638 Notes 6

The electric and magnetic field due to a dipole of length l.

$$A_z = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-l/2}^{l/2} I(z')e^{jkz'\cos\theta}dz' \qquad (1)$$

1) $E_\theta = j\omega A_z \sin\theta \qquad (2)$

2) $H_\phi = \dfrac{E_\theta}{\eta} \qquad (3)$

$$A_z = \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\left\{\underbrace{\int_{z'=-l/2}^{0} I_0 \sin\left[k\left(\frac{l}{2}+z'\right)\right]e^{jkz'\cos\theta}dz'}_{\doteq I_1} + \underbrace{\int_{z'=0}^{l/2} I_0 \sin\left[k\left(\frac{l}{2}-z'\right)\right]e^{jkz'\cos\theta}dz'}_{\doteq I_2}\right\}$$

From integral tables (CRC)

$$\int \sin(c+bz)e^{az}dz = \frac{e^{az}}{a^2+b^2}\left[a\sin(c+bz) - b\cos(c+bz)\right]$$

for I_1: $a = jk\cos\theta$, $b = k$, $c = \dfrac{kl}{2}$

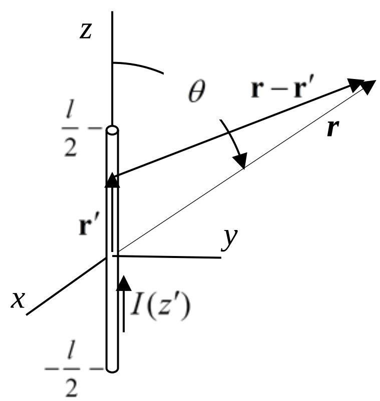
I_2: $a = jk\cos\theta$, $b = -k$, $c = \dfrac{kl}{2}$

$$A_z = \frac{\mu I_0}{k2\pi}\frac{e^{-jkr}}{r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin^2\theta}\right] \qquad (4)$$

$$E_\theta = j\omega A_z \sin\theta$$

$$E_\theta = \frac{j\eta I_0}{2\pi}\frac{e^{-jkr}}{r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta}\right] \qquad (5)$$

$$H_\phi = \frac{E_\theta}{\eta} \qquad (6)$$



ex/ $l = \dfrac{\lambda}{2}$ dipole

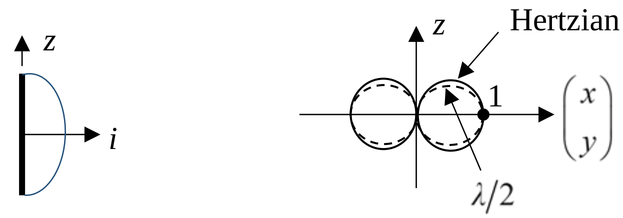
$$E_\theta = \frac{j\eta I_0}{2\pi}\frac{e^{-jkr}}{r}\left[\frac{\cos\left[\frac{1}{2}\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{2}\cos\theta\right] - \cos\left[\frac{1}{2}\left(\frac{2\pi}{\lambda}\right)\frac{\lambda}{2}\right]}{\sin\theta}\right]$$

$$= \frac{j\eta I_0}{2\pi}\frac{e^{-jkr}}{r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right] \qquad (7)$$

$$E_{\theta,\max(\theta,\phi)} = \frac{j\eta I_0}{2\pi}\frac{e^{-jkr}}{r} \quad \text{when } \theta = 90^\circ \qquad (8)$$

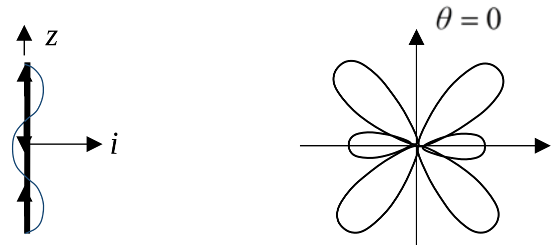
- Pattern Function from (7)

$$E_\theta^{norm} = \frac{|E_\theta|}{|E_\theta|_{\max(\theta,\phi)}} = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \qquad (9)$$

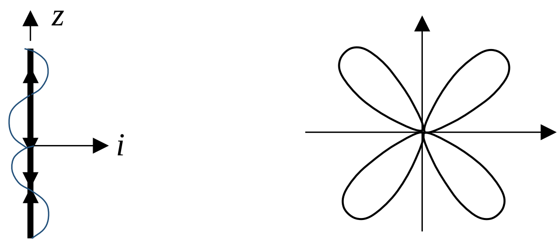


From (5):

$l = 1.5\lambda$



$l = 2\lambda$



- The power radiated, radiation resistance, radiation and reflection efficiency of a $\frac{\lambda}{2}$ dipole.

