Signals and Systems

Solutions to Homework Assignment #4

Problem 1.

(a) We can derive that

$$\begin{aligned} e^{-|t|} &\Leftrightarrow \frac{2}{1+\omega^2} \\ \cos(2t) &\Leftrightarrow \pi(\delta(\omega-2)+\delta(\omega+2)). \end{aligned}$$

Therefore,

$$e^{-|t|}\cos(2t) \Leftrightarrow \frac{1}{1+(\omega-2)^2}+\frac{1}{1+(\omega+2)^2}.$$

(b)

$$\begin{aligned} \frac{\sin(\pi t)}{\pi t} &\Leftrightarrow 1_{[-\pi,\pi)} \\ \frac{\sin(2\pi t)}{\pi t} &\Leftrightarrow 1_{[-2\pi,2\pi)} \\ \frac{\sin(2\pi(t-1))}{\pi(t-1)} &\Leftrightarrow 1_{[-2\pi,2\pi)}\mathrm{e}^{-j\omega} \\ X_2(j\omega) &= \begin{cases} \frac{\mathrm{e}^{-j\omega/2}}{\pi j}\cos(\omega/2), & -3\pi \le \omega \le \pi \\ \frac{-\mathrm{e}^{-j\omega/2}}{\pi j}\cos(\omega/2), & \pi \le \omega \le 3\pi \\ 0, & \text{else.} \end{cases} \end{aligned}$$

(c) First consider the basic signal $t[u(t)-u(t-1)]$, and use to derive the later part.

$$\begin{aligned} tx(t) &\Leftrightarrow j\frac{d}{d\omega}X(j\omega) \\ u(t) &\Leftrightarrow \frac{1}{j\omega}+\pi\delta(\omega) \end{aligned}$$

Putting it together,

$$X_2(j\omega) \Leftrightarrow \frac{2(1-\mathrm{e}^{-j\omega})}{j\omega^3}-\frac{2\mathrm{e}^{-j\omega}}{\omega^2}+\frac{1}{j\omega}+\pi\delta(\omega)+2\pi j\frac{d}{d\omega}\delta(\omega).$$

(d) Realize that $(1-|t|)u(t+1)u(1-t) = 1_{[-0.5,0.5)} * 1_{[-0.5,0.5)}$. Then,

$$(1-|t|)u(t+1)u(1-t) \Leftrightarrow \left(\frac{2\sin(\omega/2)}{\omega}\right)^2 = \frac{4}{\omega^2}\sin^2(\omega/2).$$

Problem 2.

In this section, all Fourier transforms are periodic with period 2π.

(a)

$$\begin{aligned} a^{-|n|} &\Leftrightarrow a^{-n}u[n] + a^{n}u[-n-1] \\ x_1[n] &\Leftrightarrow \frac{e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}} + \frac{0.5}{1-\frac{1}{2}e^{+j\omega}}. \end{aligned}$$

(b)

$$\begin{aligned} (n+1)a^{n}u[n] &\Leftrightarrow \frac{e^{-j\omega}}{(1-ae^{-j\omega})^2}. \\ x_2[n] &\Leftrightarrow \frac{1}{3}\frac{e^{-j\omega}}{(1-\frac{1}{3}e^{-j\omega})^2}. \end{aligned}$$

(c) Similar to previous problem 1, write as convolution of two rectangular signals,

$$\begin{aligned} \sum_{k=-5}^{5}\delta[n-k] &\Leftrightarrow \frac{\sin\omega(2.5)}{\sin\omega/2} \\ x_3[n] &\Leftrightarrow \left(\frac{\sin\omega(2.5)}{\sin\omega/2}\right)^2. \end{aligned}$$

Alternatively,

$$\begin{aligned} X_u(e^{j\omega}) &= (1/(1-e^{-j\omega})^2) + \sum_{k=-\infty}^{\infty}\pi\frac{d}{dt}(\delta(\omega-2\pi k)) \\ X_3(e^{j\omega}) &= 2(\cos(5\omega)-1)X_u(e^{j\omega}) \end{aligned}$$

(d) We get rectangles of width $2\pi/5$ centered around $-\pi/2$ and $\pi/2$.

$$x_4[n] \Leftrightarrow 1_{[-7\pi/10,-3\pi/10)} + 1_{[3\pi/10,7\pi/10)}$$

Problem 3.

