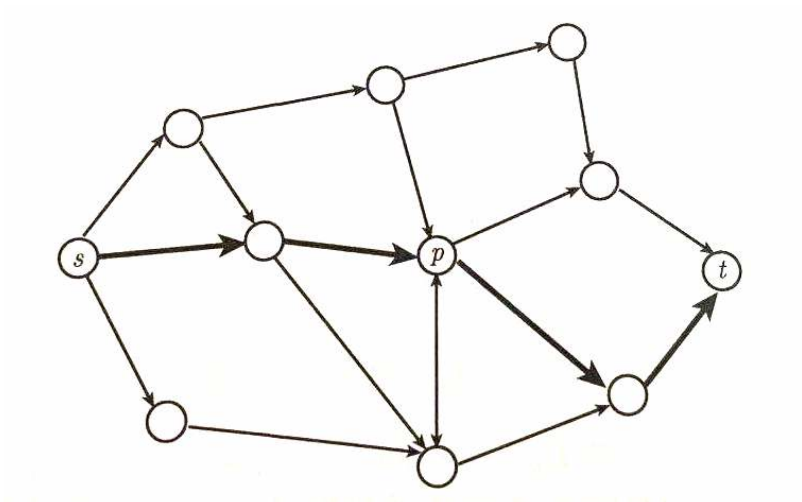
Basic Dynamic Programming

1 Shortest Path Problem

Given a directed graph which has no negative weight arc, find the shortest path from s to t.



Obsearvation: *If the s-t shortest path path passes by node p, then sub paths (s,p) and (p,t) are shortest path from s to p and from p to t respectively.*

So, we denote $D_k(j)$ = the length of shortest path from s to j, containing at most k arcs, then the recursive equation is:

$$D_k(j) = \min\left\{D_{k-1}(j), \min_{i \in V \setminus \{j\}} \{D_{k-1}(i) + c_{ij}\}\right\}$$

A simple demo written in C++ has been shown as below:

```
/**
 * Dijkstra's algorithm for shortest path problem.
 */
#include <iostream>
#include <vector>
#include <limits>
using namespace std;

class Solution {
public:
    /**
```

```cpp
 * @param graph: graph stored in adjcent matrix
 * @param s: start node
 * @param e: end node
 * @return: The maximum size
 */
int dijkstra(vector<vector<int>> graph, int s, int e) {
    int vertexNum = graph.size();
    vector<int> pre(vertexNum, 0);
    vector<bool> vis(vertexNum, 0);
    vector<int> dis(vertexNum, INT_MAX);

    // initialize distance
    for (int i = 0; i < vertexNum; ++i) {
        if (i == s) {
            dis[i] = 0;
        } else {
            dis[i] = graph[s][i];
        }
        if (graph[s][i] != -1) {
            pre[i] = s;
        } else {
            pre[i] = -1;
        }
    }
    vis[s] = true;
    // start for loop
    for (int i = 0; i < vertexNum; ++i) {
        int minimum = INT_MAX;
        int t;
        // find current shortest path
        for (int j = 0; j <vertexNum; ++j) {
            if (vis[j] == false && dis[j] < minimum) {
                t = j;
                minimum = dis[j];
            }
        }
        vis[t] = true;
        for (int j = 0; j < vertexNum; ++j) {
            if (vis[j] == false && dis[j] > dis[t] + graph[t][j]) {
                dis[j] = dis[t] + graph[t][j];
                pre[j] = t;
            }
        }
```

```
        }
        return dis[e];
    }
};
```

2 0-1 Knapsack Problem

The 0-1 knapsack problem can be modeled as:

$$f^* = \max\left\{\boldsymbol{c}^T\boldsymbol{x}\middle|\boldsymbol{a}^T\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \in \{0,1\}^n\right\}$$

Define $1 \le k \le n$ as **stages**, $0 \le \lambda \le b$ as **states**. Then, the optimal value function:

$$f_k(\lambda) = \max\left\{\sum_{j=1}^{k} c_j x_j \middle| \sum_{j=1}^{k} a_j x_j \le \lambda, \boldsymbol{x} \in \{0,1\}^n\right\}$$

And readily, $f^* = f_n(b)$. So, the recursive equation is:

$$f_k(\lambda) = \max\{f_{k-1}(\lambda), c_k + f_{k-1}(\lambda - a_k)\}$$

And initial conditions are

$$f_0(\lambda) = 0, f_1(\lambda) = \begin{cases} 0, & 0 \le \lambda < a_1 \\ \max\{c_1, 0\}, & \lambda \ge a_1 \end{cases}$$

A simple demo written in C++ is shown as below:

```
/**
 * Dynamic Programming for 0-1 knapsack problem
 */
#include <iostream>
#include <vector>
using namespace std;

class Solution {
public:
    /**
     * @param m: An integer m denotes the size of a backpack
     * @param W: Given n items with size W[i]
     * @param V: Given n items with value V[i]
     * @return: The maximum size
     */
```

```
    int zeroOneKnapsack(int m, vector<int> W, vector<int> V) {
        vector<int> f(m + 1, 0);
        for (int i = 0; i < W.size(); i++) {
            for (int j = m; j >= W[i]; j--) {
                f[j] = max(f[j], f[j - W[i]] + V[i]);
            }
        }
        return f[m];
    }
};
```

3 Integer Knapsack Problem

The integer knaspack problem can be modeled as:

$$f^* = \max\left\{\boldsymbol{c}^T\boldsymbol{x}\middle|\boldsymbol{a}^T\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \in \mathbb{Z}_+^n\right\}$$

where $c_j > 0, a_j > 0, j = 1, \ldots, n$. The optimal optimal value function:

$$g_r(\lambda) = \max\left\{\sum_{j=1}^{k} c_j x_j \middle| \sum_{j=1}^{k} a_j x_j \le \lambda, \boldsymbol{x} \in \mathbb{Z}_+^n\right\}$$

And readily, $f^* = g_n(b)$. So, the recursive equation is:

$$g_r(\lambda) = \max_{t=0,1,\ldots,\lfloor\lambda/a_r\rfloor} \{c_r t + g_{r-1}(\lambda - a_r t)\}$$

However, in this way, the time compelxity could be $O(nb^2)$. Noticed that,

$$\begin{aligned} x_r^* = 0 &\Rightarrow g_r(\lambda) = g_{r-1}(\lambda) \\ x_r^* \ge 1 &\Rightarrow x_r^* = 1 + t \Rightarrow g_r(\lambda) = g_r(\lambda - a_r) + c_r \end{aligned}$$

So, the better recursive equation is:

$$g_r(\lambda) = \max\{g_{r-1}(\lambda),\ g_r(\lambda - a_r) + c_r\}$$

And the time cpmlexity is $O(nb)$.

A simple demo written in C++ has been shown as below:

```
/**
 * Dynamic Programming for integer knapsack problem
 */
#include <iostream>
```

```cpp
#include <vector>
using namespace std;

class Solution {
public:
    /**
     * @param m: An integer m denotes the size of a backpack
     * @param W: Given n items with size W[i]
     * @param V: Given n items with value V[i]
     * @return: The maximum size
     */
    int integerKnapsack(int m, vector<int> W, vector<int> V) {
        vector<int> f(m + 1, 0);
        for (int i = 0; i < W.size(); i++) {
            for (int j = W[i]; j <= m;j++) {
                f[j] = max(f[j], f[j - W[i]] + V[i]);
            }
        }
        return f[m];
    }
};
```