$$\mathbf{P}_{\text{ave}} = \frac{1}{2}\text{Re}\left\{E_\theta\hat{\boldsymbol{\theta}} \times H_\phi^*\hat{\boldsymbol{\phi}}\right\} = \frac{|E_\theta|^2}{2\eta}\hat{\mathbf{r}}$$

$$= \underbrace{\frac{1}{2\eta}\left(\frac{\eta I_0}{2\pi}\right)^2}_{\doteq Q} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{r^2\sin^2\theta}\hat{\mathbf{r}} \qquad (10)$$

$$P_{\text{r}} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} P_{\text{ave}} r^2 \sin\theta d\theta d\phi$$

$$= Q\int_{\theta=0}^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\cancel{r^2}\sin^{\cancel{2}}\theta}\cancel{r^2}\,\cancel{\sin\theta}\,d\theta \underbrace{\int_0^{2\pi} d\phi}_{=2\pi}$$

$$= 2\pi Q \underbrace{\int_{\theta=0}^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} d\theta}_{=1.218}$$

$$\boxed{\text{from tables} \quad \left.\frac{\mathrm{Cin}(2\theta)}{2}\right|_0^{\pi} = \frac{\mathrm{Cin}(2\pi)}{2} - \cancel{\frac{\mathrm{Cin}(0)}{2}}^{0} = 1.218}$$

$$= \left[\frac{\eta}{2}\frac{I_0{}^2}{(2\pi)^{\cancel{2}}}\right]\cancel{(2\pi)}(1.218)$$

$$P_r = \frac{\eta}{2} I_0{}^2 \left(\frac{0.609}{\pi}\right) \tag{11}$$

$$R_r = \frac{P_r}{\underset{\substack{\uparrow \\ \text{max amplitude}}}{I_0{}^2}/2} = \frac{\eta(0.609)}{\pi} = \frac{120\pi(0.609)}{\pi}$$

$$R_r = 73\Omega \tag{12}$$

$$e_r = \frac{R_r}{R_r + \underbrace{R_\Omega}_{\simeq 2\Omega}} = \frac{73}{73+2} = \frac{73}{75} = 0.973$$

$$\Rightarrow 97.3\% \text{ efficient} \tag{13}$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{73-75}{73+75} = 0.0135 \quad \text{(Fed by a } 75\Omega \text{ coaxial cable)}$$

$$e_{\text{ref}} = \left(1 - |\Gamma|^2\right) = 1 - (0.0135)^2 \cong 0.99982 \simeq 1.0 \tag{14}$$

- $\frac{\lambda}{2}$ dipole maximum Directivity and maximum Gain

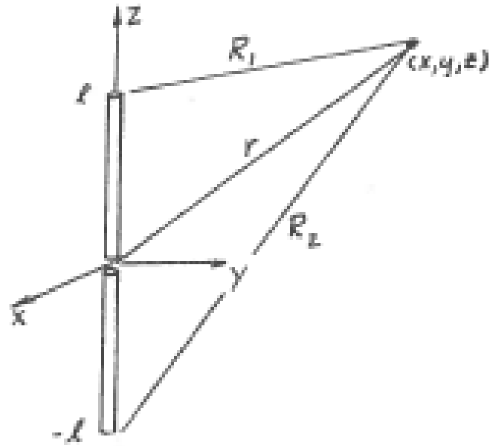
$$D_0 = \frac{\overset{\substack{\text{time ave. power } \underline{\underline{\text{density}}} \\ \downarrow}}{P_{\text{ave}}(\theta,\phi)_{\max}}}{\frac{P_r}{\underbrace{4\pi r^2}_{\substack{\text{ave. power radiated} \\ \text{through a sphere}}}}} = \frac{\overbrace{\left[r^2 P_{\text{ave}}(\theta,\phi)_{\max}\right]}^{\doteq U_{\max}(\text{rad. intensity})}}{\left[\frac{P_r}{4\pi}\right]} \doteq \frac{U_{\max}}{U_{\text{ave}}}$$

$$D_0 = \frac{4\pi \cancel{r^2} \cancel{\frac{\eta}{2}} \left(\frac{\cancel{I_0}}{2\pi}\right)^2 \cancel{\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{r^2 \sin^2\theta}}^{1}}{\cancel{\frac{\eta}{2}} \cancel{I_0}^2 \left(\frac{0.609}{\pi}\right)}$$

$$D_0 = \frac{1}{0.609} = 1.642 \tag{15}$$

$$G_0 = D_0 e_r = 1.642(0.973) = 1.598 \tag{16}$$

- <u>Exact field of a dipole</u> - with an assumed current distribution



$$I = I_m \sin[k(\ell - |z'|)]$$

$$A_z(x,y,z) = \frac{\mu}{4\pi} \int_{-\ell}^{\ell} I_m \sin[k(\ell - |z'|)] \frac{e^{-jkR}}{R} dz'$$

$$R = [x^2 + y^2 + (z-z')^2]^{1/2}$$

change sine into exponential form

$$A_z(x,y,z) = \frac{I_m \mu}{j8\pi} \Big[e^{jk\ell} \int_{-\ell}^{0} \frac{e^{-jk(R-z')}}{R} dz' - e^{-jk\ell} \int_{-\ell}^{0} \frac{e^{-jk(R+z')}}{R} dz'$$

$$+ e^{jk\ell} \int_{0}^{\ell} \frac{e^{-jk(R+z')}}{R} dz' - e^{-jk\ell} \int_{0}^{\ell} \frac{e^{-jk(R-z')}}{R} dz' \Big] \quad (1)$$

