The electric and magnetic field due to a dipole of length *l*.

$$A_z = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-l/2}^{l/2} I(z')e^{jkz'\cos\theta}dz' \qquad (1)$$

1) $E_\theta = j\omega A_z \sin\theta \qquad (2)$

2) $H_\phi = \dfrac{E_\theta}{\eta} \qquad (3)$

$$A_z = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\left\{\underbrace{\int_{z'=-l/2}^{0} I_0 \sin\left[k\left(\frac{l}{2}+z'\right)\right]e^{jkz'\cos\theta}dz'}_{\doteq I_1} + \underbrace{\int_{z'=0}^{l/2} I_0 \sin\left[k\left(\frac{l}{2}-z'\right)\right]e^{jkz'\cos\theta}dz'}_{\doteq I_2}\right\}$$

From integral tables (CRC)

$$\int \sin(c+bz)e^{az}dz = \frac{e^{az}}{a^2+b^2}\left[a\sin(c+bz)-b\cos(c+bz)\right]$$

for I_1: $a = jk\cos\theta$, $b = k$, $c = \dfrac{kl}{2}$

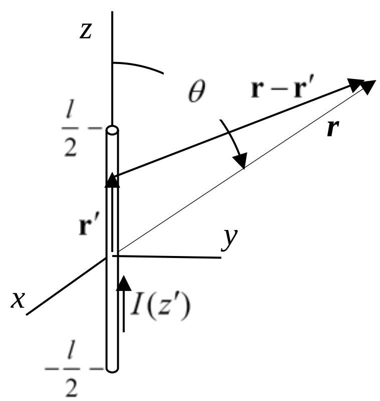
I_2: $a = jk\cos\theta$, $b = -k$, $c = \dfrac{kl}{2}$

$$A_z = \frac{\mu I_0}{k2\pi}\frac{e^{-jkr}}{r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)}{\sin^2\theta}\right] \qquad (4)$$

$$E_\theta = j\omega A_z \sin\theta$$

$$E_\theta = \frac{j\eta I_0}{2\pi}\frac{e^{-jkr}}{r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)}{\sin\theta}\right] \qquad (5)$$

$$H_\phi = \frac{E_\theta}{\eta} \qquad (6)$$



ex/ $l = \dfrac{\lambda}{2}$ dipole

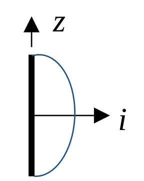
$$E_\theta = \frac{j\eta I_0}{2\pi}\frac{e^{-jkr}}{r}\left[\frac{\cos\left[\frac{1}{2}\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{2}\cos\theta\right]-\cos\left[\frac{1}{2}\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{2}\right]}{\sin\theta}\right]$$

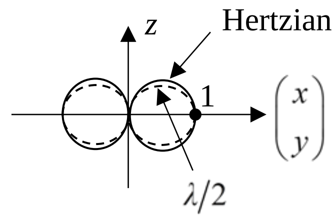
$$= \frac{j\eta I_0}{2\pi}\frac{e^{-jkr}}{r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right] \qquad (7)$$

$$E_{\theta,\max(\theta,\phi)} = \frac{j\eta I_0}{2\pi}\frac{e^{-jkr}}{r} \quad \text{when } \theta = 90° \qquad (8)$$

- Pattern Function from (7)

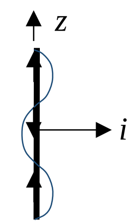
$$E_\theta^{norm} = \frac{|E_\theta|}{|E_\theta|_{\max(\theta,\phi)}} = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \qquad (9)$$

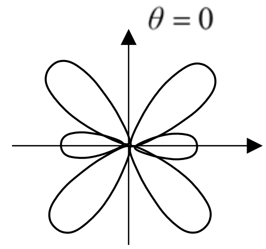




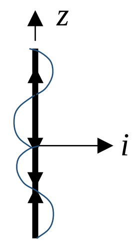
From (5):

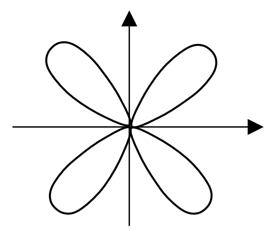
$l = 1.5\lambda$





$l = 2\lambda$





- The power radiated, radiation resistance, radiation and reflection efficiency of a $\frac{\lambda}{2}$ dipole.

$$\mathbf{P}_{\text{ave}} = \frac{1}{2}\text{Re}\left\{E_\theta \hat{\boldsymbol{\theta}} \times H_\phi^* \hat{\boldsymbol{\phi}}\right\} = \frac{|E_\theta|^2}{2\eta}\hat{\mathbf{r}}$$

$$= \underbrace{\frac{1}{2\eta}\left(\frac{\eta I_0}{2\pi}\right)^2}_{\doteq Q} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{r^2\sin^2\theta}\hat{\mathbf{r}} \qquad (10)$$

$$P_{\text{r}} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} P_{\text{ave}} r^2 \sin\theta d\theta d\phi$$

$$= Q\int_{\theta=0}^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\cancel{r^2}\sin^{\cancel{2}}\theta}\cancel{r^2}\,\cancel{\sin\theta}\, d\theta \underbrace{\int_0^{2\pi} d\phi}_{=2\pi}$$

$$= 2\pi Q \underbrace{\int_{\theta=0}^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta}_{=1.218}$$

$$\boxed{\text{from tables} \quad \left.\frac{\mathrm{Cin}(2\theta)}{2}\right|_0^{\pi} = \frac{\mathrm{Cin}(2\pi)}{2} - \cancelto{0}{\frac{\mathrm{Cin}(0)}{2}} = 1.218}$$

$$= \left[\frac{\eta}{2}\frac{I_0{}^2}{(2\pi)^{\cancel{2}}}\right]\cancel{(2\pi)}(1.218)$$

$$P_r = \frac{\eta}{2} I_0{}^2 \left(\frac{0.609}{\pi}\right) \qquad (11)$$

$$R_r = \frac{P_r}{\underset{\substack{\uparrow \\ \text{max amplitude}}}{I_0{}^2}/2} = \frac{\eta(0.609)}{\pi} = \frac{120\pi(0.609)}{\pi}$$

$$R_r = 73\Omega \qquad (12)$$

$$e_r = \frac{R_r}{R_r + \underbrace{R_\Omega}_{\simeq 2\Omega}} = \frac{73}{73+2} = \frac{73}{75} = 0.973$$

