CH1

Expression : $2x+1$

Equation : $2x^{1}+1=7$ (Linear) of line power 1

$\sin x + x^{2} = 2x$ (non-linear)

$\sin x = 2x^{1}$ (non-linear)

$\sqrt{x} = 3x+1$ (non-linear)

$x^{\frac{1}{2}}$

Def) linear equation in $x_1, x_2, \cdots, x_n$ (power 1)

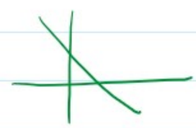
$\Leftrightarrow$ $a_1 x_1^{1} + a_2 x_2^{1} + a_3 x_3^{1} + \cdots + a_n x_n^{1} = b$,

where $\forall a_i (\in \mathbb{R})$: coefficients

$b (\in \mathbb{R})$: constant

$\forall x_i$: variables

Ex) ① $x^{1} + 2y^{1} = 3$: l. eq. in x & y



② $5^{2}x^{1} + y^{1} = \sqrt{\pi}\, y^{1} + 6$: l. eq. in x & y

③ $x + 2y + 3z = 4$: l. eq. in x, y & z

④ $\underbrace{x^{1} \cdot y^{1}}_{\text{power } 2} + e = x^{1}$: non-linear

⑤ $5x - 37y + 2z + 7w = 100$: l. eq. in x, y, z & w

Def) system of L. eq.s (i.e., Linear system)

$\Leftrightarrow$ collection of one or more L. eq.s

Ex) ① $2x + y = 3 - x$: L. system (L. eq.)

② $\begin{cases} 2x + y = 3 - x \\ x - y = 5 \\ x + 2y = 1 \end{cases}$: L. system in x & y (overdetermined)

③ $\begin{cases} x - y + z = 1 \\ 2x + y - z = 3 \end{cases}$: L. system in x, y & z (underdetermined)

Def) A solution of a L. system in $x_1, \cdots, x_n$

$\Longleftrightarrow$ $(s_1, s_2, \cdots, s_n)$: n-tuple that satisfies the L.S.

Ex) $x + 2y = 1$: l. system

ordered pair

$(1, 0)$: solution

$(0, \frac{1}{2})$: solution

$(\frac{1}{2}, \frac{1}{4})$: solution

$(2, 1)$: not a solution

$\vdots$

~~x=1~~ sol. to the L. system

(a, b) : ordered pair

Def) The solution set of a L. System

$\Longleftrightarrow$ $\{$solutions$\}$: the set of all the solutions of the system.

Ex) $x + 2y = 1$

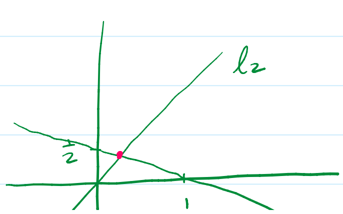
$\Rightarrow \{ (x, y) \mid x + 2y = 1 \}$

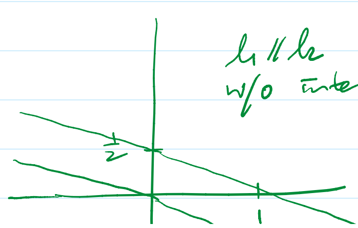
or $\{ (x, \frac{1-x}{2}) \mid \forall x \in \mathbb{R} \}$

or $\{ (1-2y, y) \mid \forall y \in \mathbb{R} \}$

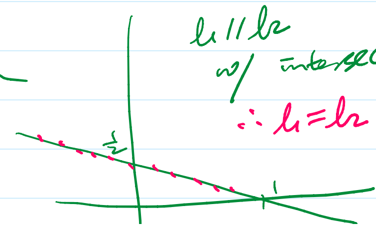
or $\{ (1,0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{4}), (-2, \frac{3}{2}), (-1, 1), \cdots \}$

Ex) $\begin{cases} x + 2y = 1 : l_1 \\ x - y = 0 : l_2 \end{cases}$ $\begin{cases} x + 2y = 1 : l_1 \\ 2x + 4y = 0 : l_2 \end{cases}$ $\begin{cases} x + 2y = 1 : l_1 \\ 2x + 4y = 2 : l_2 \end{cases}$





$l_1 // l_2$ w/o intersection



$l_1 // l_2$ w/ intersections

$\therefore l_1 = l_2$

$\exists!$ solution $\qquad \nexists$ solution $\qquad \exists\ \infty$-many solutions

Q) Given a L.S.

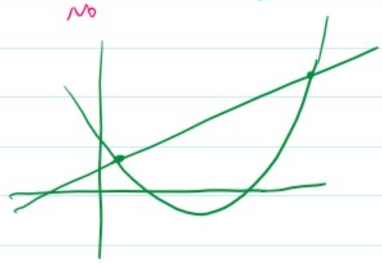
1) Existence Question : $\exists$?

2) Uniqueness Question : $\exists!$?

* L.S

- consistent : $\exists$ solution (yes)
  - Uniqueness Question
    - $\exists!$ (yes)
    - $\exists\ \infty$-many (No)
- inconsistent : $\nexists$ solution (No)

Existence Question



\# of solution-wise for a L.S.

