# Signals and Systems

Solutions to Homework Assignment # 6

**Problem 1.**

a. This is easy to do directly from the definition:

$$X_1(s) = \int_{-\infty}^{\infty} x_1(t)e^{-st}dt = \int_{-\infty}^{\infty} e^{-2t}u(t-3)e^{-st}dt = \int_{3}^{\infty} e^{-(s+2)t}dt = \left.\frac{e^{-(s+2)t}}{-(s+2)}\right|_3^{\infty}$$

$$= \frac{e^{-3(s+2)}}{s+2}; \quad \Re\{s\} > -2$$

b. Treat this as the sum of 3 signals: $x_2(t) = x_{2a}(t) + x_{2b}(t) + x_{2c}(t)$, where $x_{2a}(t) = u(t)$, $x_{2b}(t) = -e^{-3t}u(t)$, and $x_{2c}(t) = te^{-3t}u(t)$. Then:

$$X_{2a}(s) = \frac{1}{s}; \quad \Re\{s\} > 0$$

$$X_{2b}(s) = -\frac{1}{s+3}; \quad \Re\{s\} > -3$$

$$X_{2c}(s) = \frac{1}{(s+3)^2}; \quad \Re\{s\} > -3$$

Then the Laplace Transform of a sum is the sum of the Laplace transforms,

$$X_2(s) = \frac{1}{s} - \frac{1}{s+3} + \frac{1}{(s+3)^2} = \frac{4s+9}{s(s+3)^2}$$

and the region of convergence is the intersection of the 3 regions: $\Re\{s\} > 0$.

c. The signal $x_3(t)$ can be written as $te^{-t}u(t)+te^{t}u(-t)$. The transform of the first is

$$\frac{1}{(s+1)^2}; \quad \Re\{s\} > -1$$

and the transform of the second is

$$-\frac{1}{(s-1)^2}; \quad \Re\{s\} < 1.$$

Therefore

$$X_3(s) = \frac{1}{(s+1)^2} - \frac{1}{(s-1)^2} = \frac{-4s}{(s+1)^2(s-1)^2}; \quad -1 < \Re\{s\} < 1.$$

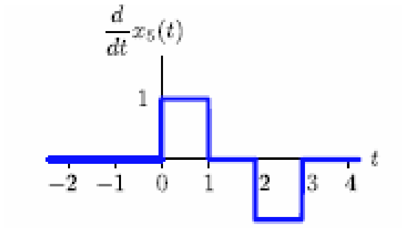
d. This is easy to do directly from the definition:

$$X_4(s) = \int_{-\infty}^{\infty} x_4(t)e^{-st}dt = \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^1 - \left.\frac{e^{-st}}{-s}\right|_1^2 = -\frac{e^{-s}}{s} + \frac{1}{s} + \frac{e^{-2s}}{s} - \frac{e^{-s}}{s}$$

$$= \frac{e^{-2s} - 2e^{-s} + 1}{s}$$

These integrals all converge for all values of $s$. Therefore the region of convergence is the entire $s$-plane.

e. This is not so easy to do directly from the definition, since this leads to integrals of $t$ times $e^{-st}$. The result can be integrated by parts, but it is messy. An easier way is to realize that the derivative of $x_5(t)$ is simple:




This function can be written as $u(t) - u(t-1) - u(t-2) + u(t-3)$ and therefore has a Laplace transform equal to

$$(1/s)(1 - e^{-s} - e^{-2s} + e^{-3s}).$$

Since $x_5(t)$ is the integral of it's derivative, the Laplace Transform of $x_5(t)$ is $1/s$ times the Laplace Transform of its derivative:

$$X_5(s) = (1/s^2)(1 - e^{-s} - e^{-2s} + e^{-3s}).$$

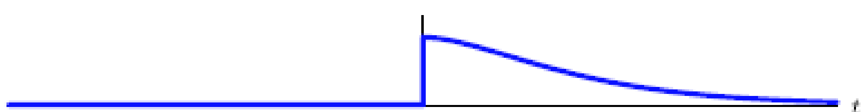
The region of convergence includes the whole $s$-plane since $x_5(t)$ has finite duration.

