming sections, our focus will be on plotting estimates of r, η_{crude}, and η_{CV} across varying inputs. Additionally, we will present the actual ETL estimates $\hat{\mu}_{crude}$ and $\hat{\mu}_{CV}$, along with the control variate expectation $\mathbb{E}[L_\infty^{(c,d)}]$, to shed light on the sensitivity of ETL concerning various parameters such as portfolio size, PD, tranche seniority, and asset correlation.

5.3 Numerical Simulation for a Homogeneous Portfolio

We will begin by examining the performance metrics using a hypothetical homogeneous portfolio. This portfolio comprises 100 loans, each with a unit NV and LGD. The asset correlation is established at $\rho_{\text{asset}} = 0.3$, while the tranche attachment and detachment points are fixed at 0.4 and 1, respectively. These parameters reflect realistic values commonly encountered in practice. Although we've set the portfolio PD to 0.1, which may seem less realistic, our sensitivity analysis will demonstrate the effectiveness of the methodology even for very small PDs, particularly within the shifted gamma framework.

It will be studied how R, η_{crude}, η_{CV}, $\hat{\mu}_{crude}, \hat{\mu}_{CV}$ and $\mathbb{E}[L_\infty^{(c,d)}]$ behave under variation of exactly one of the following inputs: portfolio size, PD, asset correlation, and attachment point. Furthermore, two tables shall be provided to offer a direct insight into how many samples iteration are spared by employing the control variate methodology. Each time the algorithm is run, we set the number of Monte-Carlo iterations to 100000, except for the sensitivity toward portfolio size. There the number of iterations is set to 10000.

5.3.1 Overall Performance

Tables 5.1 and 5.2 show estimates of the ratio R for the Gaussian and shifted gamma frameworks respectively. The control variate methodology seems to perform better for shifted gamma distributed asset values. However for the Gaussian framework there is also a significant improvement. It seems that the ratio increases with asset correlation. This is as expected, since the asset correlation controls the amount of influence the systematic factor X_ρ exerts on the tranche loss $L_n^{(c,d)}$. However, rather unexpectingly, the dependence between tranche seniority and STD ratio is negative for Gaussian factors, but positive for shifted gamma factors. Perhaps the non-zero probability of full portfolio loss in the shifted gamma framework is the reason for this behaviour, which is independent from the selected attachment point. On the other hand, in the Gaussian domain such tail events are highly unlikely, yielding ratios that are equal to one for such extreme attachment points, causing no detectable improvement. Perhaps for these scenarios other methodologies must be explored, such as importance sampling (Glasserman and J. Li 2004), which is a well established variance reduction technique in the Gaussian framework.

Table 5.1: Table of standard deviation ratio $R = \frac{\sigma_{crude}}{\sigma_{CV}}$ for normally distributed factors.

	Attachment Point						
Asset Correlation	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**
0.1	2.07	1.73	1.43	1.32	1.0	1.0	1.0
0.2	2.95	2.61	2.37	2.13	1.9	1.68	1.36
0.3	3.78	3.48	3.17	2.89	2.43	2.33	2.09
0.4	4.67	4.32	3.99	3.67	3.25	2.91	2.58
0.5	5.67	5.31	4.92	4.55	4.17	3.67	3.34
0.6	6.65	6.28	5.9	5.35	5.02	4.54	3.83
0.7	8.04	7.61	7.16	6.64	6.13	5.47	4.92

Table 5.2: Table of standard deviation ratio $R = \frac{\sigma_{crude}}{\sigma_{CV}}$ for shifted gamma distributed factors.

	Attachment Point						
Asset Correlation	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**
0.1	4.64	8.61	9.66	10.44	11.24	12.78	13.57
0.2	6.76	9.8	10.54	12.25	12.81	14.07	13.38
0.3	8.41	10.99	12.45	13.56	14.25	14.81	17.79
0.4	10.16	12.55	13.83	15.62	16.28	18.14	20.63
0.5	11.86	14.67	16.16	18.1	20.32	22.45	25.22
0.6	14.21	17.12	20.35	23.38	26.82	28.99	35.3
0.7	17.83	21.56	26.36	29.0	38.07	48.52	52.33

5.3.2 Sensitivity to Portfolio Size

Figure 5.5 depicts the crude and control variate ETL estimates under varying portfolio size. Additionally, it presents the deterministic tranched LHP approximation $\mathbb{E}[L_\infty^{(c,d)}]$. Both the Gaussian ETL and shifted gamma ETL converge towards their respective tranched LHP approximations from above. Furthermore the shifted gamma ETL is about four times higher than its Gaussian counterpart. Noticable is the higher peak between about one and 50 loans, as well as the fluctuations of the Crude Monte-Carlo estimate around the CV estimate that seem to be substantially larger in the Gaussian case.

Figure 5.6 and 5.7 delve deeper into the impact of portfolio size on the efficiency of the control variate methodology. As anticipated, the portfolio size exerts a substantial influence due to the almost sure convergence of the tranche loss to the LHP tranche loss

$$L_n^{(c,d)} \xrightarrow[a.s.]{n\to\infty} L_\infty^{(c,d)}$$

This convergence leads to a near-unity correlation between the tranche loss and tranched LHP loss, resulting in a vanishing standard deviation σ_{CV}, however in the Gaussian case, larger portfolio sizes are necesarry to observe this convergence. Conversely, the relative STDs of the CMC estimates converge to a positive constant value. In theory, this value equals the standard deviation of the tranched LHP random variable divided by $\sqrt{N}\mu$. Furthermore both the Crude and control variate Monte Carlo methods seem to function significantly better in the shifted gamma framework.

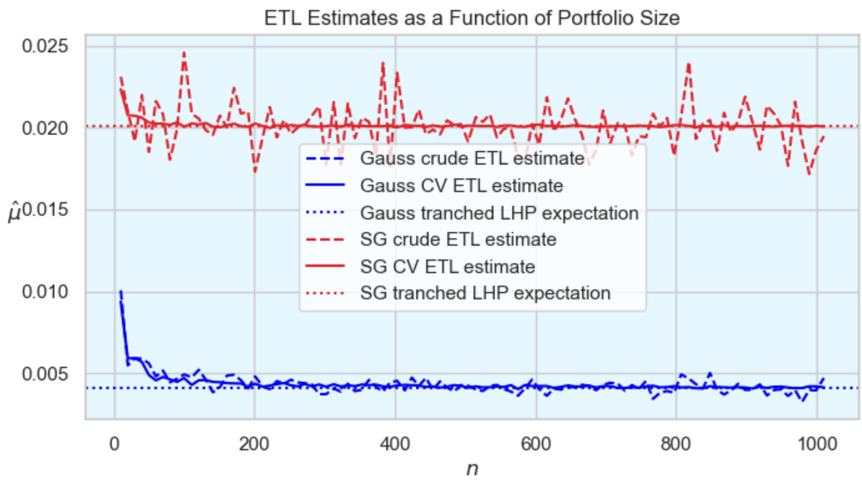

Figure 5.5: Plot of μ_{crude}, μ_{CV} and $\mathbb{E}[L_\infty^{(c,d)}]$ under varying portfolio size. The Gaussian ETL is substantially higher than the shifted gamma ETL. In both cases the estimated ETL converges to tranched LHP expectation.

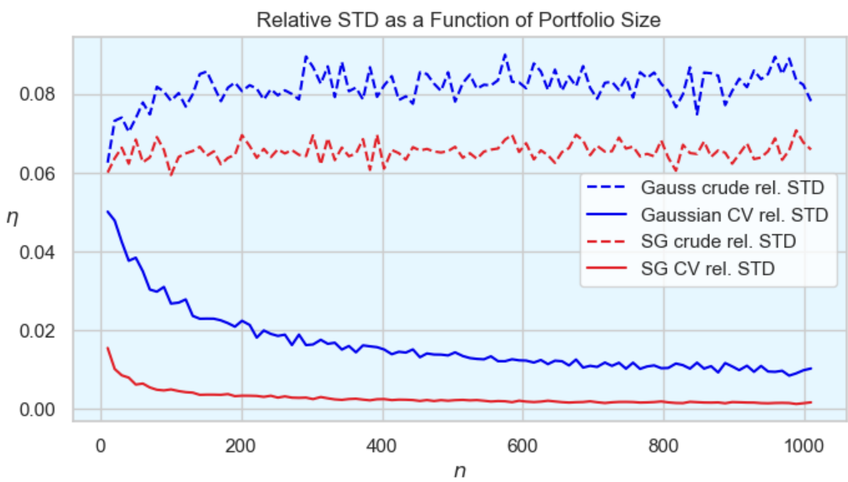

Figure 5.6: Comparison of η_{crude} and η_{CV} under varying portfolio size.

