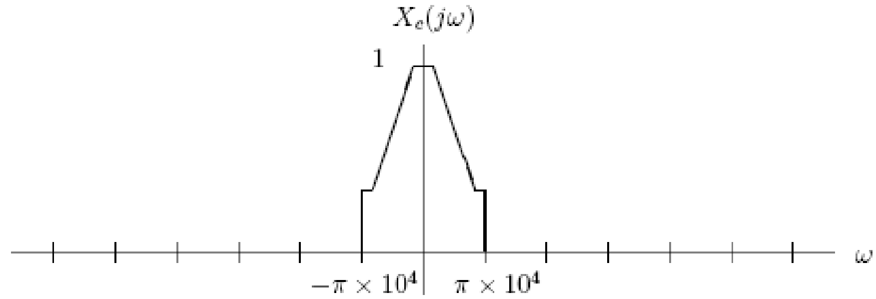
Signals and Systems

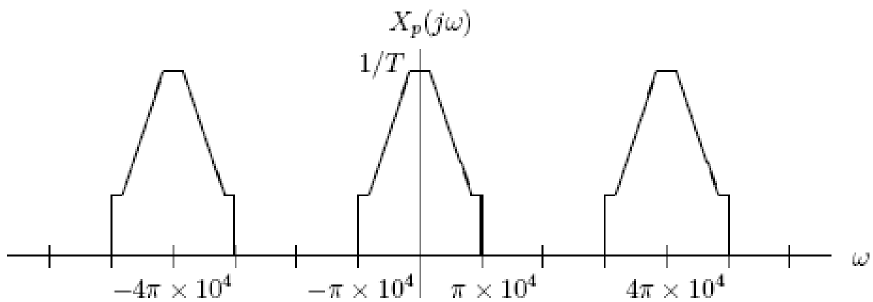
Solutions to Homework Assignment #5

Problem 1. O&W7.29

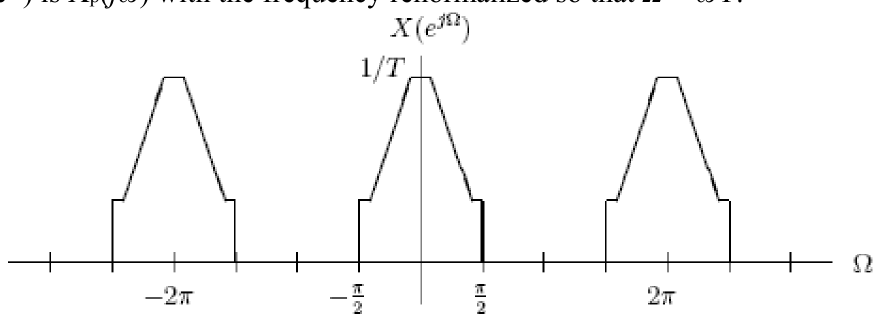
$1/T = 20\text{kHz}$, so $\omega_s = 2\pi/T = 4\pi \times 10^4$ rad/s. We are given $X_c(j\omega)$:



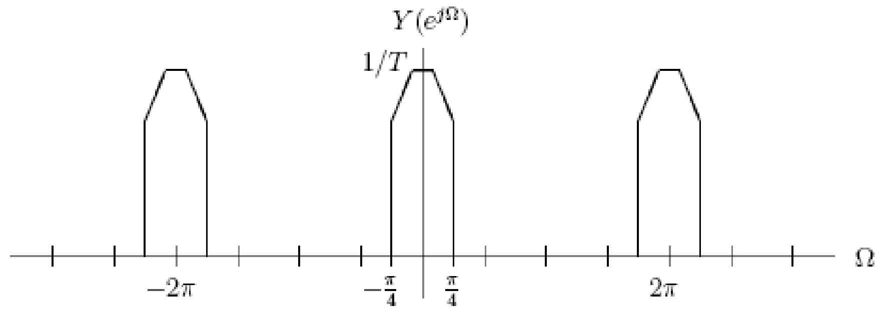
$X_p(j\omega)$ has copies of $X(e^{j\omega})$ at every multiple of ω_s and scaled down by T:



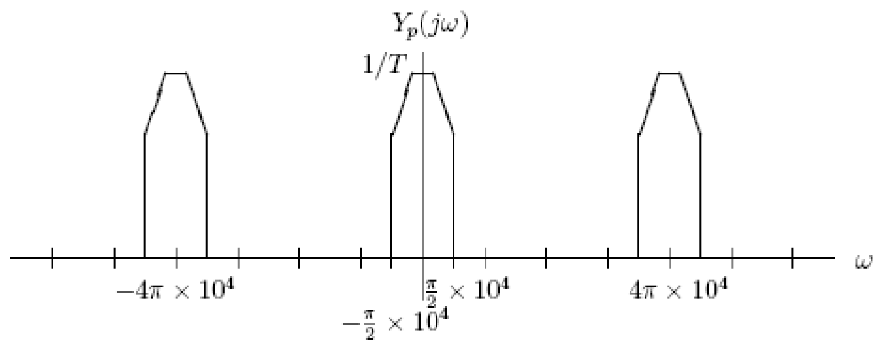
$X(e^{j\omega})$ is $X_p(j\omega)$ with the frequency renormalized so that $\Omega = \omega T$:

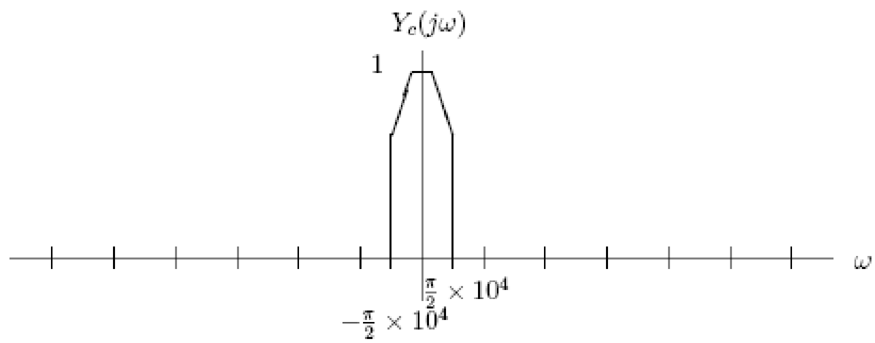


The DT filter keeps only the frequencies ω such that $|\omega| < \pi/4$, so $Y(e^{j\omega})$ is:



We can now run the CT-to-DT process in reverse to arrive at $Y_p(j\omega)$and $Y(j\omega)$:





Problem 2. O&W 7.31

It is given that $X(j\omega) = 0$ for $|\omega| \geq \pi/T$ and we have a sampling frequency, $\omega_s = 2\pi/T$, so there will be no aliasing. $x[n]$ is the sampled $x_c(t)$ with no loss and the amplitude of a particular $x[n]$is the area of the corresponding impulse.

$X(j\omega)$ is sampled and becomes $X(e^{j\Omega})$ and then is multiplied by $H_d(e^{j\Omega})$. After interpolation, $Y_c(j\omega) = X_c(j\omega)H_d(e^{j\omega T})$.

$$H_c(j\omega) = \begin{cases} H_d(e^{j\omega T}), & |\omega| < \frac{\omega_s}{2}, \\ 0, & |\omega| > \frac{\omega_s}{2}. \end{cases}$$

The difference equation for $h[n]$ is given as $y[n] = \frac{1}{2}y[n-1] + x[n]$, which gives:

$$Y(e^{j\omega}) = \tfrac{1}{2}e^{-j\Omega}Y(e^{j\Omega}) + X(e^{j\Omega}).$$

The discrete-time frequency response is:

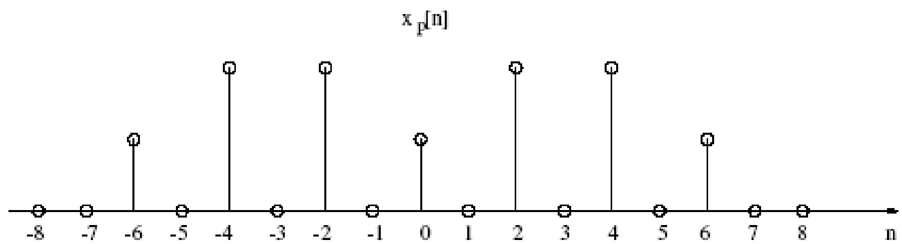
$$H_d(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}.$$

Since all of the sampling and interpolation works out, the continuous-time frequency response is:

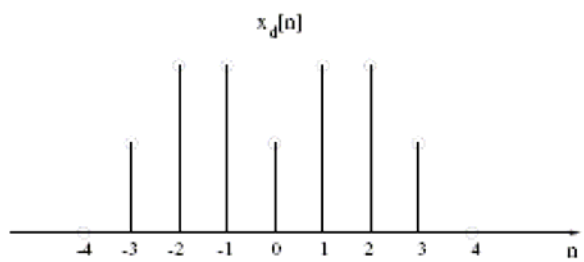
$$H_c(j\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega T}}, \quad |\omega| < \frac{\omega_s}{2}.$$

Problem 3. O&W 7.35

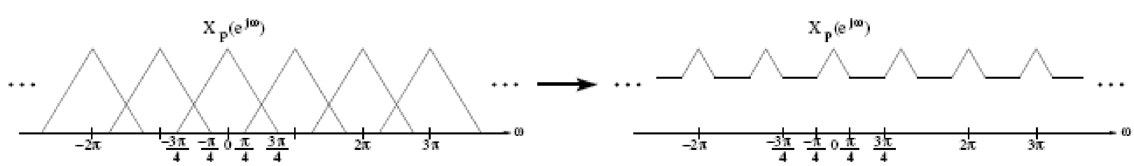
(a) $x_p[n]$ is just $x[n]$ sampled at every other point as sketched below:



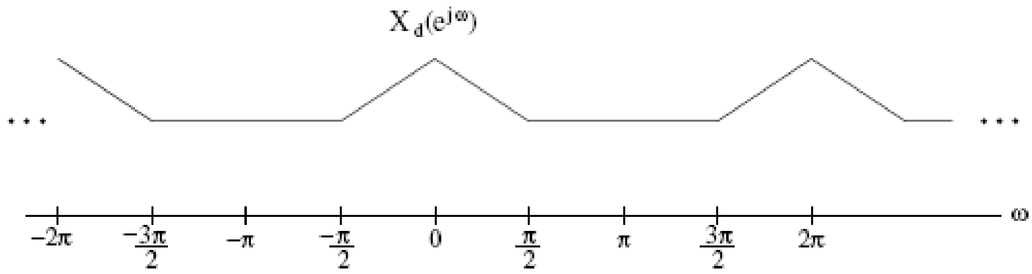
$x_d[n]$ is $x[n]$ downsampled by 2, and therefore it has all the stems in $x_p[n]$, except that all the zero points between samples are gone (i.e. $x_d[n]$ is $x_p[n]$ time-scaled by 1/2). $x_d[n]$ is sketched below:



(b) Since $x_p[n]$ is $x[n]$ sampled by a DT impulse train of period 2, $X_p(e^{j\omega})$ is $X(e^{j\omega})$ periodically convolved with an impulse train of period π. Therefore, $X_p(e^{j\omega})$ will have the triangular spectrum replicated at integer multiples of π, with height scaled by ½. $X_p(e^{j\omega})$ is graphed below:



Since $x_d[n] = x_p[2n]$, $X_d(e^{j\omega}) = X_p(e^{j\omega/2})$. $X_d(e^{j\omega})$ is graphed below:



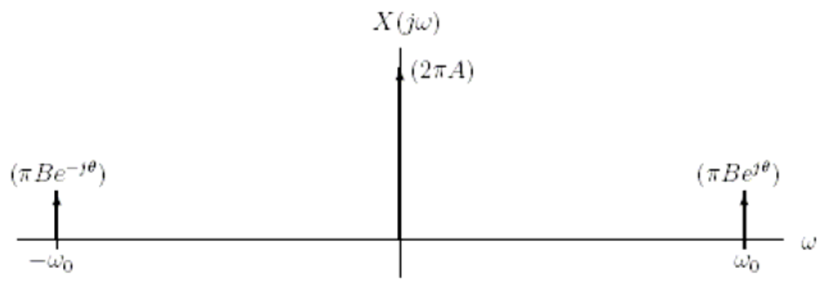
Problem 4. O&W 7.38

Let the frequency of the cosine be $\omega_0 = 2\pi/T$ and the sampling frequency be $\omega_s = 2\pi/(T+\Delta)$. Then, the Fourier transform of

$$x(t) = A + B\cos(\omega_0 t + \theta)$$

is

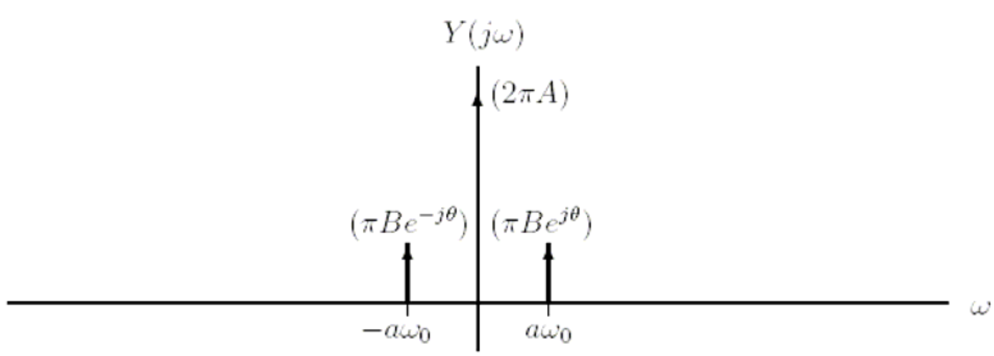
$$X(j\omega) = 2\pi A\delta(\omega) + \pi B(e^{j\theta}\delta(\omega - \omega_0) + e^{-j\theta}\delta(\omega + \omega_0))$$



In order for $y(t)$ to be proportional to $x(at)$, we need

$$\begin{aligned} y(t) &\propto x(at) \\ &\propto A + B\cos(\omega_0(at) + \theta) \\ &\propto A + B\cos((a\omega_0)t + \theta). \end{aligned}$$

So, the Fourier transform $Y(j\omega)$ of $y(t)$ must be proportional to



Note that the "B" impulses in $Y(j\omega)$ are closer to the "A" impulse than they are in $X(j\omega)$. The Fourier transform $X_p(j\omega)$ of $x_p(t) = x(t)p(t)$ is $X(j\omega)$ convolved with a train of impulses located at integer multiples of ω_s. Thus the rightmost "B" impulse in $X(j\omega)$ contributes impulses in $X_p(j\omega)$ at all frequencies $\omega_0 - n\omega_s$, where n is an integer. In order to get the desired $Y(j\omega)$ (shown above), exactly one of the frequencies $\omega_0 - n\omega_s$ must be in the interval $0 < \omega < \omega_s/2$, where $\omega_s/2$ represents the cutoff frequency of the low-pass filter. Thus,

$$0 < \omega_0 - n\omega_s < \omega_s/2$$

Solving for constraints on ω_s yields

$$\frac{\omega_0}{n+1/2} < \omega_s < \frac{\omega_0}{n}.$$

In order for the shifted frequency to be smaller than ω_0, n must be greater than zero. These ranges are indicated in the following table.