First derive the Fourier transform of the template signal:

$$x_o(t) \Leftrightarrow X_o(j\omega) = \frac{1}{1+j\omega}\left(1-e^{-(1+j\omega)}\right).$$

For the others,

(a)

$$\begin{aligned} x_1(t) &= x_0(t) + x_o(-t) \\ X_1(j\omega) &= X_o(j\omega) + X_o(-j\omega) \\ &= \frac{2Re\{\left(1-e^{-(1+j\omega)}\right)(1-j\omega)\}}{1+\omega^2}. \end{aligned}$$

(b)

$$\begin{aligned} x_2(t) &= x_0(t) - x_0(-t) \\ X_2(j\omega) &= X_o(j\omega) - X_o(-j\omega) \\ &= \frac{2jIm\{\left(1-e^{-(1+j\omega)}\right)(1-j\omega)\}}{1+\omega^2}. \end{aligned}$$

(c)

$$\begin{aligned} x_3(t) &= x_0(t) + x_0(t+1) \\ X_3(j\omega) &= X_o(j\omega) + X_o(j\omega)e^{j\omega} \\ &= \left(1+e^{j\omega}\right)\frac{\left(1-e^{-(1+j\omega)}\right)}{1+j\omega}. \end{aligned}$$

(d)

$$\begin{aligned} x_4(t) &= tx_o(t) \\ X_4(j\omega) &= j\frac{d}{d\omega}\frac{1}{1+j\omega}\left(e^{-(1+j\omega)}-1\right). \end{aligned}$$

Problem 4. O&W5.24(a),(b),(c),(d)

(a) Signal $x[n]$ is a delayed version of another signal that is real and even. Hence 3 is true, since delay corresponds to phase shift in the Fourier domain. A quick check reveals that $x[0]$ is not zero, hence 4 is not true. Property 5 always holds for DT signals. The DC term is not zero, so 6 is not true. Answer: 3, 5.

(b) We can quickly derive $X(j\omega) = 2j\sin(\omega)$. Moreover, the signal is a delayed version of another signal that is real and even. Answer: 1, 3, 4, 5, 6.

(c) We can derive,

$$X(j\omega) = \frac{e^{j\omega}}{e^{j\omega} - \frac{1}{2}}.$$

Answer: 5.

(d) Derive the Fourier transform,

$$X(j\omega) = \frac{1-(1/2)^2}{1-\cos(\omega)+1/4}.$$

Since $x[n]$ is real and even, its FT is also real and even. Answer: 2, 3, 5.

Problem 5. O&W 4.25

Let $X(j\omega)$ denote the Fourier transform of the signal $x(t)$.

(a) $X(j\omega)$can be written as $A(j\omega)e^{j\theta(j\omega)}$ where $A(j\omega)$ and $\theta(j\omega)$ are real. Find $\theta(j\omega)$. $X(j\omega) = A(j\omega)e^{j\theta(j\omega)} = |X(j\omega)|e^{j\angle X(j\omega)}$

$x(t)$ is symmetric about $t = 1$ (from Figure P4.25, redrawn above).

$\Rightarrow$ a signal $g(t) = x(t+1)$ is symmetric about $t=0$

$\Rightarrow g(t)$ is even $\rightarrow G(j\omega)$ is real.

$$x(t) = g(t-1),\ X(j\omega) = G(j\omega)e^{-j\omega(1)} = A(j\omega)e^{j\theta(j\omega)}$$

Before going through the last step to find $\theta(j\omega)$, let's underline an important subtlety: If we assume that $A(j\omega) = |X(j\omega)|$ and $\theta(j\omega) = \angle X(j\omega)$, then it might be impossible to find $\theta(j\omega)$ without actually computing $\angle G(j\omega)$. However, we are supposed to solve the problem without explicitly evaluating any Fourier Transforms. The reason is that although $G(j\omega)$ is real, that doesn't mean $\angle G(j\omega) = 0$. This is because $G(j\omega)$ might have a negative value in some range of ω. In this case, $\angle G(j\omega) = \pm\pi$, because the magnitude, by definition, has to be positive. Luckily, there is a way out of this dilemma: the only restriction we have is that $A(j\omega)$ and $\theta(j\omega)$ be real. If we include the sign of $X(j\omega)$ in $A(j\omega)$, in which case $A(j\omega)$ is still real but not necessary positive, then we are all set. In this case

$$X(j\omega) = G(j\omega)e^{-j\omega(1)} = A(j\omega)e^{j\theta(j\omega)} \text{ and } G(j\omega) \text{ is real}$$

a possible matching of the LHS and the RHS is:

$$A(j\omega) = G(j\omega) \text{ and } e^{j\theta(j\omega)} = e^{-j\omega(1)} = e^{-j\omega}$$

$$\rightarrow \theta(j\omega) = -\omega.$$

(b) Find $X(j0)$

$$\begin{aligned} X(j0) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega(0)}dt = \int_{-\infty}^{\infty} x(t)dt = (\text{total area under the curve}) \\ X(j0) &= 2[3-(-1)] - (1)(1) = 7. \end{aligned}$$

(c) Find $\int_{-\infty}^{\infty} X(j\omega)d\omega$.

$$\because x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega \rightarrow \int_{-\infty}^{\infty} X(j\omega)d\omega = 2\pi x(0) = 2\pi(2) = 4\pi.$$

(d) Evaluate $\int_{-\infty}^{\infty} X(j\omega)\frac{2\sin\omega}{\omega}e^{j2\omega}d\omega.$

