

Expression: $2x + 1$ power 1
 Equation: $2x' + 1 = 7$ (Linear)
 $\sin x + x^2 = 2x$ (non-linear)
 $\sin x = 2x'$ (non-linear)
 $\sqrt{x} = 3x + 1$ (non-linear)

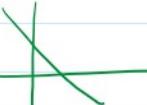
Def) linear equation in x_1, x_2, \dots, x_n

$$\Leftrightarrow a_1 x_1' + a_2 x_2' + a_3 x_3' + \dots + a_n x_n' = b,$$

where $a_i (\in \mathbb{R})$: coefficients

$b (\in \mathbb{R})$: constant

$\forall x_i$: variables

Ex) ① $x + 2y = 3$: l. eq. in x & y 

② $5x + y = \sqrt{\pi} y + b$: l. eq. in x & y

③ $x + 2y + 3z = 4$: l. eq. in x, y & z 

④ $x^2 + e = x$: non-linear
power 2 

⑤ $5x - 37y + 2z + 7w = 100$: l. eq. in x, y, z & w

Def) system of L. eq.s (i.e., Linear System)

\Leftrightarrow collection of one or more L. eq.s

Ex) ① $2x + y = 3 - x$: L. System (L. eq.)

② $\begin{cases} 2x + y = 3 - x \\ x - y = 5 \\ x + 2y = 1 \end{cases}$: L. System in x & y
(overdetermined)

$$\textcircled{3} \quad \begin{cases} x - y + z = 1 \\ 2x + y - z = 3 \end{cases} : \text{L. system in } x, y, z \\ \text{(underdetermined)}$$

Def) A Solution of a L. System in x_1, \dots, x_n

$\Leftrightarrow (s_1, s_2, \dots, s_n)$: n -tuple that satisfies the L.S.

$$\text{Ex}) \quad x + 2y = 1 : \text{l. system}$$

$$\begin{array}{ll} \text{ordered pair} & (1, 0) : \text{solution} \\ & (0, \frac{1}{2}) : \text{solution} \\ & (\frac{1}{2}, \frac{1}{2}) : \text{solution} \end{array} \quad \begin{array}{l} \cancel{x=1} \text{ sol. to the L. system} \\ (a, b) : \text{ordered pair} \end{array}$$

$$(2, 1) : \text{not a solution}$$

:

Def) The solution set of a L. System

$\Leftrightarrow \{\text{solutions}\}$: the set of all the solutions of the system.

$$\text{Ex}) \quad x + 2y = 1$$

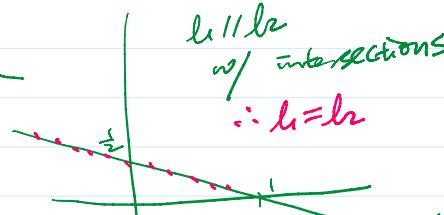
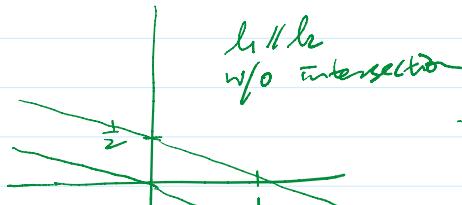
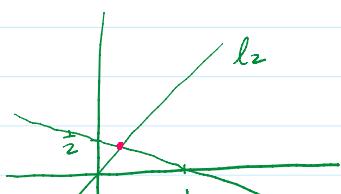
$$\Rightarrow \{(x, y) \mid x + 2y = 1\}$$

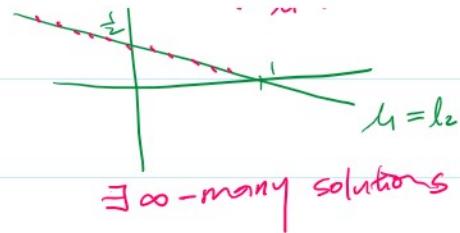
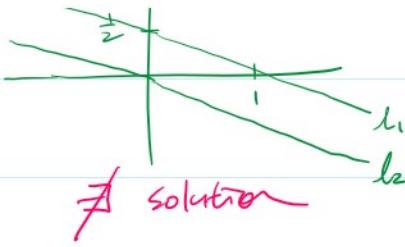
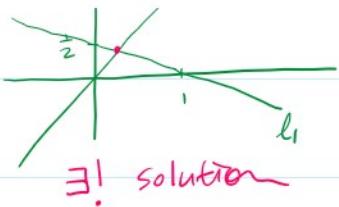
$$\text{or } \left\{ \left(x, \frac{1-x}{2} \right) \mid \forall x \in \mathbb{R} \right\}$$

