Chapter 5: Introduction to Trigonometric Functions

Life is full of phenomena that repeat at regular intervals.

Example 1:

a) The tides rise and fall in response to the gravitational pull of the moon (period: 1 day)
b) The percent of the moon that appears illuminated from the Earth (period: 1 month)
c) The pattern of the seasons repeats in response to Earth's revolution around the sun (period: 1 year)
d) Outside of nature, many stocks that mirror a company's profits are influenced by changes in the economic business cycle

In mathematics, a function that repeats its values in regular intervals is known as a **periodic function**. In this chapter, we will investigate various examples of periodic functions.

Specifically, in this chapter, we will study…

5.1 Angles
- Draw angles in standard position
- Convert between different measures of angles
- Discuss coterminal angles
- Study geometric applications: arc length, area of sector

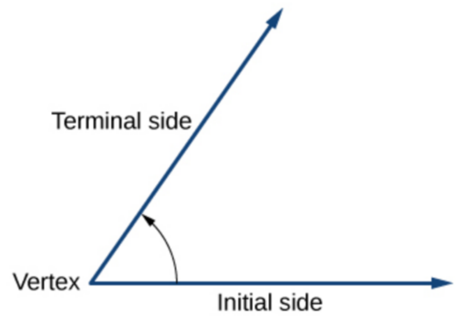
5.2 Unit Circle: Sine and Cosine Functions
- Define, study and evaluate two basic periodic functions, sine and cosine, based on the unit circle

5.3 The Other Trigonometric Functions
- Define, study and evaluate 4 other periodic functions
- Begin the study of trigonometric identities

5.4 Right Triangle Trigonometry
- Use right triangles to evaluate trigonometric functions for any angle and solve application problems

5.1: Angles

An **angle** is formed when two rays meet at a common endpoint.



We usually use Greek letters to refer to the measure of an angle:
$\theta, \phi, \alpha, \beta, \ldots$

Angles are commonly measured in **degrees**, with $360°$ representing a full revolution (or revolution)

For the purposes of this class, it's helpful to fix the initial side of an angle along the positive x-axis.

Positive angles have a counterclockwise rotation from initial side to terminal side
Negative angles have a clockwise rotation from initial side to terminal side

Example 1: Sketch each of the following angles in standard position.

a) $90°$

b) $135°$

c) $210°$

d). $-60°$

e) $400°$

Above, the angles are measured in degrees. This measure dates back to the Babylonians who used 360 days to mark a year. This would mean that 30° represents a month and 1° represents a day.

The degrees measure, then, is not a *geometric* way to measure an angle!

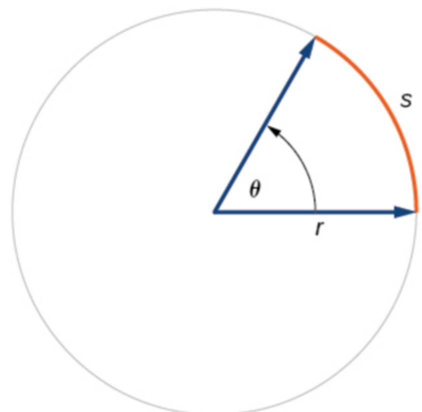
RADIAN MEASURE

Recall the Circumference C of a circle with radius r: $C = 2\pi r$

2π (radians) = 1 revolution = $360°$

The **radian** measure of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures s/r radians.

In a circle of **radius 1**, the **radian** measure corresponds to the **length of the arc**.