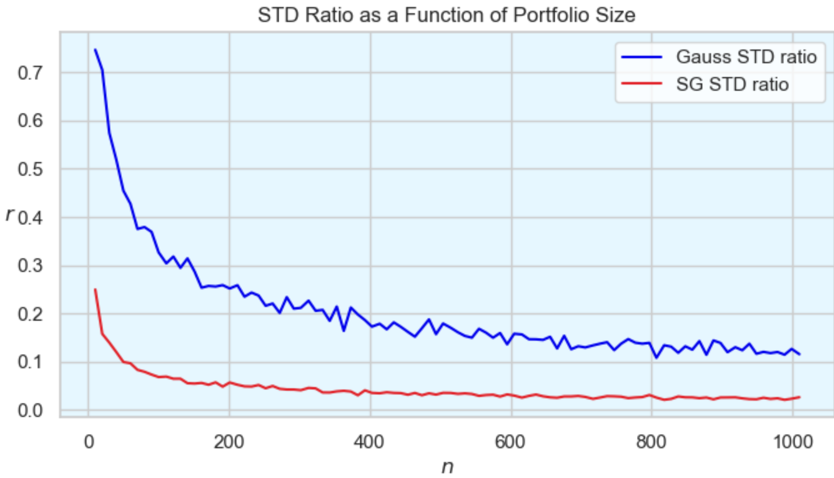

Figure 5.7: Standard deviation ratio $r = \frac{\sigma_{crude}}{\sigma_{CV}}$ under varying portfolio size.

5.3.3 Sensitivity to PD

In contrast to the impact of portfolio size, the PD exerts a considerably stronger influence on the ETL. Rather than encompassing the entire parameter space of $[0, 1]$, our sensitivity analysis focuses on PD values below 0.2. Notably, the increase in PD is directly proportional to the rise in ETL. This trend is clearly illustrated in Figure 5.8, where again the shifted gamma ETL is higher than the Gaussian ETL especially for larger PDs. Observe also the increasing difference between the ETL estimates and true control variate expectation $\mathbb{E}[L_\infty^{(c,d)}]$ for normally distributed factors. However it seems that the expectation of the control variate is for shifted gamma distributed factors already a very good analytical approximation for the true ETL.

Figure 5.9 illustrates that the relative standard deviations decrease with increasing PD. In the Gaussian setting, both the crude and control variate estimators exhibit initially high relative standard deviations for low PD values, which then rapidly decrease. Although the crude relative STDs for the shifted gamma ETL follows a similar pattern as in the Gaussian case, the control variate relative STDs consistently hover close to 0, showcasing a significant performance improvement. This is particularly evident in Figure 5.10.

The Gaussian STD ratio r initiates at 1, experiences chaotic drops, and eventually stabilizes. The reason for starting at 1, indicating no improvement with the control variate methodology, is that the observed tranche might appear random, while the tranched LHP control variate remains constant due to the extremely low PD. This lack of correlation between the tranche loss $L_n^{(c,d)}$ and the control variate $L_\infty^{(c,d)}$ results in no variance reduction.

On the other hand, the shifted gamma standard deviation ratio starts at 0 and remains consistently close. This can be attributed to the likelihood that if a tranche loss occurs, it is most likely a full loss rather than an intermediate value. Hence, there is a strong "Bernoulli" coupling between $L_n^{(c,d)}$ and $L_\infty^{(c,d)}$, leading to unity correlation. In simpler terms, it is highly improbable for one of the two samples, namely the tranched loss observations and the tranched LHP observation, to be constant while the other is random.

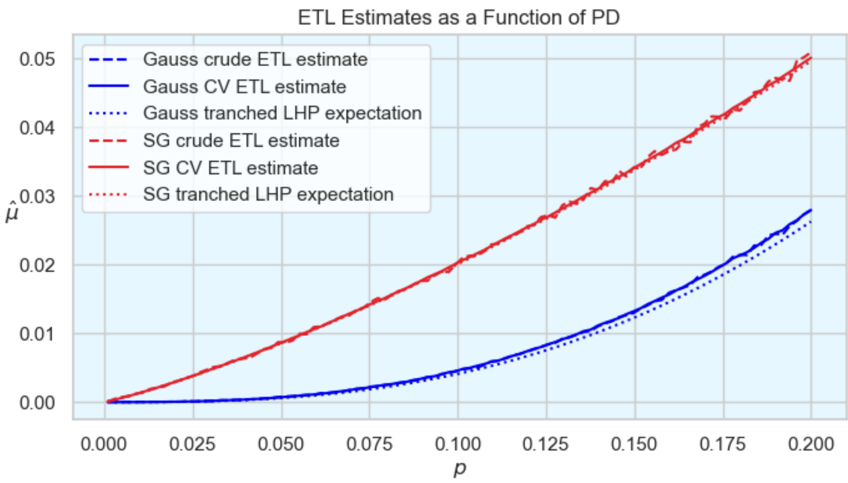

Figure 5.8: Plot of μ_{crude}, μ_{CV} and $\mathbb{E}[L_\infty^{(c,d)}]$ under varying PD. Both ETLs start at 0 and slowly rise. Notice the increasing bias between the Gaussian tranched LHP expectation and the ETL.

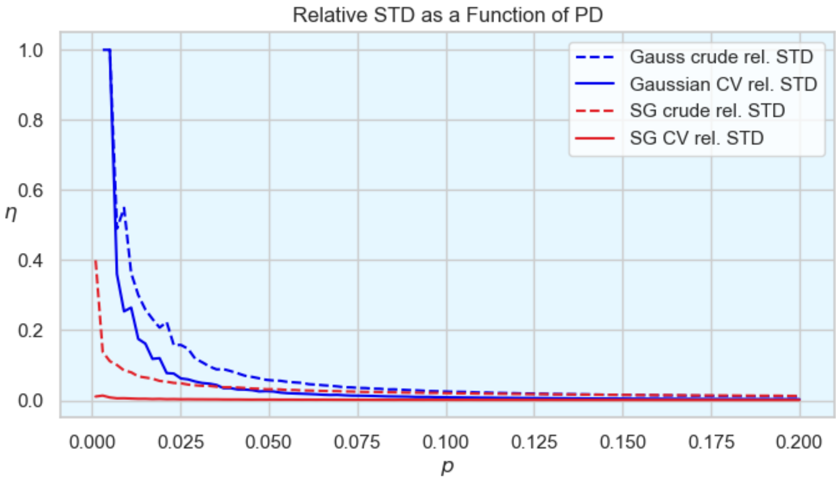

Figure 5.9: Comparison of η_{crude} and η_{CV} under varying PD. The variance reduction increases with PD, but does not approach 0.

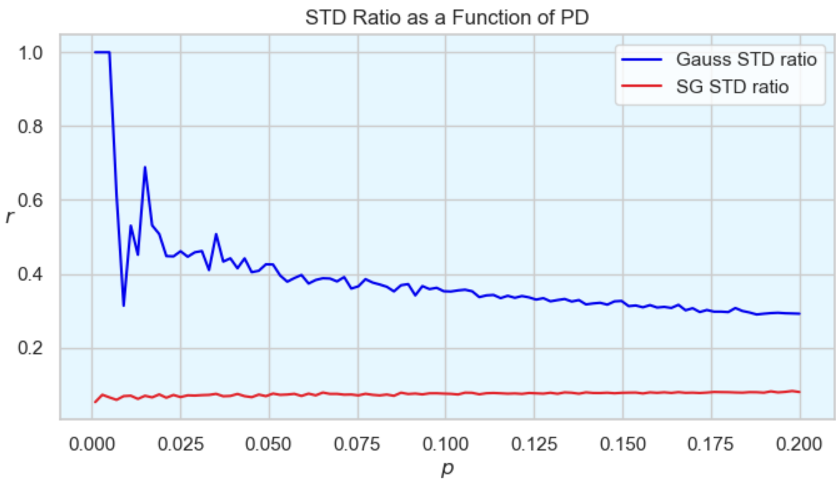

Figure 5.10: Standard deviation ratio $r = \frac{\sigma_{crude}}{\sigma_{CV}}$ under varying PD. For very small PD the methodology seems to remain stable for shifted gamma, while the Gaussian has too little loss observations that exceed 40%.

5.3.4 Sensitivity to Asset Correlation

Similar to the influence of portfolio size, we anticipate that the asset correlation parameter $\rho = \rho_{asset}$ significantly affects the correlation between the tranche loss $L_n^{(c,d)}$ and the tranched LHP loss $L_\infty^{(c,d)}$. This hypothesis is theoretically valid, as

$$\lim_{\rho \to 1} \mathrm{Corr}(L_n^{(c,d)}, L_\infty^{(c,d)}) = 1.$$

As for $\rho = 1$ the tranche loss and tranched LHP loss random variables are given by

$$L_n^{(c,d)} = \frac{\min(d, \mathbb{1}[X_1 \leq K]) - \min(c, \mathbb{1}[X_1 \leq K])}{d - c} = \mathbb{1}[X_1 \leq K]$$
$$L_\infty^{(c,d)} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}[X_1 \leq K] = \lim_{n\to\infty} \mathbb{1}[X_1 \leq K] = \mathbb{1}[X_1 \leq K]$$

Figures 5.12 and 5.13 highlight the improvement in the methodology's effectiveness with increasing asset correlation, applicable to both Gaussian and shifted gamma distributed factors. Despite minimal enhancement in the Gaussian framework for low correlations, all relative standard deviations appear to converge to 0. However, expecting the CMC estimates to similarly converge to 0 would be illogical, as they do not leverage the perfect correlation between $L_n^{(c,d)}$ and $L_\infty^{(c,d)}$. Figure 5.14 provides a detailed examination of the relative standard deviation behavior for higher correlation values, demonstrating the vanishing standard deviation for CVMC estimators, while the crude relative STD estimates remain positive at $\rho = 1$.