$\Rightarrow$ 97.3% efficient (13)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{73-75}{73+75} = 0.0135$$ (Fed by a 75Ω coaxial cable)

$$e_{\text{ref}} = \left(1 - |\Gamma|^2\right) = 1 - (0.0135)^2 \cong 0.99982 \simeq 1.0 \qquad (14)$$

- $\frac{\lambda}{2}$ dipole maximum Directivity and maximum Gain

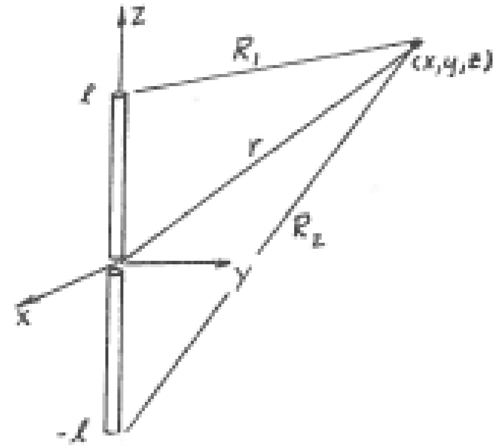
$$D_0 = \frac{\overset{\substack{\text{time ave. power density} \\ \downarrow}}{P_{\text{ave}}(\theta,\phi)_{\max}}}{\underbrace{\frac{P_r}{4\pi r^2}}_{\substack{\text{ave. power radiated} \\ \text{through a sphere}}}} = \frac{\overbrace{\left[r^2 P_{\text{ave}}(\theta,\phi)_{\max}\right]}^{\doteq U_{\max}(\text{rad. intensity})}}{\left[\frac{P_r}{4\pi}\right]} \doteq \frac{U_{\max}}{U_{\text{ave}}}$$

$$D_0 = \frac{4\pi \cancel{r^2}\, \frac{\cancel{\eta}}{\cancel{2}}\left(\frac{\cancel{I_0}}{2\pi}\right)^2 \frac{\cancelto{1}{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}}{\cancel{r^2}\sin^2\theta}}{\frac{\cancel{\eta}}{\cancel{2}}\cancel{I_0}^2\left(\frac{0.609}{\pi}\right)}$$

$$D_0 = \frac{1}{0.609} = 1.642 \qquad (15)$$

$$G_0 = D_0 e_r = 1.642(0.973) = 1.598 \qquad (16)$$

• Exact field of a dipole - with an assumed current distribution



$$I = I_m \sin[k(\ell - |z'|)]$$

$$A_z(x,y,z) = \frac{\mu}{4\pi}\int_{-\ell}^{\ell} I_m \sin[k(\ell-|z'|)]\frac{e^{-jkR}}{R}\,dz'$$

$$R = [x^2+y^2+(z-z')^2]^{1/2}$$

change sine into exponential form

$$A_z(x,y,z) = \frac{I_m\mu}{j8\pi}\Big[e^{jk\ell}\int_{-\ell}^{0}\frac{e^{-jk(R-z')}}{R}\,dz' - e^{-jk\ell}\int_{-\ell}^{0}\frac{e^{-jk(R+z')}}{R}\,dz'$$

$$+e^{jk\ell}\int_{0}^{\ell}\frac{e^{-jk(R+z')}}{R}\,dz' - e^{-jk\ell}\int_{0}^{\ell}\frac{e^{-jk(R-z')}}{R}\,dz'\Big] \quad (1)$$

$\to 0$ (no ϕ variation in the potential)

$$\bar{H} = \frac{1}{\mu}\nabla\times\bar{A} = \frac{1}{\mu}\left[\cancel{\frac{1}{\rho}\frac{\partial}{\partial\phi}}A_z\hat{\rho} - \frac{\partial}{\partial\rho}A_z\hat{\phi}\right] \quad (2)$$

so

$$H_\phi = -\frac{1}{\mu}\frac{\partial}{\partial\rho}A_z$$

For the first integral in (1)

$$H_{\phi,1} = \frac{-I_m}{j8\pi}e^{jk\ell}\int_{-\ell}^{0}\frac{\partial}{\partial\rho}\left[\frac{e^{-jk(R-z')}}{R}\right]dz' = -\frac{I_m e^{jk\ell}}{j8\pi}\int_{-\ell}^{0}\rho\left[\frac{jk}{R^2}+\frac{1}{R^3}\right]e^{-jk(R-z')}\,dz'$$

$$= \frac{I_m}{j8\pi}e^{jk\ell}\rho\int_{-\ell}^{0}\frac{\partial}{\partial z'}\left[\frac{e^{-jk(R-z')}}{R[R+z-z']}\right]dz' \quad (3)$$

$\leftarrow$ perfect differential

$$H_{\phi,1} = \frac{I_m \rho}{j8\pi} e^{jkl} \left[\frac{e^{-jkr}}{r(r+z)} - \frac{e^{-jk(R_2+l)}}{R_2(R_2+z+l)} \right] \qquad (4)$$

$R_2 = [x^2+y^2+(z+l)^2]^{1/2} = [\rho^2+(z+l)^2]^{1/2}$
$r = [x^2+y^2+z^2]^{1/2}$

or

$$H_{\phi,1} = \frac{I_m e^{jkl}}{j8\pi\rho} \left[\frac{r-z}{r} e^{-jkr} - \frac{R_2-(z+l)}{R_2} e^{-jk(R_2+l)} \right] \qquad (5)$$