$$\text{or } \left\{ (1-2y, y) \mid \forall y \in \mathbb{R} \right\}$$

$$\text{or } \left\{ (1, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}), (-2, \frac{3}{2}), (-1, 1), \dots \right\} \text{ etc}$$

$$\text{Ex}) \quad \begin{cases} x + 2y = 1 : l_1 \\ x - y = 0 : l_2 \end{cases} \quad \begin{cases} x + 2y = 1 : l_1 \\ 2x + 4y = 0 : l_2 \end{cases} \quad \begin{cases} x + 2y = 1 : l_1 \\ 2x + 4y = 2 : l_2 \end{cases}$$





~~Q2)~~

Given an L.S.

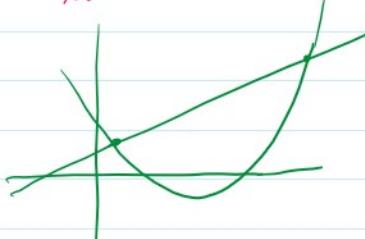
1) Existence Question : $\exists ?$

Uniqueness Question

2) Uniqueness Question : $\exists ! ?$

* L.S. consistent : \exists solution
 inconsistent : \nexists solution

$\exists !$
Yes
 $\exists \infty\text{-many}$
No



Existence Question

of
solution-W.P.
for an L.S.
 \therefore

No sol. / only one sol. / inf. many sol.s

Inconsistent

consistent

$$\begin{cases} x+y+z=1 \\ x-y-z=0 \\ x+y-z=3 \\ 2x-y+z=1 \end{cases} \Rightarrow \dots$$

* Matrix Notation

$$\begin{cases} x-2y=-1 \\ -x+3y=3 \end{cases} \Rightarrow \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

entries from
the coefficients

L.S.

Matrix Equation

$(A)(U) = (b)$
" matrix +
Unknown vector
constant vector

$(A) \text{ (l.u)} = \text{constant vector}$
 "coefficient matrix"
 unknown vector

$[A \ b]$: Augmented Matrix

$$[A \ b] = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \end{bmatrix} \text{ G.E.M. } \text{ w/ bw. s}$$

$$\Leftrightarrow \begin{cases} x - 2y = -1 \\ -x + 3y = 3 \end{cases} \quad \begin{array}{l} \\ + \end{array} \quad \begin{cases} x - 2y = -1 \\ 0x + 1y = 2 \end{cases}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad \therefore x=3 \\ y=2$$

$$\Leftrightarrow \begin{cases} x + 0y = 3 \\ 0x + 1y = 2 \end{cases}$$

$\therefore (3, 2)$: the ^{only} sol. to the L.S.

$\{(3, 2)\}$: the solution set to the L.S.

\therefore L.S. : consistent w/ uniq. sol.

Elementary Row operations on $[A \ b]$

- 1) Replacement : Add to one row a multiple of another row
- 2) Interchange : $R_i \leftrightarrow R_j$
- 3) Scaling : $kR_i, \forall k(\neq 0) \in \mathbb{R}$

Ex) L.S. in x, y, z

$$\left\{ \begin{array}{l} x - 2y + z = 0 \\ y - 4z = 4 \\ -4x + 5y + 9z = -9 \end{array} \right. \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -4 \\ -4 & 5 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -9 \end{bmatrix}$$

M.Eq.

$$A \ u = b$$

$$\Rightarrow [A \ b] = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

$$\rightarrow [A \ b] = \left[\begin{array}{ccc|c} 0 & 1 & -4 & 4 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

$\xrightarrow{fR_1+R_3}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right] \xrightarrow{3R_2+R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 13 & -9 \end{array} \right]$$

*let's not stop working here.
Continue working.

$\xrightarrow{2R_2+R_1}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$\xrightarrow{7R_3+R_1}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$\Leftrightarrow \begin{cases} 1 \cdot x + 0 \cdot y + 0 \cdot z = 29 \\ 0 \cdot x + 1 \cdot y + 0 \cdot z = 16 \\ 0 \cdot x + 0 \cdot y + 1 \cdot z = 3 \end{cases} : L.S.$

$\therefore (x, y, z) = (29, 16, 3)$

$\therefore \{(29, 16, 3)\}$

$\therefore L.S.: \text{consistent w/ unq. sol.}$

Def) L.Ss : equivalent

\Leftrightarrow The solution sets are the same.

