Mathematical Proof

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Assignment #6

Question 1

Theorem 1. *If A is a set and $\{B_i | i \in I\}$ is an indexed family of sets. $A \times (\cup_{i \in I} B_i) = \cup_{i \in I})(A \times B_i)$.*

Proof. Suppose $(a, b) \in A \times (\cup_{i \in I} B_i)$. Then $a \in A$ and $b \in \cup_{i \in I} B_i$. Because A is not affected by i, then for every $(a, b) \in A \times (\cup_{i \in I} B_i)$, $a \in (\cup_{i \in I} A)$ and $b \in (\cup_{i \in I} B_i)$. Therefore, $A \times (\cup_{i \in I} B_i) \subseteq \cup_{i \in I}(A \times B_i)$.

Suppose $(a, b) \in \cup_{i \in I}(A \times B_i)$. Then $a \in (\cup_{i \in I} A)$ and $b \in (\cup_{i \in I} B_i)$. Because A is unaffected by i, then for every $(a, b) \in \cup_{i \in I}(A \times B_i)$, $a \in A$ and $b \in (\cup_{i \in I} B_i)$ Therefore $\cup_{i \in I}(A \times B_i) \subseteq A \times (\cup_{i \in I} B_i)$.

Taken together this proves that $A \times (\cup_{i \in I} B_i) = \cup_{i \in I}(A \times B_i)$ □

Question 2

Theorem 2.

Proof. □

a $S^{-1} \circ R$

$S = \{(4, a), (4, d), (5, b), (5, c)\}$, and is a relation from B to C, so $S^{-1} = \{(a, 4), (d, 4), (b, 5), (c, 5)\}$ and is a relation from C to B. $R = \{(1, b), (2, a), (2, b), (2, c), (3, d)\}$ and is a relation from A to C. Therefore $S^{-1} \circ R = \{(1, 5), (2, 4), (2, 5), (3, 4)\}$ and is a relation from A to B. This results because we now have a relation that begins in A, connects through C and arrives at B. The final relation $S^{-1} \circ R$ becomes a set of ordered pairs with elements of A as the first coordinate and elements of B as the second.

b $R^{-1} \circ S$

$R = \{(1, b), (2, a), (2, b), (2, c), (3, d)\}$ and is a relation from A to C, so $R^{-1} = \{(b, 1), (a, 2), (b, 2), (c, 2), (d, 3)$ and is a relation from C to A. $S = \{(4, a), (4, d), (5, b), (5, c)\}$ and is a relation from B to C. Therefore $R^{-1} \circ S = \{(4, 2), (4, 3), (5, 1), (5, 2)\}$ is a relation from B to A. This results because we now have a relation that begins in

B, connects through C and arrives at A. The final relation $R^{-1} \circ S$ becomes a set of ordered pairs with elements of B as the first coordinate and elements of A as the second.

Question 3

Theorem 3.

Proof. □

a $R = Dom(R) \times Ran(R)$

Suppose $A = \{1,2,3\}$ and $B = \{a,b,c\}$ and $R = \{(1,a),(2,b),(3,c)\}$. Then R is a relation from A to B. In this scenario, $Dom(R) = \{1,2,3\}$ and Ran(R)=$\{a,b,c\}$. The Cartesian product of $Dom(R)$ and $Ran(R)$ would feature all possible combinations of the two sets. Hence, $Dom(R) \times Ran(R) = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c)\}$. Therefore, R does not necessarily equal $Dom(R) \times Ran(R)$, and the statement $R = Dom(R) \times Ran(R)$ is untrue. It would be true to state that $R \subseteq Dom(R) \times Ran(R)$.

b $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

Suppose $(b,a) \in (R \cap S)^{-1}$. Then $(a,b) \in R \cap S$. For any $(a,b) \in R, (b,a) \in R^{-1}$ and for any $(a,b) \in S, (b,a) \in S^{-1}$. Hence, $(b,a) \in R^{-1}$ and S^{-1}, and consequently $(b,a) \in R^{-1} \cap S^{-1}$. Therefore, $(R \cap S)^{-1} \subseteq R^{-1} \cap S^{-1}$