n	ω_s range	corresponding range of Δ
1	$\frac{2}{3}\,\omega_0 < \omega_s < \omega_0$	$0 < \Delta < \frac{1}{2}\,T$
2	$\frac{2}{5}\,\omega_0 < \omega_s < \frac{\omega_0}{2}$	$T < \Delta < \frac{3}{2}\,T$
3	$\frac{2}{7}\,\omega_0 < \omega_s < \frac{\omega_0}{3}$	$2T < \Delta < \frac{5}{2}\,T$
4	$\frac{2}{9}\,\omega_0 < \omega_s < \frac{\omega_0}{4}$	$3T < \Delta < \frac{7}{2}\,T$
n	$\frac{\omega_0}{n+1/2} < \omega_s < \frac{\omega_0}{n}$	$(n-1)\,T < \Delta < (n-\frac{1}{2})\,T$

The factor a is the ratio of the frequencies "*B*" impulses in $Y(j\omega)$and in $X(j\omega)$, i.e.,

$$\begin{aligned} a &= \frac{\omega_0 - n\omega_s}{\omega_0} = \frac{2\pi/T - 2\pi n/(T+\Delta)}{2\pi/T} \\ \Rightarrow a &= 1 - \frac{nT}{T+\Delta}. \end{aligned}$$

Problem 5. O&W 7.42

We want to use Parseval's relation for this problem:

$$\begin{aligned} E_d &= \sum_{k=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\Omega})|^2 d\Omega \\ E_c &= \int_{-\infty}^{\infty} |x_c(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \end{aligned}$$

Since the continuous-time signal is sampled beyond Nyquist rate, we don'*t* have to worry about aliasing. Over the period $-\pi < \Omega \le \pi$, therefore, we can say that

$$X(e^{j\Omega}) = \frac{1}{T}\,X(j\omega)|_{\omega=\Omega\frac{\omega_s}{2\pi}}$$

Now, compute the energy in the sampled signal using the equations above:

$$\begin{aligned}
E_d &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\Omega})|^2 d\Omega \\
&= \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2}\left|\frac{1}{T}X(j\omega)\right|^2 \frac{2\pi}{\omega_s}d\omega \\
&= \frac{1}{\omega_s}\int_{-\omega_M}^{\omega_M}\frac{1}{T^2}\left|X(j\omega)\right|^2 d\omega \\
&= \frac{T}{2\pi}\frac{1}{T^2}\int_{-\omega_M}^{\omega_M}\left|X(j\omega)\right|^2 d\omega \\
&= \frac{1}{T}E_c
\end{aligned}$$

Problem 6. O&W 7.45

(a) We have a band-limited signal, the Nyquist frequency

$$\omega_N = 2\omega_M = 2\times 2\pi \times 10^4$$

and we should sample at $2\pi/T > \omega_N$.

Therefore,

$$T_{max} = \frac{2\pi}{4\pi \times 10^4} = 5\times 10^{-5}\text{s}$$

(b) The impulse response would be

$$h_n[n] = Tu[n].$$

(c) We first look at $\lim_{t\to\infty}\int_{-\infty}^{t} x_c(\tau)d\tau$ what would look like.

$$\lim_{t\to\infty}\int_{-\infty}^{t} x_c(\tau)d\tau = \int_{-\infty}^{\infty} x_c(\tau)d\tau = X_c(j0)$$

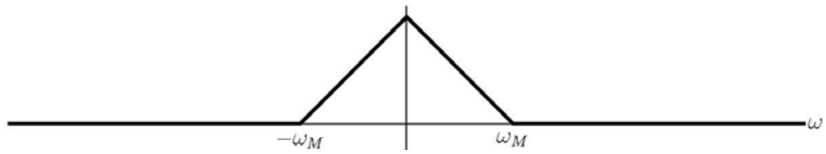
Then we simplify $\lim_{n\to\infty} y[n]$.