Example 2: Express the following in radian measure and sketch the angles in standard position

a) $180°$

b) $30°$

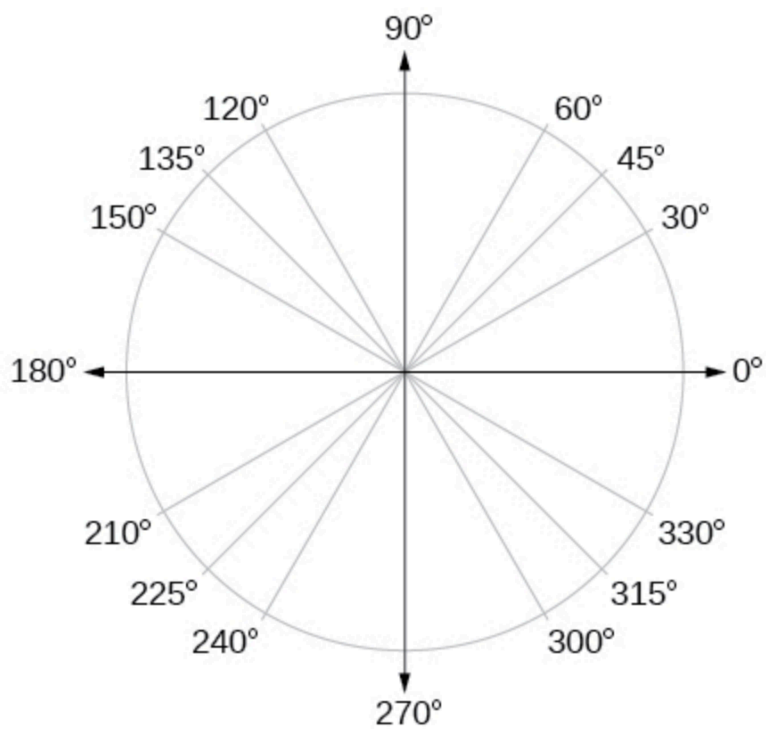
c) $120°$

c) $-45°$

Example 3: How many degrees is 1 radian?

Example 4: What quadrant does the angle measuring 4 radians in standard position lie in?

Example 5: Label the special angles with their radian measure.



CONVERTING BETWEEN RADIANS AND DEGREES

To convert between degrees and radians, use the proportion

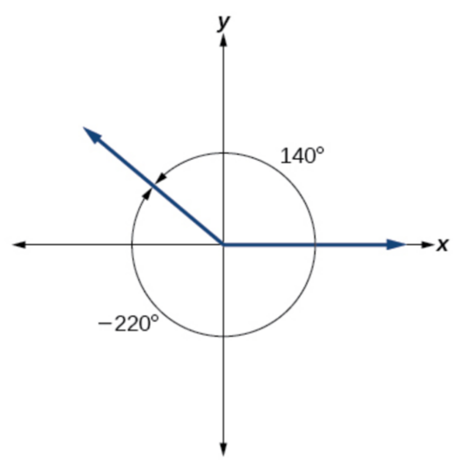
$$\frac{\theta}{180} = \frac{\theta^R}{\pi}$$

Example 6: Convert $\frac{5\pi}{12}$ into degrees.

Example 7: Convert $300°$ to radians.

COTERMINAL ANGLES

Coterminal angles are two angles in standard position that have the same terminal side.



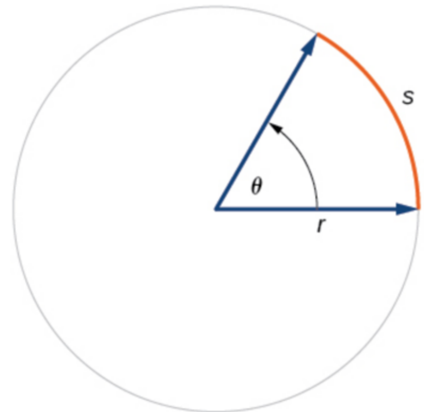
Example 8: Find two other angles (at least one positive and one negative) that are coterminal with…

a) $\mathbf{120°}$

b) $-\frac{\pi}{4}$

APPLICATION: LENGTH OF AN ARC

$s = r\theta$ $\qquad$ θ = central angle (in radians)
r = radius



<u>Example 9</u>: You are sitting in a Ferris Wheel that is 200 feet wide.

a) Suppose you board the Ferris Wheel in the "6 o'clock" position (on the clock) and move in counterclockwise direction until the "2 o'clock" position while the remaining passengers board. How much distance did you travel on the Ferris Wheel during boarding?

b) You are in the 2 o'clock position when the Ferris Wheel ride begins. It makes 10 full revolutions in the counterclockwise direction, plus a partial revolution, ending in the 9 o'clock position. How much distance did you travel during the ride?

