Expression : $2x+1$

Equation : $2x^1+1=7$ (Linear) of line power 1

$\sin x + x^2 = 2x$ (non-linear)

$\sin x = 2x^1$ (non-linear)

$\sqrt{x} = 3x+1$ (non-linear)

$x^{\frac{1}{2}}$

Def) power 1 linear equation in $x_1, x_2, \cdots, x_n$

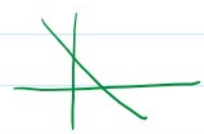
$\Leftrightarrow$ $a_1 x_1^1 + a_2 x_2^1 + a_3 x_3^1 + \cdots + a_n x_n^1 = b$,

where $\forall a_i (\in \mathbb{R})$: coefficients

$b (\in \mathbb{R})$: constant

$\forall x_i$: variables

Ex) ① $x^1 + 2y^1 = 3$: l. eq. in x & y



② $5^2 x^1 + y^1 = \sqrt{\pi}\, y^1 + 6$: l. eq. in x & y

③ $x + 2y + 3z = 4$: l. eq. in x, y & z

④ $\underbrace{x^1 \cdot y^1}_{\text{power } 2} + e = x^1$: non-linear

⑤ $5x - 37y + 2z + 7w = 100$: l. eq. in x, y, z & w

Def) system of L. eq.s (i.e., Linear System)

$\Longleftrightarrow$ collection of one or more L. eq.s

Ex) ① $2x + y = 3 - x$: L. system (L. eq.)

② $\begin{cases} 2x+y=3-x \\ x-y=5 \\ x+2y=1 \end{cases}$: L. system in x & y
(overdetermined)

③ $\begin{cases} x-y+z=1 \\ 2x+y-z=3 \end{cases}$: L. system in x, y & z
(underdetermined)

Def) A solution of a L. system in $x_1, \cdots, x_n$
$\Longleftrightarrow$ $(s_1, s_2, \cdots, s_n)$: n-tuple that satisfies the L.S.

Ex) $x+2y=1$: l. system

ordered pair
$(1,0)$: solution
$(0, \frac{1}{2})$: solution
$(\frac{1}{2}, \frac{1}{4})$: solution
$(2,1)$: not a solution
$\vdots$

$\cancel{x=1}$: sol. to the L. system
(a, b) : ordered pair

Def) The solution set of a L. system
$\Longleftrightarrow$ $\{\text{solutions}\}$: the set of all the solutions of the system.

Ex) $x+2y=1$

$\Rightarrow \{(x,y) \mid x+2y=1\}$

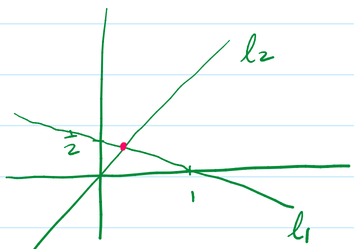
or $\{(x, \frac{1-x}{2}) \mid \forall x \in \mathbb{R}\}$

or $\{(1-2y, y) \mid \forall y \in \mathbb{R}\}$

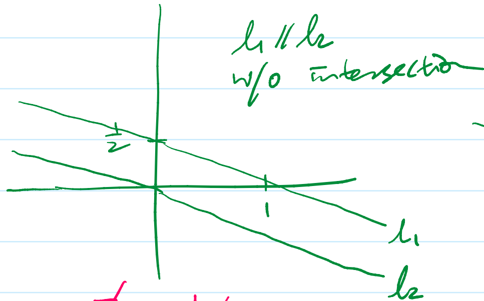
or $\{(1,0), (0,\frac{1}{2}), (\frac{1}{2}, \frac{1}{4}), (-2, \frac{3}{2}), (-1, 1), \cdots\}$

Ex) $\begin{cases} x+2y=1 : l_1 \end{cases}$ $\quad \begin{cases} x+2y=1 : l_1 \end{cases}$ $\quad \begin{cases} x+2y=1 : l_1 \end{cases}$