$\therefore$ No sol. / only one sol. / inf. many sol.s

$\underbrace{\text{No sol.}}_{\text{inconsistent}}$ / $\underbrace{\text{only one sol. / inf. many sol.s}}_{\text{consistent}}$

Ex)
$$\begin{cases} x+y+z=1 \\ x-y-z=0 \\ x+y-z=3 \\ 2x-y+z=1 \end{cases} \Rightarrow \cdots$$

* Matrix Notation

entries from the coefficients

$$\begin{cases} 1x - 2y = -1 \\ -x + 3y = 3 \end{cases} \Longrightarrow \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

L.S. $\qquad$ **Matrix Equation**

$$A\,\mathbf{u} = \mathbf{b}$$

"coefficient ... $\qquad$ unknown vector $\qquad$ constant vector

$A \; \mathbb{u} = \mathbb{b}$

"coefficient matrix" — unknown vector — constant vector

$[A \;\; \mathbb{b}]$: "Augmented Matrix"

$$[A \;\; \mathbb{b}] = \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

G.E.M. w/ bw. S

$$\Leftrightarrow \begin{cases} 1x - 2y = -1 \\ -x + 3y = 3 \end{cases} \qquad \Leftrightarrow \begin{cases} x - 2y = -1 \\ 0x + 1y = 2 \end{cases}$$

$+)$ $1y = 2$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right] \qquad \therefore \; x = 3, \; y = 2$$

$$\Leftrightarrow \begin{cases} x + 0y = 3 \\ 0 \cdot x + 1 \cdot y = 2 \end{cases}$$

$\therefore$ $(3,2)$: the only sol. to the L.S.

$\{(3,2)\}$: the solution set to the L.S.

$\therefore$ L.S : consistent w/ uniq. sol.

Elementary Row operations on $[A \; b]$

1) Replacement : Add to one row a multiple of another row
2) Interchange : $R_i \Leftrightarrow R_j$
3) Scaling : kR_i, $\forall k (\neq 0) \in \mathbb{R}$

Ex) L.S. in x, y, z

$$\begin{cases} x - 2y + z = 0 \\ y - 4z = 4 \\ -4x + 5y + 9z = -9 \end{cases} \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -4 \\ -4 & 5 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -9 \end{bmatrix}$$

M. Eq.

$A \quad \mathbb{u} = \mathbb{b}$

$$\Rightarrow [A \;\; \mathbb{b}] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

$\Rightarrow [A \;\; b] = \left[\begin{array}{ccc|c} \cdot & \cdot & \cdot & \cdot \\ 0 & 1 & -4 & 4 \\ -4 & 5 & 9 & -9 \end{array}\right]$

$\underset{\sim}{4R_1+R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \underset{\sim}{3R_2+R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

* let's not stop working here. Continue working.

$\underset{4R_3+R_2}{\underset{\sim}{2R_2+R_1}} \left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$\underset{\sim}{7R_3+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$

"equivalent"

$\Leftrightarrow \begin{cases} 1\cdot x + 0\cdot y + 0\cdot z = 29 \\ 0\cdot x + 1\cdot y + 0\cdot z = 16 \\ 0\cdot x + 0\cdot y + 1\cdot z = 3 \end{cases}$: L.S.

$\therefore (x, y, z) = (29, 16, 3)$

$\therefore \{(29, 16, 3)\}$

$\therefore$ L.S : consistent w/ uniq. sol.

Def) L.S$_s$: equivalent

$\Leftrightarrow$ The solution sets are the same.

Q) - Interchange : $R_i \leftrightarrow R_j$

- Scaling : $k\,\text{LHS} = k\,\text{RHS}, \quad \forall k (\neq 0) \in \mathbb{R}$

$\Leftrightarrow \text{LHS} = \text{RHS}$

- Replacement : ex) $[A \;\; b] = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$

$\underset{\sim}{kR_1+R_3} \begin{bmatrix} R_1^{*} \\ R_2 \\ kR_1+R_3 \end{bmatrix}$

equivalent!

$kR_1 + R_3$

R_3 : LHS = RHS

$kR_1 + R_3$: $\cancel{kR_1} + R_3$
$\cancel{-kR_1}$

ex) $\begin{cases} ① \; x+2y=1 \\ ② \; 2x-y=3 \end{cases} \xRightarrow{k① + ②} \begin{cases} x+2y=1 \\ kx+2x+k2y-y=k1+3 \end{cases}$

$\Rightarrow \begin{cases} x+2y=1 \\ \cancel{k(x+2y)}+2x-y=\cancel{k\cdot 1}+3 \end{cases}$

Ex) $\begin{cases} 2x+y=1 \\ x+y=k \end{cases}$

Determine k so that the L.S. is

① consistent w/ one sol.

② consistent w/ ∞-many sols.

③ inconsistent

Sol) L.S $\Leftrightarrow$ $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 1 & k \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & k \\ 2 & 1 & 1 \end{array}\right]$

$\xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|c} 1 & 1 & k \\ 0 & -1 & 1-2k \end{array}\right]$

$\therefore$ Cons. w/ only one sol. for $\forall k \in \mathbb{R}$

Ex) $\begin{cases} 3x-9y=4 \\ -2x+6y=k \end{cases}$

ex) $\begin{cases} \quad \cdots \\ -2x+6y = k \end{cases}$

Discuss the 2 questions!

i.e., Existence Q

& Uniq. Q.

Sol) $\begin{bmatrix} 3 & -9 & 4 \\ -2 & 6 & k \end{bmatrix} \underset{\frac{1}{2}R_2}{\overset{\frac{1}{3}R_1}{\sim}} \begin{bmatrix} 1 & -3 & \frac{4}{3} \\ -1 & 3 & \frac{k}{2} \end{bmatrix}$

$\overset{R_1+R_2}{\sim} \left[\begin{array}{cc|c} 1 & -3 & \frac{4}{3} \\ 0 & 0 & \frac{k}{2}+\frac{4}{3} \end{array}\right]$ $\qquad x-3y=\frac{4}{3}$

$0 = \frac{3k+8}{6}$?

$k=-\frac{8}{3}$: consistent $(\because R2:\ 0=0) \Rightarrow \exists\ \infty$-many.

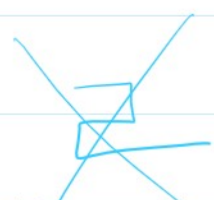
$k \neq -\frac{8}{3}$: inconsistent

Uniq. Q

Existence Q

$\boxed{1.2}$ REF, RREF

Given $A_{n \times p}$. "Step-like" :



Def) Row Echelon Form (REF)

1) . . .