(no ϕ variation in the potential)

$$\bar{H} = \frac{1}{\mu} \nabla \times \bar{A} = \frac{1}{\mu} \Big[\cancel{\frac{1}{\rho} \frac{\partial}{\partial \phi} A_z \hat{\rho}}^{\,0} - \frac{\partial}{\partial \rho} A_z \hat{\phi} \Big] \quad (2)$$

so

$$H_\phi = -\frac{1}{\mu} \frac{\partial}{\partial \rho} A_z$$

For the first integral in (1)

$$H_{\phi,1} = \frac{-I_m}{j8\pi} e^{jk\ell} \int_{-\ell}^{0} \frac{\partial}{\partial \rho} \Big[\frac{e^{-jk(R-z')}}{R} \Big] dz' = -\frac{I_m e^{jk\ell}}{j8\pi} \int_{-\ell}^{0} \rho \Big[\frac{jk}{R^2} + \frac{1}{R^3} \Big] e^{-jk(R-z')} dz'$$

$$= \frac{I_m}{j8\pi} e^{jk\ell} \rho \int_{-\ell}^{0} \frac{\partial}{\partial z'} \Big[\frac{e^{-jk(R-z')}}{R[R+z-z']} \Big] dz' \quad (3)$$

perfect differential

$$H_{\phi,1} = \frac{I_m \rho}{j8\pi} e^{jkl} \left[\frac{e^{-jkr}}{r(r+z)} - \frac{e^{-jk(R_2+l)}}{R_2(R_2+z+l)} \right] \qquad (4)$$

$R_2 = [x^2+y^2+(z+l)^2]^{1/2} = [\rho^2+(z+l)^2]^{1/2}$
$r = [x^2+y^2+z^2]^{1/2}$

or

$$H_{\phi,1} = \frac{I_m e^{jkl}}{j8\pi\rho} \left[\frac{r-z}{r} e^{-jkr} - \frac{R_2-(z+l)}{R_2} e^{-jk(R_2+l)} \right] \qquad (5)$$

The other integrals in (1) can be evaluated to give

$$H_\phi = H_{\phi,1} + H_{\phi,2} + H_{\phi,3} + H_{\phi,4}$$

$$H_\phi = -\frac{I_m}{j4\pi\rho} \left[e^{-jkR_1} + e^{-jkR_2} - (2\cos kl) e^{jkr} \right] \qquad (6)$$

$R_1 = [x^2+y^2+(z-l)^2]^{1/2} = [\rho^2+(z-l)^2]^{1/2}$

From Maxwell's equations

$$\bar{E} = \frac{1}{j\omega\epsilon_0} \nabla \times \bar{H} = \frac{1}{j\omega\epsilon_0} \left[\frac{1}{\rho} \frac{\partial}{\partial\rho} \rho H_\phi \hat{z} - \frac{1}{\rho} \frac{\partial}{\partial z} \rho H_\phi \hat{\rho} \right]$$

$$\bar{E} = -j\frac{\eta}{4\pi} I_m \left(\frac{e^{-jkR_1}}{R_1} + \frac{e^{-jkR_2}}{R_2} - 2\cos(kl) \frac{e^{-jkr}}{r} \right) \hat{z} \qquad (7)$$

$$+ j\frac{\eta}{4\pi} I_m \left(\frac{z-l}{\rho} \frac{e^{-jkR_1}}{R_1} + \frac{z+l}{\rho} \frac{e^{-jkR_2}}{R_2} - (2\cos kl) \frac{z}{\rho} \frac{e^{-jkr}}{r} \right) \hat{\rho}$$

Review:

Recall that to find the field radiated by an antenna

we had to solve the equation

$$\mathbf{E} = \frac{-j\omega}{k^2}\left(k^2\mathbf{A} + \nabla\nabla\cdot\mathbf{A}\right) \qquad (1)$$

where

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_{v'} \frac{\mathbf{J}(\mathbf{r}')e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}dv' \qquad (2)$$

However, for a thin dipole antenna where the current is restricted to travel up and down on the antenna in the z-direction (2) could be written

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int_{z'} \frac{I(z')\hat{\mathbf{z}}\ e^{-jk|\mathbf{r}-z'\hat{\mathbf{z}}|}}{|\mathbf{r}-z'\hat{\mathbf{z}}|}dz' \qquad (3)$$

Before we knew what the current was on the antenna and we wanted to find the field. Now suppose we know what the field is on the antenna and we want to find the current on the antenna. When we knew what the current was we were able to find the radiated field $\boldsymbol{E}^{rad}$ but now we will have $\boldsymbol{E}^{inc}$, the electric field incident on the antenna. The relationship between these fields on a perfect conducting antenna is the boundary condition:

$$\boldsymbol{E}^{rad} + \boldsymbol{E}^{inc} = 0 \quad \Rightarrow \quad \boldsymbol{E}^{rad} = -\boldsymbol{E}^{inc} \qquad (4)$$

Notice in (3) the vector potential can only be in the z-direction because the current is in the z-direction. Also only the z-directed component of the electric field on the surface of the antenna can create a z-directed current. So final equations are

$$E_z^{inc} = \frac{j\omega}{k^2}\left(k^2 A_z + \frac{\partial^2}{\partial z^2}A_z\right); \;\; A_z(z) = \frac{\mu}{4\pi}\int_{z'}\frac{I(z')e^{-jk|z-z'|}}{|z-z'|}dz'$$

or the integro-differential equation for $I(z)$,

$$E_z^{inc} = \frac{j\omega\mu}{4\pi k^2}\left(k^2 + \frac{\partial^2}{\partial z^2}\right)\int_{z'}\frac{I(z')e^{-jk|z-z'|}}{|z-z'|}dz' \qquad (5)$$