<u>Example 10</u>: The latitude of San Francisco is about $\mathbf{37.77°}$. The latitude of Seattle is about $\mathbf{47.61°}$. Assume that San Francisco and Seattle lie on the same longitude. (Their longitudes are *very* similar.) Assuming that the Earth is shaped like a sphere with radius 3959 miles, what is the approximate distance between San Francisco and Seattle along the surface of the Earth?

APPLICATION: LINEAR SPEED VS ANGULAR SPEED

Angular speed $\boldsymbol{= \omega = \frac{\text{angle traversed}}{\text{time}} = \frac{\theta}{t}}$ ω is the lowercase Greek letter 'omega'

Linear speed $\boldsymbol{= v = \frac{\text{distance traveled}}{\text{time}} = \frac{s}{t}}$

$\boldsymbol{v = \frac{r\theta}{t}}$ $s = r\theta$ if the object is moving along a circle

$\boldsymbol{v = r\omega}$ since $\frac{\theta}{t} = \omega$

<u>Example 11</u>: A bicycle has two different sized wheels. The wheel of radius 13 inches is rotating at a constant speed of 150 revolutions per minute as the bicycle moves.

a) Find the angular speed of the wheels in in rad/min.

b) Find the speed at which the bicycle is traveling in inches per minute.

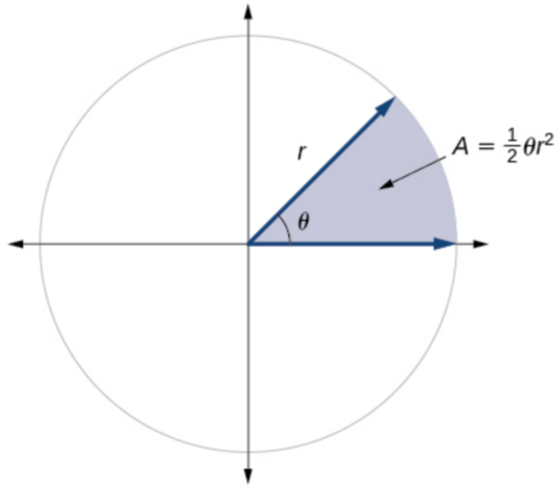
c) Convert the speed at which the bicycle is traveling to miles per hour.

d) The other wheel of the bicycle has radius 10 inches. What is the approximate RPM of that wheel?

APPLICATION: AREA OF A SECTOR

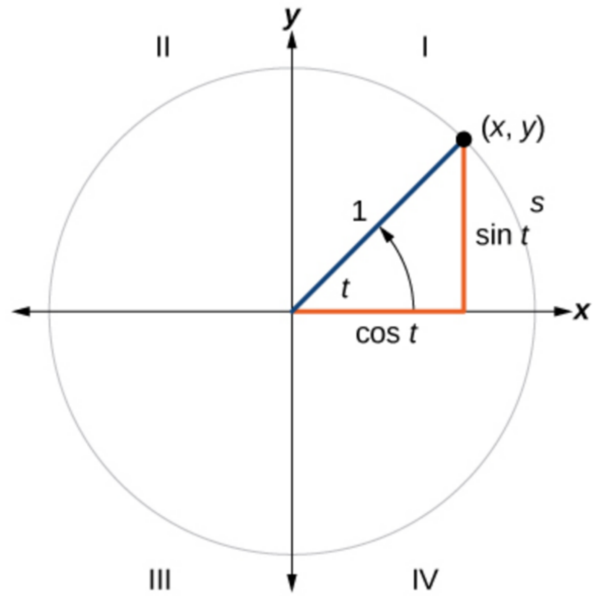
$A = \frac{1}{2}\theta r^2$ θ = central angle (in radians)