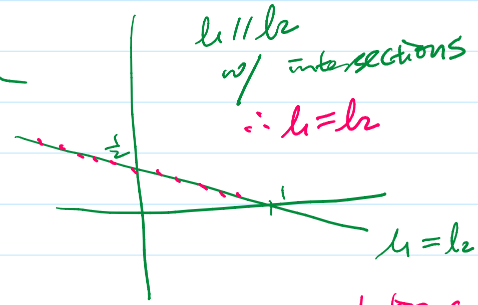
Ex) $\begin{cases} x+2y=1 : l_1 \\ x-y=0 : l_2 \end{cases}$ $\quad$ $\begin{cases} x+2y=1 : l_1 \\ 2x+4y=0 : l_2 \end{cases}$ $\quad$ $\begin{cases} x+2y=1 : l_1 \\ 2x+4y=2 : l_2 \end{cases}$



$\exists!$ solution



$\nexists$ solution



$\exists\ \infty$-many solutions

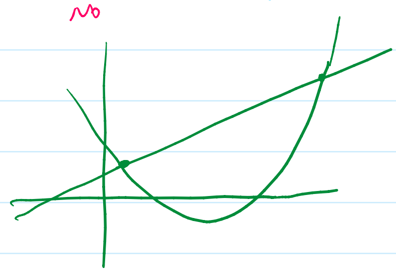
Q) Given a L.S.

1) Existence Question : $\exists$?

2) Uniqueness Question : $\exists!$?

* L.S
- consistent : $\exists$ solution (Yes)
  - Uniqueness Question
    - $\exists!$ (Yes)
    - $\exists\ \infty$-many (No)
- inconsistent : $\nexists$ solution (No)

Existence Question



# of solution-wise for a L.S.

$\therefore$ No sol. / only one sol. / inf. many sol.s

$\underbrace{\text{No sol.}}_{\text{inconsistent}}$ $\quad$ $\underbrace{\text{only one sol. / inf. many sol.s}}_{\text{consistent}}$

Ex) $\begin{cases} x+y+z=1 \\ x-y-z=0 \\ x+y-z=3 \\ 2x-y+z=1 \end{cases}$ $\Rightarrow$ ...

* Matrix Notation

entries from the coefficients

* Matrix Notation

entries from the coefficients

$$\begin{cases} 1x - 2y = -1 \\ -x + 3y = 3 \end{cases} \Rightarrow \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

L.S.

Matrix Equation

$$A \mathbb{u} = \mathbb{b}$$

"coefficient matrix" unknown vector constant vector

$[A \quad \mathbb{b}]$: "Augmented Matrix"

$$[A \ \ \mathbb{b}] = \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

G.E.M. w/ bw. s

$$\Leftrightarrow \begin{cases} 1x - 2y = -1 \\ -x + 3y = 3 \end{cases} \quad (+)$$

$$1y = 2$$

$$\Leftrightarrow \begin{cases} x - 2y = -1 \\ 0x + 1y = 2 \end{cases}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right] \quad \therefore x = 3, \ y = 2$$

$$\Leftrightarrow \begin{cases} x + 0y = 3 \\ 0 \cdot x + 1 \cdot y = 2 \end{cases}$$

$\therefore$ (3, 2) : the only sol. to the L.S.

$\{(3,2)\}$: the solution set to the L.S.

$\therefore$ L.S : consistent w/ uniq. sol.

Elementary Row operations on [A b]

1) Replacement : Add to one row a multiple of another row
2) Interchange : $R_i \leftrightarrow R_j$
3) Scaling : kR_i, $\forall k (\neq 0) \in \mathbb{R}$

3) scaling : kR_i , $\forall k(\neq 0) \in \mathbb{R}$

L.S. in x, y, z

Ex) $\begin{cases} x - 2y + z = 0 \\ y - 4z = 4 \\ -4x + 5y + 9z = -9 \end{cases} \Rightarrow$ M. Eq. $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -4 \\ -4 & 5 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -9 \end{bmatrix}$

$A \quad \mathbb{u} = \mathbb{b}$

$$\Rightarrow [A \ \ \mathbb{b}] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$$\underset{4R_1+R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \underset{3R_2+R_3}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

*Let's not stop working here. Continue working.

$$\underset{\substack{2R_2+R_1 \\ 4R_3+R_2}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\underset{7R_3+R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

"equivalent"

$$\Leftrightarrow \begin{cases} 1 \cdot x + 0 \cdot y + 0 \cdot z = 29 \\ 0 \cdot x + 1 \cdot y + 0 \cdot z = 16 \\ 0 \cdot x + 0 \cdot y + 1 \cdot z = 3 \end{cases} : \text{L.S.}$$

$\therefore (x, y, z) = (29, 16, 3)$

$\therefore \{(29, 16, 3)\}$

$\therefore$ L.S : consistent w/ uniq. sol.