Projection or Weighted-Residual Methods

Consider an integro-differential equation such as

$$\left(\frac{d^2}{dz^2}+k^2\right)\int I(z')\frac{e^{-jk|z-z'|}}{|z-z'|}dz' = E^{inc}(z)$$

written in operator equation form

$$L(z,z')I(z') = E^{inc}(z) \qquad (6)$$

operator source

current
(unknown)

where $L(z,z')=\left(\frac{d^2}{dz^2}+k^2\right)\int \frac{e^{-jk|z-z'|}}{|z-z'|}dz'$

Approximate

$$I(z') \cong \sum_{n=1}^{N} I_n\,\alpha_n(z') \qquad (7)$$

Where $\alpha_n(z')$ are basis functions chosen so as to satisfy boundary conditions and I_n are the amplitude coefficients of $\alpha_n(z')$.
Apply the operator L to (7)

$$L(z,z')I(z') \cong \sum_{n=1}^{N} L(z,z')\left(I_n\,\alpha_n(z')\right) = \sum_{n=1}^{N} I_n\,L(z,z')\alpha_n(z') \qquad (8)$$

Substitute (8) into (6)

$$\sum_{n=1}^{N} I_n L(z,z')\alpha_n(z') = E^{inc}(z) \qquad (9)$$

or

$$\sum_{n=1}^{N} I_n\left(\frac{d^2}{dz^2}+k^2\right)\int \alpha_n(z')\frac{e^{-jk|z-z'|}}{|z-z'|}dz' = E^{inc}(z) \qquad (10)$$

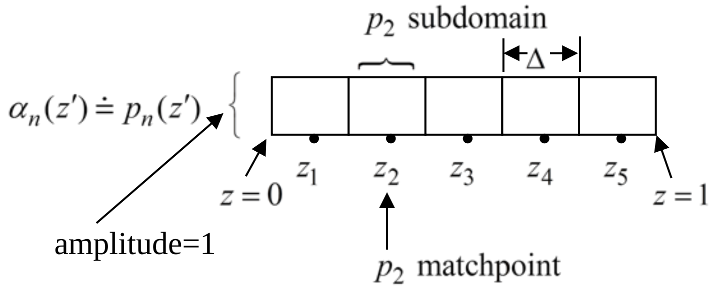
$$I_1L(z,z')\alpha_1(z')+I_2L(z,z')\alpha_2(z')+I_3L(z,z')\alpha_3(z')+\ldots+I_NL(z,z')\alpha_N(z') = E^{inc}(z) \qquad (11)$$

But this gives 1 equation with N unknowns, $I_1, I_2, I_3, \ldots I_N$.

Basis functions should be

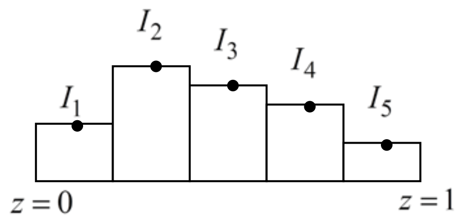
a) linearly independent (orthogonal)
b) able to approximate the function reasonably well
c) capable of satisfying boundary conditions

Ex: Pulse Basis Functions Subdomain basis functions also called basis functions of local support.



where $p_n = \begin{cases} 1, & z_n - \Delta/2 \le z \le z_n + \Delta/2 \\ 0, & \text{otherwise} \end{cases}$

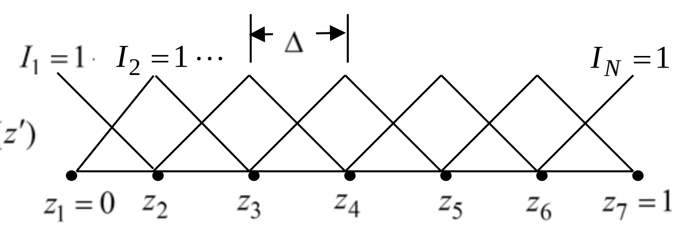
$I(z') \cong \sum I_n p_n(z') =$



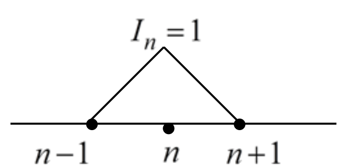
Pulse basis functions are adequate in many cases but a derivative of a pulse produces an undesirable singularity and a second derivative is undefined. A better basis is a piecewise linear (triangle) function.

