Mathematical Proof

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Assignment #6

Question 1

Theorem 1. *If A is a set and $\{B_i | i \in I\}$ is an indexed family of sets. $A \times (\cup_{i \in I} B_i) = \cup_{i \in I})(A \times B)$.*

Proof. Suppose $(a, b) \in A \times (\cup_{i \in I} B_i)$. Then $a \in A$ and $b \in \cup_{i \in I}$. so $A \times (\cup_{i \in I} B_i) \subseteq \cup_{i \in I})(A \times B)$.

Suppose $(x, y) \in \cup_{i \in I}(A \times B)$. Then $x \in$ So, $\cup_{i \in I})(A \times B) \subseteq A \times (\cup_{i \in I} B_i)$. Therefore $A \times (\cup_{i \in I} = \cup_{i \in I})(A \times B)$. □

Question 2

Theorem 2.

Proof. □

a $S^{-1} \circ R$

$S = \{(4, a), (4, d), (5, b), (5, c)\}$, and is a relation from B to C, so $S^{-1} = \{(a, 4), (d, 4), (b, 5), (c, 5)\}$ and is a relation from C to B. $R = \{(1, b), (2, a), (2, b), (2, c), (3, d)\}$ and is a relation from A to C. Therefore $S^{-1} \circ R = \{(1, 5), (2, 4), (2, 5)\}$ and is a relation from A to B. This results because we now have a relation that begins in A, connects through C and arrives at B. The final relation $S^{-1} \circ R$ becomes a set of ordered pairs that have elements of A as the first coordinate and elements of B as the second.

b $R^{-1} \circ S$

$R = \{(1, b), (2, a), (2, b), (2, c), (3, d)\}$ and is a relation from A to C, $Rso^{-1} = \{(b, 1), (a, 2), (b, 2), (c, 2), (d, 3)$ and is a relation from C to A. $S = \{(4, a), (4, d), (5, b), (5, c)\}$ and is a relation from B to C. Therefore $R^{-1} \circ S$ is a relation from B to A. This results because we now have a relation that begins in B, connects through C and arrives at A. The final relation $R^{-1} \circ S$ becomes a set of ordered pairs that have elements of B as the first coordinate and elements of A as the second.

Question 3

Theorem 3.

Proof. □

a $R = Dom(R) \times Ran(R)$

Suppose $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ and $R = \{(1, a), (2, b), (3, c)\}$. Then R is a relation from A to B. In this scenario, $Dom(R) = \{1, 2, 3\}$ and Ran(R)=$\{a, b, c\}$. The Cartesian product of $Dom(R)$ and $Ran(R)$ would feature all possible combinations of the two sets. Hence, $Dom(R) \times Ran(R) = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$. Therefore, R does not necessarily equal $Dom(R) \times Ran(R)$, and the statement $R = Dom(R) \times Ran(R)$ is untrue. It would be true to state that $R \subseteq Dom(R) \times Ran(R)$.

b $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

Suppose $(b, a) \in (R \cap S)^{-1}$. Then $(a, b) \in R \cap S$. For any $(a, b) \in R, (b, a) \in R^{-1}$ and for any $(a, b) \in S, (b, a) \in S^{-1}$. Hence, $(b, a) \in R^{-1}$ and S^{-1}, and consequently $(b, a) \in R^{-1} \cap S^{-1}$. Therefore, $(R \cap S)^{-1} \subseteq R^{-1} \cap S^{-1}$

Suppose $(b, a) \in R^{-1} \cap S^{-1}$. Then $(b, a) \in R^{-1}$ and $(b, a) \in S^{-1}$. Hence, $(a, b) \in R$ and $(a, b) \in S$, which means $(a, b) \in (R \cap S)$. Hence, $(b, a) \in (R \cap S)^{-1}$. Therefore, $R^{-1} \cap S^{-1} \subseteq (R \cap S)^{-1}$. Taken together this proves that $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$.

Question 4

Question 5

The graph depicts the following relation: $\{(a, a), (a, b), (a, c), (b, d), (d, b), (d, a)\}$.

That the relation is not reflexive is proved by the fact that there is no $(b, b), (c, c)$, or (d, d) in the relation. The only example of a reflexive relationship is (a, a). But the definition of reflexive relation requires reflexivity for all elements of the relation.

That the relation is not symmetric is proved by the fact that there is an (a, c) yet no (c, a), an (a, b) and no (b, a), and a (d, a), but no (a, d). There are examples of symmetric relationships $(b, d), (d, b)$ and reflexive relationships (i.e. (a, a)) are always symmetric. But once again, symmetry calls for all relations to be reciprocated, and the counter-examples clearly belie symmetry.

Finally, that the relation is not transitive is proved by the fact that there is an (a, b) and a (b, d), but no (a, d). There is, however, an (d, a) and (a, c), yet there is not a (d, c). There is a (a, b) and a (d, b), but again, all pairs need to be transitive, for the relation to meet the definition of a transitive relation.

Question 6