Computational Problems
&
Algorithms to Solve Them

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Sorting

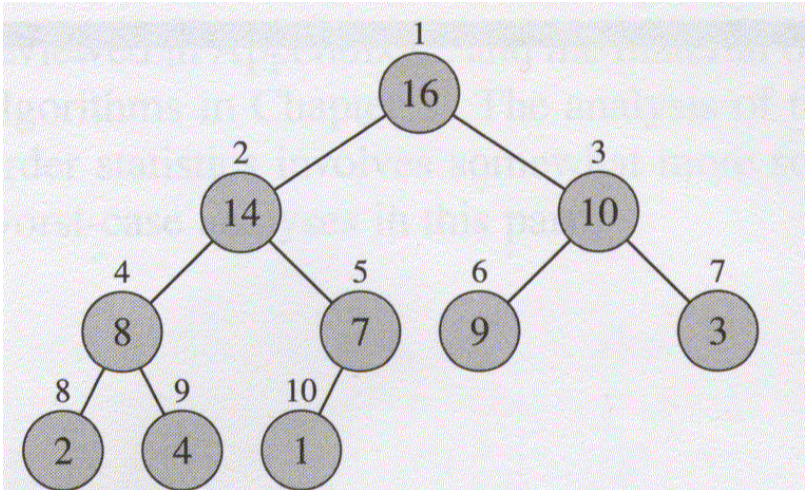
Sorting Algorithms

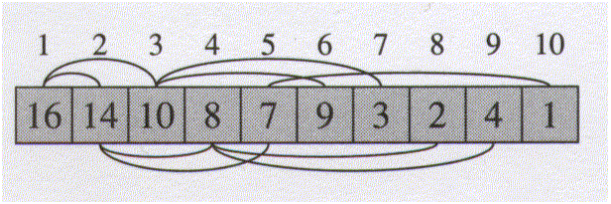
- We have already discussed three $O(n^2)$ Sorting Algorithms
  - Bubble
  - Selection
  - Insertion
- Thinking Assignment: Though both are $O(n^2)$ algorithms, Selection will always run as or more efficiently than Bubble. Why?
- And one O(nlogn) sorting algorithm
  - Merge
- Next we will discuss two more O(nlogn) sorting algorithms
  - Heap
  - Quick
- And three linear sorting algorithms
  - Counting
  - Radix
  - Bucket

The Binary Heap Data Structure
and
Heapsort

Binary Heap

- The ***binary heap*** data structure is an array object that can be viewed as a complete binary tree.





Parent(*i*)
    return $\lfloor i / 2 \rfloor$
LEFT(*i*)
    return $2i$
Right(*i*)
    return $2i+1$

Heap property

- **Max-heap** (the largest data item in the root):
  $A[\text{parent}(i)] \geq A[i]$
- **Min-heap** (the smallest data item in the root):
  $A[\text{parent}(i)] \leq A[i]$
- The ***height of a node*** in a tree: the number of edges on the longest downward path from the node to a leaf.
- The ***height of a tree***: the height of the root
- ***The height of a heap***:
  $O(\lg n)$.

Priority Queues

A Binary Heap is an instance of a data structure called a Priority Queue that allows efficient detection and extraction of the data item with the largest or smallest "key" in a set of items without having to sort them.

Priority Queue Operations

1. Build a priority queue of n data items
2. Locate the data item with the max/min key
3. Locate and extract the data item with the max/min key
4. Insert a new data item with its key into an existing Priority Queue
5. Increase or decrease the key value of a data item already in the Priority Queue

Algorithms that implement these operations on a Max Binary Heap

- Max-Heapify algorithm
- Build-Max-Heap algorithm
- Heapsort algorithm
- Heap-Extract-Max algorithm
- Heap-Maximum algorithm
- Max-Heap-Increase-Key algorithm
- Max-Heap-Insert algorithm