Ex: Triangular basis functions

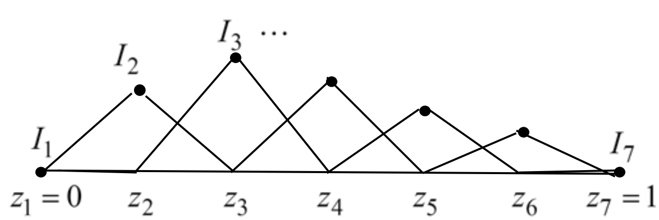
$\alpha_n(z') \doteq \Lambda_n(z')$



$$\Lambda_n(z') = \begin{cases} 1 - \dfrac{|z' - z_n|}{\Delta}, & z_{n-1} \le z' \le z_{n+1} \\ 0, & \text{otherwise} \end{cases}$$



$I \cong \sum I_n \Lambda_n(z') =$



Pulse basis functions in equation (10) gives,

$$\sum_{n=1}^{N} I_n \left(\frac{d^2}{dz^2} + k^2 \right) \int p_n(z') \frac{e^{-jk|z-z'|}}{|z-z'|} dz' = E^{inc}(z) \qquad (12)$$

$$\text{where} \quad p_n = \begin{cases} 1, & z_n - \Delta/2 \le z \le z_n + \Delta/2 \\ 0, & \text{otherwise} \end{cases}$$

which results in,

$$\sum_{n=1}^{N} I_n \left(\frac{d^2}{dz^2} + k^2 \right) \int_{z'=z_n-\Delta/2}^{z'=z_n-\Delta/2} \frac{e^{-jk|z-z'|}}{|z-z'|} dz' = E^{inc}(z) \qquad (13)$$

Expanding the summation

$$\begin{aligned} I_1 \left(\frac{d^2}{dz^2} + k^2 \right) \int_{z'=z_1-\Delta/2}^{z'=z_1-\Delta/2} \frac{e^{-jk|z-z'|}}{|z-z'|} dz' + I_2 \left(\frac{d^2}{dz^2} + k^2 \right) \int_{z'=z_2-\Delta/2}^{z'=z_2-\Delta/2} \frac{e^{-jk|z-z'|}}{|z-z'|} dz' + \cdots \\ + I_N \left(\frac{d^2}{dz^2} + k^2 \right) \int_{z'=z_N-\Delta/2}^{z'=z_N-\Delta/2} \frac{e^{-jk|z-z'|}}{|z-z'|} dz' = E^{inc}(z) \end{aligned} \qquad (14)$$

which can also be written like (11) with p_n basis functions,

$$I_1 L(z,z') p_1(z') + I_2 L(z,z') p_2(z') + ... + I_N L(z,z') p_N(z') = E^{inc}(z) \qquad (15)$$

To get N equations we use the <u>Point matching method</u> – this 'testing function' is a set of delta functions that are at the center of each pulse,

$$\delta(z - z_m), \text{ where } m = 1, 2, 3, \cdots, N \qquad (14)$$

The first delta function $\delta(z - z_1)$ multiplies each term in (15) and is integrated over the length Ω of the antenna.

$$\begin{aligned} I_1 \int_{\Omega} \delta(z-z_1) L(z,z') p_1(z') dz + I_2 \int_{\Omega} \delta(z-z_1) L(z,z') p_2(z') dz + ... \\ + I_N \int_{\Omega} \delta(z-z_1) L(z,z') p_N(z') dz = \int_{\Omega} \delta(z-z_1) E^{inc}(z) dz \end{aligned} \qquad (16)$$

The "tested" equation at $z = z_1$

$$I_1 L(z_1,z') p_1(z') + I_2 L(z_1,z') p_2(z') + \cdots + I_N L(z_1,z') p_N(z') = E^{inc}(z_1) \qquad (17)$$

Repeating this process with (15) and each delta testing function $\delta\left(z-z_m\right)$, where $m=1,2,3,\cdots,N$ gives the set of equations,

$$I_1L(z_1,z')p_1\left(z'\right)+I_2L(z_1,z')p_2\left(z'\right)+I_3L(z_1,z')p_3\left(z'\right)+\cdots+I_NL(z_1,z')p_N\left(z'\right)=E^{inc}(z_1)$$

$$I_1L(z_2,z')p_1\left(z'\right)+I_2L(z_2,z')p_2\left(z'\right)+I_3L(z_2,z')p_3\left(z'\right)+\cdots+I_NL(z_2,z')p_N\left(z'\right)=E^{inc}(z_2)$$

$$I_1L(z_3,z')p_1\left(z'\right)+I_2L(z_3,z')p_2\left(z'\right)+I_3L(z_3,z')p_3\left(z'\right)+\cdots+I_NL(z_3,z')p_N\left(z'\right)=E^{inc}(z_3)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \cdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$I_1L(p_N,z')p_1\left(z'\right)+I_2L(z_N,z')p_2\left(z'\right)+I_3L(z_N,z')p_3\left(z'\right)+\ldots+I_NL(z_N,z')p_N\left(z'\right)=E^{inc}(z_N)$$