r = radius



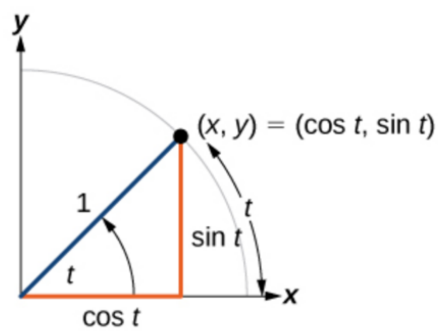
<u>Example 12</u>: A car's rear windshield wiper rotates over a central angle of $140°$. The total length of the wiping mechanism is 50 cm, with a wiping blade of 40 cm reaching to the outer tip. Find the area that is wiped by this wiper.

5.2: Unit Circle: Sine and Cosine Functions



Let’s begin with a unit circle. Let t represent a central angle in standard position. Let (x, y) be the point where the unit circle and the terminal side of the angle intersect. See figure to the right. Then, we define

$$\cos t = x$$
$$\sin t = y$$

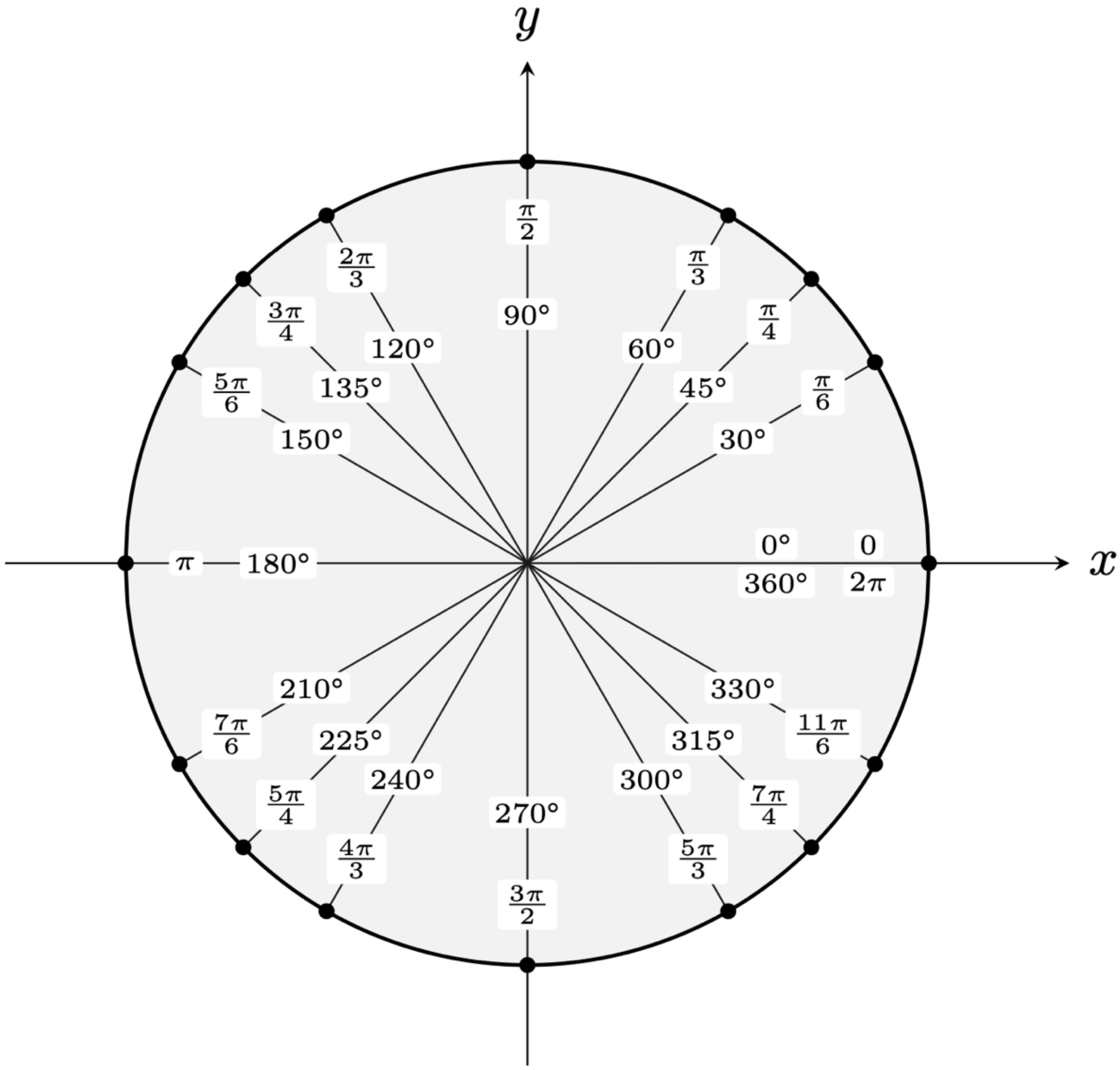


Note that since t represents the measure of an angle and we can make sense of angle with any real number measure, we can define the cosine and sine function for *any* real value t.

Note: If in a previous class, you’ve seen sine and cosine defined differently, don’t worry! There are two equivalent ways to define them. We will address the other definition later in the chapter.

EVALUATING SINE AND COSINE OF IMPORTANT ANGLES

Let's fill out this unit circle (radius = 1).



Scratch work

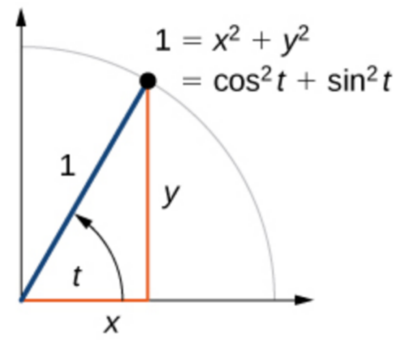
