Chapter 1 Notes

Cole Gannon

1 Linear Equations (in General)

Each variable within the equation such as x, y, or z, for example, must have either exponent 1 or 0.

- ✓ $2x + 1 = 7$
- ✗ $x^2 = 2y$
- ✗ $\sin(x) = 2x$
- ✗ $\sqrt{x} = 3x + 1$

Def: Linear Equation

$$a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_nv_n = b$$

where:

$$\forall a \in \mathbb{R}$$
$$b \in \mathbb{R}$$

$$\begin{Bmatrix} \text{linear equation } 1 = l_1 \\ \text{linear equation } 2 = l_2 \\ \text{linear equation } 3 = l_3 \\ \vdots \\ \text{linear equation } n = l_n \end{Bmatrix}$$

We group one or more equations using curly braces into a system of equations.
Though we're not going to do it all that much, we address each line inside of a system using "l_n" where "n" is the n$^{\text{th}}$ line.

Def: System of Linear Equations / Linear System / L.S.
A collection of one or more linear equations.
(One linear equation only would be kinda silly, though it technically qualifies.)

1.1 Solutions

1.1.1 A Solution

Def: *A* Solution of a Linear System
Given a L.S. in $x_1, x_2, x_3 \cdots x_n$ or $\mathbb{R}^n$, *A* solution is an n-tuple corresponding to each respective variable.
$(S_1, S_2, S_3, \cdots, S_n)$,

Let's work an example:

$$\begin{Bmatrix} x + 2y &=& 5 \\ 2x + y &=& 10 \end{Bmatrix} \quad \text{(ex. A)}$$

$x = 5 - 2y \to 2[x = 5 - 2y] + y = 10 \to \cancel{10} - 2y + y = \cancel{10} \to \mathbf{y = 0}$
$x = 5 - 2[y = 0] \to \mathbf{x = 5}$
The solution is $(x, y) = (5, 0)$

1.1.2 Solution Sets

Def: The Solution Set of a Linear System
The set of all possible solutions for a given L.S.
Two linear systems are said to be equal when their solution sets are equal.

In the example in 1.1.1, there was only one possible solution: (5,0). Therefore, the solution set of ex. A is {(5, 0)}.

$$\{x + 2y = 1\} \tag{ex. B}$$
$$\Rightarrow \ \{(x, y) \mid x + 2y = 1\}$$
$$or \ \left\{(x, \frac{1-x}{2}) \mid \forall x \in \mathbb{R}\right\}$$
$$or \ \{(1 - 2y, y) \mid \forall y \in \mathbb{R}\}$$

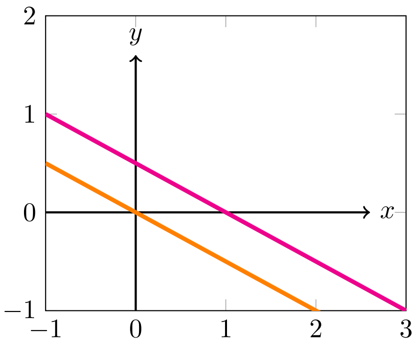
The notation you see above is the Set builder notation, not to be confused with a system of equations. $\{x \mid \forall \ x \in \mathbb{Z}, \text{modulo}(x, 2) = 0\}$ should be read as "the set of all integers x where x mod two is zero.". This is the set of all even numbers.

1.1.3 The Existence and Uniqueness Questions

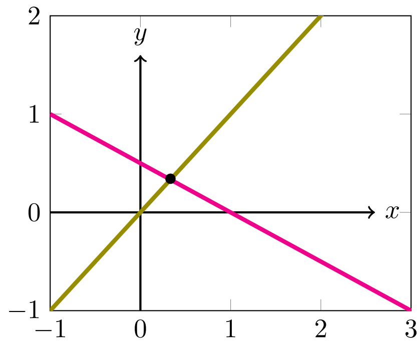
So when we have a linear system, something we like to ask is "Does a solution exist?". If a solution exists, "Does a unique solution exist?". These are *The Existence and Uniqueness Questions.* There are three solution states for any given linear system.

- No Solution $\nexists$
- Unique Solution $\exists!$
- Infinitely Many Solutions $\exists\infty$-many

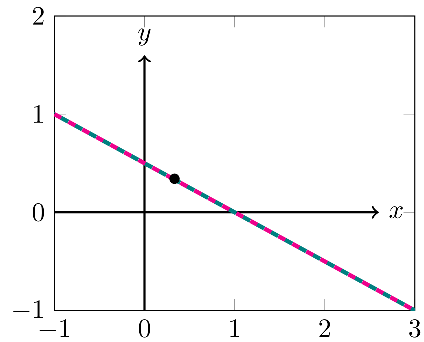
$\left\{\begin{aligned} x + 2y &= 1 \\ 2x - 4y &= 0 \end{aligned}\right\} \nexists$ sln; $l_1 \parallel l_2$.



$\left\{\begin{aligned} x + 2y &= 1 \\ x - y &= 0 \end{aligned}\right\} \exists!$ sln.



$\left\{\begin{aligned} x + 2y &= 1 \\ 2x + 4y &= 2 \end{aligned}\right\} \exists\infty$ sln; $l_1 \parallel l_2$.



Def: Inconsistent
The L.S. has no solutions. ($\nexists$)
Def: Consistent
The L.S. has at least one solution. ($\exists!$, $\exists\infty$)

1.1.4 Determination

$$\begin{Bmatrix} 2x+y &=& 3-x \\ x-y &=& 5 \\ x+2y &=& 1 \end{Bmatrix}$$

Linear System of x and y.
Overdetermined; three equations when there are only two variables.

$$\begin{Bmatrix} x-y+z &=& 1 \\ 2x+y-z &=& 3 \end{Bmatrix}$$

Linear System of x, y, and z.
Underdetermined; two equations when there are three variables.

$$\begin{Bmatrix} x-2y &=& 1 \\ 2x+4y &=& 0 \end{Bmatrix}$$

Linear System of x and y
Determined; two equations and two variables.

Keep in mind that determination alone is not a sufficient indicator of the existence of a solution. Notice that the last example in this section matches the first example in **1.1.3**, which as we can clearly see, has no solution.

What if we know our linear system is consistent like when it has a *trivial solution* $^{\textit{(we'll talk about this later)}}$? What does the determination of our L.S. mean? Well, when it's overdetermined, we don't get much more information. Things get interesting when we think about it being underdetermined, though.

Can we say that when a L.S. is underdetermined it's either $\nexists$ sln or $\exists\infty$-many sln? Let's imagine some examples because I'm not going to wrangle some 3d graphs together in LaTeX.

- L.S. in $\mathbb{R}^2$ with one equation.
  - ✓ The equation is a line; $\exists\infty$-many sln.
- L.S. in $\mathbb{R}^3$ with one equation.
  - ✓ The equation is a plane; $\exists\infty$-many sln.
  - ✓ The equation is a line; $\exists\infty$-many sln.
- L.S. in $\mathbb{R}^3$ with two lines.
  - ✓ l_1 does not intersect with l_2; $\nexists$ sln.
  - ✓ $l_1 \parallel l_2$ and $l_1 \neq l_2$; $\nexists$ sln.
  - ✓ $l_1 = l_2$; $\exists\infty$-many sln.
  - ✗ l_1 intersects l_2 at some (x, y). $\exists!$ sln.

So what can we conclude? Well, nothing, unfortunately. A shame.

2 Matrix *(why weren't we taught this sooner, huh?)*