The other integrals in (1) can be evaluated to give

$$H_\phi = H_{\phi,1} + H_{\phi,2} + H_{\phi,3} + H_{\phi,4}$$

$$H_\phi = -\frac{I_m}{j4\pi\rho} \left[e^{-jkR_1} + e^{-jkR_2} - (2\cos kl) e^{jkr} \right] \qquad (6)$$

$R_1 = [x^2+y^2+(z-l)^2]^{1/2} = [\rho^2+(z-l)^2]^{1/2}$
From Maxwell's equations

$$\bar{E} = \frac{1}{j\omega\epsilon_0} \nabla\times\bar{H} = \frac{1}{j\omega\epsilon_0} \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho H_\phi \hat{z} - \frac{1}{\rho}\frac{\partial}{\partial z}\rho H_\phi \hat{\rho} \right]$$

$$\bar{E} = -j\frac{\eta}{4\pi} I_m \left(\frac{e^{-jkR_1}}{R_1} + \frac{e^{-jkR_2}}{R_2} - 2\cos(kl)\frac{e^{-jkr}}{r} \right) \hat{z} \qquad (7)$$

$$+ j\frac{\eta}{4\pi} I_m \left(\frac{z-l}{\rho}\frac{e^{-jkR_1}}{R_1} + \frac{z+l}{\rho}\frac{e^{-jkR_2}}{R_2} - (2\cos kl)\frac{z}{\rho}\frac{e^{-jkr}}{r} \right) \hat{\rho}$$

Review:

Recall that to find the field radiated by an antenna

we had to solve the equation

$$\mathbf{E} = \frac{-j\omega}{k^2}\left(k^2\mathbf{A} + \nabla\nabla\cdot\mathbf{A}\right) \qquad (1)$$

where

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_{v'} \frac{\mathbf{J}(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}dv' \qquad (2)$$

However, for a thin dipole antenna where the current is restricted to travel up and down on the antenna in the z-direction (2) could be written

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_{z'} \frac{I(z')\hat{\mathbf{z}}\; e^{-jk|\mathbf{r}-z'\hat{\mathbf{z}}|}}{|\mathbf{r}-z'\hat{\mathbf{z}}|}dz' \qquad (3)$$

Before we knew what the current was on the antenna and we wanted to find the field. Now suppose we know what the field is on the antenna and we want to find the current on the antenna. When we knew what the current was we were able to find the radiated field $\boldsymbol{E}^{rad}$ but now we will have $\boldsymbol{E}^{inc}$, the electric field incident on the antenna. The relationship between these fields on a perfect conducting antenna is the boundary condition:

$$\boldsymbol{E}^{rad} + \boldsymbol{E}^{inc} = 0 \quad \Rightarrow \quad \boldsymbol{E}^{rad} = -\boldsymbol{E}^{inc} \qquad (4)$$

Notice in (3) the vector potential can only be in the z-direction because the current is in the z-direction. Also only the z-directed component of the electric field on the surface of the antenna can create a z-directed current. So final equations are

$$E_z^{inc} = \frac{j\omega}{k^2}\left(k^2 A_z + \frac{\partial^2}{\partial z^2}A_z\right);\;\; A_z(z) = \frac{\mu}{4\pi}\int_{z'} \frac{I(z')e^{-jk|z-z'|}}{|z-z'|}dz'$$

or the integro-differential equation for $I(z)$,

$$E_z^{inc} = \frac{j\omega\mu}{4\pi k^2}\left(k^2 + \frac{\partial^2}{\partial z^2}\right)\int_{z'} \frac{I(z')e^{-jk|z-z'|}}{|z-z'|}dz' \qquad (5)$$

Projection or Weighted-Residual Methods

Consider an integro-differential equation such as

$$\left(\frac{d^2}{dz^2}+k^2\right)\int I(z')\frac{e^{-jk|z-z'|}}{|z-z'|}dz' = E^{inc}(z)$$

written in operator equation form

$$L(z,z')I(z') = E^{inc}(z) \qquad (6)$$

operator   source

current
(unknown)

where $L(z,z') = \left(\frac{d^2}{dz^2}+k^2\right)\int \frac{e^{-jk|z-z'|}}{|z-z'|}dz'$

Approximate

$$I(z') \cong \sum_{n=1}^{N} I_n\,\alpha_n(z') \qquad (7)$$

Where $\alpha_n(z')$ are basis functions chosen so as to satisfy boundary conditions and I_n are the amplitude coefficients of $\alpha_n(z')$.
Apply the operator L to (7)

$$L(z,z')I(z') \cong \sum_{n=1}^{N} L(z,z')\left(I_n\,\alpha_n(z')\right) = \sum_{n=1}^{N} I_n\,L(z,z')\alpha_n(z') \qquad (8)$$

Substitute (8) into (6)

$$\sum_{n=1}^{N} I_n L(z,z')\alpha_n(z') = E^{inc}(z) \qquad (9)$$

or

$$\sum_{n=1}^{N} I_n \left(\frac{d^2}{dz^2}+k^2\right)\int \alpha_n(z')\frac{e^{-jk|z-z'|}}{|z-z'|}dz' = E^{inc}(z) \qquad (10)$$

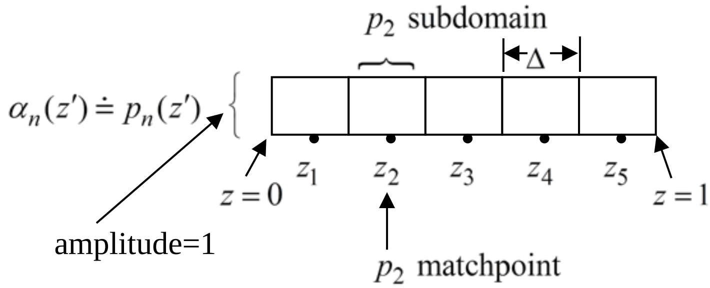
$$I_1 L(z,z')\alpha_1(z') + I_2 L(z,z')\alpha_2(z') + I_3 L(z,z')\alpha_3(z') + \ldots + I_N L(z,z')\alpha_N(z') = E^{inc}(z) \qquad (11)$$

But this gives 1 equation with N unknowns, $I_1, I_2, I_3, \ldots I_N$.