Q) - Interchange : $R_i \leftrightarrow R_j$

- Scaling : $k LHS = k RHS, \forall k (\neq 0) \in \mathbb{R}$

$$\Leftrightarrow LHS = RHS$$

- Replacement :

$$\text{ex) } [A \ b] = \left[\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \right]$$

$$\xrightarrow{kR_1+R_3} \left[\begin{array}{c} R_1 \\ R_2 \\ kR_1+R_3 \end{array} \right]$$

equivalent!

$R_3: LHS = RHS$
 $(KR_1 + R_3) \rightarrow R_3$
 $(KR_1 + R_3) - KR_1 \rightarrow R_3$

ex) $\begin{cases} ① x+2y = 1 \\ ② 2x-y = 3 \end{cases} \Rightarrow \begin{cases} x+2y = 1 \\ kx+2x+k2y-y = k+3 \end{cases}$

$\Rightarrow \begin{cases} x+2y = 1 \\ k(x+2y) + 2x-y = k+3 \end{cases}$

$\Leftrightarrow \begin{cases} 2x+y = 1 \\ x+y = k \end{cases}$

Determine k so that the L.S. \exists

① consistent w/ one sol.

② consistent w/ ∞ -many sols.

③ inconsistent

Sol) L.S. $\Leftrightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 1 & k \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & k \\ 2 & 1 & 1 \end{array} \right]$

$\xrightarrow{-2R_1 + R_2} \left[\begin{array}{cc|c} 1 & 1 & k \\ 0 & -1 & 1-2k \end{array} \right]$

\therefore Cons. w/ only one sol.
for $\forall k \in \mathbb{R}$

$\Leftrightarrow \begin{cases} 3x-9y = 4 \\ -2x+6y = k \end{cases}$

$$\text{Q1) } \begin{cases} x + y = k \\ -2x + 6y = k \end{cases}$$

Discuss the 2 questions!

re.: Existence Q
& Unq. Q.

$$\text{Sol) } \left[\begin{array}{ccc|c} 3 & -9 & 4 & k \\ -2 & 6 & k & k \end{array} \right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & -3 & \frac{4}{3} & k \\ -2 & 6 & k & k \end{array} \right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -3 & \frac{4}{3} & k \\ 0 & 0 & \frac{k}{2} + \frac{4}{3} & k \end{array} \right]$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & -3 & \frac{4}{3} & k \\ 0 & 0 & \frac{k}{2} + \frac{4}{3} & k \end{array} \right]$$

$$x - 3y = \frac{4}{3}$$

$$0 = \frac{3k+8}{6} ?$$

$k = -\frac{8}{3}$: constant ($\because R_2: 0 = 0$) $\Rightarrow \exists \text{ many.}$

$k \neq -\frac{8}{3}$: inconsistent

Unq. Q

Existence Q

1.2 REF, RREF

Given Anxp.

"Step-like":

Def) Row Echelon Form (REF)

1) ...

2) ...

3) ...

ex)
$$\left[\begin{array}{cccc|c} * & * & * & * & * \\ 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

\therefore REF

\therefore REF.
leading entry

Def) Reduced Row Echelon Form (RREF)

1) ...

2) ...

3) ...

ex)
$$\left[\begin{array}{ccccc} 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ \hline 0 & 0 & 1 & 1 & 100 \end{array} \right]$$

RREF

2) ...
3) ...
4) ...
5) ...

ex) $\left[\begin{array}{ccc|cc} 1 & -1 & 1 & b & 7 \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & 100 \end{array} \right]$ REF
✗ REF

$\left[\begin{array}{cccc|cc} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 1 & 100 \end{array} \right]$ REF
✗ REF

$\left[\begin{array}{cccc|cc} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 100 \end{array} \right]$ RREF

Ex) $\begin{cases} x + 4y + 3z = -1 \\ 3x + 14y + 10z = -5 \\ 2y + 6z = 8 \end{cases}$: L. system

$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 3 & 14 & 10 & -5 \\ 0 & 2 & 6 & 8 \end{array} \right] \begin{matrix} x \\ y \\ z \end{matrix} = \begin{bmatrix} -1 \\ -5 \\ 8 \end{bmatrix}$: Matrix Equation