In addition to the improved effectiveness with correlation, we also anticipate an increase in ETL. Figure 5.11 indeed confirms this expectation. Both Gaussian and shifted gamma ETL values commence near 0 and ascend together towards the portfolio PD of 0.1, as confirmed by

$$\mathbb{E}[L_n^{(c,d)}] = \mathbb{E}[\mathbb{1}[X_1 \leq K]] = H(K) = p \quad \text{if } \rho = 1.$$

Additionally, as the correlation skews the loss distribution towards the right (see, e.g., Figure 5.3), it is reasonable to anticipate an increase in ETL with higher asset correlation.

The observation that η_{crude} converges to some positive constant, stems from $\sigma_{tranche}$ experiencing a slight increase with asset correlation (due to the variance of the loss L increasing with ρ). However, the reason η_{crude} still decreases with ρ is that the ETL μ grows faster than $\sigma_{tranche}$. Thus, in relation to the actual ETL, the precision of the CMC estimate also appears to improve with higher asset correlations.

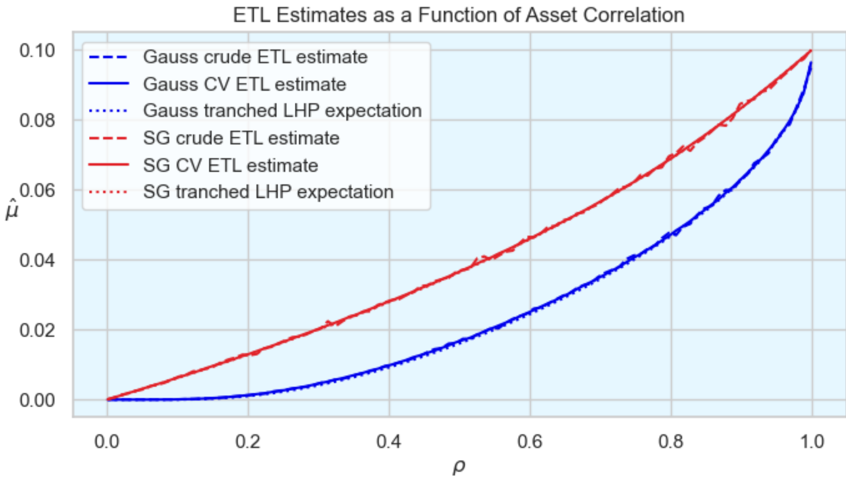

Figure 5.11: Plot of μ_{crude}, μ_{CV} and $\mathbb{E}[L_\infty^{(c,d)}]$ under varying asset correlation. Both start at 0 and end at the portfolio PD of 0.1.

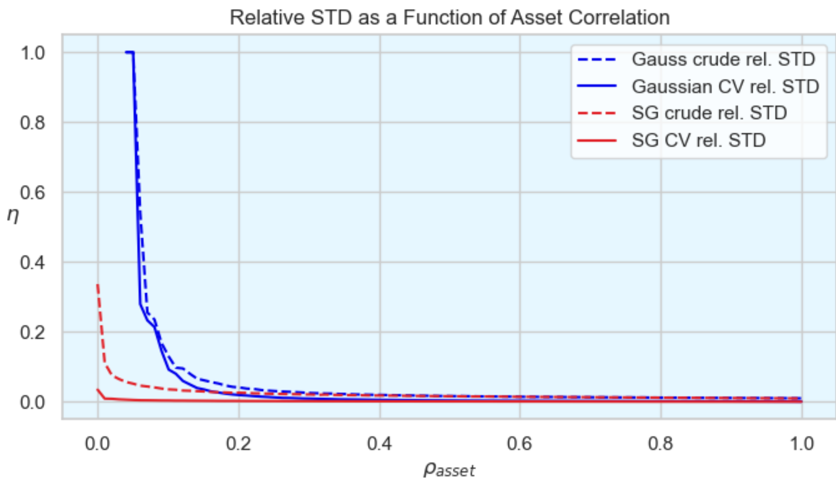

Figure 5.12: Comparison of η_{crude} and η_{CV} under varying asset correlation. The relative STD for the crude estimates do not converge to 0, while the CV estimates do.

5.3.5 Sensitivity to Tranche Seniority

The attachment point $c \in [0, 1]$ defines the seniority of the tranche. Figure 5.15 illustrates that the ETL decreases with tranche seniority, a logical outcome as the tranching transformation can only decrease its input. Both plots initiate at 0.1 for $c = 0$, corresponding to the portfolio PD set at 0.1. For low attachment points, the shifted gamma ETL decreases slightly faster than the Gaussian ETL, resulting in a higher Gaussian ETL

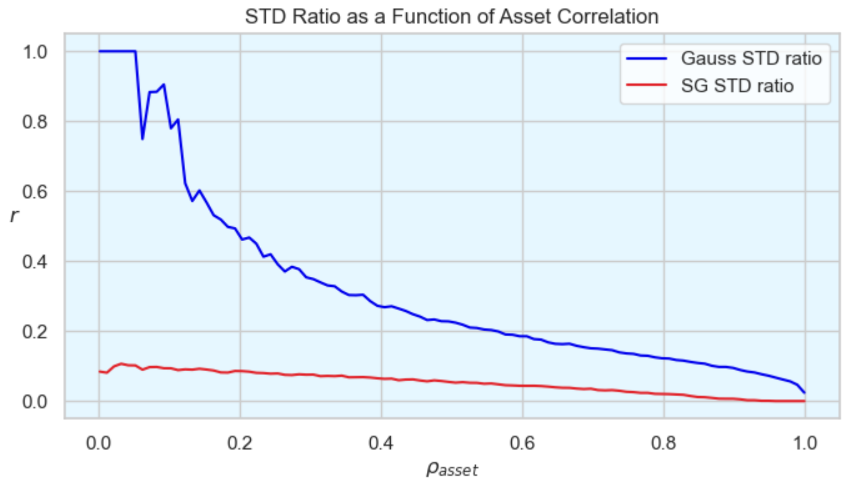

Figure 5.13: Standard deviation ratio $r = \frac{\sigma_{crude}}{\sigma_{CV}}$ under varying asset correlation. The method performs again significantly better in the shifted gamma framework.

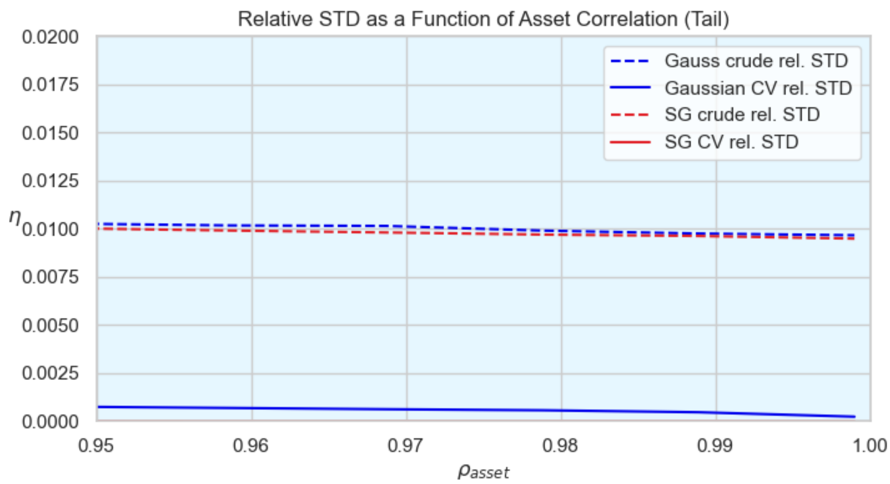

Figure 5.14: Comparison of η_{crude} and η_{CV} for very high asset correlation. Notice that the crude relative STDs are not 0 for $\rho = 1$.

for small attachment points. However, this difference diminishes rapidly, and the shifted gamma ETL does not drop much below 0.02. Conversely, the Gaussian ETL approaches 0 for high attachment points, while the shifted gamma ETL consistently remains above 0. Notably, for an attachment point of 0.4, the Gaussian ETL is significantly lower than the shifted gamma ETL, a observation consistent with the earlier analysis.

One of the most prominent disparities between the Gaussian and shifted gamma frameworks emerges in Figure 5.16. Notably, the relative standard deviation remains close to zero even for high attachment points in the shifted gamma framework. In contrast, the Gaussian relative STD experiences rapid growth for higher tranche seniorities, a consequence of its swift ETL decay. The shifted gamma framework not only maintains a low relative standard deviation but also exhibits a decreasing standard deviation ratio with tranche seniority, as shown in Figure 5.17. Meanwhile, the Gaussian standard deviation ratio grows with seniority and becomes numerically unstable for higher attachment points.