2) . . .

3) . . .

ex) $\begin{bmatrix} * & * & * & * \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\leftarrow$ leading entry

$=1$, $\neq 0$

$\therefore$ ~~RREF~~

: REF.

Def) Reduced Row Echelon Form (RREF)

1) . . .

2) . . .

3) . . .

ex) $\begin{bmatrix} 1 & 2 & 3 & 0 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 1 & 100 \end{bmatrix}$ REF

2) . . . .

3) . . . .

& 4) . . . .

5) . . . .

ex)

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 1 & 100 \end{bmatrix}$$ REF, ~~R~~REF

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 6 & 7 \\ 0 & 0 & 0 & 1 & 100 \end{bmatrix}$$ REF, ~~R~~REF

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 100 \end{bmatrix}$$ RREF

Ex)
$$\begin{cases} x + 4y + 3z = -1 \\ 3x + 14y + 10z = -5 \\ 2y + 6z = 8 \end{cases}$$: L. system

$$\Leftrightarrow \begin{bmatrix} 1 & 4 & 3 \\ 3 & 14 & 10 \\ 0 & 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 8 \end{bmatrix}$$: Matrix Equation

$$\Rightarrow [A \ \ b] = \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 3 & 14 & 10 & -5 \\ 0 & 2 & 6 & 8 \end{array}\right] \underset{\frac{1}{2}R_3}{\overset{-3R_1+R_2}{\sim}} \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 1 & -2 \\ 0 & 1 & 3 & 4 \end{bmatrix} \overset{R_2 \Leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 1 & -2 \end{bmatrix}$$

$$\overset{-2R_2+R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -5 & -10 \end{array}\right] \underset{\frac{1}{-5}R_3}{\overset{-4R_2+R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & -9 & -17 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

REF, ~~RREF~~ REF, ~~RREF~~

$$\underset{-3R_3+R_2}{\overset{9R_3+R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{cases} 1x + 0y + 0z = 1 \\ 0x + 1y + 0z = -2 \\ 0x + 0y + 1z = 2 \end{cases}$$

REF, RREF $\therefore \{(1, -2, 2)\}$

Existence Q ?

You may stop here to answer "Yes, $\exists$ sol."

Q) Given $A_{n \times p}$. $\exists\ \infty$-many REFs
but $\exists$ only one RREF

* $[A \mid b]$: consistent

$\sim$ $\iff \nexists$ row w/ $(0 \cdots 0\ \overset{\ne 0}{\star})$

Def) pivot position
pivot
pivot column

Def) basic variable : any variable that corresponds to a pivot column in a coeff. matrix

free variable : non-basic variable
i.e., any variable that corresponds non-pivot column in a coeff. M.

Ex) $[A \mid b] \sim \cdots \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 1 \\ 0 & 9 & 1 & 0 & 2 \\ 0 & 0 & 0 & 5 & 7 \end{array}\right]$

: L.S w/ 3 equations & 4 unknowns i.e., 4 variables

pivots ∴ pivot columns : 1, 2, 4

∴ basic variable : x_1, x_2, x_4
free variable : x_3

Ex) $[A\ \ b] \sim \cdots \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$ RREF $\leftarrow \nexists\ (0\ 0\ 0 \cdots 0\ \overset{\ne 0}{\star})$

∴ consistent
Yes to Exist. Q

free variable : x_3

$$\begin{cases} x_1 = 3 - 2x_3 \\ x_2 = 4 - 1x_3 \\ x_3 : \text{free} \\ x_4 = 5 \end{cases} \quad \therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \\ 0 \\ 5 \end{bmatrix}, \quad \forall x_3 \in \mathbb{R}$$

$\therefore \exists\ \infty$-many sol.s ($\because \exists$ free variable for a cons. system)

$$\therefore \left\{ k \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \\ 0 \\ 5 \end{bmatrix} \;\middle|\; \forall k \in \mathbb{R} \right\} : \text{sol. set}$$

~~R~~REF

Ex) $$[A \ \ b] \sim \cdots \sim \left[\begin{array}{cccc|c} 1 & 6 & 8 & 2 & 3 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right]$$

$0 \neq 6$

NoNoNo!!!

~~$\exists$ free v. : x_2, x_3~~

~~$\therefore$ cons. w/ ∞-many sol.s~~

: not even a consistent system

($\because \exists$ (0 0 0 0 6), $0 \neq 6$)

$\because 0 \neq 6$

: inconsistent system

i.e., $\nexists$ sol.

Ex) L.S. w/ $A_{3\times 4}$

$$[A \ \ b] = \left[\begin{array}{cccc|c} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right] \quad \text{or} \quad \left[\begin{array}{cccc|c} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free, f (left); free, free (right)

$\exists$ at most 1 pivot in a row/column

$\therefore \exists$ at least 1 free variable

$\therefore \exists$ at least 1 free var.

~~$\therefore$ cons. w/ ∞-many sol.s~~

may not be a consistent system

c.ex) $[A\ b] \sim \begin{bmatrix} * & - & - & - & \vdots & * \\ 0 & * & - & - & \vdots & * \\ 0 & 0 & 0 & 0 & \vdots & * \end{bmatrix}$: inconsistent.

$\nexists$ sol.

$* \neq 0$

free variables

$\boxed{1.3}$ vector equations

Def) (column) vector : $n \times 1$ matrix

i.e., matrix w/ one column.