Def) L.Ss : <u>equivalent</u>

$\Leftrightarrow$ The solution sets are the same.

Q) - Interchange : $R_i \Leftrightarrow R_j$

- scaling : k LHS $= k$ RHS , $\forall k(\neq 0) \in \mathbb{R}$

- interchange ...

- Scaling : $k\,LHS = k\,RHS$, $\forall k (\neq 0) \in \mathbb{R}$

$\Leftrightarrow$ $LHS = RHS$

- Replacement : ex) $[A \ b] = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \overset{kR_1+R_3}{\sim} \begin{bmatrix} R_1 \\ R_2 \\ kR_1+R_3 \end{bmatrix}$ equivalent!

R_3 : $LHS = RHS$

$kR_1 + R_3$: $\cancel{kR_1} + R_3$
$- \cancel{kR_1}$

ex) $\begin{cases} ① \ x+2y=1 \\ ② \ 2x-y=3 \end{cases} \overset{k①+②}{\Rightarrow} \begin{cases} x+2y=1 \\ kx+2x+k2y-y=k1+3 \end{cases}$

$\Rightarrow \begin{cases} x+2y=1 \\ k(\cancel{x+2y}) + 2x-y = k\cdot\cancel{1}+3 \end{cases}$

Ex) $\begin{cases} 2x+y=1 \\ x+y=k \end{cases}$

Determine k so that the L.S. is

① consistent w/ one sol.

② consistent w/ ∞-many sols.

③ inconsistent

sol) L.S $\Leftrightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 1 & k \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & k \\ ② & 1 & 1 \end{array}\right]$

Sol) L.S $\iff$ $\begin{bmatrix} & & \\ 1 & 1 & k \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$

$\overset{-2R_1+R_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & k \\ 0 & -1 & 1-2k \end{array}\right]$

$\therefore$ Cons. w/ only one sol. for $\forall k \in \mathbb{R}$

Ex) $\begin{cases} 3x-9y=4 \\ -2x+6y=k \end{cases}$

Discuss the 2 questions!

i.e., Existence Q & Uniq. Q.

Sol) $\begin{bmatrix} 3 & -9 & 4 \\ -2 & 6 & k \end{bmatrix} \underset{\frac{1}{2}R_2}{\overset{\frac{1}{3}R_1}{\sim}} \begin{bmatrix} 1 & -3 & \frac{4}{3} \\ -1 & 3 & \frac{k}{2} \end{bmatrix}$

$\overset{R_1+R_2}{\sim} \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ 0 & 0 & \frac{k}{2}+\frac{4}{3} \end{array}\right]$ $\qquad x-3y=\frac{4}{3}$

$0 = \frac{3k+8}{6}$?

$k=-\frac{8}{3}$: constant $\left(\because R_2 : 0=0\right) \Rightarrow \exists\ \infty$-many.

$k \neq -\frac{8}{3}$: inconsistent

Uniq. Q

Existence Q

$\boxed{1.2}$ REF, RREF

Given $A_{n \times p}$, "Step-like" :

Def) Row Echelon Form (REF)

$\neq 0$ $\therefore$ ~~RREF~~

Def) Row Echelon Form (REF)

1) . . .

2) . . .

3) . . .

ex)
$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\neq 0$

$\therefore$ ~~R~~REF

: REF.

leading entry

:

Def) Reduced Row Echelon Form (RREF)

1) . . .

2) . . .

3) . . . .

& 4) . . .

5) . . . .

ex)
$$\begin{bmatrix} 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 1 & 100 \end{bmatrix}$$

REF

~~R~~REF

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 1 & 100 \end{bmatrix}$$

REF

~~R~~REF

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 100 \end{bmatrix}$$

RREF