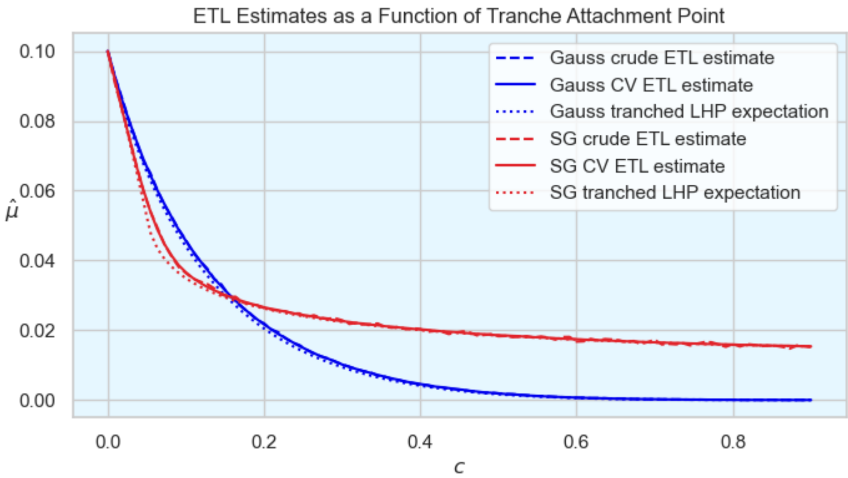

Figure 5.15: Plot of μ_{crude}, μ_{CV} and $\mathbb{E}[L_\infty^{(c,d)}]$ under varying attachment point c. Both ETLs start at the portfolio PD of 0.1. The shifted gamma ETL decays faster, but then remains above 0. In contrast the Gaussian ETL decays to 0.

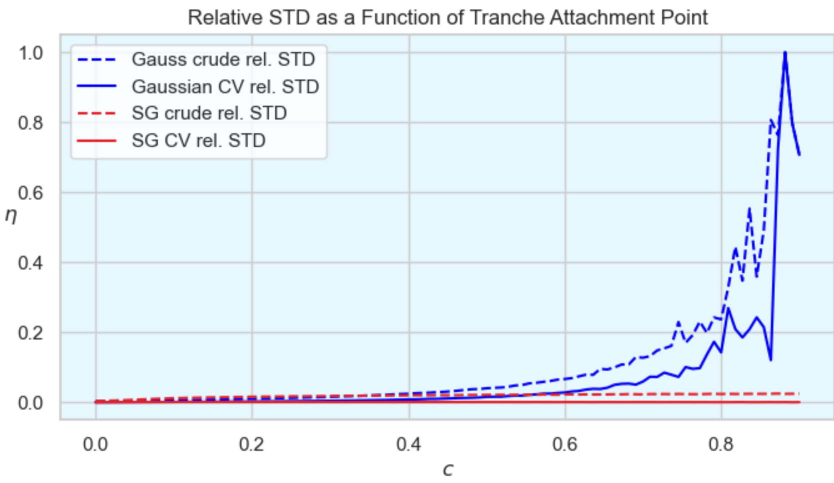

Figure 5.16: Comparison of η_{crude} and η_{CV} under varying attachment point c. The methodology becomes numerically unstable for high attachment points in the Gaussian framework, while for shifted gamma factors the performance remains stable.

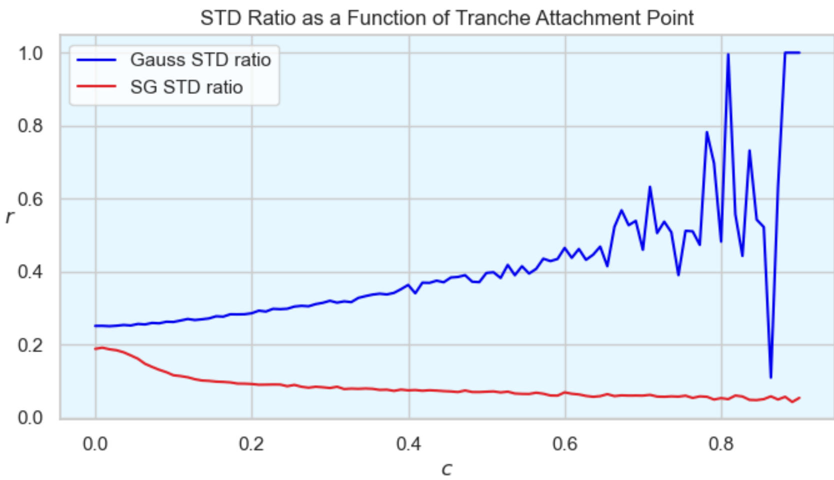

Figure 5.17: Standard deviation ratio $r = \frac{\sigma_{crude}}{\sigma_{CV}}$ under varying attachment point c. For shifted gamma factors the ratio decreases, meaning that the method becomes better with more extreme seniority.

5.4 Numerical Analysis for Inhomogeneous Portfolios

Though the analysis of a homogeneous portfolio is of theoretical importance, it is not very realistic. In practice portfolios consist of loans that each may have its own NV, LGD and PD. Even multiple asset correlations might be considered, rather than a single ρ_{asset}, as the dependence of defaults between different counterparties might vary. The numerical study presented here will solely focus on inhomogeneous NV, LGD and PD, but still considers homogeneous correlation across default indicators, prescribed by $\rho = \rho_{asset}$.

Besides studying the performance of the control variate methodology for inhomogeneous portfolios, there is another question that the numerical study here aims to address. This question refers to the computational complexity of the control variate methodology for inhomogeneous portfolios, and how it might be resolved. Recall that for a homogeneous portfolio, a single tranched LHP random variable acts as the asymptotically optimal control variate, i.e.

$$L_{\infty}^{(c,d)} = \frac{\min(d, L_{\infty}) - \min(c, L_{\infty})}{d-c},$$

with expectation equal to (3.9) and where $L_{\infty} = \mathbb{E}[L_n|X_\rho] = H_{1-\rho}(K - X_\rho)$. On the other hand for an inhomogeneous portfolio, a tranched sum of LHPs, weighted by exposure, acts as the control variate, i.e

$$L_{\infty}^{(c,d)} = \frac{\min(d, \sum_{i=1}^{n} w_i H_{1-\rho}(K_i - X_\rho)) - \min(c, \sum_{i=1}^{n} w_i H_{1-\rho}(K_i - X_\rho))}{d-c},$$

which has expectation equal to (3.11). In the homogeneous case, computing the exact expectation of the tranched LHP random variable is highly efficient. The computational time for evaluating (3.9) using numerical integration is negligible in relation to the overall algorithm duration, particularly for larger portfolios, as its computation remains independent of portfolio size. Moreover, the integration bounds x_c and x_d can be readily determined through inversion of $H_{1-\rho}$.

Conversely, calculating the tranched LHP expectation in (3.11) for an inhomogeneous portfolios is slower. Here, a sum of n elements must be computed for each integration point, leading to a time complexity that scales with portfolio size n. Additionally, determining the integration bounds x_c and x_d is more challenging since there is no explicit expression for the inverse of f as provided in (3.10).However this issue could be resolved by using a numerical root finding procedure to obtain the roots of the functions $f(x)-c$ and $f(x)-d$, yielding the desired bounds of integration x_c, x_d. This step would have to be done after having defined all input parameters of the model, such as PDs, exposures and asset correlation, since the shape of f is influenced by all three of those parameters.

Another way to circumvent the additional computational effort to find the integration bounds x_c and x_d, is to consider a single tranched LHP approximation as a control variate, despite the fact that the portfolio is inhomogeneous. In other words, for an inhomogeneous portfolio with exposures and PD threshholds $e_1, \ldots, e_n, K_1, \ldots, K_n$, we would employ the control variate

$$\tilde{L}_\infty^{(c,d)} = \frac{\min(d, \tilde{L}_\infty) - \min(c, \tilde{L}_\infty)}{d - c}, \tag{5.12}$$

where

$$\tilde{L}_\infty = H_{1-\rho}(\tilde{K} - X_\rho)$$

and

$$\tilde{K} = g(e_1, \ldots, e_n, K_1, \ldots, K_n)$$

for some function g. The problem of efficiently finding $\mathbb{E}[L_\infty^{(c,d)}]$ is now replaced by finding a suitable $\tilde{K}$, such that the correlation between $L_n^{(c,d)}$ and $\tilde{L}_\infty^{(c,d)}$ is maximized. One way to proceed might be to choose $\tilde{K}$ such that the first moments of $\tilde{L}_\infty$ and L_n align. This would imply

$$\tilde{p} = \sum_{i=1}^{n} w_i p_i \implies \tilde{K} = H_1^{-1}\Big(\sum_{i=1}^{n} w_i p_i\Big). \tag{5.13}$$

While it's not established as the optimal single tranched LHP approximation for the tranche loss, it has the potential to yield significant variance reduction. We will conduct a comparison of the variance reduction achievable using either the inhomogeneous tranched LHP (3.10) or the (approximated) homogeneous tranched LHP (5.12) as control variates. For the homogeneous tranched LHP, the parameter is selected by aligning the first moment with the full portfolio loss, drawing inspiration from a similar approach outlined in (Tchistiakov, Smet, and Hoogbruin 2004).