Maintaining the heap property

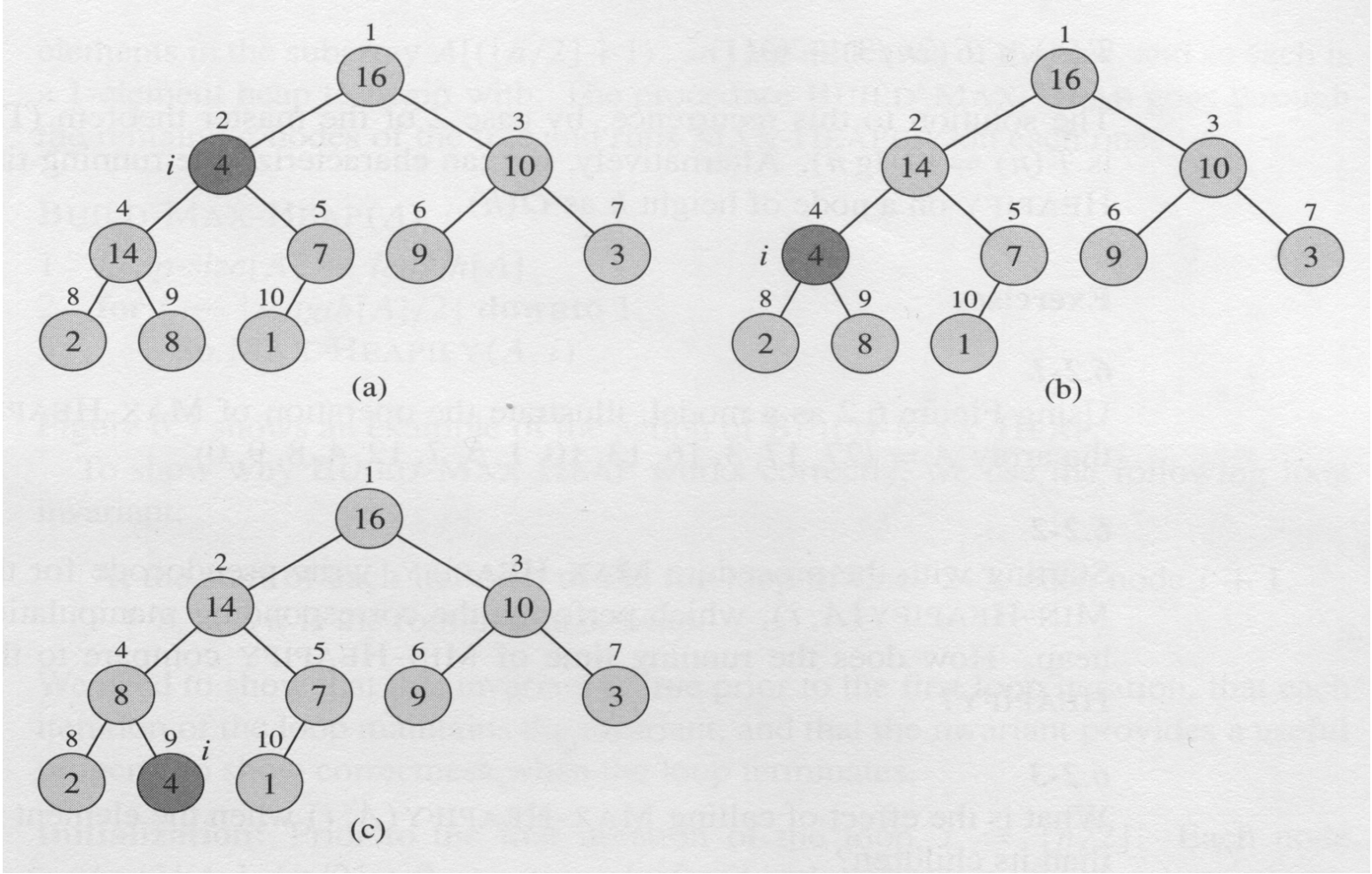
Max-Heapify is an important algorithm for manipulating heaps. Its inputs are an array A and an index i in the array. When Max-Heapify is called, it is expected that the binary trees rooted at LEFT(i) and RIGHT(i) are correct max heaps, but that A[i] may be smaller than its children, thus violating the max heap property at node i.