Using the chart above, fill out the values of the sine and cosine function for all of the special angles between 0° and 360°.

θ (degrees)	θ (radians)	t	$\sin t$	$\cos t$
0°	0	0		
30°	$\frac{\pi}{6}$	$\frac{\pi}{6}$		
45°	$\frac{\pi}{4}$	$\frac{\pi}{4}$		
60°	$\frac{\pi}{3}$	$\frac{\pi}{3}$		
90°	$\frac{\pi}{2}$	$\frac{\pi}{2}$		
120°	$\frac{2\pi}{3}$	$\frac{2\pi}{3}$		
135°	$\frac{3\pi}{4}$	$\frac{3\pi}{4}$		
150°	$\frac{5\pi}{6}$	$\frac{5\pi}{6}$		
180°	π	π		
210°	$\frac{7\pi}{6}$	$\frac{7\pi}{6}$		
225°	$\frac{5\pi}{4}$	$\frac{5\pi}{4}$		
240°	$\frac{4\pi}{3}$	$\frac{4\pi}{3}$		
270°	$\frac{3\pi}{2}$	$\frac{3\pi}{2}$		
300°	$\frac{5\pi}{3}$	$\frac{5\pi}{3}$		
315°	$\frac{7\pi}{4}$	$\frac{7\pi}{4}$		
330°	$\frac{11\pi}{6}$	$\frac{11\pi}{6}$		
360°	2π	2π		

PYTHAGOREAN IDENTITY

Note that for any angle t, we have that:

$$(\cos t)^2 + (\sin t)^2 = 1$$



This is known as the **Pythagorean Identity,** written more commonly as

$$\boldsymbol{\cos^2 t + \sin^2 t = 1} \quad \text{or} \quad \boldsymbol{\sin^2 t + \cos^2 t = 1}$$

<u>Example 1</u>: Suppose that for an angle t that falls in Quadrant II, $\sin t = \frac{2}{3}$. Find $\cos t$.

<u>Example 2</u>: Suppose that for an angle t that falls in Quadrant III, $\cos t = -\frac{1}{4}$. Find $\sin t$.

