Mathematical Proof

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Assignment #2

Question 1

a. Any of:
   - {x | x is a positive integer that is divisible by 5}
   - $\{x \in \mathbb{Z}^+ \mid x \equiv 0 \pmod 5\}$
   - $\{x \in \mathbb{Z}^+ \mid 5|x\}$
   - $\{5x \mid x \in \mathbb{Z}^+\}$

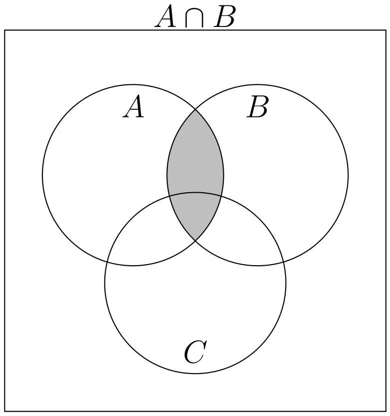
b. Any of:
   - {x | x is an integer between -4 and 3 inclusive}
   - $\{x \in \mathbb{Z} \mid -4 \leq x \leq 3\}$

c. Any of:
   - {x | x is a prime number}
   - $\{x \mid \pi(x)\}$

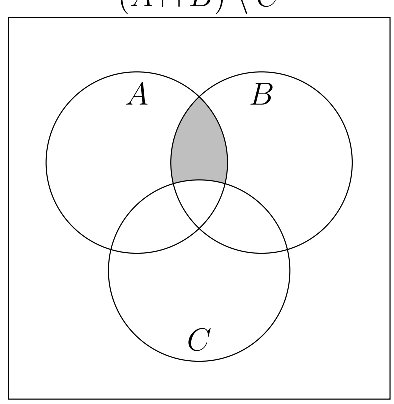
Question 2

a. To show that $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$, we begin by building the left side of the statement. First, we look at $(A \cap B)$, and the we remove C.

   $A \cap B$

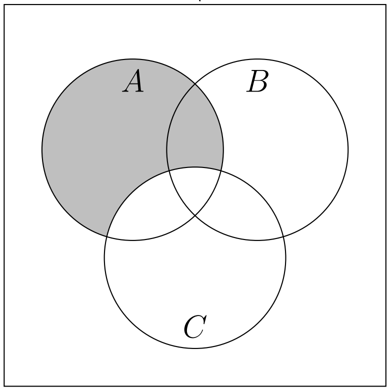
   

   $(A \cap B) \setminus C$

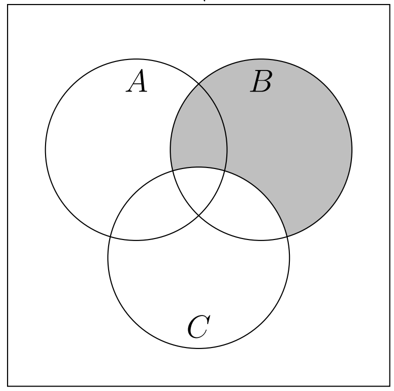
   

   We then look at $A \setminus C$ and $B \setminus C$, and then take their intersection.

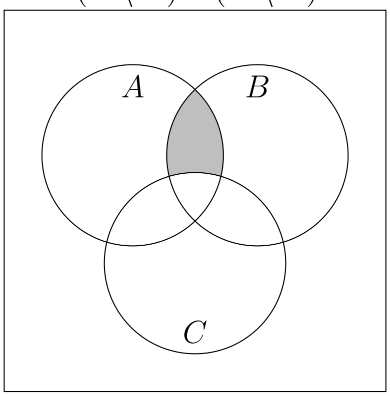
$A \setminus C$



$B \setminus C$



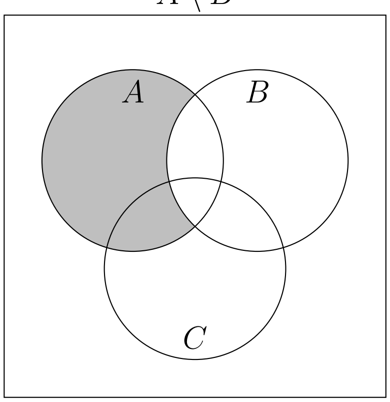
$(A \setminus B) \cap (B \setminus C)$



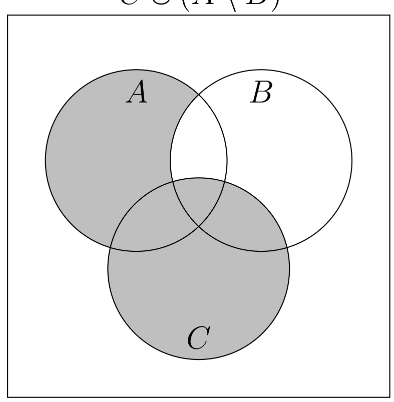
Clearly, the above Venn Diagrams demonstrate that $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$.

b. To show that $C \cup (A \setminus B) = ((A \cup C) \setminus B) \cup (B \cap C)$, we begin with the left side. First, we diagram $A \setminus B$, and then we add C.