The matrix equation is therefore

$$\begin{bmatrix} L(z_1,z')p_1\left(z'\right) & L(z_1,z')p_2\left(z'\right) & L(z_1,z')p_3\left(z'\right) & \ldots & L(z_1,z')p_N\left(z'\right) \\ L(z_2,z')p_1\left(z'\right) & L(z_2,z')p_2\left(z'\right) & L(z_2,z')p_3\left(z'\right) & \ldots & L(z_2,z')p_N\left(z'\right) \\ L(z_3,z')p_1\left(z'\right) & L(z_3,z')p_2\left(z'\right) & L(z_3,z')p_3\left(z'\right) & \ldots & L(z_3,z')p_N\left(z'\right) \\ \vdots & \vdots & \vdots & & \vdots \\ L(z_N,z')p_1\left(z'\right) & L(z_N,z')p_2\left(z'\right) & L(z_N,z')p_3\left(z'\right) & \cdots & L(z_N,z')p_N\left(z'\right) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} E^{inc}\left(z_1\right) \\ E^{inc}\left(z_2\right) \\ E^{inc}\left(z_3\right) \\ \vdots \\ E^{inc}\left(z_N\right) \end{bmatrix} \tag{18}$$

or symbolically

$$\bar{\boldsymbol{Z}}\,\boldsymbol{I}=\boldsymbol{E}$$

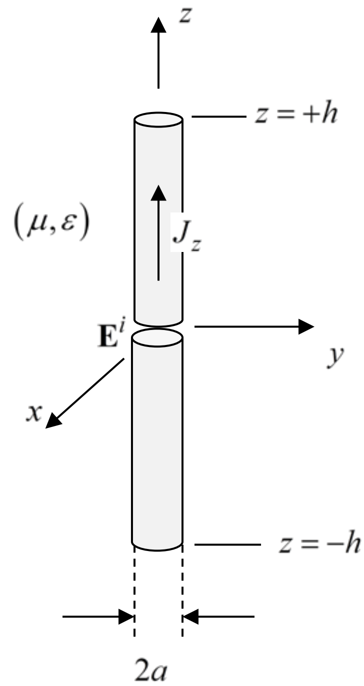
which can be solved by matrix inversion

$$\boldsymbol{I}=\bar{\boldsymbol{Z}}^{-1}\boldsymbol{E} \tag{19}$$

The current is therefore approximately

$$I\left(z\right)\cong\sum_{n=1}^{N}I_np_n\left(z\right) \tag{20}$$

In general the integral equation must include ϕ integration.



$\hat{\mathbf{n}}\times(\mathbf{E}_1-\mathbf{E}_2)=0$

where

$\mathbf{E}_1=\mathbf{E}^i+\mathbf{E}^s$, the total field outside the conductor (wire)

$\mathbf{E}_2=0$, the total field inside the conductor

$\Rightarrow \hat{\mathbf{n}}\times\left(\mathbf{E}^{\mathrm{i}}+\mathbf{E}^s\right)=0$

Since

$$\hat{\mathbf{n}}\times\left(\mathbf{E}^i+\mathbf{E}^s\right)=\hat{\boldsymbol{\rho}}\times\left(E^i_\rho\hat{\boldsymbol{\rho}}+E^i_z\hat{\mathbf{z}}+E^s_\rho\hat{\boldsymbol{\rho}}+E^s_z\hat{\mathbf{z}}\right)$$
$$=\left(E^i_z+E^s_z\right)\hat{\boldsymbol{\varphi}}=0$$

then

$$E^i_z+E^s_z=0 \quad\Rightarrow\quad E^i_z=-E^s_z$$

which is entirely scalar.

Before we had $\mathbf{E}^s=\dfrac{-j\omega}{k^2}\left(k^2\mathbf{A}+\nabla\nabla\cdot\mathbf{A}\right)$,

but because

$$\mathbf{J}=J_z\hat{\mathbf{z}} \;\Rightarrow \mathbf{A}=A_z\hat{\mathbf{z}} \text{ and from boundary conditions } E^i_z=-E^s_z$$

this equation becomes

$$E^i_z=\frac{j\omega}{k^2}\left(k^2A_z+\frac{\partial^2}{\partial z^2}A_z\right)$$

on a thin wire scatterer with

$$A_z=\frac{\mu}{4\pi}\int_{-h}^{h}J_z(z')\int_{-\pi}^{\pi}\frac{e^{-jkR}}{R}a\,d\phi' dz'$$

where J_z is a surface current (A/m). Since $I(z)=2\pi a\,J_z(z)$ this equation can be written

$$A_z = \frac{\mu}{4\pi}\int_{-h}^{h} I(z')G(z-z')dz'$$

where

$$G(z-z') = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-jkR}}{R}d\phi' \quad \text{and}$$

$$R = \left[(z-z')^2 + 4a^2\sin^2\frac{\phi'}{2}\right]^{1/2}$$

or

$$G(z-z') = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-jk\left[(z-z')^2+4a^2\sin^2\frac{\phi'}{2}\right]^{1/2}}}{\left[(z-z')^2+4a^2\sin^2\frac{\phi'}{2}\right]^{1/2}}d\phi'$$

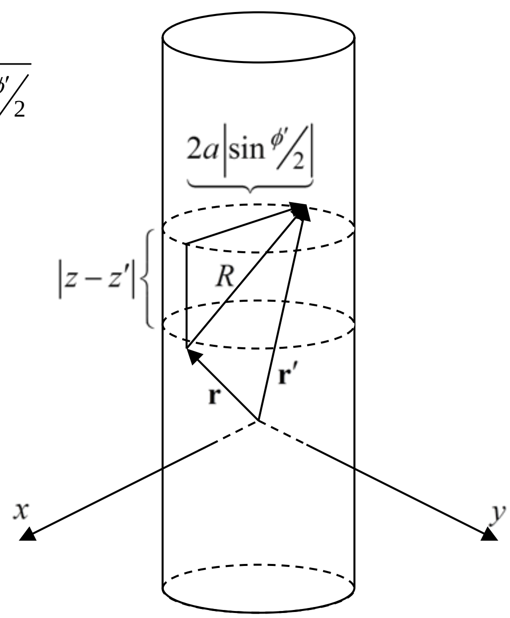
This integral can be shown to have a log singularity

at the lower limit $\frac{\phi'}{2}\to 0,\ \frac{(z-z')}{2a}\to 0,\ \Rightarrow \lim G(z-z')\to -\frac{1}{\pi a}\ln\left(\frac{|z-z'|}{8a}\right)$

