Signals and Systems

Homework Solution #9

Problem 1. Determine the Z Transform (including region of convergence) for each of the following signals:

a. $X_1(z) = \sum_n x_1[n]z^{-n} = \sum_{n=3}^{\infty}(1/2)^n z^{-n} = \sum_{n=3}^{\infty}\left(\frac{z^{-1}}{2}\right)^n = \frac{(z^{-1}/2)^3}{1-(z^{-1}/2)}$. The ROC is $|z| > 1/2$.

b. We use the Z-transform property for this one:

$$(1/3)^n u[n] \Leftrightarrow \frac{z}{z-1/3}, \quad |z| > 1/3.$$

$$n(1/3)^n u[n] \Leftrightarrow -z\frac{d}{dz}\left(\frac{z}{z-1/3}\right) = \frac{1/3z^{-1}}{(1-1/3z^{-1})^2}, \quad |z| > 1/3.$$

$$X_2(z) = \frac{z}{z-1/3} + \frac{1/3z}{(z-1/3)^2} = 1/(1-1/3*z^{-1})^2, \quad \text{ROC:}|z| > 1/3$$

c. As before, we split the signal up into two regimes, and use the result of the previous part:

$$x_3[n] = \underbrace{n(1/2)^n u[n]}\,A + \underbrace{n(1/2)^{-n}u[-n-1]}\,B.$$

$$n(1/2)^n u[n] \Leftrightarrow \frac{(1/2)z^{-1}}{(1-(1/2)z^{-1})^2}, \quad ROC|z| > 1/2.$$

$$n(1/2)^{-n}u[-n-1] \Leftrightarrow \frac{-2z^{-1}}{(1-2z^{-1})^2}, \quad ROC|z| < 2.$$

$$\begin{aligned} X_3(z) &= \frac{(1/2)z^{-1}}{(1-(1/2)z^{-1})^2} + \frac{-2z^{-1}}{(1-2z^{-1})^2} \\ &= \frac{\frac{-3}{2}z(z-1)(z+1)}{(z-2)^2(z-1/2)^2}, \quad ROC: 1/2 < |z| < 2. \end{aligned}$$

d. Use straightforward Z-transform:

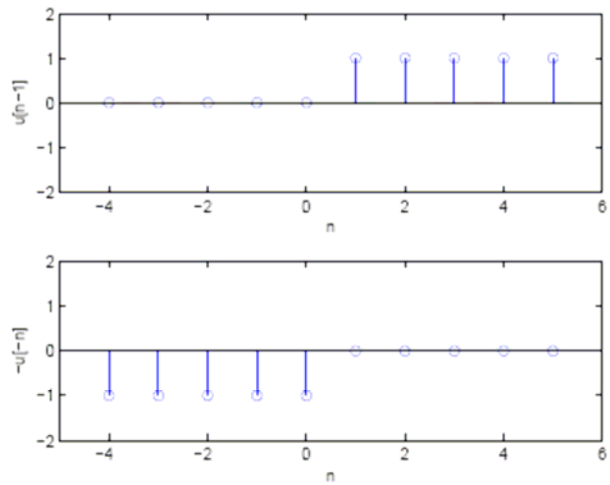
$$\begin{aligned} X_4(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} = -x_4[-1]z + x_4[1]z^{-1} + x_4[2]z^{-2} \\ &= -z + z^{-1} + z^{-2}, \quad \text{ROC: all z, except} z = 0, \infty. \end{aligned}$$

Problem 2. Determine and sketch all possible signals with Z Transforms of the following forms:

a. The system has a pole at $z = 1$. The possible signals are,

$x_1[n] = u[n\text{-}1]$, ROC : $|z| > 1$,

$x_1[n] = -u[-n]$, ROC : $|z| < 1$.



