bits. It is important to note that for virtual qubits only one type of error is possible, so these qubits require only one bit per step.

For a given error string E, calculating the values of the check generators, thus the corresponding three-dimensional syndrome ∂E, is straightforward. We need to check the parity of the overlapping qubits between the error string and each check generator. To decode the three-dimensional syndrome ∂E and get a correction operator $C_{\partial E}$, I constructed the syndrome graph and ran minimum weight perfect matching decoder on it as it is implemented in PyMatching [37, 38]. By checking the parity of error + correction string endings at each boundary I identified a logical Pauli error based on Tab. 2.

To obtain probabilities for each logical Pauli error I sampled error strings from two inequivalent error distributions a.k.a. error models.

The independent error model contains independent and equally probable X and Z errors. This model is described by the following single qubit error channel:

$$\begin{aligned} \varepsilon(\rho) &= (1-p)^2\rho + p(1-p)(X\rho X + Z\rho Z) + p^2 Y\rho Y; \\ q &= p, \end{aligned} \tag{21}$$

where q is the probability of readout errors.

The other phenomenological error model, the depolarizing error model is described by the following single qubit error channel and readout error strength:

$$\begin{aligned} \varepsilon(\rho) &= (1-p)\rho + \frac{p}{3}(X\rho X + Z\rho Z + Y\rho Y); \\ q &= \frac{2}{3}p. \end{aligned} \tag{22}$$

In each model I set q to a value that ensures equal timelike and spacelike error rates, thus the physical error rate p fully specifies independent and depolarizing errors.

6.2 Circuit-level noise and process tomography

Syndrome extraction circuits of surface code-based logical operations are Clifford circuits, therefore, in the case of Pauli errors (more generally Clifford errors) these circuits are efficiently simulatable due to the well-known Gottesman-Knill theorem [27]. In this work I used Stim [36] an efficient Clifford simulator to simulate circuit-level noise during the logical CNOT gate.

I considered a circuit-level noise model with a single parameter p (physical error rate), which is the strength of both initialization, gate, idling and readout errors. I used minimum weight perfect matching to decode the three-dimensional syndrome in this case as well.

The determination of logical Pauli error probabilities is a little bit more involved in the case of circuit-level noise. In contrast to phenomenological errors here, I could not investigate the check graphs, thus the error + correction string endings directly; instead I had to run logical quantum circuits with different initial states and final measurements and perform a process tomography for the logical CNOT gate. Note that the timelike boundaries for control and target patches were still perfect boundaries, I just needed to specify the perfectly initialized logical states and the error-free final measurements. The initial states and the final measurements, determining the logical circuit, are summarized in Table 3.

Initial state	Final measurements	Obtained probabilities
$\lvert\overline{00}\rangle$	$\bar{Z}_1$, $\bar{Z}_2$	$P_{00}^{ZZ}(-,+)$, $P_{00}^{ZZ}(+,-)$, $P_{00}^{ZZ}(-,-)$
$\lvert\overline{0+}\rangle$	$\bar{Z}_1$, $\bar{X}_2$	$P_{0+}^{ZX}(-,+)$, $P_{0+}^{ZX}(+,-)$, $P_{0+}^{ZX}(-,-)$
$\lvert\overline{0i}\rangle$	$\bar{Z}_1$, $\bar{Y}_2$	$P_{0i}^{ZY}(-,+)$, $P_{0i}^{ZY}(+,-)$, $P_{0i}^{ZY}(-,-)$
$\lvert\overline{i+}\rangle$	$\bar{Y}_1$, $\bar{X}_2$	$P_{i+}^{YX}(-,+)$, $P_{i+}^{YX}(+,-)$, $P_{i+}^{YX}(-,-)$
$\lvert\overline{++}\rangle$	$\bar{X}_1$, $\bar{X}_2$	$P_{++}^{XX}(-,+)$, $P_{++}^{XX}(+,-)$, $P_{++}^{XX}(-,-)$
$\lvert\overline{+0}\rangle$	$\bar{Y}_1\bar{Y}_2$	$P_{+0}^{YY}(-)$
$\lvert\overline{+i}\rangle$	$\bar{Y}_1\bar{Z}_2$	$P_{+i}^{YZ}(-)$
$\lvert\overline{i0}\rangle$	$\bar{X}_1\bar{Y}_2$	$P_{i0}^{XY}(-)$
$\lvert\overline{ii}\rangle$	$\bar{X}_1\bar{Z}_2$	$P_{ii}^{XZ}(-)$