<u>The reduced kernal</u> method gets rid of the singularity in R and still has a reasonably accurate solution. Starting with

$$G(z-z') = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-jkR}}{R}d\phi' \quad ; \quad R = \sqrt{(z-z')^2 + 4a^2\sin^2 \phi'/2}$$

notice $2a\left|\sin \phi'/2\right|$ ranges from 0 to $2a$ as ϕ' goes from $-\pi$ to π. When $|z-z'| >> a$ we take the average value of $2a\left|\sin \phi'/2\right|$ which is approximately $2a\left|\sin \phi'/2\right| \approx a$,



Using this observation approximate

$$\Rightarrow R \to R_r = \sqrt{(z-z')^2 + a^2}$$

and

$$G(z-z') \approx \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-jkR_r}}{R_r} d\phi'$$

$$\Rightarrow G(z-z') = \frac{e^{-jkR_r}}{R_r}$$

We now want to solve the integral equation on a thin wire antenna where

$$\frac{j\omega}{k^2}\left(k^2 + \frac{\partial^2}{\partial z^2}\right)A_z = E_z^i \qquad \text{with}$$

$$A_z = \frac{\mu}{4\pi}\int_{-h}^{h} I(z') \frac{e^{-jk\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}} dz'$$

Now move the differentiation inside the integration

$$\frac{j\omega\mu}{k^2 4\pi}\int_{-h}^{h} I(z')\left(k^2 + \frac{\partial^2}{\partial z^2}\right)\frac{e^{-jk\sqrt{(z-z')^2+a^2}}}{\sqrt{(z-z')^2+a^2}} dz' = E_z^i$$

and differentiate

$$\frac{j\eta}{k}\int_{z'=-h}^{h} I(z')\frac{e^{-jkR_r}}{4\pi R_r^5}\left[(1-jkR_r)^2(2R_r - 3a^2) + (kaR_r)^2\right]dz' = E_z^i$$

One last change is that we want to know the input impedance of the antenna, so the incident field will be coming from a source feeding the antenna in the feed gap from a transmission line. For a linear wire antenna with a with a feed point gap at the center, the integral equation for the current *I*(*z*) is obtained by applying $E_z = -\frac{\partial V}{\partial z}$ which comes from $\mathbf{E} = -\nabla V$, a static field approximation which ignores the $-j\omega\mathbf{A}$ time harmonic contribution to E_z. This creates the numerical replacement $E_z(a,z) \simeq -V(z=0)/\Delta z$ in our equation,

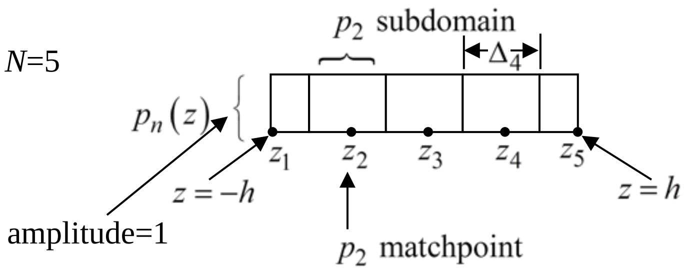
$$\frac{j\eta}{k}\int_{z'=-h}^{h} I(z')\frac{e^{-jkR_r}}{4\pi R_r^5}\left[(1+jkR_r)(2R_r^2 - 3a^2) + (kaR_r)^2\right]dz' = -\frac{V}{\Delta z} \quad *$$

Homework:

Eqn. (*) is a form of Pocklington's integral equation.

(a) Solve the integral equation * for the current on a thin wire antenna of length $2h$ using pulse expansion functions with ½ pulses at the two ends and delta function testing. Use a frequency f = 300 MHz, and a feed point voltage $V = 1$ volt, which means the feed point electric field is $1/\Delta$ V/m. Compare your results, real and imaginary parts, with those in the first figure on the following page. All of the needed specifications are in the figure.

ex/ N=5



Force the pulse amplitudes at the ends to zero $(I_1 = I_N = 0)$, so the numerical code should just solve for the current amplitudes $I_2 \to I_{N-1}$, but your plot of the current should include the zeros at the ends.

(b) Obtain the input admittance and compare with that in second figure on the following page. The input admittance is found by dividing the gap current by the gap voltage, which is $V = 1$ volt in your code.

(c) Obtain the normalized amplitude of the far electric field of the antenna on a polar plot when $2h = 4\lambda/5$ and $2h = \lambda$.

