Mathematical Proof

John Shea

February 03, 2019

Assignment #4

Question 1

Theorem 1. *If* $0 < \frac{1}{a} < \frac{1}{b}$, *then* $b < a$.

Proof. Suppose $0 < \frac{1}{a} < \frac{1}{b}$ Because $\frac{1}{a}$ and $\frac{1}{b}$ are both greater than zero, both a and b must be positive. Therefore, all terms can be multiplied by a and b without needing to change the direction of the inequality. Thus, when all three terms are multiplied by a, the result is $0 < 1 < \frac{a}{b}$. Then when all three terms are multiplied by b, the result is $0 < b < a$. Therefore b is less than a. $\square$

A more visual depiction:

$$
\begin{aligned}
&0 < \frac{1}{a} < \frac{1}{b} \\
&\qquad \equiv 0 \cdot a < \frac{1}{a} \cdot a < \frac{1}{b} \cdot a && \text{(multiply all terms by a)} \\
&\qquad \equiv 0 < 1 < \frac{a}{b} && \text{(simplify)} \\
&\qquad \equiv 0 \cdot b < 1 \cdot b < \frac{a}{b} \cdot b && \text{(multiply all terms by b)} \\
&\qquad \equiv 0 < b < a && \text{(simplify)} \\
&\qquad b < a && \text{(conclusion)}
\end{aligned}
$$

Question 2

Theorem 2. *Given* $A \subseteq B$ *and* $x \in A$, *if* $x \notin B \setminus C$ *then* $x \in C$.

Proof. Suppose $A \subseteq B$ and $x \in A$. Suppose $x \notin B \setminus C$ and $x \notin C$. Then the only way for x to not be a member of $B \setminus C$ would be to not be an element of B. If not a member of B, then x cannot be a member of any subset of B, which creates a contradiction with the given $x \in A$ and $A \subseteq B$. Therefore, x must be in C or in $B \setminus C$. Therefore, if $x \notin B \setminus C$, then $x \in C$. $\square$

Question 3

Theorem 3. *Given $A \setminus B \subseteq C \cap D$ and $x \in A$, if $x \notin C$, then $x \in B$*

Proof. Suppose $A \setminus B \subseteq C \cap D$ and $x \in A$. Suppose $x \notin C$ and $x \notin B$. It is given that $x \in A$. Subsequently, if $x \notin B$, then $x \in A \setminus B$. However, $A \setminus B$ is a subset of $C \cap D$. Any element which is a member of $C \cap D$, must be in C. Hence, we have a contradiction. Therefore, if $A \setminus B \subseteq C \cap D$ and $x \in A$, then $x \notin C$ implies $x \in B$. □

Question 4

Theorem 4. *If x is a negative real number and $x < \frac{1}{x}$, then $x < -1$.*

Proof. Suppose a is the absolute value of x. Then the given statement can be written as $-a < -\frac{1}{a}$. Then multiply both sides by $-a$ and we get $a^2 > 1$. We then take the principle square root of both sides (because a is known to be an absolute value, it is not necessary to take the plus or minus square root) and arrive at $a > 1$. Then we multiply both sides by -1 and get $-a < -1$. Finally, we insert x for $-a$ and we get the desired $x < -1$. □

A more visual depiction:

$$x < 0 \land x < \frac{1}{x}$$

$$\begin{aligned}
& a = |x| && \text{(stipulation)} \\
& -a < -\frac{1}{a} && \text{(restatement)} \\
& -a \cdot -a < -\frac{1}{a} \cdot -a && \text{(mutliply both sides by -a)} \\
& a^2 > 1 && \text{(result of multiplication)} \\
& \sqrt{a^2} > \sqrt{1} && \text{(take principal square root of both sides)} \\
& a > 1 && \text{(result)} \\
& a \cdot -1 < 1 \cdot -1 && \text{(mutliply both sides by -1)} \\
& -a < -1 && \text{(result)} \\
& x < -1 && \text{(substitute x for -1)}
\end{aligned}$$

Question 5

Theorem 5. *If x is a real number and $x \neq 2$, then there is a real number y such that $x = \frac{2y+1}{y-1}$.*

Proof. Suppose $\frac{2y+1}{y-1} = x$ and $x \neq 2$ Therefore, $\frac{2y+1}{y-1} \neq 2$. To solve, we first multiply both sides by $y - 1$ with the stipulation that $y \neq 1$ because $y = 1$ would create a division by zero, which is undefined. With that stipulation, we arrive at $2y + 1 \neq 2y - 2$. We then subtract 1 from both sides, resulting in $2y \neq 2y - 3$. We then subtract $2y$ from both sides, which results in $0 \neq -3$. Because $0 \neq -3$ for all values of y, this provides for an infinite number of solutions to the second step of the equation. However, we began withe the stipulation that $y \neq 1$. Therefore, we have shown that $\frac{2y+1}{y-1} \neq 2$ for all real numbers except 1, which proves that there exists a real number y, in fact infinitely many, for which $\frac{2y+1}{y-1} = x$ and $x \neq 2$. $\square$

A more visual depiction:

$$\frac{2y+1}{y-1} \neq 2, y \neq 1$$

$$\begin{aligned}
&\frac{2y+1}{y-1} \cdot (y-1) \neq 2 \cdot (y-1) && \text{(mutliply both sides by y-1)} \\
&2y + 1 \neq 2(y-1) && \text{(simplify)} \\
&2y + 1 \neq 2y - 2 && \text{(distribute)} \\
&2y + 1 - 1 \neq 2y - 2 - 1 && \text{(subtract 1 from both sides)} \\
&2y \neq 2y - 3 && \text{(simplify)} \\
&2y - 2y \neq 2y - 3 - 2y && \text{(subtract 2y from both sides)} \\
&0 \neq -3 && \text{(result)} \\
&y = (-\infty, 1) \cup (1, \infty) && \text{(solution)}
\end{aligned}$$

Question 6

Theorem 6. *If $\mathscr{F}$ is a non-empty family of sets, B is a set, and $\forall A \in \mathscr{F}(A \subseteq B)$, then $\cup\mathscr{F} \subseteq B$.*

Proof. Suppose $x \in \cup\mathscr{F}$, then x must be a member of at least one of the sets that comprise $\mathscr{F}$. As we are given that every set in $\mathscr{F}$ is a subset of B, x must exist in B. $\square$