Let $Y(j\omega) = \frac{2\sin\omega}{\omega}e^{j2\omega}$,therefore:

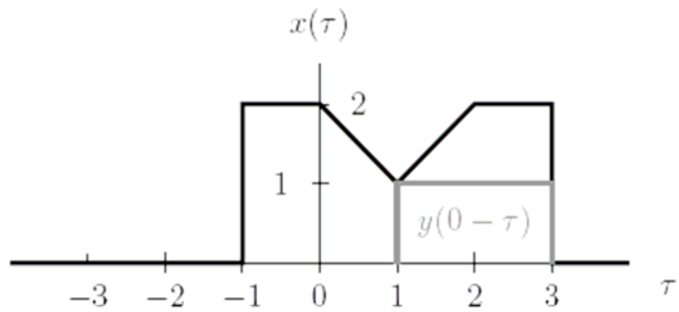
$$\begin{aligned} \int_{-\infty}^{\infty} X(j\omega)\frac{2\sin\omega}{\omega}e^{j2\omega}d\omega &= \int_{-\infty}^{\infty} X(j\omega)Y(j\omega)d\omega \\ &= 2\pi\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y(j\omega)e^{j\omega(0)}d\omega\right] \\ &= 2\pi\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)Y(j\omega)e^{j\omega t}d\omega\right]_{t=0} \\ &= 2\pi\left[x(t) * y(t)\right]_{t=0} \quad (\text{see O \& W, Sec. 4.4, p.314}) \end{aligned}$$

Knowing that $g(t) = \begin{cases} 0, & |t| < T_1 \\ 1, & |t| > T_1 \end{cases} \Leftrightarrow \frac{2\sin\omega T_1}{\omega}$

(From O & W, Table 4.2, p.329 or Example 4.4, p. 293), therefore:

$$y(t) = \begin{cases} 1, & -3 < t < -1 \\ 0, & \text{otherwise} \end{cases} \Leftrightarrow Y(j\omega) = \frac{2\sin\omega(1)}{\omega} e^{j\omega(2)}$$

$$\rightarrow x(t) * y(t)|_{t=0} = \int_1^3 x(\tau)d\tau = 3.5$$ (as seen in the figure, below, depicting the convolution)



$$\rightarrow \int_{-\infty}^{\infty} X(j\omega)\frac{2\sin\omega}{\omega} e^{j2\omega} d\omega = 2\pi(3.5) = 7\pi.$$

(e) Evaluate $\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$

From Parseval's theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \qquad \text{(O \& W, Section 4.3.7, p. 312)}$$

$$\begin{aligned} \rightarrow \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega &= 2\pi \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= 2\pi \left[\int_{-1}^{0} (2)^2 dt + \int_0^1 (2-t)^2 dt + \int_1^2 (t)^2 dt + \int_2^3 (2)^2 dt \right] \\ &= 2\pi \left[4 + \frac{(2-t)^3}{-3}\Big|_0^1 + \frac{t^3}{3}\Big|_1^2 + 4 \right] = 2\pi \left[4 - (\frac{1}{3} - \frac{8}{3}) + \frac{8}{3} - \frac{1}{3} + 4 \right] \\ &= 2\pi \left(\frac{38}{3}\right) = \frac{76\pi}{3}. \end{aligned}$$

Note that a useful Fourier transform property that we have used several times now is the following:

$$2\pi x(0) \Leftrightarrow \int_{-\infty}^{\infty} X(j\omega)d\omega, \text{ and by duality: } \int_{-\infty}^{\infty} x(t)dt \Leftrightarrow X(j0).$$

(f) Sketch the inverse Fourier transform of $Re\{X(j\omega)\}$.

The key to answering this question is recalling that the real part of a Fourier trans-form corresponds to the even part of the signal:

$Ev\{x(t)\} \Leftrightarrow Re\{X(j\omega)\}$, $Od\{x(t)\} \Leftrightarrow jIm\{X(j\omega)\}$ (O & W, Section 4.3.3, p. 303).

To resolve the even part, we use the following formulae:

$$x_e = \mathcal{E}v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)] \quad , \quad x_o = \mathcal{O}d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

You might want to double-check that $x_o(t)+x_e(t)= x(t)$. Note that the sketch for the odd part of $x(t)$ is included here for illustration purposes, and was not required in the original problem.

As a last note, one might be tempted to find the inverse Fourier transform of $Re\{X(j\omega)\}$ by shifting $x(t)$ to the left by one unit, and hence making it even symmetric which would have a real Fourier transform. It will be easy to convince yourself of the falsity of that method, if you remember that shifting a signal in time changes its Fourier transform's angle but does not affect the magnitude. This means that the real part of the Fourier transform of a signal changes with the time-shifting of that signal. (Hint: $Ae^{j\theta} = A\cos\theta + jA\sin\theta$).