b. We write this as

$$X_2(z) = \frac{1}{z}\left(\frac{zz^{-1}}{(z-1)^2}\right)$$

,in the time domain,

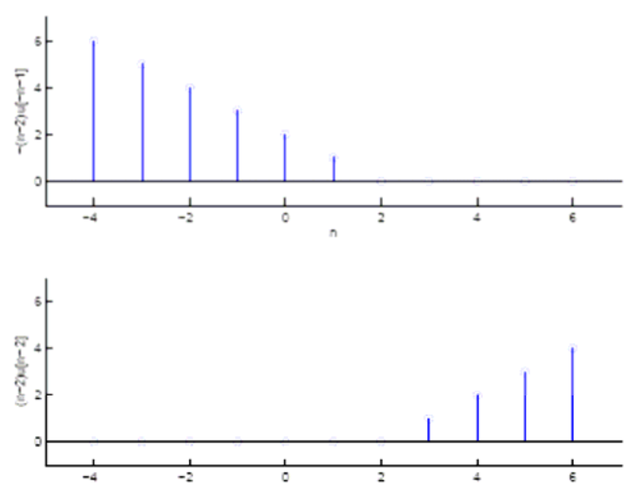
$x_2[n] = \delta[n\text{-}1]^* \text{-}nu[\text{-}n\text{-}1] = \text{-}(n\text{-}2)u[\text{-}n\text{+}1]$, ROC: $|z| < 1$,

$x_2[n] = \delta[n\text{-}2]^*nu[n] = (n\text{-}2)u[n\text{-}2]$, ROC: $|z| > 1$.

Equivalently, we can write,

$x_2[n] = \delta[n\text{-}1]^* \text{-}nu[\text{-}n\text{-}1] = \text{-}(n\text{-}2)u[\text{-}n\text{ +}2]$, ROC: $|z| < 1$,

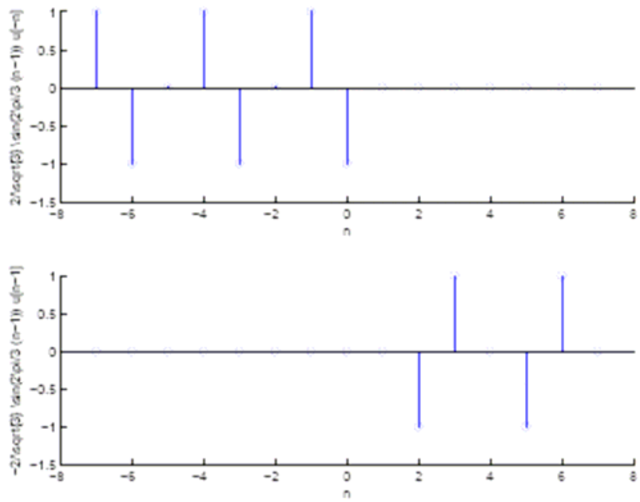
$x_2[n] = \delta[n\text{-}2]^*nu[n] = (n\text{-}2)u[n\text{-}3]$, ROC: $|z| > 1$.



c. From the text, we know that the signal contains a harmonic term. The poles are complex conjugates on the unit circle.

$$x_3[n] = \frac{2}{\sqrt{3}}\sin(2\pi/3(n-1))u[-n], \quad \text{ROC:}|z| < 1$$

$$x_3[n] = -\frac{2}{\sqrt{3}}\sin(2\pi/3(n-1))u[n-1], \quad \text{ROC:}|z| > 1.$$



d. In finding the inverse Z-transform, we normally have 2 options. If the Z-transform is a rational function, we use Partial Fraction Expansion (PFE) to break up it up into simpler terms whose Z-transforms are familiar to us. However, when the Z-transform is non-rational, as in this case, we should look into messaging the expression a bit to put it in a form that we know how to deal with. In this case, we have e^{-z} term in the Z-transform that we don't know its inverse. However, we can use Taylor's Series to express e^{-z} in powers of z which we can take care of easily. Hence, starting with

$$X_4(z) = (\frac{1-e^{-z}}{z})^2 = z^{-2}(1 - 2e^{-z} + e^{-2z})$$

We can now replace exponential terms with their Taylor Series expansion (recall that $e^{\theta} = \sum_{k=0}^{\infty} \frac{\theta^k}{k!}$). In other words,

$$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!}$$

$$e^{-2z} = 1 - 2z + \frac{(2z)^2}{2!} - \frac{(2z)^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{(-2z)^k}{k!}$$