$$\begin{aligned}
X(e^{j\Omega}) &= \sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega k} \\
\lim_{n\to\infty} y[n] &= T\lim_{n\to\infty}\sum_{k=-\infty}^{n} x[k] \\
&= T\sum_{k=-\infty}^{\infty} x[k] \\
&= TX(e^{j0})
\end{aligned}$$

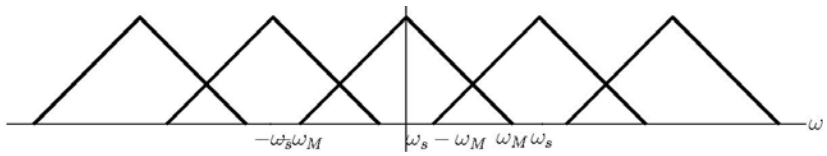
As we know already, $X(e^{j\omega}) = (1/T)X(j\Omega/T)$ if there is no aliasing at any particular Ω.

For $\lim_{n\to\infty} y[n] = \lim_{t\to\infty}\int_{-\infty}^{t} x_c(\tau)d\tau$ to be true, $TX(e^{j0}) = X_c(j0)$ has to be true, i.e., there can be no aliasing at $\omega = 0$. We do not mind aliasing as long as $X_p(j0) = X_c(j0)$. Suppose $X_c(j\omega)$ is a triangular waveform with band-limit frequency ω_M.