$* \ \mathbb{R}^n : \mathbf{u}, \vec{u}$

"lower case u in bold"

Ex) $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}_{3\times 1}$: vector

Ex) $\mathbf{u} (\in \mathbb{R}^n) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}_{n\times 1}$

Def) Given vectors $\underset{\in \mathbb{R}^n}{\mathbf{u}_1}, \underset{\in \mathbb{R}^n}{\mathbf{u}_2}, \cdots, \underset{\in \mathbb{R}^n}{\mathbf{u}_p} \in \mathbb{R}^n$,

$\forall\ \mathbf{u}_k = \begin{bmatrix} u_{1k} \\ u_{2k} \\ \vdots \\ u_{nk} \end{bmatrix} \in \mathbb{R}^n$, $\forall u_{ik} \in \mathbb{R}$

$\sum_{i=1}^{p} c_i \mathbf{u}_i = c_1 \mathbf{u}_1^{①} + c_2 \mathbf{u}_2^{①} + c_3 \mathbf{u}_3^{①} + \cdots + c_p \mathbf{u}_p^{①}$

: linear combination of $\mathbf{u}_1, \cdots, \mathbf{u}_p$

of line

power 1

Ex) $2\mathbf{u} + 3\mathbf{v} + 0\mathbf{w}$: linear comb. of $\mathbf{u}$ & $\mathbf{v}$ & $\mathbf{w}$

Ex) $2\mathbb{u} + 3\mathbb{v} + 0\mathbb{w}$: linear comb. of $\mathbb{u}$ & $\mathbb{v}$ & $\mathbb{w}$

Ex) $\mathbb{a} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbb{b} = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, $\mathbb{c} = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$, $\mathbb{d} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$

1) Is $\mathbb{d}$ a L.C. of $\mathbb{a}, \mathbb{b}, \mathbb{c}$?

Sol) $\mathbb{d} = k_1 \mathbb{a} + k_2 \mathbb{b} + k_3 \mathbb{c}$ w/ some k_1, k_2, k_3 ?

$\exists$?

L. system in k_1, k_2, k_3

L.S : $\left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{array}\right] \sim \begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -9 & -17 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & -5 & -10 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -9 & -17 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$

$\therefore k_1 = 1,\ k_2 = -2,\ k_3 = 2$

ie, L.S : consistent

$\therefore \mathbb{d} = 1\mathbb{a} + (-2)\mathbb{b} + 2\mathbb{c}$

: $\mathbb{d}$ as a linear comb. of $\mathbb{a}, \mathbb{b}, \mathbb{c}$

check) $\begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + (-2)\begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$

$\Longleftrightarrow$ Is the L.S w/ $[\mathbb{a}\ \mathbb{b}\ \mathbb{c} \mid \mathbb{d}]$ consistent ?

Def) Given $\mathbb{v}_1, \cdots, \mathbb{v}_p \in \mathbb{R}^n$,

$\text{Span}\{\mathbb{v}_1, \cdots, \mathbb{v}_p\}$: the set of all the linear combinations of $\mathbb{v}_1, \cdots, \mathbb{v}_p$

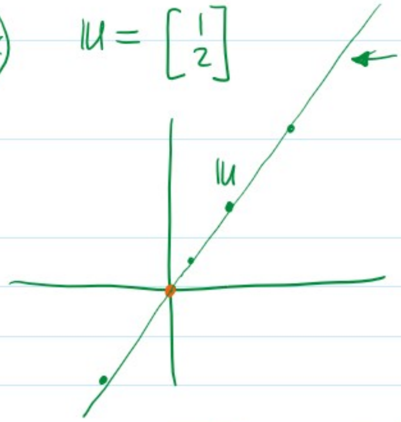
" the set spanned (generated) by $v_1, \cdots, v_p$ "

$\equiv \{ k_1\mathbb{v}_1 + k_2\mathbb{v}_1 + \cdots + k_p\mathbb{v}_p \mid \forall k_i \in \mathbb{R} \}$

Ex) $\mathbb{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ← $\text{span}\{\mathbb{u}\} = \{ k\mathbb{u} \mid \forall k \in \mathbb{R} \}$

: line in $\mathbb{R}^2$ passing through 0 always ($\because$ w/ $k=0$)

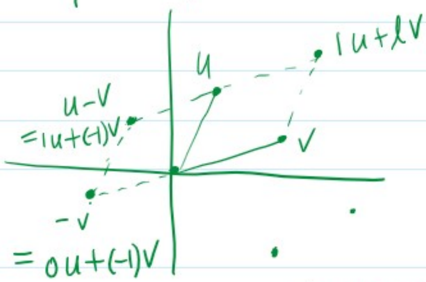
// to $\mathbb{u}$



Ex) $\mathbb{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbb{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\text{span}\{u, v\} = \{ ku + lv \mid \forall k, l \in \mathbb{R} \}$ = the xy-plane $= \mathbb{R}^2$

$(\because \underbrace{u \nparallel v}\ \in \mathbb{R}^2)$



$v = 3u \therefore u \parallel v$

Ex) $\mathbb{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbb{v} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$\text{span}\{\mathbb{u}, \mathbb{v}\} = \text{span}\{\mathbb{u}\} = \text{span}\{v\}$

$\because$ $u \parallel v \Leftrightarrow \exists\, c \in \mathbb{R}$ s.t. $v = cu$

$\Rightarrow \text{span}\{u, v\} = \{ ku + lv \mid \forall k, l \in \mathbb{R} \}$

$= \{ ku + l(cu) \mid \forall k, l \in \mathbb{R} \text{ w/ } c \in \mathbb{R} \}$

$= \{ \underbrace{(k + lc)}\, u \mid \forall p = k + lc \in \mathbb{R} \}$

$= \{ pu \mid \forall p \in \mathbb{R} \}$: line w/ 0, u

* $\mathbb{R}^2$

1) $\text{span}\{\mathbb{u}\}$
- ① $\{0\}$ if $\mathbb{u} = 0$
- ② line through 0 & $\mathbb{u}$ if $u \neq 0$