$\Rightarrow [A \ b] = \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 3 & 14 & 10 & -5 \\ 0 & 2 & 6 & 8 \end{array} \right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 1 & -2 \\ 0 & 1 & 3 & 4 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 1 & -2 \end{array} \right]$

$\xrightarrow{-2R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -5 & -10 \end{array} \right] \xrightarrow{\frac{1}{-5}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -9 & -17 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{array} \right]$

REF, RREF

REF, RREF

$\xrightarrow{9R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array} \right]$

REF, RREF

$\therefore \{(1, -2, 2)\}$

Existence Q?

You may stop here to answer
 "Yes, \exists sol."

② Given $A_{n \times p}$. $\exists \infty$ -many REFs
but \exists only one RREF

* $[A | b]$: consistent

$\sim \Leftrightarrow \nexists$ row w/ $(0 \dots 0 \star^*)$

Def) pivot position
pivot
pivot column

Def) **basic** variable : any variable that corresponds to a pivot column in a coeff. matrix

free variable : non-basic variable

i.e., any variable that corresponds to a non-pivot column in a coeff. M.

$$\text{Ex)} [A | b] \sim \dots \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 0 & 9 & 1 & 0 & 2 \\ 0 & 0 & 0 & 5 & 7 \end{array} \right]$$

: L.S w/ 3 equations & 4 unknowns \Rightarrow 4 variables

pivots \therefore pivot columns : 1, 2, 4

\therefore basic variable : x_1, x_2, x_4

free variable : x_3

$$\text{Ex)} [A | b] \sim \dots \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array} \right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 5 \end{array} \right] \leftarrow \nexists (0 \ 0 \ 0 \dots 0 \star^*)$$

free variable : x_3

\therefore consistent
Yes to Exist. Q

$$\begin{cases} x_1 = 3 - 2x_3 \\ x_2 = 4 - x_3 \\ x_3 : \text{free} \\ x_4 = 5 \end{cases} \quad \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \\ 0 \\ 5 \end{bmatrix}, \quad \forall x_3 \in \mathbb{R}$$

$\therefore \exists \infty\text{-many sol.s } (\because \exists \text{ free variable for a cons. system})$

$$\therefore \left\{ k \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \\ 0 \\ 5 \end{bmatrix} \mid \forall k \in \mathbb{R} \right\} = \text{sol. set}$$

RREF

$$\text{E)} [A \ b] \sim \dots \sim \left[\begin{array}{cccc|c} 1 & 6 & 8 & 2 & 3 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{array} \right]$$

$\exists \text{ free v. : } x_2, x_3$
 $\therefore \text{cons. w/ } \infty\text{-many sol.s}$

: not even a consistent system

$$(\because \exists (0 \ 0 \ 0 \ 0 \ \cancel{6}))$$

$\therefore 0 \neq 6$

: inconsistent system
 $\text{rep. } \not\models \text{ sol.}$

L.S. w/
 $A_{3 \times 4}$

$$[A \ b] = \left[\begin{array}{cccc|c} \cancel{1} & \cancel{2} & \cancel{3} & \cancel{4} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right] \text{ or } \left[\begin{array}{cccc|c} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right]$$

$\exists \text{ at most 1 pivot in a row/column}$

$\therefore \exists \text{ at least 1 free variable}$

... = at least 1 true ...

~~∴ cons. w/ ∞ many sol.s~~

may not be a consistent system

c.ex) $[A \ b] \sim \left[\begin{array}{cccc|c} \textcircled{*} & \textcircled{*} & \textcircled{*} & \textcircled{*} & * \\ 0 & \textcircled{*} & \textcircled{*} & \textcircled{*} & * \\ 0 & 0 & 0 & 0 & * \\ \hline & & & & * \end{array} \right]$: inconsistent.
 \nearrow free variables

1.3 vector equations

Def) vector: $n \times 1$ matrix
(column)

i.e., matrix w/ one column.