In contrast to the sensitivity analysis that has been done for a hypothetical homogeneous portfolio, the analysis presented here approaches it from a different angle. Of primarily interest is the influence of inhomogenuity on the methodology. Thus, rather than plotting the ETL $\hat{\mu}$, relative STD η and STD ratio r over parameter inputs such as portfolio size, PD, correlation and attachment point, we proceed differently here. Two portfolios, that differ only in PD and exposure are considered. Both chosen in such a way that their portfolio ELs are both equal to 0.1. Solely the influence of the attachment point c on the performance of the control variate methodology will be analyzed, for both portfolios.

There are three main questions that the analysis aims to tackle. Firstly, how much variance reduction can be obtained using control variates for inhomogeneous portfolios? Secondly, how does the performance differ between Gaussian and shifted gamma distributed factors? Finally, how does the effectiveness of a homogeneous tranched LHP control variate compare to the effectiveness of an inhomogeneous LHP control variate?

5.4.1 Portfolio 1: Two Homogeneous Groups

First we consider a hypothetical portfolio with the following characteristics:

- portfolio size $n = 100$
- For $1 \leq i \leq 50$:
  - exposure $e_i = 1$
  - PD $p_i = 0.15$
- For $51 \leq i \leq 100$:
  - $e_i = 2$
  - $p_i = 0.075$
- asset correlation $\rho_{asset} = 0.3$
- detachment point $d = 1$

Notice that this portfolio consists of two homogeneous groups, call them group A and B. Group A has loans with low exposure of 1 and high PD of 0.15, while group B has high exposure of 2 and low PD of 0.075. but high exposure, the other with high PD and low exposure. The resulting EL of the portfolio is given by 0.1, identical to the EL of the previously presented homogeneous portfolio. The inhomogeneous tranched LHP random variable would in this case be given by

$$L_\infty = \frac{1}{3} H_{0.7}(K_A - X_{0.3}) + \frac{2}{3} H_{0.7}(K_B - X_{0.3}),$$

where $K_A = H_1^{-1}(0.15)$ and $K_2 = H_1^{-1}(0.075)$. On the other hand, the single LHP approximation for this inhomogeneous portfolio, that matches the first moment to the full portfolio loss, is given by

$$\tilde{L}_\infty = H_{0.7}(\tilde{K} - X_{0.3}),$$

where $\tilde{K}$ is computed as in (5.13).

Figures 5.18, 5.19 and 5.20 show the results. Figure 5.18 displays the ETL for various attachment points c. Notably the behaviour is is very similar to the ETL for a fully homogeneous portfolio, as shown in Figure 5.15. For small c, the Gaussian ETL is slightly higher than the shifted gamma ETL, and vice versa for higher c.

Figures 5.19 shows the relative standard deviations for the CMC estimate, inhomogeneous CV estimate (using $L_\infty^{(c,d)}$ as a control variate) and homogeneous CV estimate (using $\tilde{L}_\infty^{(c,d)}$ as a control variate). For Gaussian distributed factors, both the homogeneous and inhomogeneous CVs seem to be identical in performance, though both clearly outperforming the CMC estimate. Since the inhomogeneous CV requires more extensive computational effort in comparison to the homogeneous CV, using a single tranched LHP random variable as a control variate seems preferable here. On the other hand, the inhomogeneous CV performs drastically better for shifted gamma distributed factors. For a $(0.4, 1)$ tranche and shifted gamma factors, employement of the inhomogeneous CV $L_\infty^{(c,d)}$ results in a STD ratio that is around 0.1, hence reducing the required number of MC iterations by a factor 100. On the other hand, the homogeneous CV $\tilde{L}_\infty^{(c,d)}$ has an STD ratio of 0.2 at $c = 0.4$, which would reduce the number of iterations by 50.

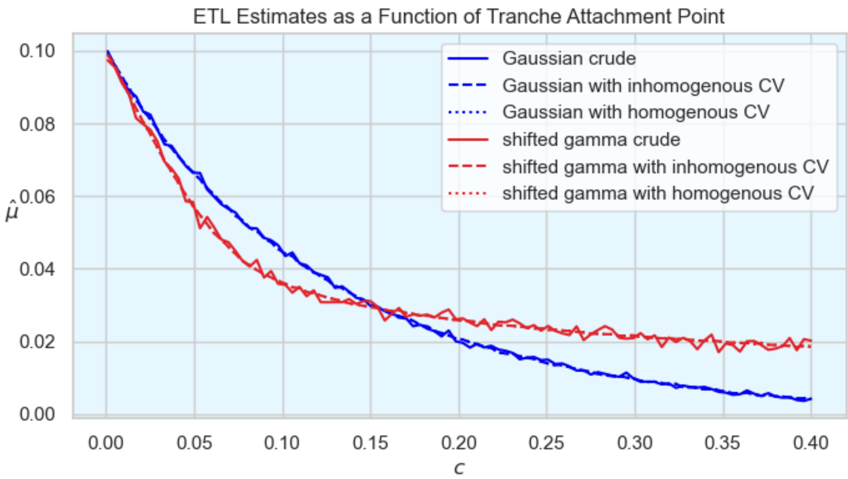

Figure 5.18: Plot of μ_{crude}, μ_{CV} and $\mathbb{E}[L_\infty^{(c,d)}]$ under varying attachment point c for an inhomogeneous portfolio consisting of two homogeneous groups. Notice the similarity in behaviour to the homogeneous portfolio.

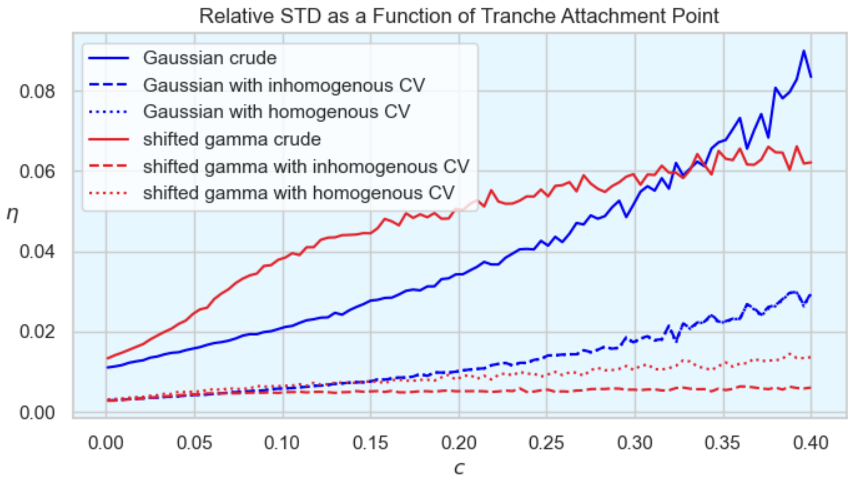

Figure 5.19: Comparison of η_{crude} and η_{CV} under varying attachment point c for an inhomogeneous portfolio consisting of two homogeneous groups. In the Gaussian framework, the homo- and inhomogeneous tranched LHP perform similarly, while in the shifted gamma framework the inhomogeneous tranched LHP clearly performs better.

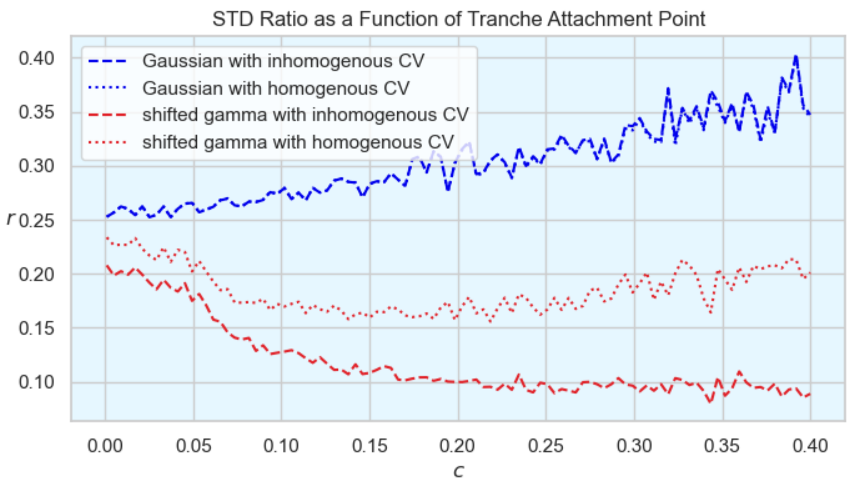

Figure 5.20: Standard deviation ratio $r = \frac{\sigma_{crude}}{\sigma_{CV}}$ under varying attachment point c for an inhomogeneous portfolio consisting of two homogeneous groups. Observe the drastic improvement of the inhomogeneous CV over the homogeneous CV for shifted gamma factors.