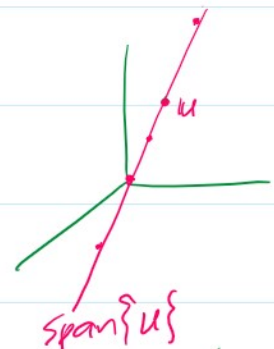
① $\{0\}$ if $u = v = 0$

② line through 0 & u

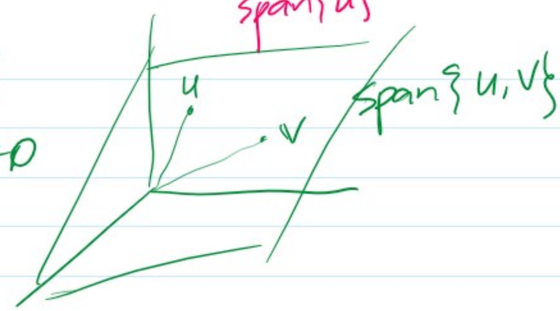
2) span$\{u, v\}$
- ① $\{0\}$ if $u = v = 0$
- ② line through 0 & u/v if $u \parallel v$
- ③ plane, $\mathbb{R}^2$, if $u \not\parallel v$

* $\mathbb{R}^3$, u, v, w : non-zero vectors

1) span$\{u\}$: line in $\mathbb{R}^3$ through 0 & u
ie, $\parallel u$



2) span$\{u, v\}$
- ① plane if $u \not\parallel v$
- ② line if $u \parallel v$



3) Span$\{u, v, w\}$
- ⓪ pt. if $u = v = w = 0$
- ① line
- ② plane
- ③ $\mathbb{R}^3$

$\not\parallel$ $\therefore$ span$\{a_1, a_2\}$: plane w/ 0, a_1, a_2

Ex) $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$

Is b in the plane spanned by the columns of A ?

Sol) $\Leftrightarrow$ $b \overset{?}{\in} \text{span}\{a_1, a_2\}$

$= \{ k_1 a_1 + k_2 a_2 \mid \forall k_1, k_2 \in \mathbb{R} \}$

$\Leftrightarrow [a_1 \; a_2] \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = b$ with $\underset{\exists ?}{\text{some}}$ $k_1, k_2 \in \mathbb{R}$

$\Rightarrow [A \; b] = \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 5 & 17 \end{array}\right]$ can stop.

$\sim \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & 5 & 21 \\ 0 & 0 & 1 \end{array}\right]$ REF $\therefore$ inconsistent

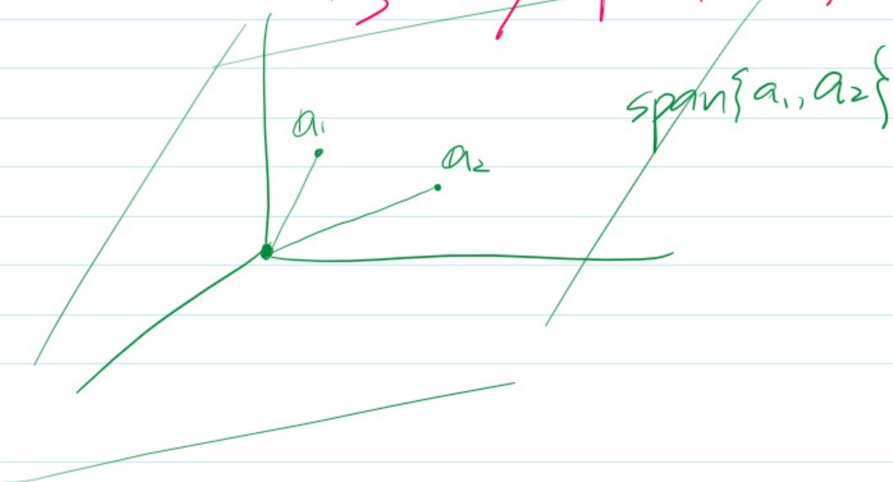
$$\begin{bmatrix} 0 & 5 & \cdots \\ 0 & 0 & \neq 1 \end{bmatrix}$$

$0 \neq 1$

$\therefore \nexists\, k_1, k_2$ s.t. $k_1 a_1 + k_2 a_2 = b$

$b \notin \mathrm{span}\{a_1, a_2\}$

i.e., $b \notin \mathrm{span}\{a_1, a_2\}$



$\boxed{1.4}$ Matrix Eq.

$\underset{n\times p}{A}\, \underset{p\times 1}{x} = b$: L.S. w/ n equations & $\underline{p}$ unknowns

Def) matrix multiplication by a vector

$$\underbrace{\underset{n\times p}{A}\ \underset{p\times 1}{x}} = [a_1\ a_2\ \cdots\ a_p]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

$$\equiv x_1 a_1 + x_2 a_2 + \cdots + x_p a_p$$

$$= \sum_{k=1}^{p} x_k a_k$$: l. comb. of a_i w/ k_i

Ex) $$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 4 \end{bmatrix} + z\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} + \begin{bmatrix} 3y \\ 4y \end{bmatrix} + \begin{bmatrix} 5z \\ 6z \end{bmatrix}$$

$$= \begin{bmatrix} x + 3y + 5z \\ 2x + 4y + 6z \end{bmatrix}$$

OR

$$= \begin{bmatrix} (1, 3, 5) \cdot (x, y, z) \\ (2, 4, 6) \cdot (x, y, z) \end{bmatrix}$$

II

$$= \begin{bmatrix} x + 3y + 5z \end{bmatrix}$$

$$= \begin{bmatrix} x+3y+5z \\ 2x+4y+6z \end{bmatrix}$$

☆ 3 equivalents Questions

1) L.S.

$$\begin{cases} x_1 a_{11} + x_2 a_{12} + \cdots + x_p a_{1p} = b_1 \\ x_1 a_{21} + x_2 a_{22} + \cdots + x_p a_{2p} = b_2 \\ \vdots \\ x_1 a_{n1} + x_2 a_{n2} + \cdots + x_p a_{np} = b_n \end{cases}$$

$\Leftrightarrow$ Vector Equations

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_p \mathbf{a}_p = \mathbf{b}$$

$\Leftrightarrow$ Matrix Equation

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_p] \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = \mathbf{b}$$

$$\therefore A \quad \mathbf{x} = \mathbf{b}$$

Q) $\underset{n\times p}{A}x = b$ is consistent? ie; $Ax=b$ has a sol.?