$\star \mathbb{R}^n : \vec{u}, \vec{\vec{u}}$

"lower case u "
in bold"

Ex) $u = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$: vector

Ex) $u \in \mathbb{R}^n = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}_{n \times 1}$

Def)

Given vectors

$u_1, u_2, \dots, u_p \in \mathbb{R}^n$,

$\forall u_k = \begin{bmatrix} u_{1k} \\ u_{2k} \\ \vdots \\ u_{nk} \end{bmatrix} \in \mathbb{R}^n, \forall u_{ik} \in \mathbb{R}$

$\sum_{i=1}^p c_i u_i = c_1 u_1 + c_2 u_2 + c_3 u_3 + \dots + c_p u_p$

: linear combination of u_1, \dots, u_p
of line
power 1

$\Rightarrow a u + b v + c w$: linear comb. of u, v, w

Ex) $2U + 3V + 0W$: linear comb. of $U \in \mathbb{W}$ & $V \in \mathbb{W}$

$$\text{Ex)} \quad a = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}, \quad d = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$$

1) Is d a L.C. of a, b, c ?

Sol) $d = k_1 a + k_2 b + k_3 c$ w/ some k_1, k_2, k_3 ?
L.S. system in k_1, k_2, k_3

$$\text{L.S.: } \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 0 & 14 & 10 & -5 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 1 & -2 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -9 & -17 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -5 & -10 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -9 & -17 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\therefore k_1=1, k_2=-2, k_3=2$$

∴ L.S. is consistent

$$\therefore d = 1a + (-2)b + 2c$$

: d as a linear comb. of a, b, c

$$\text{check: } \left[\begin{array}{c} -1 \\ 8 \\ -5 \end{array} \right] = 1 \left[\begin{array}{c} 1 \\ 0 \\ 3 \end{array} \right] + (-2) \left[\begin{array}{c} 4 \\ 2 \\ 14 \end{array} \right] + 2 \left[\begin{array}{c} 3 \\ 6 \\ 10 \end{array} \right]$$

\Leftrightarrow Is the L.S. w/ $[a \ b \ c \ | \ d]$ consistent?

(*)

Given $v_1, \dots, v_p \in \mathbb{R}^n$,

Span $\{v_1, \dots, v_p\}$: the set of (all) the linear combinations
of v_1, \dots, v_p
"the set spanned by v_1, \dots, v_p "

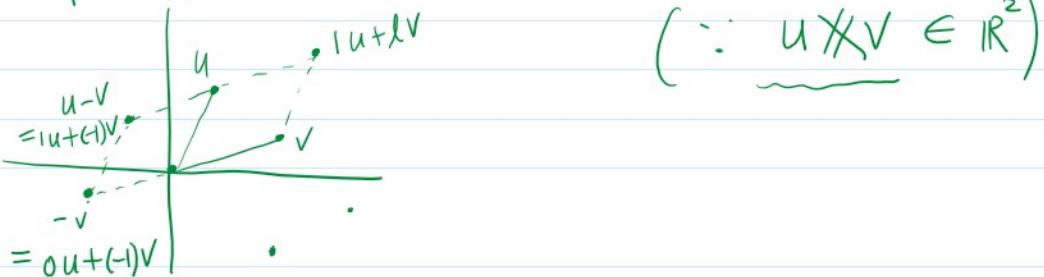
$$\equiv \left\{ k_1 v_1 + k_2 v_2 + \dots + k_p v_p \mid \forall k_i \in \mathbb{R} \right\}$$

$\Leftrightarrow \mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\leftarrow \text{Span}\{\mathbf{u}\} = \{k\mathbf{u} \mid k \in \mathbb{R}\}$

: line in \mathbb{R}^2 passing through 0 always ($\because \frac{0}{k} = 0$)

$\Leftrightarrow \mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\text{Span}\{\mathbf{u}, \mathbf{v}\} = \{k\mathbf{u} + l\mathbf{v} \mid k, l \in \mathbb{R}\} = \text{the } xy\text{-plane} = \mathbb{R}^2$



$\Leftrightarrow \mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$\text{Span}\{\mathbf{u}, \mathbf{v}\} = \text{Span}\{\mathbf{u}\} = \text{Span}\{\mathbf{v}\}$

$\therefore \mathbf{u} \parallel \mathbf{v} \Leftrightarrow \exists c \in \mathbb{R} \text{ st. } \mathbf{v} = c\mathbf{u}$

$$\begin{aligned} \Rightarrow \text{Span}\{\mathbf{u}, \mathbf{v}\} &= \{k\mathbf{u} + l\mathbf{v} \mid k, l \in \mathbb{R}\} \\ &= \{k\mathbf{u} + l(c\mathbf{u}) \mid k, l \in \mathbb{R} \wedge c \in \mathbb{R}\} \\ &= \{(k+lc)\mathbf{u} \mid \forall p = k+lc \in \mathbb{R}\} \end{aligned}$$

$$= \{p\mathbf{u} \mid \forall p \in \mathbb{R}\} = \text{line w/ } 0, \mathbf{u}$$

