# Signals and Systems

Solutions to Homework Assignment #3

**Problem 1.**

(a)

$$
\begin{aligned}
x_1[n] &= \sin\left(\frac{\pi}{4}n\right) = \frac{1}{2j}e^{\frac{\pi}{4}jn} - \frac{1}{2j}e^{-\frac{\pi}{4}jn} \\
a_1 &= \frac{1}{2j} \\
a_{-1} &= -\frac{1}{2j}
\end{aligned}
$$

(b)

$$
\begin{aligned}
a_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{8}\sum_{n=0}^{7} x[n]e^{-j\frac{\pi}{4}kn} \\
&= \frac{1}{8}\left\{\left(1 + e^{-j\frac{\pi}{4}k} + e^{-j\frac{\pi}{2}k}\right) - \left(e^{-j\frac{3\pi}{4}k} + e^{-j\pi k} + e^{-j\frac{5\pi}{4}k}\right) + \frac{1}{2}\left(e^{-j\frac{3\pi}{2}k} + e^{-j\frac{7\pi}{4}k}\right)\right\} \\
&= \frac{1}{8}\left\{e^{-j\frac{\pi}{4}k}\left(e^{j\frac{\pi}{4}k} + 1 + e^{-j\frac{\pi}{4}k}\right) - e^{-j\pi k}\left(e^{j\frac{\pi}{4}k} + 1 + e^{-j\frac{\pi}{4}k}\right) + \frac{1}{2}\left(e^{-j\frac{3\pi}{2}k} + e^{-j\frac{7\pi}{4}k}\right)\right\} \\
&= \frac{1}{8}\left\{e^{-j\frac{\pi}{4}k}\left(2\cos(\frac{\pi}{4}k) + 1\right) - e^{-j\pi k}\left(2\cos(\frac{\pi}{4}k) + 1\right) + \frac{1}{2}\left(e^{j\frac{\pi}{2}k} + e^{j\frac{\pi}{4}k}\right)\right\} \\
&= \frac{1}{8}\left\{\left(1 + 2\cos\left(\frac{\pi}{4}k\right)\right)\left(e^{-j\frac{\pi}{4}k} - e^{-j\pi k}\right) + \frac{1}{2}\left(e^{j\frac{\pi}{2}k} + e^{j\frac{\pi}{4}k}\right)\right\}, \text{for} \quad k = 0, 1, 2, \ldots 7.
\end{aligned}
$$

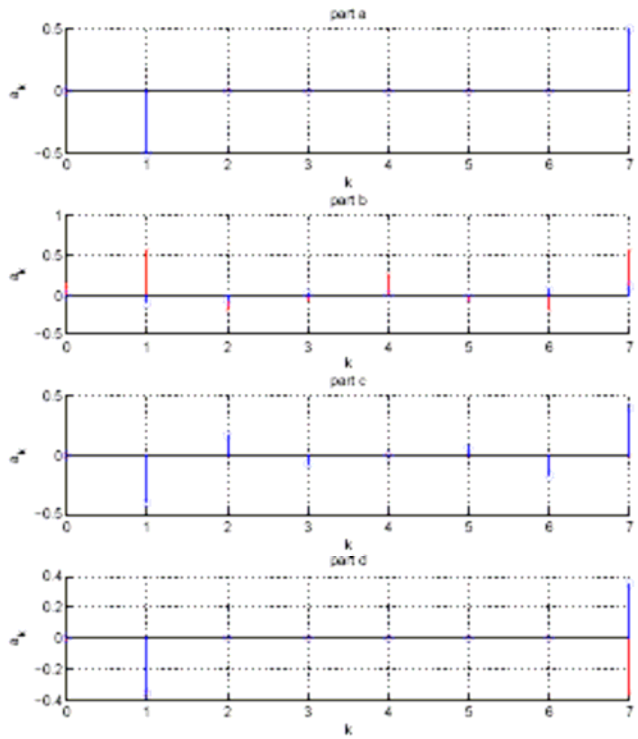
(c)

$$
\begin{aligned}
a_k &= \frac{1}{N}\sum_{n=<N>} x[n]e^{-j\frac{2\pi}{N}kn} = \frac{1}{8}\sum_{n=-3}^{3}\frac{1}{3}ne^{-j\frac{\pi}{4}kn} \\
&= \frac{1}{12}e^{-\frac{1}{2}\pi jk} - \frac{1}{12}e^{\frac{1}{2}\pi jk} + \frac{1}{24}e^{-\frac{1}{4}\pi jk} - \frac{1}{24}e^{\frac{1}{4}\pi jk} + \frac{1}{8}e^{-\frac{3}{4}\pi jk} - \frac{1}{8}e^{\frac{3}{4}\pi jk} \\
&= -\frac{1}{6}j\sin(\frac{\pi}{2}k) - \frac{1}{12}j\sin(\frac{\pi}{4}k) - \frac{1}{4}j\sin(\frac{3\pi}{4}k)
\end{aligned}
$$

(d) $x_4[n] = x_1[n-1]$

Using the shifting property we have:

$$
\begin{aligned}
a_1 &= \frac{1}{2j}e^{-\frac{\pi}{4}j} \\
a_{-1} &= -\frac{1}{2j}e^{\frac{\pi}{4}j}
\end{aligned}
$$