**Problem 2.**

a. We can expand $X_1(s)$ using partial fractions as

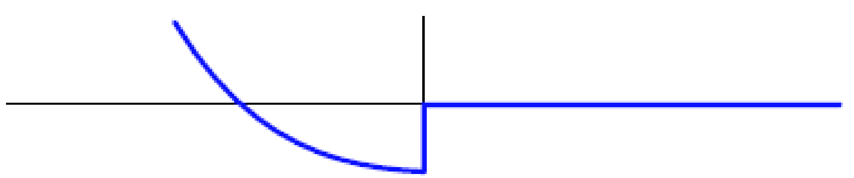
$$X_1(s) = \frac{s+2}{(s+1)^2} = \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

Because both poles are at $s = -1$, there are just two possible regions of convergence: $s > -1$ and $s < -1$. For the right-sided region, $x_1(t) = (1+t)e^{-t}u(t)$




and for the left-sided region,

$$x_1(t) = -(1+t)e^{-t}u(-t).$$

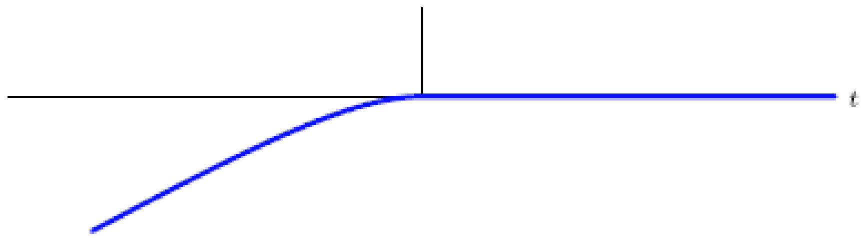



b. Using partial fractions,

$$X_2(s) = \frac{1}{s^2(s-1)} = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2}.$$

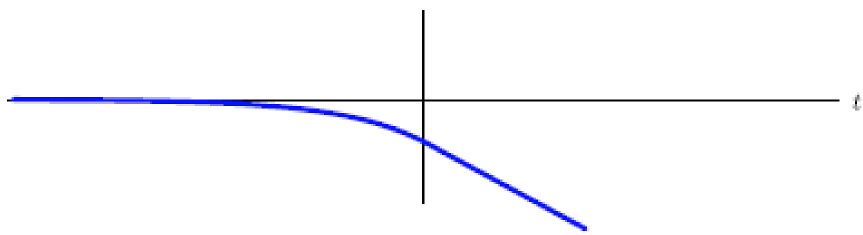
The two poles at $s = 0$ and the pole at $s = 1$ break the $s$-plane into 3 possible regions of convergence: $s < 0$, $0 < s < 1$, and $s > 1$. For $s < 0$, all of the terms are left-sided, so

$$x_2(t) = -e^{t}u(-t) + u(-t) + tu(-t).$$



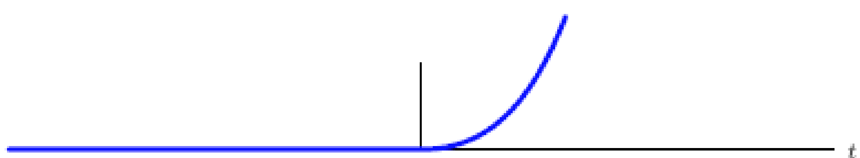

For $0 < s < 1$, exponential term is left-sided and the others are right-sided, so

$$x_2(t) = -e^{t}u(-t) - u(t) - tu(t).$$




For $s > 1$, all of the terms are right-sided, so

$$x_2(t) = e^{t}u(t) - u(t) - tu(t).$$

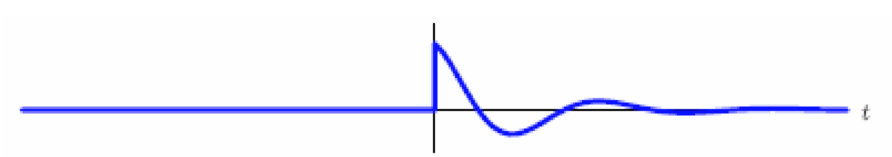



c. The factors of the denominator of $X_3(s)$ are complex-valued. Nevertheless, partial fractions still work.

$$X_3(s) = \frac{s+1}{s^2+2s+2} = \frac{1/2}{s+1+j} + \frac{1/2}{s+1-j}.$$

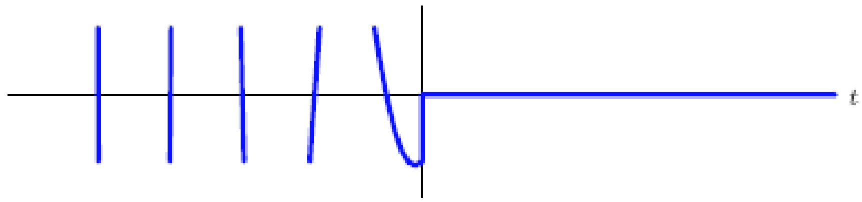
Both of these poles have the same real part. Therefore, there are two possible regions of convergence: $s > -1$ and $s < -1$. Both terms are right-sided for $s > -1$, so

$$x_3(t) = \frac{1}{2}e^{-(1+j)t}u(t) + \frac{1}{2}e^{-(1-j)t}u(t) = e^{-t}\cos(t)u(t).$$