* \mathbb{R}^2

- 1) $\text{Span}\{\mathbf{u}\}$
 - ① $\{0\}$ if $\mathbf{u} = 0$
 - ② line through $0 \neq \mathbf{u}$ if $\mathbf{u} \neq 0$

② line through $0 \sim u$

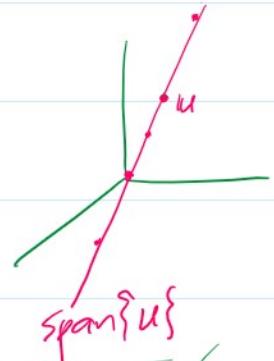
2) $\text{span}\{u, v\}$ ① {0} if $u=v=0$

② line through $0 \& u/v$ if $u \parallel v$

③ plane, \mathbb{R}^2 , if $u \times v$

* \mathbb{R}^3 , u, v, w : non-zero vectors

1) $\text{span}\{u\}$: line in \mathbb{R}^3 through $0 \& u$
 $\Rightarrow u \parallel u$



2) $\text{span}\{u, v\}$ ① plane if $u \times v$

② line if $u \parallel v$

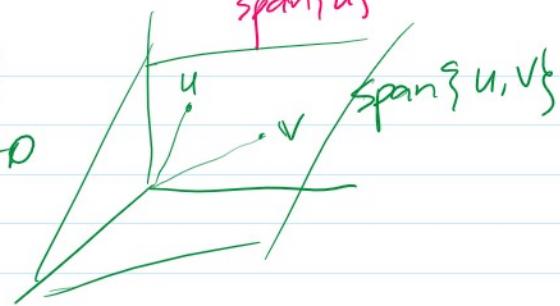
③ pt. if $u=v=w=0$

④ line

3) $\text{Span}\{u, v, w\}$ ② plane

③ \mathbb{R}^3

$\times \because \text{span}\{\alpha_1, \alpha_2\}$: plane $\nparallel 0, \alpha_1, \alpha_2$



$$E) A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$$

Is b in the plane spanned by the columns of A ?

Sol) $\Leftrightarrow b \in \text{span}\{\alpha_1, \alpha_2\}$

$$\therefore \{k_1\alpha_1 + k_2\alpha_2 \mid k_1, k_2 \in \mathbb{R}\}$$

$$\Leftrightarrow [\alpha_1 \ \alpha_2] \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = b \text{ with some } k_1, k_2 \in \mathbb{R}$$

$$\Rightarrow [A \ b] = \begin{bmatrix} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 5 & 17 \end{bmatrix} \text{ can stop.}$$

$$\sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & 5 & -21 \\ 0 & 0 & 1 \end{bmatrix} \text{REF} \therefore \text{inconsistent}$$

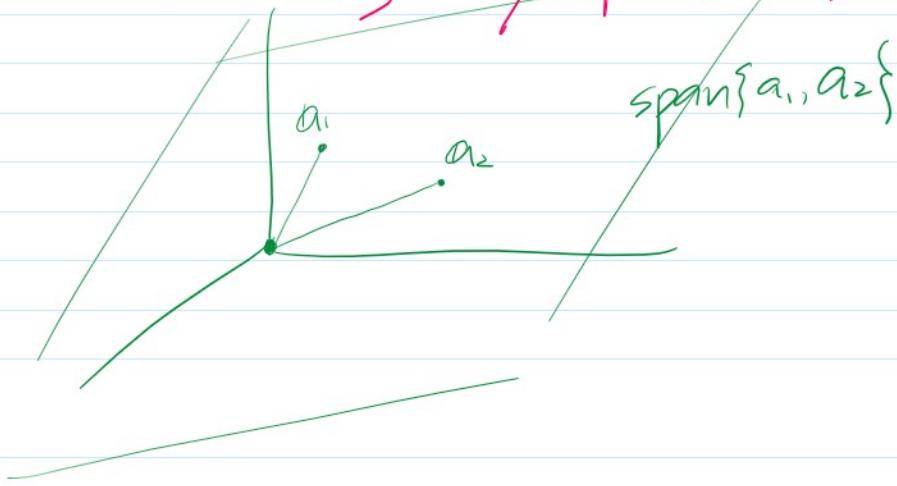
$$\left[\begin{array}{ccc} 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right]$$