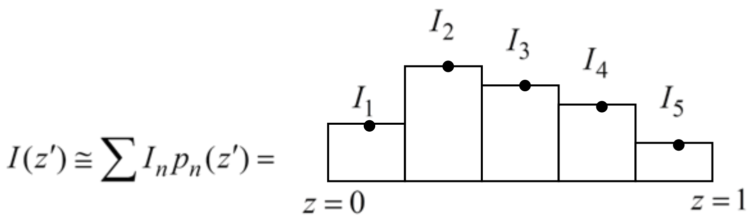
Basis functions should be

a) linearly independent (orthogonal)
b) able to approximate the function reasonably well
c) capable of satisfying boundary conditions

Ex: <u>Pulse Basis Functions</u> <u>Subdomain basis functions</u> also called basis functions of <u>local support</u>.

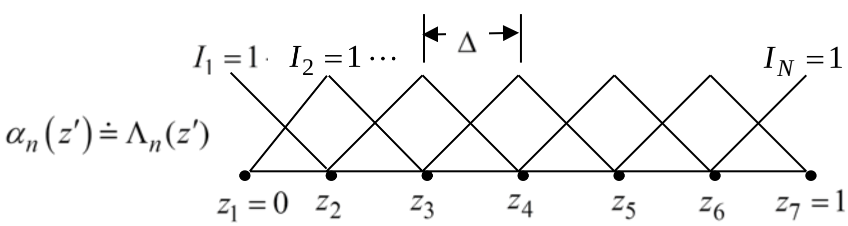


where $p_n = \begin{cases} 1, & z_n - \Delta/2 \le z \le z_n + \Delta/2 \\ 0, & \text{otherwise} \end{cases}$

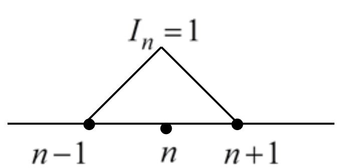


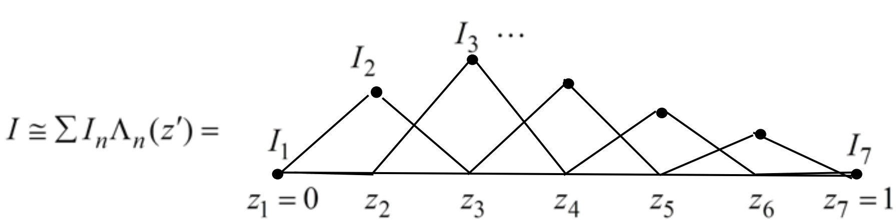
Pulse basis functions are adequate in many cases but a derivative of a pulse produces an undesirable singularity and a second derivative is undefined. A better basis is a piecewise linear (triangle) function.

Ex: <u>Triangular basis functions</u>



$$\Lambda_n(z') = \begin{cases} 1 - \dfrac{|z' - z_n|}{\Delta}, & z_{n-1} \le z' \le z_{n+1} \\ 0, & \text{otherwise} \end{cases}$$





Pulse basis functions in equation (10) gives,

$$\sum_{n=1}^{N} I_n \left(\frac{d^2}{dz^2} + k^2 \right) \int p_n(z') \frac{e^{-jk|z-z'|}}{|z-z'|} dz' = E^{inc}(z) \qquad (12)$$

$$\text{where } \; p_n = \begin{cases} 1, & z_n - \Delta/2 \le z \le z_n + \Delta/2 \\ 0, & \text{otherwise} \end{cases}$$

which results in,

$$\sum_{n=1}^{N} I_n \left(\frac{d^2}{dz^2} + k^2 \right) \int_{z'=z_n-\Delta/2}^{z'=z_n-\Delta/2} \frac{e^{-jk|z-z'|}}{|z-z'|} dz' = E^{inc}(z) \qquad (13)$$

Expanding the summation

$$\begin{aligned} I_1 \left(\frac{d^2}{dz^2} + k^2 \right) \int_{z'=z_1-\Delta/2}^{z'=z_1-\Delta/2} \frac{e^{-jk|z-z'|}}{|z-z'|} dz' + I_2 \left(\frac{d^2}{dz^2} + k^2 \right) \int_{z'=z_2-\Delta/2}^{z'=z_2-\Delta/2} \frac{e^{-jk|z-z'|}}{|z-z'|} dz' + \cdots \\ + I_N \left(\frac{d^2}{dz^2} + k^2 \right) \int_{z'=z_N-\Delta/2}^{z'=z_N-\Delta/2} \frac{e^{-jk|z-z'|}}{|z-z'|} dz' = E^{inc}(z) \end{aligned} \qquad (14)$$

which can also be written like (11) with p_n basis functions,

$$I_1 L(z,z') p_1(z') + I_2 L(z,z') p_2(z') + \ldots + I_N L(z,z') p_N(z') = E^{inc}(z) \qquad (15)$$

To get N equations we use the Point matching method – this 'testing function' is a set of delta functions that are at the center of each pulse,

$$\delta(z - z_m), \text{ where } m = 1, 2, 3, \cdots, N \qquad (14)$$

The first delta function $\delta(z - z_1)$ multiplies each term in (15) and is integrated over the length Ω of the antenna.

$$\begin{aligned} I_1 \int_\Omega \delta(z-z_1) L(z,z') p_1(z') dz + I_2 \int_\Omega \delta(z-z_1) L(z,z') p_2(z') dz + \ldots \\ + I_N \int_\Omega \delta(z-z_1) L(z,z') p_N(z') dz = \int_\Omega \delta(z-z_1) E^{inc}(z) dz \end{aligned} \qquad (16)$$