<u>Example 3</u>: Evaluate sine and cosine for the following angles.

a) $480°$

b) $-\frac{\pi}{4}$

5.3: The Other Trigonometric Functions

Review of important information from last section: Two Key Triangles

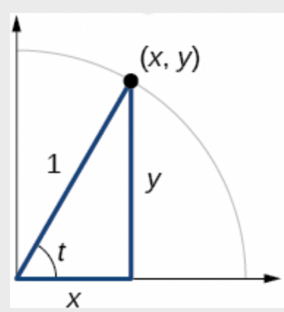
30-60-90 triangle **45-45-90 triangle**

ADDITIONAL TRIGONOMETRIC FUNCTIONS

TANGENT, SECANT, COSECANT, AND COTANGENT FUNCTIONS

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\tan t = \frac{y}{x}, x \neq 0$$
$$\sec t = \frac{1}{x}, x \neq 0$$
$$\csc t = \frac{1}{y}, y \neq 0$$
$$\cot t = \frac{x}{y}, y \neq 0$$



Example 1: Evaluate tangent, secant, cosecant, and cotangent functions at each of the following values.

a) $t = \frac{\pi}{4}$

$\tan t$	$\sec t$	$\csc t$	$\cot t$

b) $t = \frac{5\pi}{6}$

$\tan t$	$\sec t$	$\csc t$	$\cot t$

c) $t = \frac{3\pi}{2}$

$\tan t$	$\sec t$	$\csc t$	$\cot t$

c) $t = -\pi$

$\tan t$	$\sec t$	$\csc t$	$\cot t$

FUNDAMENTAL IDENTITIES

Quotient Identities: $\tan t = \frac{\sin t}{\cos t}$

$\cot t = \frac{\cos t}{\sin t}$

Reciprocal Identities: $\sec t = \frac{1}{\cos t}$ $\quad$ $\cos t = \frac{1}{\sec t}$

$\csc t = \frac{1}{\sin t}$ $\quad$ $\sin t = \frac{1}{\csc t}$

$\cot t = \frac{1}{\tan t}$ $\quad$ $\tan t = \frac{1}{\cot t}$

<u>Example 2</u>: Suppose the terminal side of t is in Quadrant I and $\sec t = \frac{3}{2}$. Use diagrams and/or identities to evaluate each of the following.

a) $\cos t$

b) $\sin t$

c) $\cot t$

d) $\sec(-t)$

e) $\sin(-t)$

ALTERNATE FORMS OF PYTHAGOREAN IDENTITY

From the Pythagorean Identity $\cos^2 t + \sin^2 t = 1$, we can derive two others:

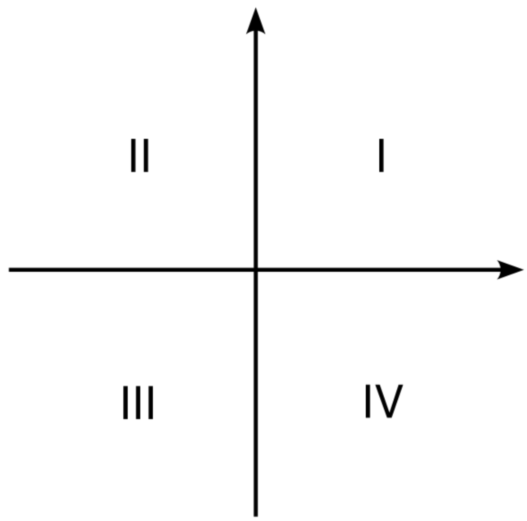
$1 + \tan^2 t = \sec^2 t$ and $\cot^2 t + 1 = \csc^2 t$

Why are the above identities true?

<u>Example 3</u>: Suppose the terminal side of t is in Quadrant II and $\tan t = -\frac{1}{2}$. Use ONLY trigonometric identities to evaluate the other five trigonometric functions at t. DO NOT draw a diagram.