5.4.2 Portfolio 2: Uniform Exposures and PDs

The second portfolio that we consider is assumed to have the following characteristics:

- portfolio size $n = 100$
- exposures $e_i \sim \text{Unif}(0, 1)$
- PD $p_i \sim \text{Unif}(0, 0.2)$
- asset correlation $\rho_{asset} = 0.3$
- detachment point $d = 1$

Under these characteristics, the (expected) portfolio EL is again equal to 0.1. Even though the exposures and PDs are randomly generated, it is expected that the observed outcome of the simulation is not significantly influenced by the randomness of exposures and PDs. For much smaller portfolio sizes, however, that influence might be greater. Figures 5.21, 5.22 and 5.23 show the simulation results. The ETLs that are displayed in Figure 5.21 are similar in shape as in Figure 5.18, though decaying less quickly, resulting in the ETL for the shifted gamma staying below the Gaussian ETL until approximately $c = 0.25$, but also ETLs that are higher than in Figure 5.18.

Furthermore Figures 5.22 and 5.23 display a notable difference in performance between the Gaussian and shifted gamma frameworks. First of all, similar as to the portfolio with two homogeneous groups, the difference in performance between the inhomogeneous and homogeneous CVs is negligable for Gaussian factors. This is quite surprising, since it means that the random variable $\tilde{L}_{\infty}^{(c,d)}$ approximates the tranche loss $L_n^{(c,d)}$ just as well as $L_{\infty}^{(c,d)}$. Hence, for Gaussian distributed factors, the homogeneous CV would be a better choice, as the inhomogeneous CV calculation is a slower algorithm. On the other hand for shifted gamma distributed factors, a significant improvement can be obtained. The inhomogeneous CV has a relative STD that is over two times lower than the homogeneous CV. Interestingly, the crude relative STD for the shifted gamma framework is higher than in the Gaussian case. However it does seem that after attachment point $c = 0.4$, the Gaussian crude relative STD dominates, as we have seen as well in Figure 5.16.

Different from what we have seen for the previous inhomogeneous portfolio, is that Figure 5.22 and 5.23 seem to exhibit much more stable results for higher attachment points, compared to 5.19 and 5.20. The reason this occurs is that the Gaussian ETL decays much quicker for a portfolio of loans consisting of two homogeneous groups, causing chaotic fluctuations for high tranche attachment points. For a portfolio that has uniformly distributed exposures and PDs, this problem does not occur.

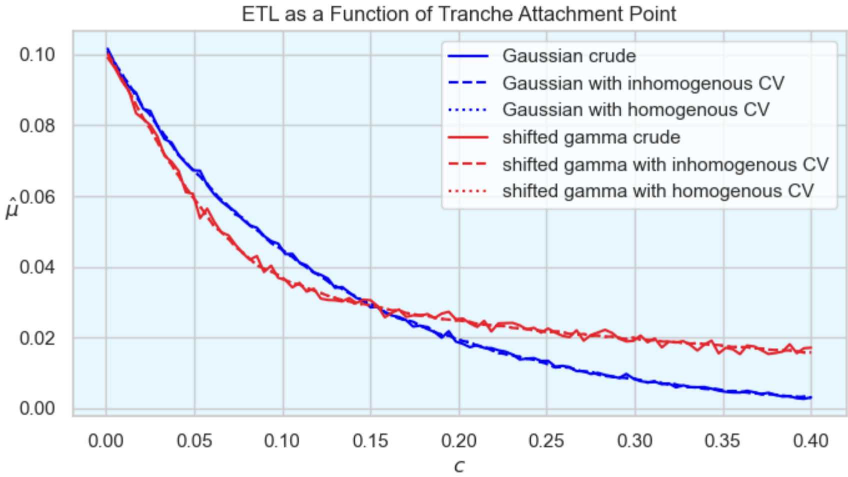

Figure 5.21: Plot of μ_{crude}, μ_{CV} and $\mathbb{E}[L_\infty^{(c,d)}]$ under varying attachment point c for an inhomogeneous portfolio consisting of uniform exposures and LGDs.

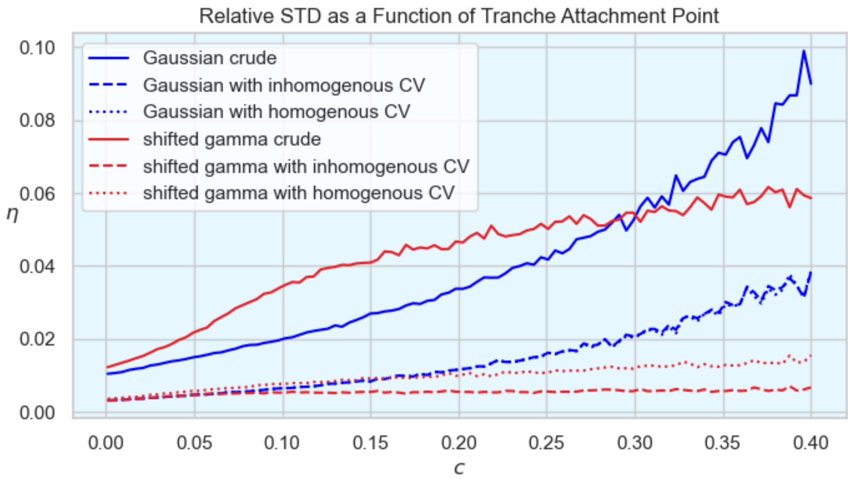

Figure 5.22: Comparison of η_{crude} and η_{CV} under varying attachment point c for an inhomogeneous portfolio consisting of uniform exposures and LGDs. Notice the improved stability in comparison to the previous inhomogeneous portfolio.

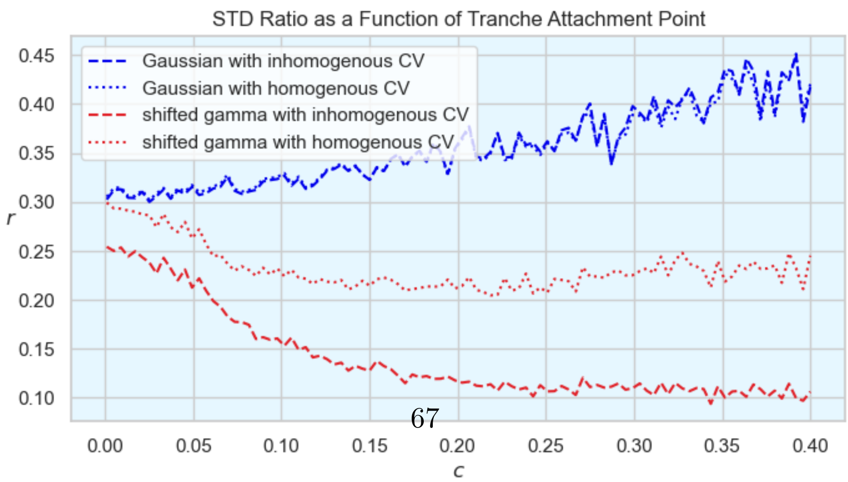

Figure 5.23: Standard deviation ratio $r = \frac{\sigma_{crude}}{\sigma_{CV}}$ under varying attachment point c for an inhomogeneous portfolio consisting of uniform exposures and LGDs.

6 Conclusion

The primary objective of this thesis was to devise a novel control variate variance reduction technique specifically tailored for the estimation of ETL, within the domain of a Lévy one-factor threshold model for correlated defaults. The method's formulation leveraged a tranched version of well-known LHP approximation, building upon the current state of research. Furthermore through a simple argument involving conditional expectations, the method is shown to be optimal for very large portfolio sizes. The methodology underwent a thorough numerical performance and sensitivity analysis, encompassing evaluation under the conventional Gaussian framework as well as the lesser known shifted gamma framework. Across both frameworks, the numerical results validate the efficacy of the proposed control variate methodology, attesting to its formidable capability in substantially reducing variance. It acts as a potent tool with broad applicability for factor-based credit risk models. Consequently, the method exhibits promising potential for profound impact within the realm of real-world financial risk management.

6.1 Contribution to the Literature

In Chapter 1, a comprehensive literature review delves into the contemporary landscape of variance reduction techniques within credit risk models, with a specific focus on CDO pricing. The chapter provides insights into the historical evolution of credit risk models, methodologies for approximating loss distributions, and variance reduction techniques pertinent to Monte Carlo simulation. Conclusively, it addresses the utilization of control variates in the domain of credit risk modeling. While existing studies have demonstrated the efficacy of control variates as a successful variance reduction technique, their application in the context of ETL estimation remained relatively unexplored until this thesis. An illustrative example is found in (Tchistiakov, Smet, and Hoogbruin 2004), where the use of an LHP as a control variate is considered, albeit in the context of estimating VaR. In the realm of ETL estimation, (Chen and Glasserman 2007) briefly mentions the possibility of using approximations of loss random variables as control variates, though without specific emphasis on the LHP. Notably, their approach employs a discrete loss proxy as a control variate rather than a continuous one. This thesis builds up upon both of those works, by developing theory to establish a control variate methodology and confirming its potential with numerical simulation.