$$0 \neq 1$$

$\therefore \nexists k_1, k_2 \text{ s.t. } k_1 a_1 + k_2 a_2 = b$

$b \notin \text{span}\{a_1, a_2\}$

$\Rightarrow b \notin \text{span}\{a_1, a_2\}$



1.4 Matrix Eq.

$A \times = b$: L.S. w/ n equations
 $\underset{n \times p}{A} \underset{p \times 1}{\times}$ w/ p unknowns

Def) matrix multiplication by a vector

$$A \times = [a_1 \ a_2 \ \dots \ a_p] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

$$\equiv x_1 a_1 + x_2 a_2 + \dots + x_p a_p$$

$$= \sum_{k=1}^p x_k a_k : \text{l. comb. of } a_i \text{ w/ } k_i$$

$$\textcircled{2} \quad \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 4 \end{bmatrix} + z \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} + \begin{bmatrix} 3y \\ 4y \end{bmatrix} + \begin{bmatrix} 5z \\ 6z \end{bmatrix}$$

$$\text{OR} = \left[\begin{array}{c} (1, 3, 5) \cdot (x, y, z) \\ (2, 4, 6) \cdot (x, y, z) \end{array} \right]$$

$$= \left[\begin{array}{c} x + 3y + 5z \\ 2x + 4y + 6z \end{array} \right]$$

(1)

$$= \begin{bmatrix} x+3y+5z \\ 2x+4y+6z \end{bmatrix} \quad \textcircled{1}$$

★ 3 equivalents Questions

1) L.S.

$$\left\{ \begin{array}{l} x_1 a_{11} + x_2 a_{12} + \dots + x_p a_{1p} = b_1 \\ x_1 a_{21} + x_2 a_{22} + \dots + x_p a_{2p} = b_2 \\ \vdots \\ x_1 a_{n1} + x_2 a_{n2} + \dots + x_p a_{np} = b_n \end{array} \right.$$

\Leftrightarrow Vector Equations

$$x_1 a_1 + x_2 a_2 + \dots + x_p a_p = b$$

\Leftrightarrow Matrix Equation

$$[a_1 \ a_2 \ \dots \ a_p] \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = b$$

$$\therefore A \ x = b$$

② $Ax=b$ is consistent? i.e; $Ax=b$ has a sol.?

\Leftrightarrow Is b in the set spanned by the col.s of A ?
i.e., $b \in \text{Span}\{a_1, a_2, \dots, a_p\}$?

\Leftrightarrow Is b a linear comb. of the col.s in A ?

~~Ex~~

$$A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

1) Is $Ax=b$ consistent for all b ? No

- - - - - If A is non-singular, then $Ax=b$ is consistent.

1) Is $Ax=b$ consistent for all b ? No

2) Find b that makes the system $Ax=b$ consistent.

$$\text{SOL: } 1) [A \ b] = \left[\begin{array}{ccc|c} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{array} \right] \xrightarrow{\begin{matrix} 3R_1+R_2 \\ -2R_1+R_3 \end{matrix}} \left[\begin{array}{ccc|c} 1 & 4 & 5 & b_1 \\ 0 & 1 & 1 & 3b_1+b_2 \\ 0 & 0 & 0 & -2b_1+b_3 \end{array} \right]$$

0?

If $-2b_1+b_3 \neq 0$,
 $Ax=b$ is inconsistent.

$$2) -2b_1+b_3=0 \Rightarrow Ax=b : \text{consistent}$$

$$\Rightarrow \text{if } b \text{ s.t. } -2b_1+b_3=0 \Rightarrow b_3=2b_1$$

w/ E.A $\left[\begin{array}{c|c} b_1 \\ b_2 \\ b_3 \end{array} \right]$, where $b_3=2b_1$

w/ L.A : b s.t. $\underline{-2b_1+b_3=0}$ L.Eq.

$$\left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \end{array} \right] \text{ee}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \end{array} \right] \text{ee}$$