Suppose $(b,a) \in R^{-1} \cap S^{-1}$. Then $(b,a) \in R^{-1}$ and $(b,a) \in S^{-1}$. Hence, $(a,b) \in R$ and $(a,b) \in S$, which means $(a,b) \in (R \cap S)$. Hence, $(b,a) \in (R \cap S)^{-1}$. Therefore, $R^{-1} \cap S^{-1} \subseteq (R \cap S)^{-1}$. Taken together this proves that $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.

Question 4

The graph depicts the following relation: $\{(a,a),(a,b),(a,c),(b,d),(d,b),(d,a)\}$.

That the relation is not reflexive is proved by the fact that there is no $(b,b),(c,c)$, or (d,d) in the relation. The only example of a reflexive relationship is (a,a). But the definition of reflexive relation requires reflexivity for all elements of the relation.

That the relation is not symmetric is proved by the fact that there is an (a,c) yet no (c,a), an (a,b) and no (b,a), and a (d,a), but no (a,d). There is an example of a symmetric relationships $(b,d),(d,b)$. But once again, symmetry calls for all relations to be reciprocated, and the counter-examples clearly belie symmetry.

Finally, that the relation is not transitive is proved by the fact that there is an (a,b) and a (b,d), but no (a,d). There is also a (d,a) and (a,c), yet there is not a (d,c). There is, however, a (a,b) and a (d,b), but again, all pairs need to be transitive, for the relation to meet the definition of a transitive relation.

Question 5

Theorem 4. *R is a partially ordered set.*

Proof. R is reflexive because for any x, $f(x) = f(x)$ and hence renders $f(x) \leq f(x)$ is true. Therefore for an decimal x, $(x, x) \in R$, and R is reflexive.

R is also transitive because for any $(a, b) \in R$ and $(b, c) \in R$, $f(a) \leq f(b) \leq f(c)$. Hence $f(a) \leq f(c)$. Therefore $(a, c) \in R$ and R is transitive.

Finally, R is antisymmetric because for any $(a, b) \in R$, if $a \neq b$ (which would be reflexivity), then $f(a) < f(b)$. Hence $f(b) > f(a)$, and therefore $(b, a) \notin R$.

Taken together, R is reflexive, transitive and antisymmetric and therefore meets every requirement of a partially ordered relation. □

Question 6

Theorem 5. *The relation R on $\mathbb{Z}$, such that $(a, b) \in R$ if and only if a and b, when written out, have the same number of $5s$ is an equivalence relation.*

Proof. Suppose $\{B_i | i \in \mathbb{Z}^+\}$ is an indexed family of sets such that B_1 contains all of the infinitely many numbers which contain one 5, and B_2 contains all of the infinitely many numbers which contain two $5s$, and so on for an infinitely number of sets B_i. Clearly, no number can reside in more than one of the subsets of B. Therefore, we can restate the definition of R as $(a, b) \in R$ if, and only if, $a \in B_i$ and $b \in B_i$ for some unique $i \in \mathbb{Z}^+$

Relation R is such that if, and only if, for a given (a, b), both a and b are taken from the same subset of B, then $(a, b) \in R$. Thus, for every a in a given subset of B, (a, a) would always meet the criteria for being an element of R and therefore, R is reflexive.

Also, because for any given $(a, b) \in R$, both a and b reside in the same subset of B, (b, a) would meet the criteria for inclusion in R. Therefore, R is symmetric.

Lastly, if (a, b) and (b, c) are elements of R, then a, b, and c are all elements of the same subsets of B. Therefore (a, c) meets the requirement of a transitive relation.

Taken together, R is proven to be reflexive, symmetric, and transitive; and therefore, R is an equivalence relation. □