$A \setminus B$

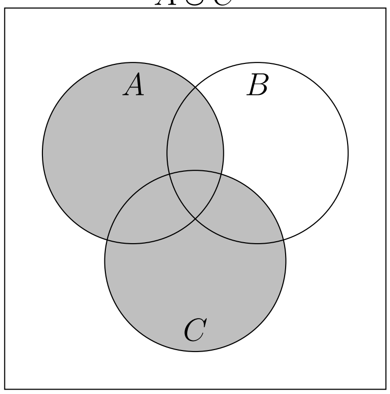


$C \cup (A \setminus B)$

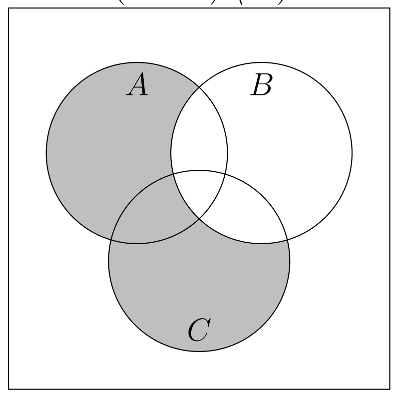


Next, we move to the right side of the statement. We will first diagram $A \cup C$, then remove B. Next, we will diagram $B \cap C$, and finally we will put them together.

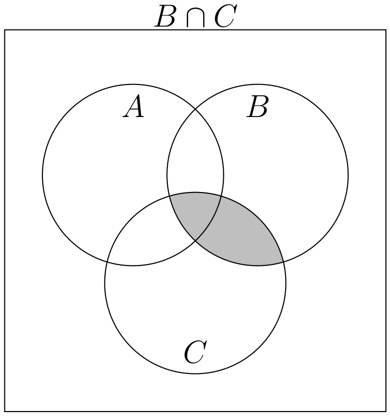
$A \cup C$



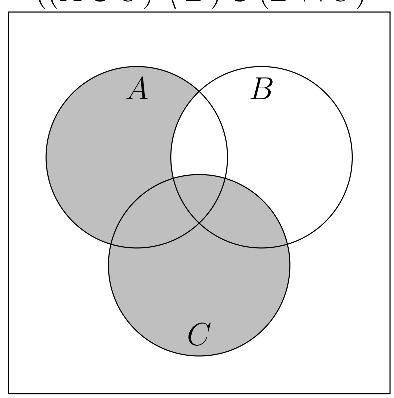
$(A \cup C) \setminus B)$



$B \cap C$



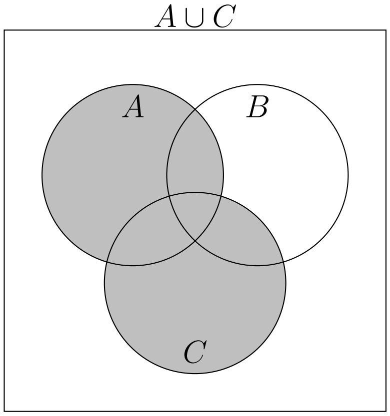
$((A \cup C) \setminus B) \cup (B \cap C)$



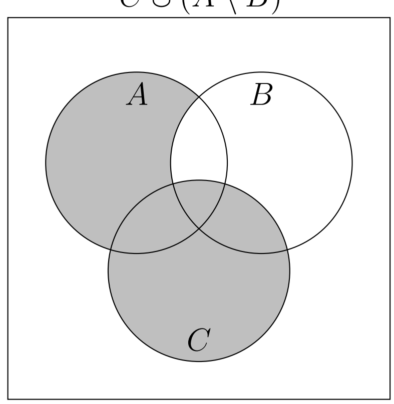
From the above diagrams, it's easy to see that $C \cup (A \setminus B) = ((A \cup C) \setminus B) \cup (B \cap C)$