$\Leftrightarrow$ Is b in the set spanned by the col.s of A?

ie, $\mathbf{b} \in \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_p\}$?

$\Leftrightarrow$ Is b a linear comb. of the col.s in A?

☆ $A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -11 & -14 \\ 2 & 8 & 10 \end{bmatrix}$, $\quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

1) Is $Ax=b$ consistent for all $\mathbf{b}$? No

1) Is $Ax=b$ consistent <u>for all $\mathbb{b}$</u> ? No

2) Find $\mathbb{b}$ that makes the system $Ax=b$ consistent.

sol) 1) $[A \;\; b] = \left[\begin{array}{ccc|c} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{array}\right] \xrightarrow[-2R_1+R_3]{3R_1+R_2} \left[\begin{array}{ccc|c} 1 & 4 & 5 & b_1 \\ 0 & 1 & 1 & 3b_1+b_2 \\ 0 & 0 & 0 & -2b_1+b_3 \end{array}\right]$

$-2b_1+b_3 \overset{?}{=} 0$

If $-2b_1+b_3 \neq 0$,
$Ax=b$ is inconsistent.

2) $-2b_1+b_3=0 \Rightarrow Ax=b$: consistent

$\Rightarrow \mathbb{b}$ s.t. $-2b_1+b_3=0$
$\Rightarrow b_3=2b_1$

w/ E.A $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, where $b_3=2b_1$

w/ L.A : $\mathbb{b}$ s.t. $\underline{-2b_1+b_3=0}$ L.Eq.

$\left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \end{array}\right]$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \end{array}\right]$

basis free variables

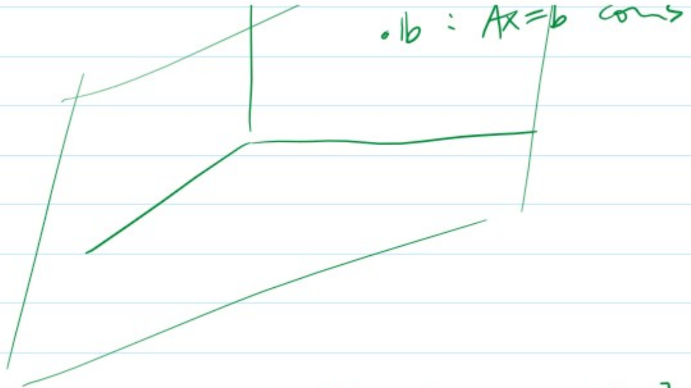
$\mathbb{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}, \quad \forall b_2, b_3 \in \mathbb{R}$

$\therefore \{\mathbb{b}\} = \left\{ k \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + l \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \;\middle|\; \forall k, l \in \mathbb{R} \right\}$

$= \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$: plane $\subset \mathbb{R}^3$

$\cdot \mathbb{b}$: $Ax=b$ incons.

$\cdot \mathbb{b}$: $Ax=b$ cons.



Def) Given $\underset{n\times p}{A} = [a_1 \; a_2 \; \cdots \; a_p]$, $\forall a_i \in \mathbb{R}^n$

$S \equiv \text{Span}\{a_1, a_2, \cdots, a_p\} \subseteq \mathbb{R}^n$

If $S = \mathbb{R}^n$, then "the col.s of A span $\mathbb{R}^n$."

Ex) $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} \subsetneq \mathbb{R}^2$

$\therefore \begin{bmatrix}1\\2\end{bmatrix}$ does not span $\mathbb{R}^2$.

Ex) $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\} = \mathbb{R}^2$

$\therefore \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}$ span $\mathbb{R}^2$.

Ex) $\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}5\\7\end{bmatrix}\right\} = \mathbb{R}^2$

Thm 4) $\cdots$

$\boxed{1.5}$

Def) $Ax = b$: homogeneous L.S.

$\Leftrightarrow b = 0$ i.e., $Ax = 0$

otherwise, nonhomogeneous L.S

i.e., $b \neq 0$

Ex) $\begin{cases} 2x - y = z \\ x + y - z = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - y - z = 0 \\ x + y - z = 0 \end{cases}$

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ H.L.S.

Thm) $Ax = 0$ is consistent. ie, Yes to Exist. Q

Pf) $\exists\ x = \mathbb{0}$ s.t. $A\mathbb{0} = \mathbb{0}$

$\therefore$ A sol. exists.

$\therefore$ consistent.

"trivial solution"

* 2 Questions for $Ax = 0$

① Existence Q : Yes ($\because \exists$ trivial sol. $\mathbb{0}$)

② Uniqueness Q
- Yes : only the trivial sol. ?
- No : $\exists$ ∞-many non-trivial sol.s

Ex) $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: H.L.S ie, cons. w/ at least trivial sol.

sol) $[A\ b] = \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right]$ No need to carry $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

free variable $\therefore$ cons. w/ free variable

$\therefore \exists\ \infty$-many sols

(one trivial sol. & ∞-many non-trivial sols

(one trivial sol.
& ∞-many non-trivial sols

$\therefore$ Given HLS, always consistent.

No to uniq. Q.

$\exists$ free variable $\Longrightarrow$ $\exists$ non-trivial sol.s

$\nexists$ free variable $\Longrightarrow$ $\exists!$ sol. i.e., only the trivial sol.

