Mathematical Proof: Final Assignment

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Proving

$$\lim_{x \to 4} \frac{1}{x} = \frac{1}{4}, \{x \in \mathbb{R} | 1 < x < 7\}$$

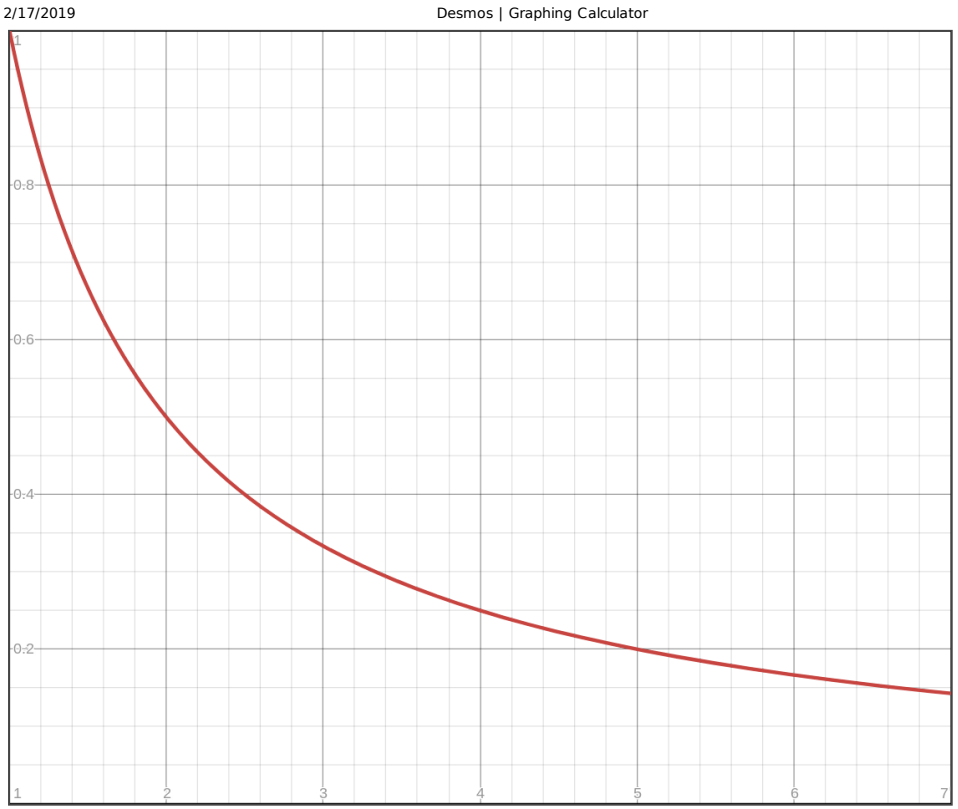
Formal Definition of a Limit

$\lim_{x \to a} f(x) = L$ if for every $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$.

Applying Formal Definition to the Current Proof

$\lim_{x \to 4} \frac{1}{x} = \frac{1}{4}$ if for every $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that $0 < |x - 4| < \delta \Rightarrow |\frac{1}{x} - \frac{1}{4}| < \epsilon$.

Intuition - Graph



$\frac{1}{x}$

Preliminaries

- $|\frac{1}{x} - \frac{1}{4}| < \epsilon$

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- $\left|x - 4\right| < \delta$
- $\delta = 4\epsilon$
- $1 < x < 7$
- $\frac{1}{7} < \frac{1}{x} < 1$

Formal Proof

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$$\equiv |\frac{4-x}{4x}| < \epsilon \qquad \text{(common denominator)}$$

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&|\frac{1}{x} - \frac{1}{4}| < \epsilon \\
&\qquad \equiv |\frac{4-x}{4x}| < \epsilon && \text{(common denominator)} \\
&\qquad \equiv \frac{|4-x|}{|4x|} < \epsilon && \text{(rewrite)}
\end{aligned}
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&\qquad \equiv \frac{|x-4|}{4} \cdot \frac{1}{x} < \epsilon && \text{(rewrite)}
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&\equiv \frac{|x-4|}{4|x|} < \epsilon && \text{(rewrite)} \\
&\equiv \frac{|x-4|}{4x} < \epsilon && (1 < x < 7) \\
&\equiv \frac{|x-4|}{4} \cdot \frac{1}{x} < \epsilon && \text{(rewrite)} \\
&\equiv \frac{|x-4|}{4} \cdot 1 < \epsilon && (\frac{1}{7} < \frac{1}{x} < 1)
\end{aligned}
$$

Formal Proof (continued)

$$\equiv \frac{|x-4|}{4} \cdot 1 < \epsilon \qquad \left(\frac{1}{7} < \frac{1}{x} < 1\right)$$

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Formal Proof (continued)

$$\equiv \frac{|x-4|}{4} \cdot 1 < \epsilon \qquad \left(\frac{1}{7} < \frac{1}{x} < 1\right)$$

$$\equiv \frac{|x-4|}{4} < \epsilon \qquad \text{(simplify)}$$

$$\equiv \frac{|x-4|}{4} \cdot 4 < \epsilon \cdot 4 \qquad \text{(mutliply both sides by } 4\text{)}$$

Formal Proof (continued)

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&\equiv \frac{|x-4|}{4} \cdot 1 < \epsilon && \left(\frac{1}{7} < \frac{1}{x} < 1\right) \\
&\equiv \frac{|x-4|}{4} < \epsilon && (\text{simplify}) \\
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Formal Proof (continued)

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&\equiv \frac{|x-4|}{4} < \epsilon && (\text{simplify}) \\
&\equiv \frac{|x-4|}{4} \cdot 4 < \epsilon \cdot 4 && (\text{mutliply both sides by } 4) \\
&\equiv |x-4| < 4\epsilon && (\text{simplify}) \\
&\equiv |x-4| < \delta && (\delta = 4\epsilon)
\end{aligned}
$$

Questions