Now, by linearity and by noticing that z^k has inverse Z-transform of $\delta[n+k]$, we know that the inverse Z-transform of $(1 - 2e^{-z} + e^{-2z})$ is

$$v[n] = \delta[n] + \sum_{k=0}^{\infty}(\frac{(-2)^k - 2(-1)^k}{k!}\delta[n+k])$$

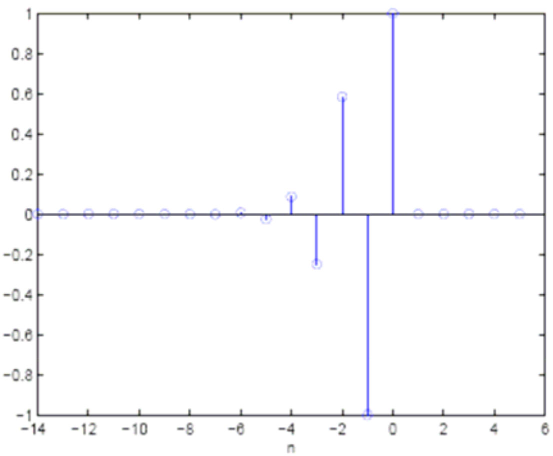
Finally, since $X_4(z) = z^{-2}$ times the above Z-transform, its inverse Z-transform is simply $v[n]$

shifted to the right by 2 samples (recall that $x[n\text{-}n_0]$ has Z-transform of $z^{-n_0}X(z)$). So we have,

$$x_4[n] = \delta[n-2] + \sum_{k=0}^{\infty}\left(\frac{(-2)^k - 2(-1)^k}{k!}\delta[n+k-2]\right)$$

It may appear that the signal has a pole at $z = 0$ and a zero at $z = 0$, but using Taylor's expansion we can see that as $z \to 0$ the numerator goes to zero faster than the denominator goes to infinity. Hence the signal has no poles, and the ROC is the entire Z-plane. The signal can be better expressed as:

$$x_4[n] = \begin{cases} 0, & n \geq 2 \\ \frac{(-2)^{2-n} - 2(-1)^{2-n}}{(2-n)!}, & n < 2 \end{cases}$$



Problem 3. Diagram 1 is a unit step signal. As we move along the $j\omega$ axis, we move away from the pole at the origin and the log-magnitude decays linearly. The phase is constant since the angle between the pole and any point along positive side of the $j\omega$ axis remains constant.

Diagram 4 moves the pole to the left of the origin. Then the signal becomes damped. As we move along the $j\omega$ axis, we move away from the pole at the origin, and the log-magnitude will eventually decay linearly. Because the pole is not exactly at the origin, this decay does not start until some later. The phase starts at 0, and eventually moves to $-\pi/2$. Note that as we move farther up the $j\omega$ axis, this system behaves like the system of diagram1.

Diagram 3 adds a zero at the origin. When ω is small, the zero is dominant. As we move away from $\omega = 0$, the effect of the zero diminishes and the log-magnitude increases linearly. For sufficiently large ω we are far enough that the zero and pole appear to cancel each other. Hence the magnitude becomes a constant. The zero at the origin adds a constant term to the derivative of the log-magnitude everywhere. A zero at the origin means taking the derivative, so we take the impulse response of diagram 4 and take its derivative. A zero at the origin means that we take the phase response of diagram 4 and add π to it.

Diagram 2 contains complex conjugate poles, and has a harmonic term. Since they are not exactly on the $j\omega$ axis, the impulse response is damped (remember diagram 1 and diagram 4). The magnitude response will eventually decay twice as fast as that of diagram

4. Since there are two poles, there will be a bump at around $\omega = 1$. At the origin, the angular

contributions of the two poles cancel each other out, hence the angle is zero. As we move up the jωaxis, the angles add up to $-\pi$, with each pole contributing $-\pi/2$.