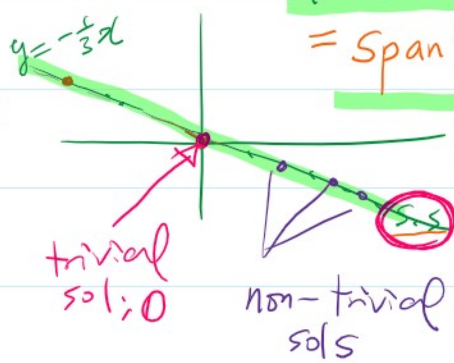
Yes to uniq. Q.

$$A \sim \begin{array}{c} \begin{bmatrix} 1 & 3 \\ \hline 0 & 0 \end{bmatrix} \\ \quad\ \ \uparrow \\ \quad\ \ f \end{array} \qquad \therefore\ x = x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix},\ \forall x_2 \in \mathbb{R}$$

$$\therefore\ S.S:\ \left\{ k \begin{bmatrix} -3 \\ 1 \end{bmatrix} \,\middle|\, \forall k \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\} : \text{line}$$

$$\begin{cases} x + 3y = 0 \\ 2x + 6y = 0 \end{cases}$$

$\iff$ $y = -\frac{1}{3}x$:



$$S.S = \boxed{\text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}} \ni 0$$

Thm) * H.L.S. $Ax = 0$ has non-trivial sol.s

$\iff$ A has at least one free variable.

Ex) $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$: NHLS

Sol) $\left[\begin{array}{cc|c} 1 & 3 & 1 \\ 2 & 6 & 8 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 1 \\ \hline 0 & 0 & 1 \end{array}\right]$: Not consistent

$\uparrow$ f

~~$\therefore \exists\ \infty$-many sol.s~~

Ex) $\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix}$: NHLS

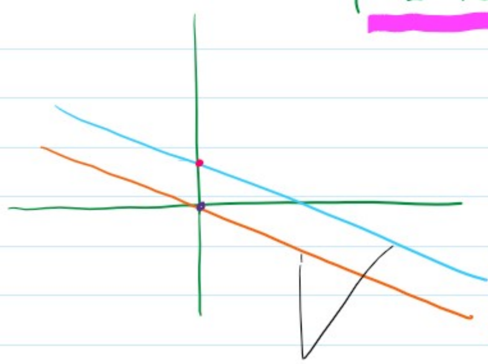
Ex) $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$: NHLS

Sol) $\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & 6 & 4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 0 & 0 \end{array}\right]$: consistent

$\underset{f}{\uparrow}$ (second column) $\quad \therefore \exists\ \infty$-many sol.s

$$\begin{bmatrix} x \\ y \end{bmatrix} = y\begin{bmatrix} -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\therefore \text{s.set} = \left\{ k\begin{bmatrix} -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \;\middle|\; \forall k \in \mathbb{R} \right\}$$



span{ }

not possible!

s.set of NHLS (doesnot have $\mathbb{0}$)

s.set of HLS (does have $\mathbb{0}$)

* parallel

* Relationship between $Ax = 0$ & $Ax = b \neq 0$

same.

H.L.S : $\mathbb{X}_h$

N.H.L.S : $\mathbb{X} = \mathbb{X}_h + \mathbb{P}$

Ex) H. system 1 : $x + 2y + 3z = 0$: 2 systems

N.H. System 2 : $x + 2y + 3z = 1$

Sol) ① H.L.S. : $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

$\uparrow \quad \uparrow$ free

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad \forall y, z \in \mathbb{R}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} \\ 0 \\ 1 \end{bmatrix},$$

$$\therefore \text{ s. set : } \left\{ k\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + l\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \,\middle|\, \forall k, l \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

: plane through $\mathbb{0}$, $(-2,1,0)$, $(-3,0,1)$ in $\mathbb{R}^3$

plane through $\mathbb{0}$ containing $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$

② N.H.L.S : $x + 2y + 3z = 1$

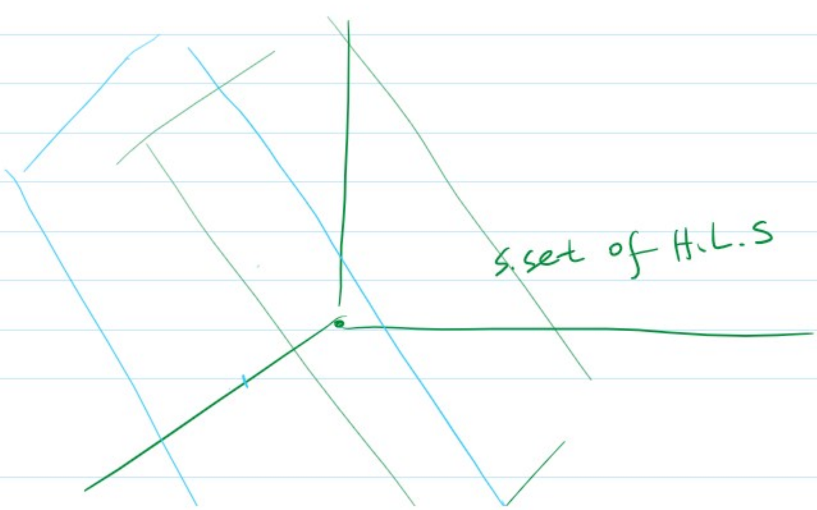
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \end{array}\right]$$

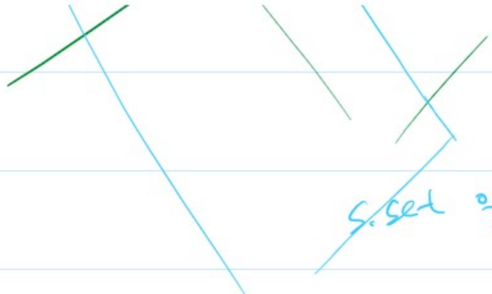
$\uparrow f \uparrow$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{s. set : } \left\{ k\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + l\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \,\middle|\, \forall k, l \in \mathbb{R} \right\}$$