The far field of a dipole antenna is given by

$$E_\theta \cong j\omega A_z \sin\theta$$

where

$$A_z \cong \frac{\mu}{4\pi r} e^{-jkr} \int_{-h}^{h} I(z') e^{jkz'\cos\theta} dz'$$

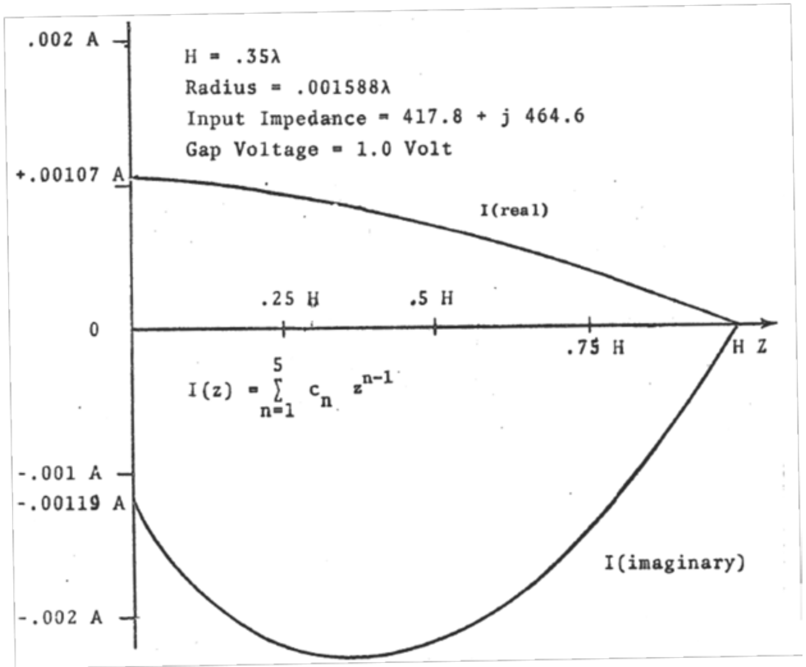
Since the current has been found by the approximation

$$I(z) \doteq \sum_n I_n p_n(z)$$

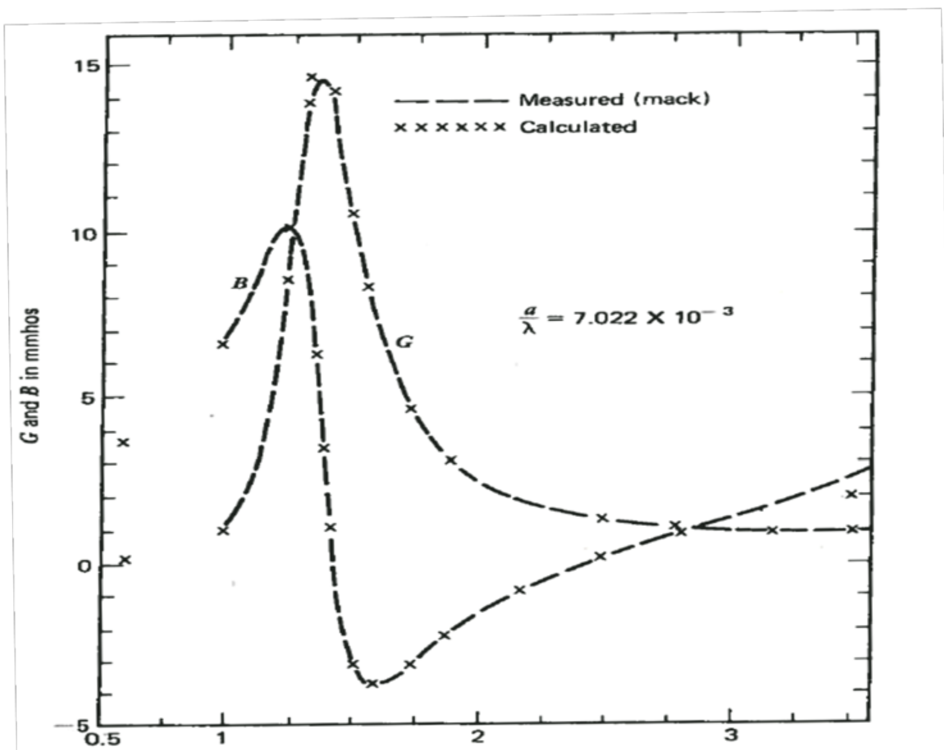
the potential breaks down into the simple series

$$A_z \cong \frac{\mu}{4\pi r} e^{-jkr} \sum_n I_n e^{jkz_n \cos\theta} \Delta_n$$

Therefore: $$E_\theta \cong \frac{j\omega\mu}{4\pi r} e^{-jkr} \sin\theta \sum_n I_n e^{jkz_n \cos\theta} \Delta_n$$



Real and imaginary current (above) and input impedance (below) of a 2H = 0.7λ antenna. In the past we have found that N=99 pulse functions worked best.



Calculated and measured dipole admittance with data calculated using pulse basis functions (N=99).

$$\frac{\beta L}{2} = \frac{2\pi}{\lambda}\frac{L}{2} = \pi\frac{L}{\lambda}$$

$$0.5 \le \frac{\beta L}{2} \le 3.5$$

$$\frac{0.5}{\pi} \le \frac{L}{\lambda} \le \frac{3.5}{\pi}$$