Recall:	$\sin t = y$	$\cos t = x$	$\tan t = \frac{y}{x}$, $x \neq 0$
	$\csc t = \frac{1}{y}$, $y \neq 0$	$\sec t = \frac{1}{x}$, $x \neq 0$	$\cot t = \frac{x}{y}$, $y \neq 0$

WHICH TRIGONOMETRIC FUNCTIONS ARE POSITIVE IN EACH QUADRANT?



DOMAIN AND RANGE OF TRIGONOMETRIC FUNCTIONS

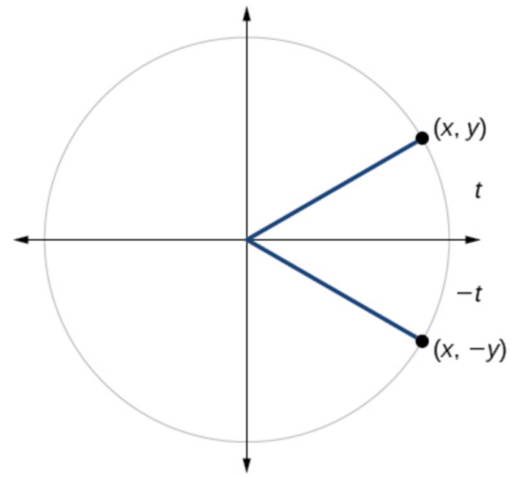
	$f(t) = \sin t$	$f(t) = \cos t$	$f(t) = \tan t$
Domain			
Range			

	$f(t) = \csc t$	$f(t) = \sec t$	$f(t) = \cot t$
Domain			
Range			

EVEN AND ODD FUNCTIONS

Recall that:

- A function f is **even** if $f(-x) = f(x)$ for all possible inputs x
- A function f is **odd** if $f(-x) = -f(x)$ for all possible inputs x



Are the six trigonometric functions even, odd or neither? Let's check…

1) $f(t) = \sin t$

2) $f(t) = \cos t$

3) $f(t) = \tan t$

4) $f(t) = \csc t$

5) $f(t) = \sec t$

6) $f(t) = \cot t$

Even/Odd Identities

<u>Even</u> <u>Odd</u>

PERIOD OF TRIGONOMETRIC FUNCTIONS

PERIOD OF A FUNCTION

The **period** P of a repeating function f is the number representing the interval such that $f(x+P) = f(x)$ for any value of x.

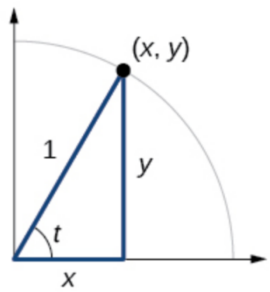
The period of the cosine, sine, secant, and cosecant functions is 2π.

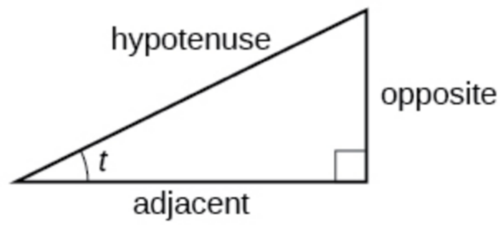
The period of the tangent and cotangent functions is π.

Why?

5.4 Right Triangle Trigonometry

Now, let's extend the definition of the trigonometric functions so they make sense on any triangle.

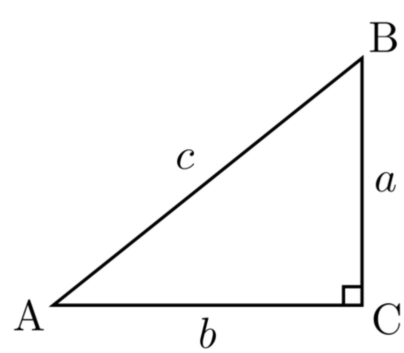




$\sin t = y = \frac{y}{1}$	$\sin t = \frac{\text{opposite}}{\text{hypotenuse}}$
$\cos t = x = \frac{x}{1}$	$\cos t = \frac{\text{adjacent}}{\text{hypotenuse}}$
$\tan t = \frac{y}{x}$	$\tan t = \frac{\text{opposite}}{\text{adjacent}}$
$\csc t = \frac{1}{y}$	$\csc t = \frac{\text{hypotenuse}}{\text{opposite}}$
$\sec t = \frac{1}{x}$	$\sec t = \frac{\text{hypotenuse}}{\text{adjacent}}$
$\cot t = \frac{x}{y}$	$\cot t = \frac{\text{adjacent}}{\text{opposite}}$