: plane through $(1,0,0)$
// to the plane above





S. set of N.H.L.S.

$\boxed{1.6}$ Applications

M.M :

$\boxed{1.7}$

Def) $S = \{ \overset{\in \mathbb{R}^n}{v_1}, \overset{\in \mathbb{R}^n}{v_2}, \cdots, \overset{\in \mathbb{R}^n}{v_p} \} \underset{\text{subset}}{\subset} \mathbb{R}^n$

: linearly indep. set

$\overset{\text{iff}}{\Longleftrightarrow}$ if $\underbrace{\sum_{i=1}^{p} k_i v_i}_{\text{linear comb. of } v_i\text{s}} = 0 \in \mathbb{R}^n$, then $\forall \underset{\text{weight}}{k_i} = 0 \ (\in \mathbb{R})$

all zero

Otherwise, S : l. dependent. $\therefore$ not all zero

$\therefore$ some k_i, nonzero

i.e, $\sum k_i v_i = 0$ w/ $\underset{\text{(existence)}}{\text{some}}$ $k_i \neq 0$

~~all nonzero~~ NoNoNo

Ex) $\left\{ \overset{u}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \overset{v}{\begin{bmatrix} 2 \\ 4 \end{bmatrix}} \right\}$: l. dep. $\left(\because \underset{\neq 0}{2} u + \underset{\neq 0}{(-1)} v = 0 \right)$

Ex) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$: l. indep.

$\because \quad k \begin{bmatrix} 1 \\ 2 \end{bmatrix} + l \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: H.L.S. (always cons.)

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

no free variable

$\therefore$ only the trivial sol.

$\therefore k = l = 0$

$\therefore k = l = 0$
all zero
$\therefore u, v$: l. indep.

Ex) $S = \left\{ \begin{array}{c} u \\ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \end{array}, \begin{array}{c} v \\ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \end{array}, \begin{array}{c} w \\ \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \end{array} \right\}$: $\underset{\neq 0}{1}u + \underset{\neq 0}{1}v + \underset{\neq 0}{(-1)}w = 0$

$\therefore S$: l. dep.

$u \not\parallel v$, $u \not\parallel w$, $v \not\parallel w$ $\not\Rightarrow$ u, v, w : l. indep — wrong!!!

* Special cases

1) S : set of one vector, ie, $S = \{u\}$
singleton set

① $u = 0$: $ku = k0 = 0$ w/ $k = 37 \neq 0$
$\therefore \{0\}$: l. dep.

② $u \neq 0$: $k\underset{\neq 0}{u} = 0 \Rightarrow k = 0$
$\therefore \{\overset{\neq 0}{u}\}$: l. indep.

2) S : set of only 2 vectors

① $u \parallel v$ ie, $u = kv$ for some k
$\therefore S = \{u, v\}$: l. dep.

② $u \not\parallel v$ ie, $\nexists\, k \in \mathbb{R}$ s.t. $u = kv$
$\therefore S = \{u, v\}$: l. indep.

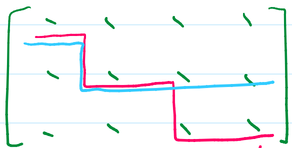
3) A set w/ 0
ie, $S = \{u, v, w, \cdots, 0\}$: l. dep.

$\because$ $ku + lv + mw + \cdots + p0 = 0$
$\Rightarrow k = l = m = \cdots = 0$ & $p = \underset{\neq 0}{37}$

4) A set w/ too many vectors : l. dep.

ie, $S = \{v_1, \cdots, v_p\} \subset \mathbb{R}^n$

too many : $p > n$

∵ HLS : 

2 variables

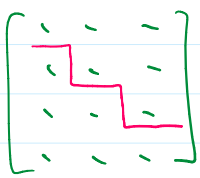
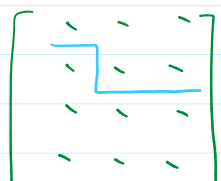
at least one free variable

∴ ∃ non-trivial sol.s

ie, at-least one entry not zero

∴ at-least one weight is not zero.

Q) What if a set w/ less vectors ? inconclusive

HLS: $A \sim$  or $A \sim$ 

only the trivial sol. : l. indep.

f

∃ non-trivial sols. : l. dep.

$\boxed{1.8}$ Transformations

1) Young : $y = f(x)$: $\mathbb{R} \longrightarrow \mathbb{R}$

real/scalar -variable real/scalar -valued ft.

2) MIC : $z = f(x, y)$: $\mathbb{R}^2 \longrightarrow \mathbb{R}$

vector/multi -variable real/scalar -valued ft.

vector) multi) -variable   real) scalar) -valued ft.

3) M1D : $\vec{r}(t) : \mathbb{R} \longrightarrow \mathbb{R}^2$

"parameter"   real) scalar) -variable   vector-valued ft.

ex) $\vec{r}(t) = \langle \cos t , \sin t \rangle$: parametrization

same space

4) M1D : $\vec{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

vector field $(x, y) \longmapsto \langle F_1(x,y) , F_2(x,y) \rangle$

vector-variable   vector-valued ft.

* M2B) : $T : \mathbb{R}^n \longrightarrow \mathbb{R}^p$

ex) $T(x,y) = (x+y, 2x-y, x-y)$

$T(x,y) = (2x+y, 3y)$ — linear T.

$T(x,y) = (xy, x^2+y)$? Not linear T.

Def) $T : \mathbb{R}^p \longrightarrow \mathbb{R}^n$ defined by

$\mathbf{x} \longmapsto T(\mathbf{x}) = A\mathbf{x}$

$\Rightarrow$ T : matrix transformation,

where $A_{n \times p}$: "standard matrix" of T