$\uparrow \uparrow \uparrow$
basis free variables

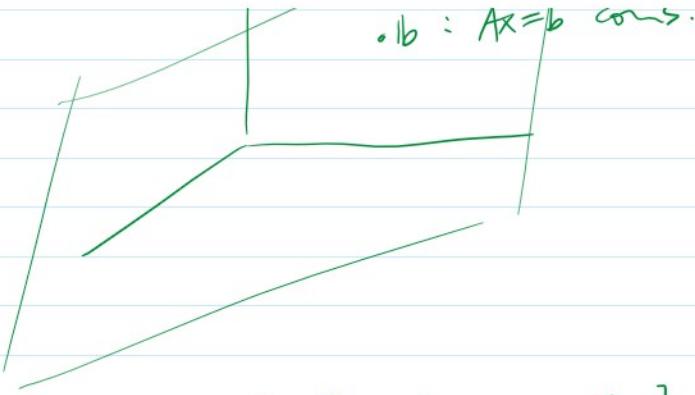
$$b = \left[\begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} \right] = b_2 \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right] + b_3 \left[\begin{array}{c} \frac{1}{2} \\ 0 \\ 1 \end{array} \right], \quad \forall b_2, b_3 \in \mathbb{R}$$

$$\therefore \{b\} = \left\{ k \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right] + \lambda \left[\begin{array}{c} \frac{1}{2} \\ 0 \\ 1 \end{array} \right] \mid \forall k, \lambda \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right], \left[\begin{array}{c} \frac{1}{2} \\ 0 \\ 1 \end{array} \right] \right\} : \text{plane} \subset \mathbb{R}^3$$

$\cdot b : Ax=b$ incons.

$\cdot b : Ax=b$ cons.



Def) Given $A = [a_1 \ a_2 \ \dots \ a_p]_{n \times p}$, $\forall a_i \in \mathbb{R}^n$

$$S \equiv \text{Span}\{a_1, a_2, \dots, a_p\} \subseteq \mathbb{R}^n$$

If $S = \mathbb{R}^n$, then "the col.s of A span \mathbb{R}^n ".

Ex) $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} \subsetneq \mathbb{R}^2$
 $\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ does not span \mathbb{R}^2 .

Ex) $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\} = \mathbb{R}^2$
 $\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ span \mathbb{R}^2 .

Ex) $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \end{bmatrix}\right\} = \mathbb{R}^2$

~~Ex~~ (contd) ...

1.5

Def) $Ax = b$: homogeneous L.S.
 $\Leftrightarrow b = 0$ i.e., $Ax = 0$

otherwise, nonhomogeneous L.S
 i.e., $b \neq 0$

$$\text{Ex}) \begin{cases} 2x - y = z \\ x + y - z = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - y - z = 0 \\ x + y - z = 0 \end{cases}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

\therefore H.L.S.

Thn) $Ax=0$ is consistent. i.e., Yes to Exist. Q.

Pf) $\exists x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ s.t. $A\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$

\therefore A sol. exists.

\therefore consistent.

"trivial solution"

* 2 questions for $Ax=0$

① Existence Q : Yes ($\because \exists$ trivial sol. ①)

② Uniqueness Q

- Yes : only the trivial sol.
- No : \exists non-trivial sol.s
or many

$$\text{Ex}) \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} : \text{H.L.S. i.e., cons. w/ at least trivial sol.}$$

Sol) $[A|b] = \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array} \right]$ $\xrightarrow{\text{No need to carry } \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

↑ free variable \therefore cons. w/ free variable

$\therefore \exists$ many sols

(one trivial sol.
& co-many non-trivial sols)

\therefore Given LHS,
 always consistent.
 \exists free variable $\Rightarrow \exists$ non-trivial sols
 \nexists free variable $\Rightarrow \exists!$ sol. \Rightarrow only the trivial sol.
 No to Uniq. Q.
 Yes to Uniq. Q.

$$A \sim \left[\begin{array}{cc} 1 & 3 \\ 0 & 0 \end{array} \right]$$

\uparrow
 f

$$\begin{cases} x+3y=0 \\ 2x+6y=0 \end{cases}$$

$\Leftrightarrow y = -\frac{1}{3}x$

$$\therefore x = x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \forall x_2 \in \mathbb{R}$$

$$\therefore S.S: \left\{ k \begin{bmatrix} -3 \\ 1 \end{bmatrix} \mid \forall k \in \mathbb{R} \right\}$$

$$= \text{Span} \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\} = \text{line}$$