Question 3

$A \cup C$



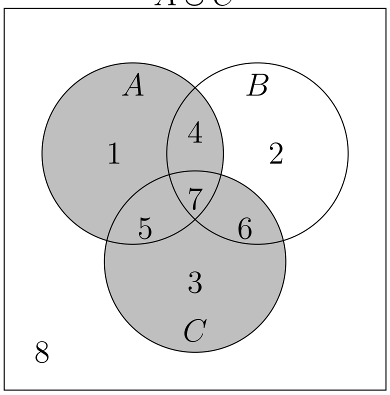
$C \cup (A \setminus B)$



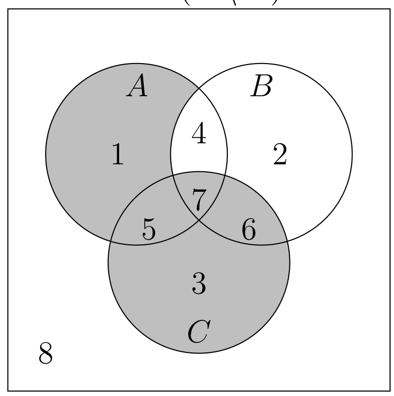
Explanation

$A \cup B$ is similar to $C \cup (A \setminus B)$, with the exception of any element that is in A and B, but not C (i.e. $(A \cap B) \setminus C$). For example: if we have sets $A = \{1, 4, 5, 7\}$, $B = \{2, 4, 6, 7\}$, and $C = \{3, 5, 6, 7\}$. $A \cup C = \{1, 3, 4, 5, 6, 7\}$. Whereas $C \cup (A \setminus B) = \{1, 3, 5, 6, 7\}$. This is created by taking $A \setminus B = \{1, 5\}$ and adding C. The difference between the two sets is $A \cap B \setminus C = \{4\}$. All of which is depicted below.

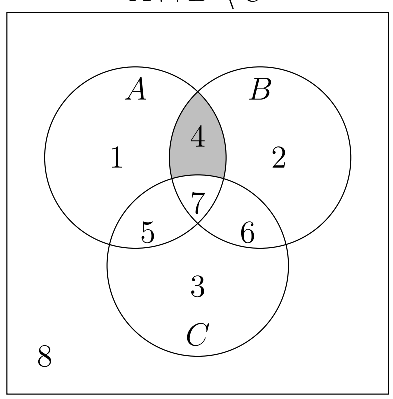
$A \cup C$



$C \cup (A \setminus B)$



$A \cap B \setminus C$



Question 4

a. $W \rightarrow (S \wedge \neg M)$

b. $S \rightarrow (W \wedge \neg M)$

- This statement is different than a and c in that it specifies what conditions can be determined from the knowledge that I will go shopping. While a and c detail what information can be gleaned from the information that I will go shopping.

c. $W \rightarrow (S \wedge \neg M)$

- equivalent to statement a. Both statements allow for the conclusion that I will got shopping and I will not go to a movie based on the information that it is Wednesday.

Question 5

$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$

$$
\begin{aligned}
& P \leftrightarrow Q && (Begin) \\
& \equiv (P \rightarrow Q) \wedge (Q \rightarrow P) && \text{(definition of bi-directionality)} \\
& \equiv (\neg P \vee Q) \wedge (\neg Q \vee P) && \text{(Conditional Law)} \\
& \equiv (Q \vee \neg P) \wedge (P \vee \neg Q) && \text{(Commutative Law)} \\
& \equiv (\neg Q \rightarrow \neg P) \wedge (\neg P \rightarrow \neg Q) && \text{(Conditional Law)} \\
& \equiv \neg P \leftrightarrow \neg Q && \text{(definition of bi-directionality)}
\end{aligned}
$$

This can also be shown with truth tables.

$P \leftrightarrow Q$

P	Q	$P \rightarrow Q$	$Q \leftarrow P$	$P \leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

$\neg P \leftrightarrow \neg P$

$\neg P$	$\neg Q$	$\neg P \rightarrow \neg Q$	$\neg Q \leftarrow \neg P$	$\neg P \leftrightarrow \neg Q$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

$P \leftrightarrow Q \wedge \neg P \leftrightarrow \not Q$

P	Q	$P \leftrightarrow Q$	$\neg P \leftrightarrow \neg Q$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

Question 6

a. $P \vee Q \equiv \neg P \to Q$

- By the rules of implication, $\neg P \to Q$ will be true whenever P is true. This accounts for the left side of the statement $P \vee Q$. When P is false, the implication requries Q to be true in order to be a true statement (which is the conditional law), thus perfectly replicating the original disjunct term, as the folowing truth table demonstrates.

P	Q	$P \vee Q$	$\neg P \to Q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

b. $P \wedge Q \equiv \neg(P \to \neg Q)$

- The $\neg$ on the outside of the parenthesis reverses the meaning of the inside term (i.e. $P \to Q$). Therefore, to make this statement equivelant to $P \wedge Q$, this statement needs to return fase only when P and Q are both true. $P \leftarrow Q$ will return true whenever P is false, so this gaurentees that P must be true. By setting the right-hand side of the implication to $\neg Q$, This gaurantees that if P is true and Q false $P \to Q$ will return true. Therefore, the only way the inside statement fails is if P and Q are both true. Thus return a true for the overall statement. This is best demonstrated with the following equivalences:

$$
\begin{aligned}
& P \wedge Q && \text{(Begin)} \\
& \equiv \neg(\neg P \vee \neg Q) && \text{(Negation Rule)} \\
& \equiv \neg(P \to \neg Q) && \text{(Conditional Law)}
\end{aligned}
$$