Both terms are left-sided for $s < -1$, so

$$x_3(t) = -e^{-t}\cos(t)u(-t).$$

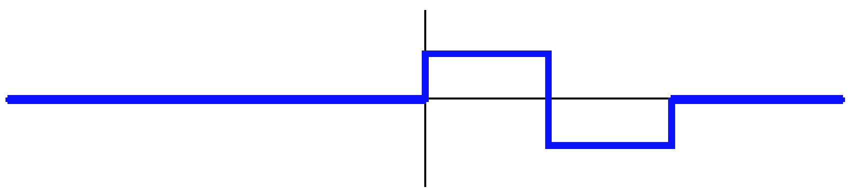



The values of $|x_3(t)|$ are so large that many regions in this plot are clipped.

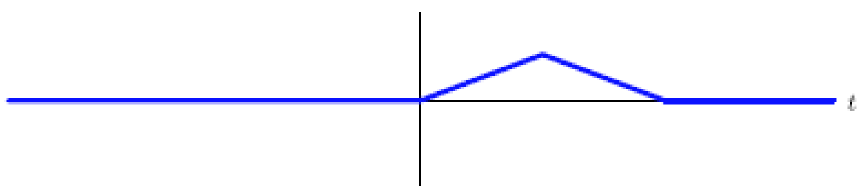
d. This expands to

$$X_4(s) = \left(\frac{1-e^{-s}}{s}\right)^2 = \frac{1-2e^{-s}+e^{-2s}}{s^2}.$$

If the denominator had be $s$ instead of $s^2$, then this Laplace Transform would correspond to the sum of three step functions as shown below.




Dividing a transform by $s$ is equivalent to integrating the time waveform, as shown below.




Since $X_4(s)$ converges everywhere in $s$, there is a single region of convergence, so the previous function is the only inverse transform of $X_4(s)$.

**Problem 3.**

a. The input $x_1(t) = \cos(t)$ can be written as

$$x_1(t) = \tfrac{1}{2}e^{jt} + \tfrac{1}{2}e^{-jt}$$

Both $e^{jt}$ and $e^{-jt}$ are eigenfunctions, so the corresponding outputs are $H(j)e^{jt}$ and $H(-j)e^{-jt}$, where

$$H(j) = \frac{j+2}{(j+1)(j+3)} = \frac{j+2}{4j+2}$$

and

$$H(-j) = \frac{-j+2}{(-j+1)(-j+3)} = \frac{-j+2}{-4j+2}.$$

Thus

$$y_1(t) = \frac{1}{2} \times \frac{j+2}{4j+2} e^{jt} + \frac{1}{2} \times \frac{-j+2}{-4j+2} e^{-jt} = \frac{1}{2} \times \frac{j+2}{4j+2} e^{jt} + \frac{1}{2} \times \left( \frac{j+2}{4j+2} e^{jt} \right)^* .$$

$$= \Re\left\{ \frac{j+2}{4j+2} e^{jt} \right\} = \Re\left\{ \frac{4-3j}{10} (\cos t + j \sin t) \right\}$$

$$= 0.4 \cos t + 0.3 \sin t.$$

b. The input $x_2(t) = 1$ is also an eigenfunction with $s = 0$. Therefore the output is

$$y_2(t) = H(0) = \frac{2}{(1)(3)} = \frac{2}{3}.$$

c. The function $x_3(t) = e^{-5t}$ is equal to $e^{st}$ for $s = -5$. Thus, it has the form of an eigenfunction. However, this value of $s$ is not in the region of convergence. Since the system is given to be causal, the region of convergence must be to the right side of all of the poles. There is a pole at $-1$ and at $-3$. Therefore, the region of convergence is $\Re\{s\} > -1$ which does not include $s = -5$. Since the signal is not in the region of convergence, the output does not converge.