Diagram 6 a zero at the origin, meaning that we take the derivative of the impulse response of diagram 2. The zero at the origin increases the derivative of the log-magnitude plot everywhere. It also adds π to the phase response everywhere.

Diagram 5 complex conjugate poles and zeros at the same frequency ω. This means that the effect on magnitude cancel each other out, and we get a constant log-magnitude plot. The phase response at $\omega = 0$ is zero, as the contributions cancel each other out. As we move past $\omega = 1$ where the conjugates are located, the phase moves in the negative direction faster, but eventually settles back at 0 as we move farther and the contributions again cancel each other out. The zeros correspond to differentiation, hence we can derive the impulse response from the previous case.

	$h(t)$	Magnitude	Angle
PZ diagram 1 =	3	5	4
PZ diagram 2 =	1	2	3
PZ diagram 3 =	4	3	6
PZ diagram 4 =	2	6	2
PZ diagram 5 =	6	1	1
PZ diagram 6 =	5	4	5

Problem 4.

a. Since the signal is two-sided, only B could represent the given signal. We write the signal as

$$x[n] = a^{-|n|}, \qquad 0 < a < 1.$$

Then the Z-transform is given by:

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} a^{-n} z^{-n} + \sum_{n=-\infty}^{-1} a^{n} z^{-n} \\ &= \frac{(a - 1/a)z}{(z-a)(z-1/a)}, \quad \text{ROC:} 1/a < |z| < a. \end{aligned}$$

The Z-transform then has poles at $z = a$, $z = 1/a$, and a zero at $z = 0$. Therefore the answer is none of the above.

b. The signal $x[n] = 1$ can be written as $x[n] = (1)^{n}$. Therefore $z = 1$ must be in the ROC on the z-plane. This is possible for A,B,C depending on the choice of ROC but not possible for D. Therefore the answer is that $x[n] = 1$ for all n cannot be an eigenfunction of all 4 systems.

c. For stability, the ROC has to include the unit circle. A causal system must also be right-sided. A right-sided system has its ROC to be of form $|z| > a$, therefore only A,C can be such a system. A causal system has $h[n] = 0$, $n < 0$, therefore its Z-transform has only negative powers of z, meaning that it has more poles than zeros.

Problem 5. For this problem consider the following input signal:

$$x[n] = \cos(\Omega_1 n) = \frac{\exp(j\Omega_1 n) + \exp(-j\Omega_1 n)}{2} = \frac{1}{2}\left\{\exp(j\Omega_1)^n + \exp(-j\Omega_1)^n\right\}.$$

This signal consists of eigenfunctions of an LTI system, which are $\exp(j\Omega_1)$ and $\exp(-j\Omega_1)$. Both of these are points on the unit circle, with angles given by $\pm\Omega_1$. Given a system with two poles, suppose the two poles are at $z_A = r\exp(j\Omega_0)$ and $z_B = r\exp(j\Omega_0)$. Then we can write

$$\frac{Y(z)}{X(z)} = \frac{1}{(z - z_A)(z - z_B)} = \frac{1}{z^2 - (z_A + z_B)z + z_A z_B},$$

or in other words, as a difference equation:

$$z^2 Y(z) - (z_A + z_B)zY(z) + z_A z_B Y(z) = X(z),$$

$$y[n+2] - (z_A + z_B)y[n-1] + z_A z_B y[n] = x[n].$$

a. Consider a system given by two poles at $z = r\exp(j\Omega_0)$. The frequency response of this system will have its magnitude peak at frequency $\pm\Omega_0$. Hence the signal $x[n]$ given above will be amplified more when $\Omega_0 \approx \Omega_1$. We use this to discriminate the three possible tones, by setting $\Omega_0 = 2\pi 0.1209$.

```
r = 0.99;
w_0 = 2*pi*0.1209;
zA = r*exp(j*w_0);
zB = r*exp(-j*w_0);
y = zeros(1,3000);
for n = 1:(length(y)-2)
    y(n+2) = (zA+zB)*y(n+1) - (zA*zB)*y(n) + x(n);
end
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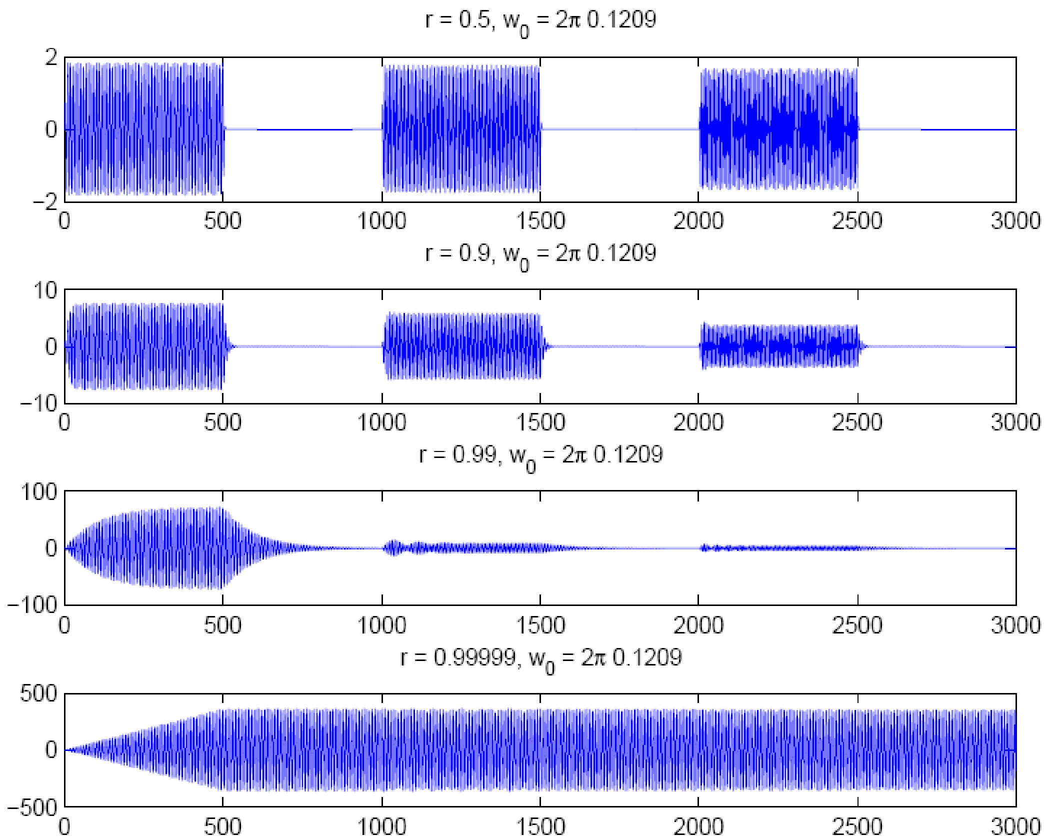
b. The closer Ω_0 is to $2\pi 0.1209$, the larger the magnitude at the output corresponding to that tone is. If Ω_0 approaches one of the other tones, then the magnitude at the output corresponding to those respective tones are larger.

c. We wish to choose r to be close to 1 but without making the system unstable. Choosing r to be between 0.9 and 0.99 gives good discrimination of the tones without incurring instability. This is shown in the following figure. When r is chosen to be too close to 1, or greater than 1, the system becomes unstable.



r = 0.5, w$_0$ = 2π 0.1209
2
0
−2
0 500 1000 1500 2000 2500 3000
r = 0.9, w$_0$ = 2π 0.1209
10
0
−10
0 500 1000 1500 2000 2500 3000
r = 0.99, w$_0$ = 2π 0.1209
100
0
−100
0 500 1000 1500 2000 2500 3000
r = 0.99999, w$_0$ = 2π 0.1209
500
0
−500
0 500 1000 1500 2000 2500 3000