The "tested" equation at $z = z_1$

$$I_1 L(z_1,z') p_1(z') + I_2 L(z_1,z') p_2(z') + \cdots + I_N L(z_1,z') p_N(z') = E^{inc}(z_1) \qquad (17)$$

Repeating this process with (15) and each delta testing function $\delta\left(z-z_m\right)$, where $m=1,2,3,\cdots,N$ gives the set of equations,

$$
\begin{aligned}
&I_1 L(z_1,z')p_1\left(z'\right)+I_2 L(z_1,z')p_2\left(z'\right)+I_3 L(z_1,z')p_3\left(z'\right)+\cdots+I_N L(z_1,z')p_N\left(z'\right)=E^{inc}(z_1)\\
&I_1 L(z_2,z')p_1\left(z'\right)+I_2 L(z_2,z')p_2\left(z'\right)+I_3 L(z_2,z')p_3\left(z'\right)+\cdots+I_N L(z_2,z')p_N\left(z'\right)=E^{inc}(z_2)\\
&I_1 L(z_3,z')p_1\left(z'\right)+I_2 L(z_3,z')p_2\left(z'\right)+I_3 L(z_3,z')p_3\left(z'\right)+\cdots+I_N L(z_3,z')p_N\left(z'\right)=E^{inc}(z_3)\\
&\qquad\vdots \qquad\qquad\qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \ldots \qquad\qquad \vdots \qquad\qquad\qquad \vdots\\
&I_1 L(p_N,z')p_1\left(z'\right)+I_2 L(z_N,z')p_2\left(z'\right)+I_3 L(z_N,z')p_3\left(z'\right)+\ldots+I_N L(z_N,z')p_N\left(z'\right)=E^{inc}(z_N)
\end{aligned}
$$

The matrix equation is therefore

$$
\begin{bmatrix}
L(z_1,z')p_1\left(z'\right) & L(z_1,z')p_2\left(z'\right) & L(z_1,z')p_3\left(z'\right) & \ldots & L(z_1,z')p_N\left(z'\right)\\
L(z_2,z')p_1\left(z'\right) & L(z_2,z')p_2\left(z'\right) & L(z_2,z')p_3\left(z'\right) & \ldots & L(z_2,z')p_N\left(z'\right)\\
L(z_3,z')p_1\left(z'\right) & L(z_3,z')p_2\left(z'\right) & L(z_3,z')p_3\left(z'\right) & \ldots & L(z_3,z')p_N\left(z'\right)\\
\vdots & \vdots & \vdots & & \vdots\\
L(z_N,z')p_1\left(z'\right) & L(z_N,z')p_2\left(z'\right) & L(z_N,z')p_3\left(z'\right) & \cdots & L(z_N,z')p_N\left(z'\right)
\end{bmatrix}
\begin{bmatrix} I_1\\ I_2\\ I_3\\ \vdots\\ I_N \end{bmatrix}
=
\begin{bmatrix} E^{inc}\left(z_1\right)\\ E^{inc}\left(z_2\right)\\ E^{inc}\left(z_3\right)\\ \vdots\\ E^{inc}\left(z_N\right) \end{bmatrix}
\tag{18}
$$

or symbolically

$$\bar{\boldsymbol{Z}}\,\boldsymbol{I}=\boldsymbol{E}$$

which can be solved by matrix inversion

$$\boldsymbol{I}=\bar{\boldsymbol{Z}}^{-1}\boldsymbol{E} \tag{19}$$

The current is therefore approximately

$$I\left(z\right)\cong\sum_{n=1}^{N} I_n p_n\left(z\right) \tag{20}$$

In general the integral equation must include ϕ integration.

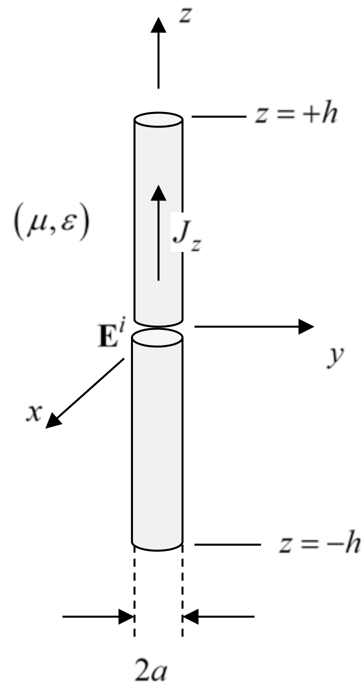
$\hat{\mathbf{n}}\times(\mathbf{E}_1-\mathbf{E}_2)=0$

where

$\mathbf{E}_1=\mathbf{E}^i+\mathbf{E}^s$, the total field outside the conductor (wire)

$\mathbf{E}_2=0$, the total field inside the conductor

$\Rightarrow \hat{\mathbf{n}}\times\left(\mathbf{E}^{\mathrm{i}}+\mathbf{E}^s\right)=0$



Since

$$\hat{\mathbf{n}}\times\left(\mathbf{E}^i+\mathbf{E}^s\right)=\hat{\boldsymbol{\rho}}\times\left(E_\rho^i\hat{\boldsymbol{\rho}}+E_z^i\hat{\mathbf{z}}+E_\rho^s\hat{\boldsymbol{\rho}}+E_z^s\hat{\mathbf{z}}\right)$$
$$=\left(E_z^i+E_z^s\right)\hat{\boldsymbol{\varphi}}=0$$

then

$$E_z^i+E_z^s=0 \quad\Rightarrow\quad E_z^i=-E_z^s$$

which is entirely scalar.