This same conclusion follows from convolution. This system function corresponds to an impulse response of

$$h(t) = \frac{1}{2} e^{-t} u(t) + \frac{1}{2} e^{-3t} u(t).$$

Consider calculating the output by convolution:

$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau = \int_{-\infty}^{\infty} \left( \frac{1}{2} e^{-\tau} u(\tau) + \frac{1}{2} e^{-3\tau} u(\tau) \right) e^{-5(t-\tau)} d\tau.$$

$$= e^{-5t} \int_0^{\infty} \left( \frac{1}{2} e^{-\tau} + \frac{1}{2} e^{-3\tau} \right) e^{5\tau} d\tau = e^{-5t} \int_0^{\infty} \left( \frac{1}{2} e^{4\tau} + \frac{1}{2} e^{2\tau} \right) d\tau.$$

This integral doesn'*t* converge, therefore $y(t)$does not converge for any finite $t$.

**Problem 4.** If $x(t)$ is an even function of time, then

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \int_{-\infty}^{\infty} x(-t) e^{-st} dt = \int_{-\infty}^{\infty} x(t) e^{st} dt = X(-s).$$

Thus the corresponding system function must be an even function of $s$. If $X(s)$ is an even function of $s$, then the pole-zero diagram must be symmetric about the $j\omega$ axis. This means that $X_2(s)$ cannot correspond to an even function of time.

For $X(s)$ to be an even function of $s$, its region of convergence must also be symmetric about the $j\omega$ axis. Symmetry is not possible for $X_4(s)$, which must be either right-sided or left-sided but not both. Both $X_1(s)$ and $X_3(s)$ can have symmetric regions of convergence if the center regions $-1 < s < 1$ are chosen, but both can have either odd-or even-symmetry. Specifically for $X_3(s)$ is can be quickly determined that $X_3(s) = -X_3(-s)$. Therefore $x_3(t)$ is cannot be even in the time domain.

Both $X_1(s)$and $X_3(s)$ have similar partial fraction expansions:

$A/(s+1) + B/(s-1)$

To get a zero at zero, $A = B$. The corresponding time function is

$$x_3(t) = A(e^{-t}u(t) - e^{t}u(-t))$$

which is an odd function of time for all values of $A$. To get no finite zeros, $A = -B$. The corresponding time function is

$$x_1(t) = A(e^{-t}u(t) + e^{t}u(-t))$$

which is an even function of time for all values of $A$. Thus, the only case that corresponds to an even function of time is case 1.

**Problem 5.** It is helpful to look at this problem in the transform domain. The Laplace Transform of $h(t) = \delta(t) - 2e^{-t}u(t)$ is

$$H(s) = 1 - \frac{2}{s+1} = \frac{s-1}{s+1}$$

which converges for $\Re\{s\} > -1$ since $h(t)$ is right-sided. The Laplace Transform of the output signal $y(t) = e^{-2t}u(t)$ is

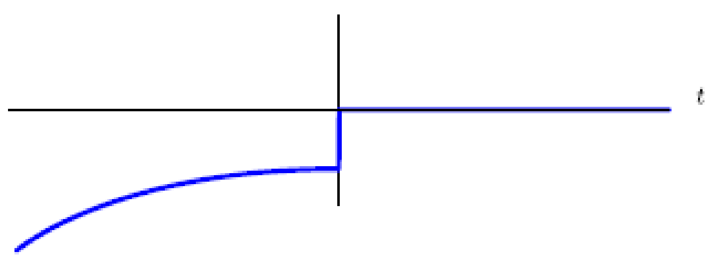
$$Y(s) = 1/(s+2)$$

Thus the input signal must be

$$X(s) = \frac{Y(s)}{H(s)} = \frac{\frac{1}{s+2}}{\frac{s-1}{s+1}} = \frac{s+1}{(s+2)(s-1)} = \frac{1/3}{s+2} + \frac{2/3}{s-1}.$$

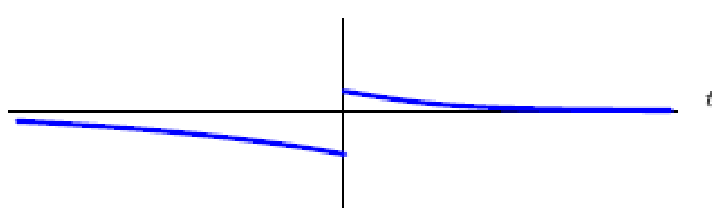
Thus $X(s)$has two poles, which gives rise to three possible regions of convergence: $s < -2$, $-2 < s < 1$, and $s > 1$. For $s < -2$, both terms must be left-sided:

$$x_1(t) = -\frac{1}{3}e^{-2t}u(-t) - \frac{2}{3}e^{t}u(-t).$$



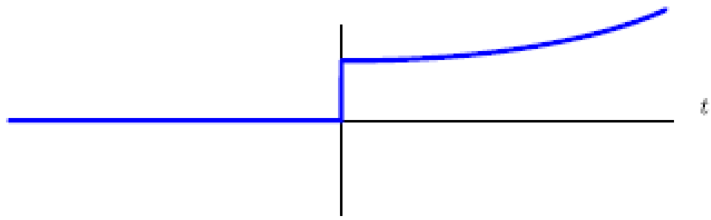

For $-2 < s < 1$, the first term is right-sided and the second is left-sided:

$$x_2(t) = \frac{1}{3}e^{-2t}u(t) - \frac{2}{3}e^{t}u(-t).$$



For $s > 1$, both terms must be right sided:

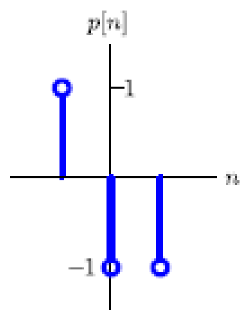
$$x_3(t) = \frac{1}{3}e^{-2t}u(t) + \frac{2}{3}e^{t}u(t).$$




To determine which of these signals could generate the given output, we must consider whether the regions of convergence make sense. The system $H(s)$ converges for $\Re\{s\} > -1$. Since $X_1(s)$ only converges for $\Re\{s\} < -2$, there is no overlap between its region of convergence and that of $H(s)$. Therefore, $x_1(t)$ cannot be the input. The region of convergence of $X_2(s)$ is $-2 < \Re\{s\} < 1$, which does intersect with that of $H(s)$. Thus $x_2(t)$ could be the input. The region of convergence of $X_3(s)$ is $\Re\{s\} > 1$, which also intersects with that of $H(s)$. Thus $x_3(t)$ could also be the input. Either $x_2(t)$ or $x_3(t)$ could be the input.

**Problem 6.**

a.




If we flip this signal about $n = 0$ and then shift it to the right by two, we get $p[n]$ to line up with the desired pattern in $x[n]$.

b. The following code was used to verify that the matched filter worked:

*z* = sign(randn(1,50)); *y* = conv(*z*,[1 -1 -1]); % note: the matched filter is flippedfigure(1); stem(1:50,z); figure(2); stem(1:52,*y*); find(*y*==max(*y*))

Figure 1 shows the random sequence of 1's and −1's used for the test.

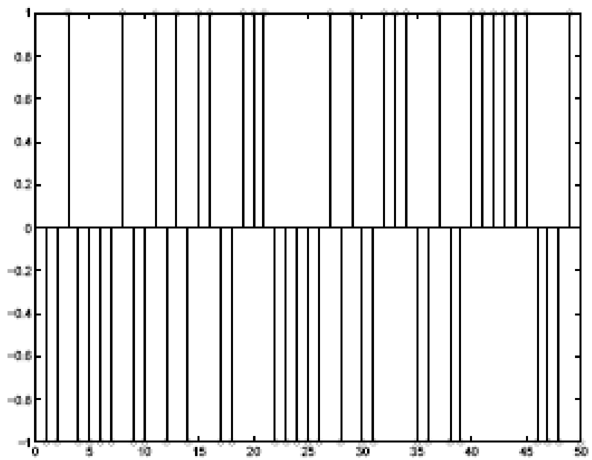
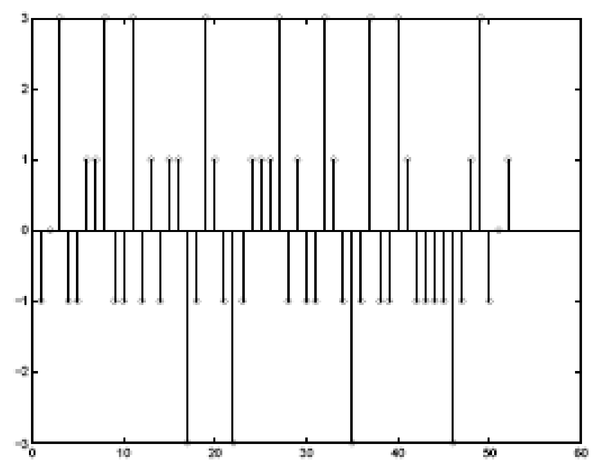



Figure 2 shows the output of the matched filter.




The find command located maxima at $n = 3$, 8, 11, 19, 27, 32, 37, 40, and 40. These points are displaced to the right by one because the conv command does not allow the matched filter to start at $n = -1$.