$X_c(j\omega)$



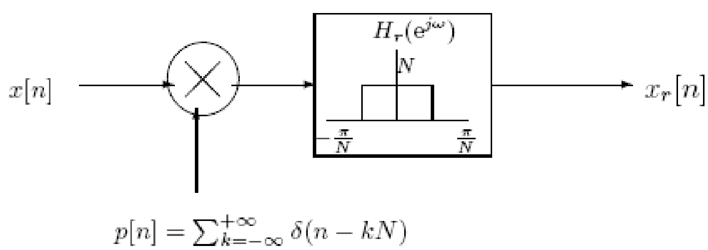
$X_p(j\omega)$



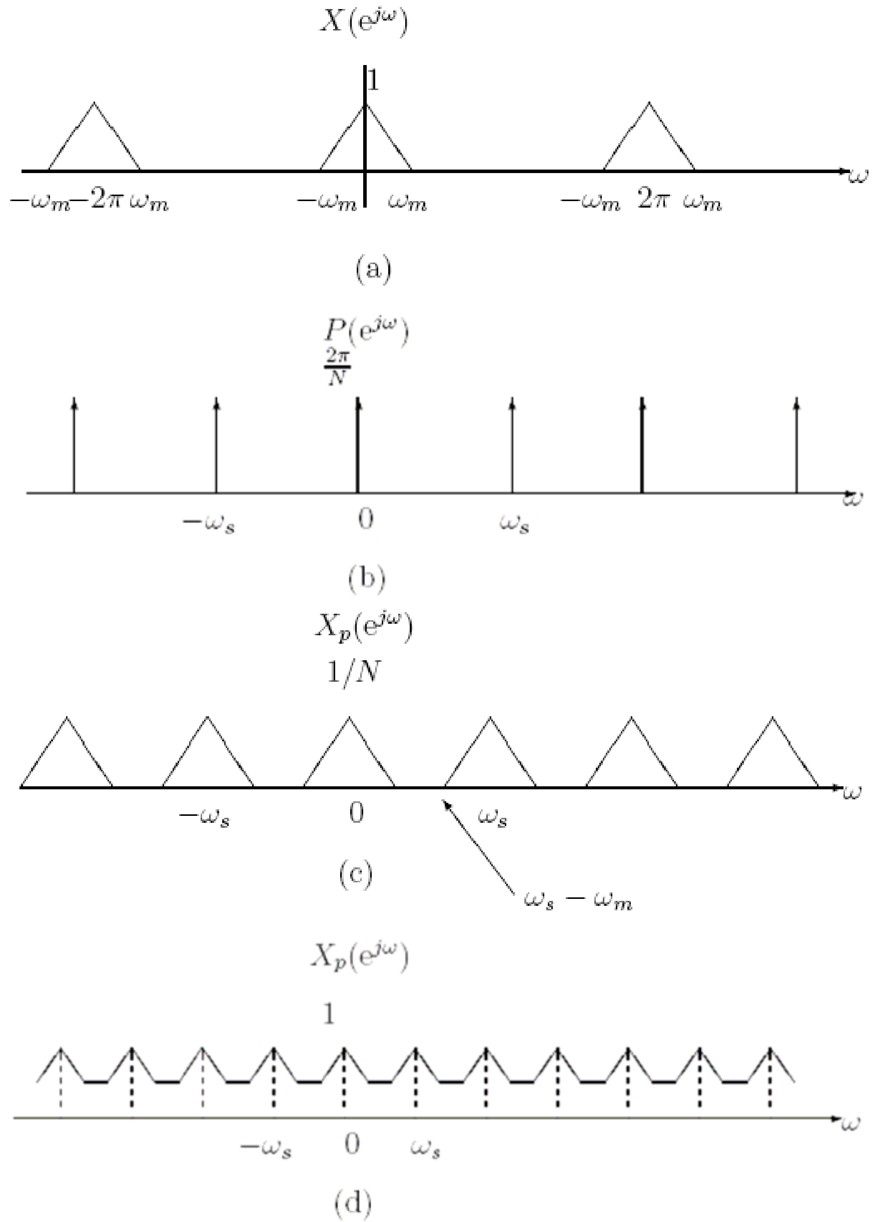
where $\omega_s = 2\pi/T$. Thus, we do not want the copies at $k\omega_0$ for non-zero k values to overlap with the copy at 0. want $\omega_s - \omega_M > 0$, i.e., $\omega_s > \omega_M$. Therefore, $T < 2\pi/\omega_M$, $T_{max} = 10^{-4}$ s.

Problem 7. O&W 7.46

Note that the filter should have gain N and cutoff frequency $\omega_c = \pi/N$. The system is given by the figure below:



Assume that the frequency response of $x[n]$ is given in figure (a) below, and the frequency response of the sampling sequence $p[n]$ is given as in figure (b). If $\omega_s > 2\omega_m$, then there is no aliasing (i.e the non-zero replicas of $X(e^{j\omega})$ do not overlap), as shown in figure (c), whereas if $\omega_s < 2\omega_m$, then over lapping occurs and aliasing results, as shown in figure (d). In the absence of aliasing, $X(e^{j\omega})$ is faithfully reproduced around $\omega = 0$ and integer multiples of 2π. Hence $x[n]$ can be completely recovered from $x_r[n]$. In the case of aliasing, $x_r[n]$ will no longer be equal to $x[n]$. However, the two sequences will be equal at multiples of the sampling period. This is shown in (d) by the dotted lines at multiple values of ω_s, the values occuring at these locations are the same as that in $X_p(e^{j\omega})$. Hence, $x_r[mN] = x[mN]$ regardless of aliasing.



$X(e^{j\omega})$
1
ω
$-\omega_m$ -2π ω_m
$-\omega_m$ ω_m
$-\omega_m$ 2π ω_m
(a)
$P(e^{j\omega})$
$\frac{2\pi}{N}$
ω
$-\omega_s$
0
ω_s
(b)
$X_p(e^{j\omega})$
$1/N$
ω
$-\omega_s$
0
ω_s
(c)
$\omega_s - \omega_m$
$X_p(e^{j\omega})$
1
ω
$-\omega_s$
0
ω_s
(d)