Before we had $\mathbf{E}^s=\dfrac{-j\omega}{k^2}\left(k^2\mathbf{A}+\nabla\nabla\cdot\mathbf{A}\right)$,

but because

$\mathbf{J}=J_z\hat{\mathbf{z}} \;\Rightarrow \mathbf{A}=A_z\hat{\mathbf{z}}$ and from boundary conditions $E_z^i=-E_z^s$

this equation becomes

$$E_z^i=\frac{j\omega}{k^2}\left(k^2A_z+\frac{\partial^2}{\partial z^2}A_z\right)$$

on a thin wire scatterer with

$$A_z=\frac{\mu}{4\pi}\int_{-h}^{h}J_z(z')\int_{-\pi}^{\pi}\frac{e^{-jkR}}{R}a\,d\phi'dz'$$

where J_z is a surface current (A/m). Since $I(z)=2\pi a\,J_z(z)$ this equation can be written

$$A_z = \frac{\mu}{4\pi}\int_{-h}^{h} I(z')G(z-z')dz'$$

where

$$G(z-z') = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-jkR}}{R}d\phi' \quad \text{and}$$

$$R = \left[(z-z')^2 + 4a^2\sin^2\frac{\phi'}{2}\right]^{1/2}$$

or

$$G(z-z') = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-jk\left[(z-z')^2+4a^2\sin^2\frac{\phi'}{2}\right]^{1/2}}}{\left[(z-z')^2+4a^2\sin^2\frac{\phi'}{2}\right]^{1/2}}d\phi'$$

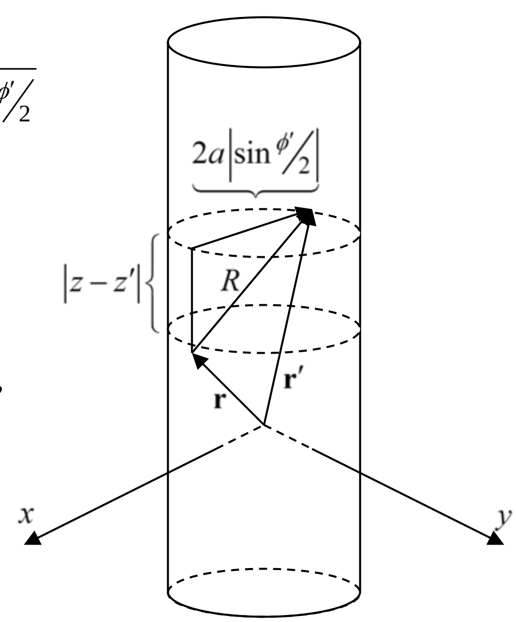
This integral can be shown to have a log singularity

at the lower limit $\frac{\phi'}{2}\to 0,\ \frac{(z-z')}{2a}\to 0,\quad \Rightarrow \lim G(z-z')\to -\frac{1}{\pi a}\ln\left(\frac{|z-z'|}{8a}\right)$

<u>The reduced kernal</u> method gets rid of the singularity in R and still has a reasonably accurate solution. Starting with

$$G(z-z') = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-jkR}}{R}d\phi' \quad ; \quad R = \sqrt{(z-z')^2 + 4a^2\sin^2 \phi'/2}$$

notice $2a\left|\sin \phi'/2\right|$ ranges from 0 to $2a$ as ϕ' goes from $-\pi$ to π. When $|z-z'| >> a$ we take the average value of $2a\left|\sin \phi'/2\right|$ which is approximately $2a\left|\sin \phi'/2\right| \approx a$,



Using this observation approximate

$$\Rightarrow R \to R_r = \sqrt{(z-z')^2 + a^2}$$

and

$$G(z-z') \approx \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-jkR_r}}{R_r} d\phi'$$

$$\Rightarrow G(z-z') = \frac{e^{-jkR_r}}{R_r}$$

We now want to solve the integral equation on a thin wire antenna where

$$\frac{j\omega}{k^2}\left(k^2 + \frac{\partial^2}{\partial z^2}\right) A_z = E_z^i \qquad \text{with}$$

$$A_z = \frac{\mu}{4\pi}\int_{-h}^{h} I(z') \frac{e^{-jk\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}} dz'$$

Now move the differentiation inside the integration

$$\frac{j\omega\mu}{k^2 4\pi}\int_{-h}^{h} I(z')\left(k^2 + \frac{\partial^2}{\partial z^2}\right)\frac{e^{-jk\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}} dz' = E_z^i$$

and differentiate

$$\frac{j\eta}{k}\int_{z'=-h}^{h} I(z')\frac{e^{-jkR_r}}{4\pi R_r^5}\left[(1-jkR_r)^2(2R_r - 3a^2) + (kaR_r)^2\right] dz' = E_z^i$$

One last change is that we want to know the input impedance of the antenna, so the incident field will be coming from a source feeding the antenna in the feed gap from a transmission line. For a linear wire antenna with a with a feed point gap at the center, the integral equation for the current $I(z)$ is obtained by applying $E_z = -\frac{\partial V}{\partial z}$ which comes from $\mathbf{E} = -\nabla V$, a static field approximation which ignores the $-j\omega\mathbf{A}$ time harmonic contribution to E_z. This creates the numerical replacement $E_z(a,z) \simeq -V(z=0)/\Delta z$ in our equation,

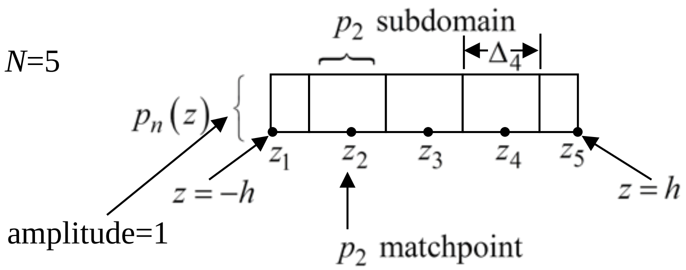
$$\frac{j\eta}{k}\int_{z'=-h}^{h} I(z')\frac{e^{-jkR_r}}{4\pi R_r^5}\left[(1+jkR_r)(2R_r^2 - 3a^2) + (kaR_r)^2\right] dz' = -\frac{V}{\Delta z} \qquad *$$

Homework:

Eqn. (*) is a form of Pocklington's integral equation.