The standard notation for labeling angles and sides of a right triangle is as shown to the right. Note that the angle at vertex C is a right angle, or 90°.

Please note that α and A are often used interchangeably for the angle measure, as with β and B.



<u>Example 1</u>: Sketch the right triangle given by the following information, label all the sides, and evaluate the six trigonometric functions at α and β.

a) $a = 3, b = 4$

$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\csc\alpha$	$\sec\alpha$	$\cot\alpha$

$\sin\beta$	$\cos\beta$	$\tan\beta$	$\csc\beta$	$\sec\beta$	$\cot\beta$

b) $a = 2, c = 3$

$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\csc\alpha$	$\sec\alpha$	$\cot\alpha$

$\sin\beta$	$\cos\beta$	$\tan\beta$	$\csc\beta$	$\sec\beta$	$\cot\beta$

<u>Key Facts</u>:

- There is a relationship between the trig functions evaluated at α and β, given by:

  ____

- These relationships are known as ____________________ Identities, and generally written:

$$\cos t = \sin\left(\frac{\pi}{2} - t\right) \qquad \sin t = \cos\left(\frac{\pi}{2} - t\right)$$

$$\tan t = \cot\left(\frac{\pi}{2} - t\right) \qquad \cot t = \tan\left(\frac{\pi}{2} - t\right)$$

$$\sec t = \csc\left(\frac{\pi}{2} - t\right) \qquad \csc t = \sec\left(\frac{\pi}{2} - t\right)$$

<u>Example 2</u>: Sketch the right triangle given by the following information, and label all the sides.

a) $a = 5,\ \alpha = 30°$

b) $c = 7,\ \beta = 45°$

For angles other than the special angles, we use technology to evaluate the trigonometric functions.

When using technology (such as calculator), be sure that the calculator is in the correct angle mode. The standard options are DEG (decimal degree), RAD (radian), DMS (degrees-minutes-seconds).

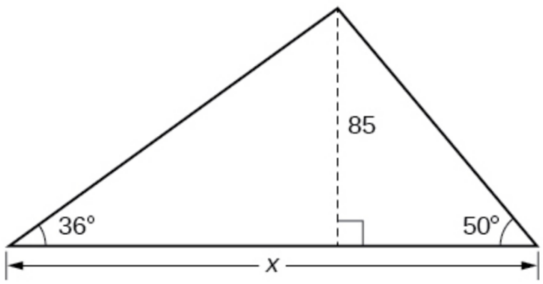
<u>Example 3</u>: Sketch the right triangle given by the following information, and label all the sides. Round to two decimal places.

a) $c = 6,\ \beta = 50°$

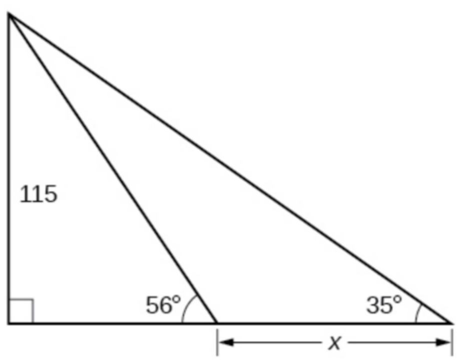
b) $b = 7, \alpha = 20°$

<u>Example 4</u>: Find the value of x in each of the following problems.

a)



b)



APPLICATIONS

Example 5: The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

Example 6: There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be $40°$. From the same location, the angle of elevation to the top of the antenna is measured to be $43°$. Find the height of the antenna.