**Problem 2.**

(a) $a_0^1 = \frac{1}{10}\exp -j\frac{\pi}{10}k$
$a_k^1 = \frac{\sin(k\pi/10)}{k\pi}\exp -j\frac{\pi k}{10}$

(b) $a_0^2 = \frac{1}{5}\exp -j\frac{\pi}{5}k$
$a_k^2 = \frac{\sin(k\pi/5)}{k\pi}\exp -j\frac{\pi}{5}k$

(c) $x_3(t) = x_1(t) - x_1(t-2) \leftrightarrow a_k^3 = a_k^1 - a_k^1\exp -j\frac{2\pi}{10}(2)k$
$a_k^3 = a_k^1(1 - \exp -j\frac{2\pi}{5}k) = a_k^1\exp -j\frac{\pi}{5}k(2j)\sin(k\pi/5)$

(d) $x_4(t) = x_1(t) * x_2(t) \leftrightarrow a_k^4 = Ta_k^1a_k^2$
$a_0^4 = \frac{1}{5}\exp(-j\frac{3\pi}{10}k)$
$a_k^4 = 10\frac{\sin(k\pi/10)}{k\pi}\frac{\sin(k\pi/5)}{k\pi}\exp(-j\frac{3\pi}{10}k)$

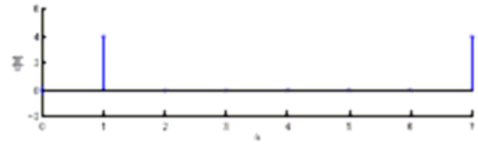
**Problem 3.**

(a) We can derive that $a_k$ real, even => $x[n]$, real, even. Further, $x[n]$ periodic in 8 => $a_k$ also periodic in 8. Write:

$$a_k = \cos(2\pi k/8).$$

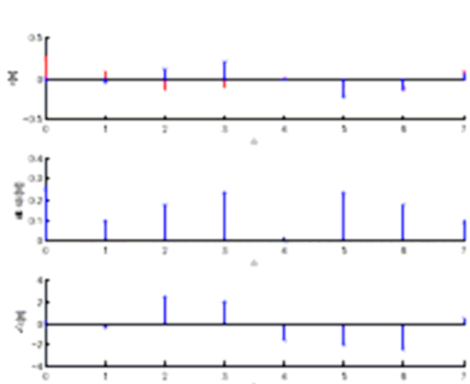
Then clearly,

$$
\begin{aligned}
x[n] &= \frac{8}{2}\delta[n-1] + \frac{8}{2}\delta[n-6] \\
x[n] &= \frac{8}{2}\delta[n-1] + \frac{8}{2}\delta[n+1]
\end{aligned}
$$



(b) By the synthesis formula,

$$
\begin{aligned}
x[n] &= \sum_{k=0}^{7} a_k \exp(j2\pi kn/8) \\
&= \exp(j2\pi(-2)n/8) + \exp(j2\pi 1n/8) \\
&= \exp(j2\pi(-0.5)n/8)\,[\exp(j2\pi(-1.5)n/8) + \exp(j2\pi(1.5)n/8)] \\
&= \exp(j2\pi(-0.5)n/8)2\cos(2\pi(1.5)n/8)
\end{aligned}
$$



**Problem 4.**

(a)

$y_1(t) = x(t - 0.5)$

$b_k = a_k\exp(-j2\pi k(0.5)) = a_k \exp(-j\pi k) = a_k(-1)^k$.

(b)

$y_2(t) = \text{odd}(x(t)) = ½(x(t) - x(-t))$

$b_k = ½(a_k - a_k^*) = jIm(a_k)$.

(c)

$$
\begin{aligned}
y_3(t) &= x(t) + \frac{dx(t)}{dt} + 7 \\
b_k &= a_k + jk\frac{2\pi}{1}a_k + 7\delta[k] \\
&= \begin{cases} a_0 + 7, & k = 0 \\ (1 + j2\pi k)a_k, & k \neq 0 \end{cases}
\end{aligned}
$$

(d) We have $y_4(t) = x(2t)$, periodic with period 1/2. The FS coefficients are still the same $b_k = a_k$.

(e)

$y_5(t) = x_2(t)$
$b_k = a_k * a_k$

**Problem 5.**

(a) The system described in an LTI system. Let's call this system $H(s)$. Using the synthesis equation, we have that

$$x(t) = \sum_k a_k \exp(j\omega_0 kt).$$

Recall that $\exp(j\omega_0 kt)$ is an eigenfunction of an LTI system, hence we can write

$$y(t) = \sum_k a_k H(\exp(j\omega_0 k)) \exp(j\omega_0 kt).$$

Note that $\omega_0$ does not change, hence $y(t)$ is also periodic with the same period $T$.

(b) By direct evaluation,

$$a_k = \int_0^T x(t) \exp(j\omega_0 kt),$$

we obtain

$$a_k = \exp(-j\pi/2k)\frac{\sin(\pi k/2)}{k\pi}, k \neq 0$$

and $a_k = 1/2$, $k = 0$. Using the eigenfunction property, we get the FS coefficients of the output:

$$b_k = a_k/(1 + j0.2\pi k/5).$$