(a) Solve the integral equation * for the current on a thin wire antenna of length $2h$ using pulse expansion functions with ½ pulses at the two ends and delta function testing. Use a frequency f = 300 MHz, and a feed point voltage $V = 1$ volt, which means the feed point electric field is $1/\Delta$ V/m. Compare your results, real and imaginary parts, with those in the first figure on the following page. All of the needed specifications are in the figure.

ex/ N=5



Force the pulse amplitudes at the ends to zero $(I_1 = I_N = 0)$, so the numerical code should just solve for the current amplitudes $I_2 \to I_{N-1}$, but your plot of the current should include the zeros at the ends.

(b) Obtain the input admittance and compare with that in second figure on the following page. The input admittance is found by dividing the gap current by the gap voltage, which is $V = 1$ volt in your code.

(c) Obtain the normalized amplitude of the far electric field of the antenna on a polar plot when $2h = 4\lambda/5$ and $2h = \lambda$.

The far field of a dipole antenna is given by

$$E_\theta \cong j\omega A_z \sin\theta$$

where

$$A_z \cong \frac{\mu}{4\pi r} e^{-jkr} \int_{-h}^{h} I(z') e^{jkz'\cos\theta} dz'$$

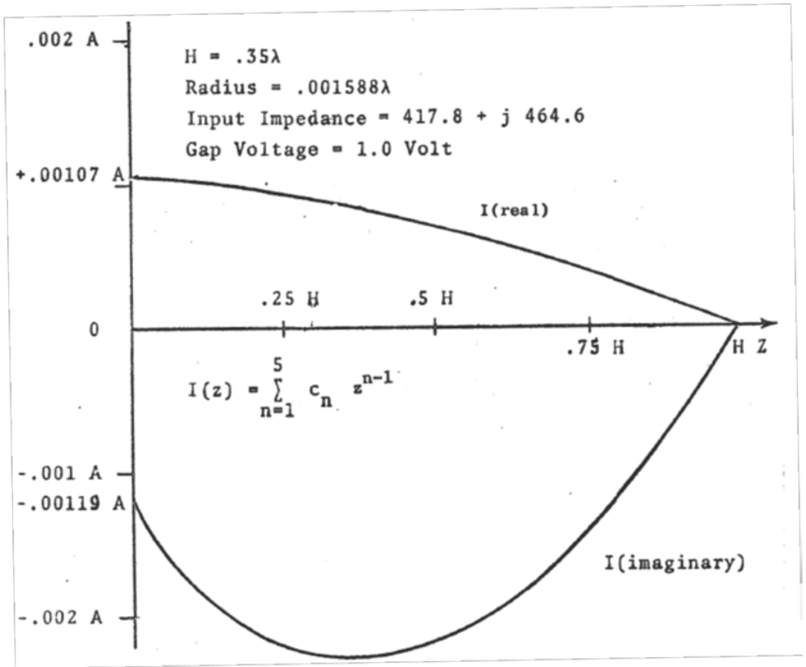
Since the current has been found by the approximation

$$I(z) \doteq \sum_n I_n p_n(z)$$

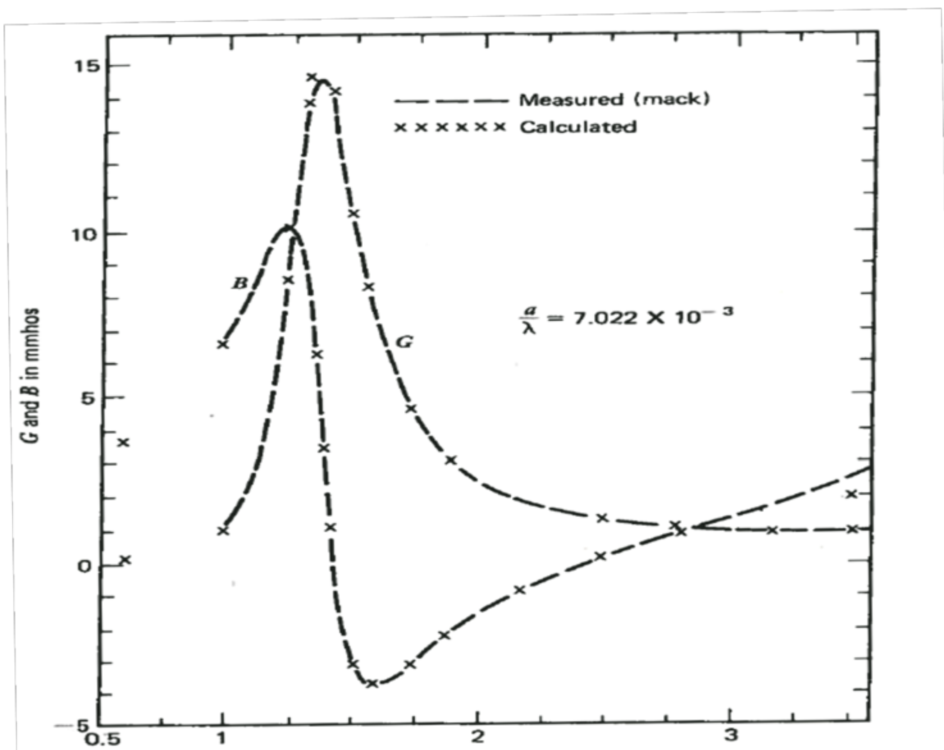
the potential breaks down into the simple series

$$A_z \cong \frac{\mu}{4\pi r} e^{-jkr} \sum_n I_n e^{jkz_n \cos\theta} \Delta_n$$

Therefore: $E_\theta \cong \dfrac{j\omega\mu}{4\pi r} e^{-jkr} \sin\theta \sum_n I_n e^{jkz_n \cos\theta} \Delta_n$



Real and imaginary current (above) and input impedance (below) of a 2H = 0.7λ antenna. In the past we have found that N=99 pulse functions worked best.