6.2 Credit Risk Framework and Tranched LHP

Chapter 2 serves as an exploration into critical risk measures and the credit risk model employed for modeling the loss distribution. The chosen model adheres to a general Lévy specification, as initially proposed by (Albrecher, Ladoucette, and Schoutens 2006). Contributing to the general applicability of the discussed methodology. Within this framework, the loan conditional PD π_i emerges as a pivotal factor influencing both the loss distribution and the conditional EL $\mathbb{E}[L_n|X_\rho]$. In the context of homogeneous portfolios, the conditional EL corresponds to a single LHP, while in the case of inhomogeneous portfolios, it manifests as a sum of LHPs, weighted by exposure. This random variable, in a sense, represents a noiseless rendition of the actual portfolio loss, devoid of idiosyncratic contributions. Additionally, Lemmas 2.6.1 and 2.6.2 underscore that the conditional EL, and consequently the LHP, serves as the optimal approximation for the portfolio loss, irrespective of the portfolio size. However, this result holds only asymptotically for tranched portfolio losses and the associated tranched LHP. For such cases, the most correlated function of the systematic factor is encapsulated by $\mathbb{E}[L_n^{(c,d)}|X_\rho]$, yet an explicit expression for this function of the systematic factor remains infeasible at present.

6.3 Asymptotic Optimality of the Control Variate

Chapter 3 introduced the realm of CDOs and the underlying mathematical intricacies involved in determining the fair premium of a CDO tranche. The focal challenge in pricing a CDO tranche lies in computing the ETL, a risk measure not easily derived directly from the loss distribution due to its inherent complexity. As a result, approximation methodologies play a crucial role, with randomized techniques like Monte Carlo simulation proving particularly valuable. The exploration into ETL estimation begins with the introduction of a tranched LHP approximation. Supported by Lemmas 3.2.1, 2.6.1, and 2.6.2, it is established that for large portfolios, the tranched LHP stands as the most accurate approximation. Subsequently, the chapter demonstrates the computability of the exact expectation of the tranched LHP for both homogeneous and inhomogeneous portfolios, laying the foundation for a control variate methodology. It is acknowledged that determining the exact expectation for inhomogeneous portfolios involves a slightly more intricate process, given the absence of an explicit formulation for the distribution function of the inhomogeneous LHP. Nevertheless, the exact expectation can be obtained, albeit necessitating an additional numerical root-finding step to ensure the correct bounds of integration. The decision to invest effort in this additional step is dependent upon the selected distribution for the factors, as the shifted gamma distribution poses some problems for numerical integration due to its vertical asymptote. However it seems the extra computational effort pays of, since it has been observed homogeneous tranched LHP for an inhomogeneous portfolio performs significantly worse then the inhomogeneous tranched LHP, which is not the case for Gaussian distributed factors.

6.4 Numerical Sensitivity Analysis and Results

The concluding chapter of this thesis undertakes a comprehensive numerical investigation into the proposed control variate methodology, considering both Gaussian and shifted gamma distributed factors, but also homogeneous and inhomogeneous portfolios. The choice of Gaussian factors, which prescribe dependency between defaults through the Gaussian copula, faced significant criticism post the 2008 financial crisis for its inability to realistically model the joint probabilistic nature of extreme events. Seeking an alternative, the shifted gamma distribution emerges as a solution, introducing a distinct peak at the occurrence of a full portfolio loss due to the upper boundedness of idiosyncratic factors. A comparative study on performance and sensitivity to inputs is conducted for both distribution choices, revealing that the methodology exhibits notably superior performance within the shifted gamma framework, particularly for highly senior tranches. For homogeneous portfolios, a sensitivity analysis is conducted regarding inputs such as portfolio size, PD, asset correlation, and tranche seniority. The results of this analysis lead to the conclusion that portfolio size and asset correlation exert a strong positive influence on the method's efficacy. Contrarily, tranche seniority negatively impacts efficacy in the context of Gaussian factors. Interestingly, the effectiveness of the method within the shifted gamma framework exhibits a slight positive dependency on tranche seniority, contrary to expectations. The discontinuity of the shifted gamma LHP is suspected to be the reason for this behaviour.

While the sensitivities to portfolio size and asset correlation remain unexplored for inhomogeneous portfolios, an analysis of tranche seniority has been conducted across two portfolios exhibiting varying degrees of inhomogeneity. The initial portfolio comprises two distinct homogeneous groups—one with low and the other with high PD, while the second portfolio comprises loans with uniformly distributed exposures and PDs. It is noted that the optimal proposed methodology involves a slower procedure, attributed to the numerical integration of a sum of LHPs. An alternative procedure is introduced, employing a single tranched LHP with its PD parameter set to match the first moment of the full portfolio loss. This method, albeit cruder, is quicker and is compared to the original approach of utilizing a tranched weighted sum of LHPs as a control variate. An important conclusion drawn from the comparative analysis is that within the Gaussian framework, employing the simpler single tranched LHP $\tilde{L}_{\infty}^{(c,d)}$ as a control variate appears sufficient. Minimal improvement is observed in comparison to the tranched weighted sum LHP $L_{\infty}^{(c,d)}$. In stark contrast, within a shifted gamma setting, the cruder approach distinctly proves inferior. In this context, the additional computational effort required to calculate the expectation of the tranched weighted sum LHP appears to be justified.

6.5 Discussion

Although the method has demonstrated successful variance reduction with minimal additional computational effort, several crucial aspects remain unaddressed and require further discussion. Foremost among these is the limitation that only two out of the numerous potential distribution choices have been explored. The method's general applicability opens avenues to investigate its effectiveness with alternative copula choices, such as student-t or Archimedean copulas. It raises the question whether the method might derive substantial benefits from discontinuous LHP distribution functions, as evident in the case of the shifted gamma framework.

Secondly, the method's efficacy has been demonstrated within a simplified one-period framework, while practical applications often involve time-dependent setups to account for loan amortization. In such cases, default indicators evolve into default times, influenced by PD values determined through survival functions and exposures dictated by specific payment schedules. While there are approaches to address these additional layers of complexity, this thesis intentionally refrains from delving into that specific discussion. Future research may explore the method's adaptability and performance within intricate temporal structures, offering a more comprehensive understanding of its practical applicability.

An additional possibility for future theoretical development involves a more rigorous exploration of employing the cruder single tranched LHP $\tilde{L}_{\infty}^{(c,d)}$ as a control variate. The current approach of selecting the threshold parameter for this LHP via first-moment matching to the full portfolio loss appears somewhat simplistic, given that the crucial factor is the correlation between the tranche loss and the tranched LHP. Unfortunately, establishing how this dependency could be explicitly captured remains an unresolved question, hopefully answered with further research.

Conclusively and notably, the LHP loss random variable holds potential as a control variate for other pivotal risk metrics, including VaR and ES. Unlike the ETL estimate, which pertains to the estimation of an expectation, these alternative risk metrics are derived from quantiles of the loss distribution and are estimated through order statistics. Fortunately, control variates can be applied in such contexts as well, necessitating a distinct methodology, as extensively discussed in (Hesterberg and Nelson 1998). While techniques such as importance sampling are firmly established in this domain, augmenting them with additional control variates could enhance their effectiveness. The incorporation of control variates in estimating quantiles and order statistics represents a promising direction to enhance the precision and efficiency of risk metrics. This contribution holds the potential to advance risk management methodologies within a broader scope.

Popular Summary

Credit risk refers to the probability that a borrower may not fulfill loan obligations. Whether it's individuals, businesses, or governments borrowing money, there's a constant risk of difficulties in repayment. Managing credit risk is vital for financial institutions to ensure stability and safeguard against potential losses in their loan portfolios. Quantitative methods play a crucial role in empowering credit risk managers with a robust foundation for decision-making. The developer of those models faces the challenge of constructing a model that accurately resembles the probabilities associated to particular losses.

CDOs are financial tools whose value is linked to an underlying portfolio of loans. Their goal is to divide the risk tied to the portfolio by categorizing potential losses into separate segments known as tranches. Investors can then invest in these tranches, earning a portion of the revenue generated by the portfolio but also assuming the risk of losses from defaulted loans, thus compensating any losses that might be resulting from them. The seniority of the tranche dictates not only the share of revenues received by the investor but also the likelihood of absorbing losses due to defaults. The question of how much share of the revenues is received by tranche holder, boils down to finding what fraction of defaulting loans it will have to compensate, or in technical terms, finding the expected tranche loss, which is a function of the loss distribution. However, the resultant loss distribution from credit risk models is often intricate and lacks a straightforward expression for evaluation. Consequently, determining the expected tranche loss isn't a matter of straightforward formulaic calculation. Monte Carlo simulation is a popular methodology in this context, providing a means to estimate expectations and, specifically, to estimate the expected tranche loss. The objective of this thesis is to enhance the existing Monte Carlo methodology, seeking more accurate and expeditious estimates for expected tranche loss. This improvement is achieved by approximating the portfolio with an infinitely sized set of loans, simplifying the loss distribution description through a more accessible formula.

