Mathematical Proof

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Assignment #5

Question 1

Theorem 1. *If $\mathscr{F}$ and $\mathscr{G}$ are families of sets, and A in $\mathscr{F}$ and B in $\mathscr{G}$, then $\cup\mathscr{F}$ and $\cup\mathscr{G}$ are not disjoin if A and B are not disjoint.*

Proof. Suppose A and B are not disjoint. Then there exists at least one x which exists in A and B. Since A is an element of $\mathscr{F}$ and $\cup\mathscr{F}$ consists of every member of the elements of $\mathscr{F}$, then x exists in $\cup\mathscr{F}$. And since B is an element of $\mathscr{G}$ and $\cup\mathscr{G}$ consists of every member of every element of $\mathscr{G}$, then x exists in $\cup\mathscr{G}$. Therefore, $\cup\mathscr{F}$ and $\cup\mathscr{G}$ have x in common and are subsequently not disjoint. $\square$

Question 2

Theorem 2. *For every integer n, $30|n$ if, and only if, $5|n$ and $6|n$.*

Proof. Suppose $30|n$. Then there exists an integer k such that $30k = n$. Therefore $n = 30k = 5(6k)$, so $5|n$. Similarly $n = 30k = 6(5k)$, so $6|n$.

Suppose $5|n$ and $6|n$. Then there exists integers j and k such that $n = 5j$ and $n = 6k$, which means $n = 5j = 6k$. Therefore $30(j - k) = 30j - 30k = 6(5j) - 5(6k) = 6n - 5n = n$, so $30|n$. $\square$

Question 3

Theorem 3. *There is a unique real number x such that for every real number y, $xy+x-17 = 17y$*

Proof. First, take $xy + x - 17 = 17y$ and add 17 to both sides, the result is $xy + x = 17y + 17$. Then factor $x+1$ out of both sides and get $x(y+1) = 17(y+1)$. Then dived both sides by $y+1$ and get $x = 17$. This proves that $x = 17$ for all real values of y except -1. Because, if $y = -1$ then dividing by $y + 1$ constitutes dividing by zero which is undefined. To prove that $x = 17$ holds as true for $y = -1$, take the point where the division by zero would occur and insert

$x = 17$ and $y = -1$ to test for truth. That results in the statement $17(-1+1) = 17(-1+1)$, which is clearly identical and leads to the true statement that $0 = 0$. $\square$

Visual proof that $x = 17$ for all real numbers $y \neq -1$.

$$
\begin{aligned}
zy + x - 17 = 17y & \\
&\equiv xy + x - 17 + 17 = 17y + 17 && \text{(add 17 to both sides)} \\
&\equiv xy + x = 17y + 17 && \text{(simplify)} \\
&\equiv x(y+1) = 17(y+1) && \text{(factor both sides)} \\
&\equiv \frac{x(y+1)}{y+1} y + 1 = \frac{17(y+1)}{y+1} && \text{(divide both sides by y-1)} \\
&\equiv x = 17 && \text{(conclusion)}
\end{aligned}
$$

Visual proof that $x = 17$ for $y = -1$.

$$
\begin{aligned}
zy + x - 17 = 17y & \\
&\equiv xy + x - 17 + 17 = 17y + 17 && \text{(add 17 to both sides)} \\
&\equiv xy + x = 17y + 17 && \text{(simplify)} \\
&\equiv x(y+1) = 17(y+1) && \text{(factor both sides)} \\
&\equiv 17(-1+1) = 17(-1+1) && \text{(insert } x = 17 \text{ and } y = -1\text{)} \\
&\equiv 17(0) = 17(0) && \text{(simplify)} \\
&\equiv 0 = 0 && \text{(true statement)}
\end{aligned}
$$

Question 4

Theorem 4. *For any set U, for every $B \in \wp(U)$ there is a unique D such that for every $C \in \wp(U), C \setminus B = C \cap D$.*

Proof. Because B, D, and C are within $\wp(U)$, $\wp(U)$ is the universal set within the context of this problem. Suppose D is the complement of B (i.e. B^c) within set U. Then D is the set of all $x \notin B$ within $\wp(U)$. Therefore, for any possible $C \in \wp(U)$ if the members of C which also exists in B are removed (i.e. $C \setminus B$), the remainder of the set will exist in both C and D. Therefore, there exists a unique set D such that for all C, $C \setminus B = C \cap D$. Furthermore, that unique set D is the compliment of B, B^c. $\square$

Question 5

Theorem 5. *For every positive integer n, there is a sequence of $2n$ consecutive positive integers containing no primes.*

Proof. Suppose n is a positive integer. Suppose $x = (2n+1)! + 2$. $x = 1 \cdot 2 \cdot 3 \cdot 4 ... (2n+1) + 2)$. Two can be factored out to create $2 \cdot (1 \cdot 3 \cdot 4 ... (2n+1) + 1)$. If 2 can be factored out of x, then x is divisible by 2 and therefore not prime. Similarly, $x + 1 = 1 \cdot 2 \cdot 3 \cdot 4 ... (2n+1) + 3$. Three can be factored out leaving $3 \cdot (1 \cdot 2 \cdot 4 ... (2n+1) + 1)$. With 3 factorizable, $x + 1$ is divisible by 3 and therefore not prime. This pattern repeats for all the numbers in the sequence proving that for any positive integer n, There is a sequence of $2n$ consecutive positive integers containing no primes. □

A more visual depiction:

$$
\begin{aligned}
x &= 1 \cdot 2 \cdot 3 \cdot 4 ... (2n+1) + 2 \\
&= 2 \cdot (1 \cdot 3 \cdot 4 ... (2n+1) + 1) && (x \text{ not prime}) \\
x + 1 &= 1 \cdot 2 \cdot 3 \cdot 4 ... (2n+1) + 3 \\
&= 3 \cdot (1 \cdot 2 \cdot 4 ... (2n+1) + 1) && (x + 1 \text{ not prime}) \\
x + 2 &= 1 \cdot 2 \cdot 3 \cdot 4 ... (2n+1) + 4 \\
&= 4 \cdot (1 \cdot 2 \cdot 3 ... (2n+1) + 1) && (x + 2 \text{ not prime}) \\
&... && (\text{pattern repeats for the remainder of the sequence})
\end{aligned}
$$

Question 6

Theorem 6. *For $f(x) = x^2$ with domain $0 \leq x \leq 10$, then $\lim_{x \to 5} f(x) = 25$*

Proof. Given $\epsilon > 0$ and $0 < |x - 5| < \delta$. Suppose $\delta = min\{1, \frac{\epsilon}{11}\}$. With a δ no greater than 1, x can be no greater than 6, and $|x + 5|$ can be no greater than 11. Therefore:

$$
\begin{aligned}
|x^2 - 25| &= |x - 5||x + 5| \\
&< |x - 5|11 \\
&< (\frac{\epsilon}{11}) \cdot 11 = \epsilon
\end{aligned}
$$

□