Table 3: Initial logical states and final measurement bases for the process tomography of the logical CNOT gate. A comma separating final measurement bases, e.g., $\bar{Z}_1$, $\bar{X}_2$, means that both measurements are deterministic, therefore, logical errors can cause 3 different measurement outcomes ((-,+),(+,-),(-,-)). Non-separated final measurement bases, e.g., $\bar{Y}_1\bar{Y}_2$ means that only the product of the measurement outcome is deterministic, therefore, the only erroneous case is when the measurement outcomes are not the same (-). The probabilities of the erroneous cases are shown in the third column.

The outcomes of the final measurements in logical quantum circuits summarized in Table 3 are deterministic in the noiseless case, therefore, by sampling the logical quantum circuits we gain information about the logical noise. The probabilities, which are directly obtained from sampling the logical circuits ($P_{00}^{ZZ}(-,+), P_{00}^{ZZ}(+,-), \ldots$) can be expressed with the probabilities of logical Pauli errors (p_{IX},$p_{IY}, \ldots$) with the following system of independent equations:

$$\begin{pmatrix} P_{00}^{ZZ}(-,+) \\ P_{00}^{ZZ}(+,-) \\ P_{00}^{ZZ}(-,-) \\ P_{0+}^{ZX}(-,+) \\ P_{0+}^{ZX}(+,-) \\ P_{0i}^{ZY}(-,+) \\ P_{0i}^{ZY}(+,-) \\ P_{i+}^{YX}(-,+) \\ P_{i+}^{YX}(+,-) \\ P_{++}^{XX}(-,+) \\ P_{++}^{XX}(+,-) \\ P_{+0}^{YY}(-) \\ P_{+i}^{YZ}(-) \\ P_{i0}^{XY}(-) \\ P_{ii}^{XZ}(-) \end{pmatrix} = \begin{pmatrix} 0&0&0&1&0&0&1&1&0&0&1&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&1&1&0&0&1&1&0&0&0&0&0 \\ 0&0&0&1&1&0&0&1&1&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0&0&0&0&0&1&1 \\ 0&0&0&1&0&1&0&1&0&1&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0&0&0&1&0&1 \\ 0&0&0&1&1&0&0&0&0&0&0&1&1&0&0 \\ 0&1&1&0&0&0&0&0&0&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&1&0&0&1&1&0&0 \\ 0&1&1&0&0&1&1&0&0&0&0&0&0&0&0 \\ 1&0&1&1&0&1&0&0&1&0&1&1&0&1&0 \\ 1&1&0&1&0&0&1&0&1&1&0&1&0&0&1 \\ 1&0&1&0&1&0&1&1&0&1&0&1&0&1&0 \\ 1&1&0&0&1&1&0&1&0&0&1&1&0&0&1 \end{pmatrix} \begin{pmatrix} p_{IX} \\ p_{IY} \\ p_{IZ} \\ p_{XI} \\ p_{XX} \\ p_{XY} \\ p_{XZ} \\ p_{YI} \\ p_{YX} \\ p_{YY} \\ p_{YZ} \\ p_{ZI} \\ p_{ZX} \\ p_{ZY} \\ p_{ZZ} \end{pmatrix}. \tag{23}$$

By solving this system of equations one can uniquely determine the probabilities of logical Pauli errors. Note that in Eq. 23 I did not use all the 19 probabilities which can be determined by sampling the logical circuits summarized in Table 3. The remaining 4 probabilities give us 4 non-independent equations.