Calculated and measured dipole admittance with data calculated using pulse basis functions (N=99).

$$\frac{\beta L}{2} = \frac{2\pi}{\lambda}\frac{L}{2} = \pi\frac{L}{\lambda}$$

$$0.5 \le \frac{\beta L}{2} \le 3.5$$

$$\frac{0.5}{\pi} \le \frac{L}{\lambda} \le \frac{3.5}{\pi}$$

Matlab program comments:

Access to Matlab:

1. Students can download Matlab free from the SELL campus webpage https://sell.tamu.edu/ by using your TAMU ID,
2. TAMU VOAL, is a TAMU virtual open lab site which gives Matlab access remotely. I have had trouble with maintaining the connection which is sometimes quirky.
3. Download Octave, which is a free Matlab clone produced by GNU, an old and reliable free software provider. I have used it often, it is a little slower than Matlab, but an Octave code will run in Matlab and vice versa. https://www.gnu.org/software/octave/. Some of Matlab's more recent functions may not be available. The last time I used it, a couple of years ago, the plotter I commonly use in Matlab was not yet available.

Just a few comments on some coding problems that have shown up in this homework in the past:

4. There are several methods that will force the half pulses at the ends of the antenna to zero. The easiest and most accurate is to implement it in the impedance matrix before solving for the current. Just initialize the matrix Z(np+2,np+2)=0 and fill it starting at Z(2,2) and ending at Z(np-1,np-1), where np is the number of full pulses. Then set Z(1,1)=1 and Z(np+2,np+2)=1. In the matrix equation you have created this means you have set *I*(1)=0. and *I*(np+2)=0, which are the 2 half pulse amplitudes at the ends.

5. In this problem the matrix is Toeplitz. This means that all of the matrix terms are symmetric around the matrix diagonal. So, you only have to calculate the first row and fill in the rest of the matrix with the terms you have calculated on the first row. Unfortunately Matlab's function by default gives the complex conjugate. In other words T_ij = conj(T_ji). The easiest work around seems to be to do answer = toeplitz(real(X)) +1I* toeplitz(imag(X)). Another possibly less intuitive solution that gives a good answer was answer = toeplitz(x,x)

6. An irritating error I have seen (and done myself) is to incorrectly write I=V\Z rather than I=Z\V when solving for the current matrix I.

7. Notice that you can get part (b) by just rerunning part (a) code over a range of frequencies (or wavelengths). np = 99 seems to get a reasonable result.

8. There can be small differences in the results obtained with different codes due to how you define some variables. For example, eta can be 120*pi or 377 or sqrt(mu*epsilon).

9. The original codes for these figures were likely written with single precision accuracy, whereas we now use double precision by default. So your results may be slightly more accurate than the figures.

Code Organization

```
clc;
clear;
close all; %clears the command window and closes the plot window when
           % the code is rerun
```

Specify the constants

H = 0.35;   % half-length of the dipole

ns = 91;   % number of segments on the dipole

etc.

Set all variables to zero, which also allocates space for the variables

Vm = zeros(ns+2, 1) % voltage amplitude vector

Z1n = zeros (1, ns) % 1st row vector for the initial matrix

Z2n = zeros (ns,ns) % initial matrix

Zmn = zeros (ns+2, ns+2) % complete matrix with half pulses

etc.

Calculate the 1st row vector

Create the toeplitz matrix

```
Z2n = toeplitz(real(Z1n))+1i*toeplitz(imag(Z1n));   % impedance matrix
```

Create the final matrix Zmn

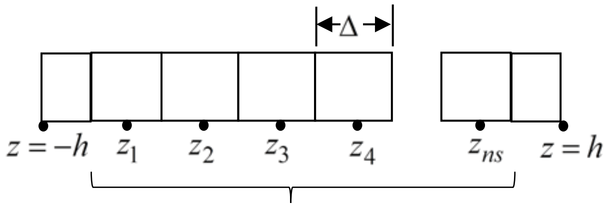
Calculate the current

```
In = Zmn\Vm;
```

Plot

```
%figure(1)       % Create a separate first figure window
cr = real(In);   % Real part of the current
plot(z,cr);
hold on % Allows another graph to be plotted on the same figure
%figure(2)       %Create a separate second figure window
ci = imag(In);   %Imaginary part of current
plot(z,ci,':')   % ':' designates a dotted line graph
%figure(3)       %plot an axis line at 0 amps
plot(z,cline)
xlabel('Position in Wavelengths')
ylabel('Current Amplitude')
hold off % Allows another graph to be plotted on the same figure
legend('Real','Imag') %used when plotting the 2 figs. together
```

Create Z1n = zeros (1, ns) % 1st row vector for the initial matrix



Create a row vector with
these pulses

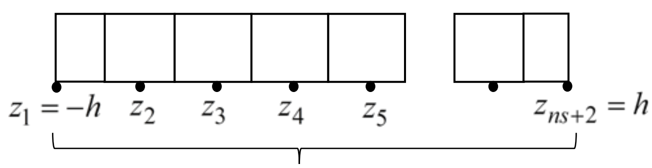
Create the toeplitz matrix using the row vector above

```
Z2n = toeplitz(real(Z1n))+1i*toeplitz(imag(Z1n));   % impedance matrix
```

or

$Z2n$ = toeplitz(Z1n,Z1n)

Create the final matrix Zmn



Create a final matrix with
all of these grid points

In the final matrix force the half-pulse values to zero

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & Toplitz & \vdots & 0 \\ 0 & \vdots & Matrix & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{ns+2} \end{bmatrix} = \begin{bmatrix} V_1 = 0 \\ V_2 \\ \vdots \\ V_{ns+2} = 0 \end{bmatrix}$$

Calculate the current

```
In = Zmn\Vm;
```