7 Appendix A

7.1 Proof of Theorem 2.2.1

Proof. By definition

$$C(u_1, u_2, \dots, u_n) = \mathbb{P}(X_1 \le F_1^{-1}(u_1), \dots, X_n \le F_n^{-1}(u_n))$$

satisfies the decomposition and is uniquely given by continuity of the marginals.

7.2 Proof of Lemma 2.2.2

Proof. Let (A, K) and $(\tilde{A}, \tilde{K})$ be any two threshold models. Let G_A and $G_{\tilde{A}}$ be the associated CDFs of the asset values with marginals G_i and $\tilde{G}_i$ respectively. By Sklar's theorem and assumption (ii), G_A and $G_{\tilde{A}}$ both admit the same unique copula C, given by

$$G_A(G_1^{-1}(u_1), \dots, G_n^{-1}(u_n) = C(u_1, \dots, u_n) = G_{\tilde{A}}(\tilde{G}_1^{-1}(u_1), \dots, \tilde{G}_n^{-1}(u_n).$$

Let $F_D, F_{\tilde{D}}$ and $F_i, \tilde{F}_i$ denote the multivariate and marginal CDF of D and $\tilde{D}$ respectively. Assumption (i) implies that $F_i = \tilde{F}_i$ for all $i \in \{1, \dots, n\}$, which in turn is equivalent that for all $i \in \{1, \dots, n\}$ we have

$$G_i(K_i) = F_i(1) = \tilde{F}_i(1) = \tilde{G}_i(\tilde{K}_i).$$

Let $d_1, \dots, d_n \in \{0, 1\}$ be given. Let $I = \{i \in \{1, \dots, n\} : d_i = 1\}$ and $J = \{1, \dots, n\} \setminus I$. Let C_I and C_J be the copula C restricted to the inputs I and J respectively. Then

$$\begin{aligned}
F_D(d_1, \dots, d_n) &= \mathbb{P}(D_1 = d_1, \dots, D_n = d_n) \\
&= \mathbb{P}(\forall i \in I : A_i \le K_i, \forall j \in J : A_i > K_i) \\
&= \mathbb{P}(\forall i \in I : A_i \le K_i) - \mathbb{P}(\forall j \in J : A_i \le K_i) \\
&= C_I((G_i(K_i))_{i \in I}) - C_J((G_j(K_j))_{j \in J}) \\
&= C_I((\tilde{G}_i(\tilde{K}_i))_{i \in I}) - C_J((\tilde{G}_j(\tilde{K}_j))_{j \in J}) \quad \text{by assumption (i)} \\
&= \mathbb{P}(\forall i \in I : \tilde{A}_i \le \tilde{K}_i) - \mathbb{P}(\forall j \in J : \tilde{A}_i \le \tilde{K}_i) \quad \text{by assumption (ii)} \\
&= \mathbb{P}(\forall i \in I : \tilde{A}_i \le \tilde{K}_i, \forall j \in J : \tilde{A}_i > \tilde{K}_i) \\
&= \mathbb{P}(\tilde{D}_1 = d_1, \dots, \tilde{D}_n = d_n) \\
&= F_{\tilde{D}}(d_1, \dots, d_n).
\end{aligned}$$

$\square$

7.3 Proof of Lemma 2.2.3

Proof. Let C be any copula. By definition, C is a multivariate uniform CDF on some underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The upper bound follows from the fact that for any $i \in \{1, \ldots, n\}$ we have

$$C(u_1, \ldots, u_n) = \mathbb{P}(U_1 \leq u_1, \ldots, U_n \leq u_n) \leq \mathbb{P}(U_i \leq u_i) = u_i.$$

And thus

$$C(u_1, \ldots, u_n) \leq \min(u_1, \ldots, u_n).$$

The lower bound is established in a similar way. We write

$$\begin{aligned}
C(u_1, \ldots, u_n) &= \mathbb{P}(U_1 \leq u_1, \ldots, U_n \leq u_n) \\
&= 1 - \mathbb{P}\Big(\bigcup_{i=1}^{n} \{U_i > u_i\}\Big) \\
&\geq 1 - \sum_{i=1}^{n} \mathbb{P}(U_i > u_i) \\
&= 1 - \sum_{i=1}^{n} (1 - u_i) \\
&= 1 - n + \sum_{i=1}^{n} u_i.
\end{aligned}$$

Since $C(u_1, \ldots, u_n)$ is also a probability, we conclude

$$\max(1 - n + \sum_{i=1}^{n} u_i, 0) \leq C(u_1, \ldots, u_n).$$

$\square$

7.4 Proof of Lemma 2.6.1

Proof. Abbreviate $M = \mathbb{E}[Y|X]$ and $f = f(X)$. First observe that

$$\begin{aligned}
\mathbb{E}[Yf] &= \int_X \int_Y y f(x) \phi_{XY}(x,y) dx dy \\
&= \int_X \int_Y y f(x) \phi_{Y|X}(y|x) \phi_X(x) dx dy \\
&= \int_X \Big[\int_Y y \phi_{Y|X}(y|x) dy \Big] f(x) \phi_X(x) dx \\
&= \int_X \mathbb{E}[Y|X = x] f(x) \phi_X(x) dx \\
&= \mathbb{E}[Mf].
\end{aligned}$$

And also by Law of Total Expectation $\mathbb{E}[Y] = \mathbb{E}[M]$, thus

$$\mathrm{Cov}(Y, f) = \mathrm{Cov}(M, f).$$

And hence in particular for $f = M$,

$$\mathrm{Corr}(Y, M) = \frac{\mathrm{Cov}(Y, M)}{\sigma_Y \sigma_M} = \frac{\mathrm{Cov}(M, M)}{\sigma_Y \sigma_M} = \frac{\sigma_M}{\sigma_Y}.$$

From this we can deduce that

$$\begin{aligned}
\mathrm{Corr}(Y, f)^2 &= \frac{\mathrm{Cov}(Y, f)^2}{\sigma_Y^2 \sigma_{f^2}} \\
&= \frac{\mathrm{Cov}(M, f)^2}{\sigma_f^2 \sigma_M^2} \frac{\sigma_M^2}{\sigma_Y^2} \\
&= \mathrm{Corr}(M, f)^2 \mathrm{Corr}(Y, M)^2 \leq \mathrm{Corr}(Y, M)^2.
\end{aligned}$$

Since $\mathrm{Corr}(Y, M) \in [0, 1]$, we conclude

$$|\mathrm{Corr}(Y, f)| \leq \mathrm{Corr}(Y, M)$$

$\square$

7.5 Proof of Lemma 2.6.2

Proof. The proof relies on the following chain of equalities, here $f(X)$ and $\mathbb{E}[Y|X]$ are abbreviated to f and M respectively, then

$$\begin{aligned}\mathbb{E}[(Y-f)^2] &= \mathbb{E}[(Y-M+M-f)^2] \\ &= \mathbb{E}[(Y-M)^2] + \mathbb{E}[(M-f)^2] + 2\mathbb{E}[(Y-M)(M-f)] \\ &= \mathbb{E}[(Y-M)^2] + \mathbb{E}[(M-f)^2],\end{aligned}$$

where the last equality follows from the Law of Total Expectation, since

$$\begin{aligned}\mathbb{E}[(Y-M)(M-f)] &= \mathbb{E}[(Y-\mathbb{E}[Y|X])(\mathbb{E}[Y|X]-f)] \\ &= \mathbb{E}[\mathbb{E}[(Y-\mathbb{E}[Y|X])(\mathbb{E}[Y|X]-f)|X]] \\ &= \mathbb{E}[(\mathbb{E}[Y|X]-\mathbb{E}[Y|X])(\mathbb{E}[Y|X]-\mathbb{E}[f|X])] \\ &= 0.\end{aligned}$$

Thus we conclude,

$$\mathbb{E}[(Y-f)^2] = \mathbb{E}[(Y-M)^2] + \mathbb{E}[(M-f)^2] \geq \mathbb{E}[(Y-M)^2].$$

□

7.6 Proof of Lemma 3.2.1

Proof. Let $X_1, X_2, \ldots$ be a bounded sequence of non-negative random variables converging almost surely to X, i.e. $0 \leq X_i \leq M$ for all i. Let Y be any random variable. Application of the conditional Fatou's lemma yields

$$\begin{aligned}\mathbb{E}[M-X|Y] &\leq \liminf_n \mathbb{E}[M+X_n|Y] \quad \text{a.s.} \\ \mathbb{E}[X-M|Y] &\leq \liminf_n \mathbb{E}[X_n-M|Y] \quad \text{a.s.},\end{aligned}$$

which implies that

$$\limsup_n \mathbb{E}[X_n|Y] \leq \mathbb{E}[X|Y] \leq \liminf_n \mathbb{E}[X_n|Y] \quad \text{a.s.},$$

hence $\lim_{n\to\infty} \mathbb{E}[X_n|Y] = \mathbb{E}[X|Y]$ almost everywhere.

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Code

https://github.com/cnvermeulen/ETL_estimation.git

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