Max-Heapify (A[1...n]: array of number, i: int 1≤i≤n)

```
 1 l =2i
 2 r =2i+1
 3 if l ≤ A.heap-size and A[l ] > A[i ]
 4         then largest  = l
 5         else largest  = i
 6 if r ≤ A.heap-size and A[r] > A[largest]
 7         then largest = r
 8 if largest ≠ i
 9         then swap A[i] and A[largest]
10                   Max-Heapify (A, largest)
```

Thinking Assignment: Modify Max-Heapify to get Min-Heapify for min heaps

Max-Heapify(A,2)
heap-size[A] = 10



Max-Heapify Complexity

- It has to be O(height) – why?
- Height of a heap is O(lgn)
- So Max-Heapify is O(lgn)

Alternately

- What is the base case?
- What are the recurrence relations? The size of the tree reduces by at least one third in each recursive call. So:
  - T(base case) = c
  - T(n) = T(2n/3) + c
- These can be solved to show T(n)=O(lgn)
- Thinking Assignment: Try it using the Master Theorem or Recursion Tree to solve these

Building a heap

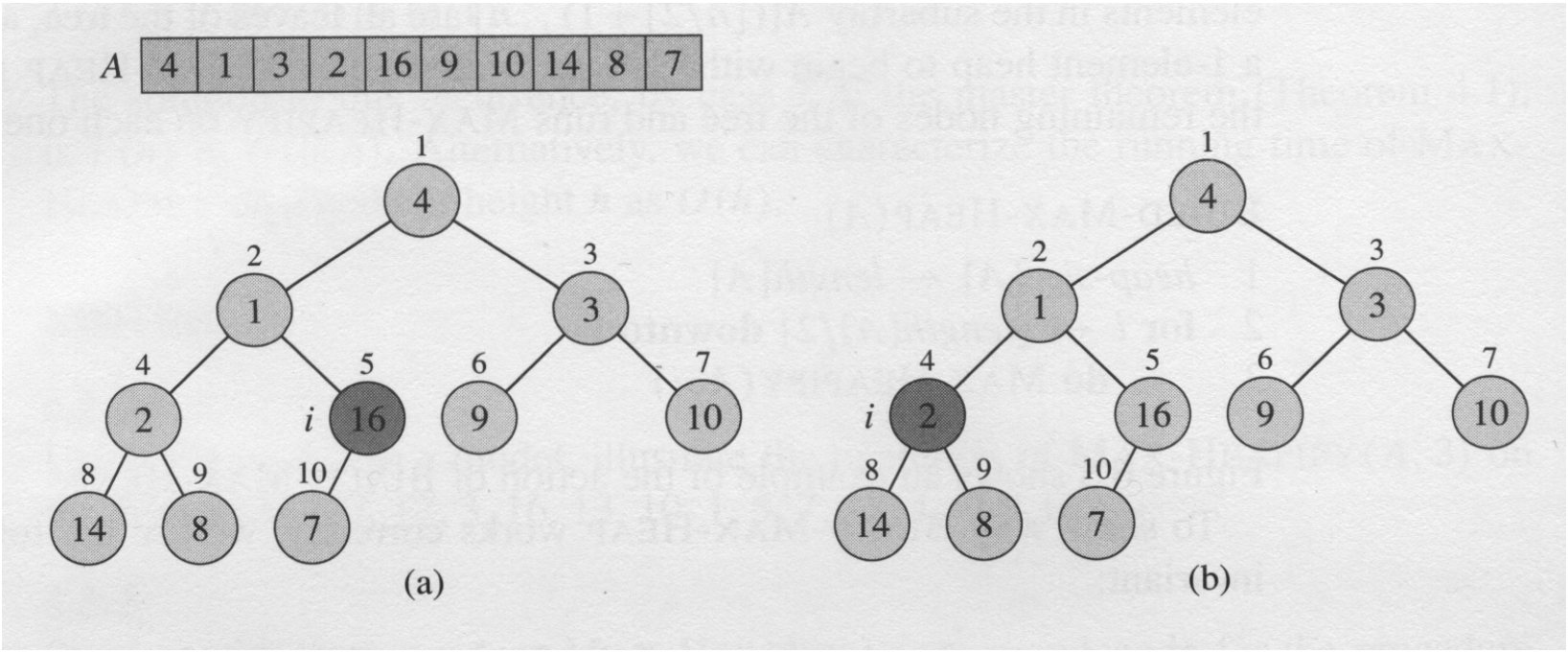
```
Build-Max-Heap(A[1...n]: array of number)
1   A.heap-size = A.length {A.heap-size used by Max-Heapify}
2   for i = ⌊A.length/2⌋ downto 1
3       Max-Heapify(A, i)
```

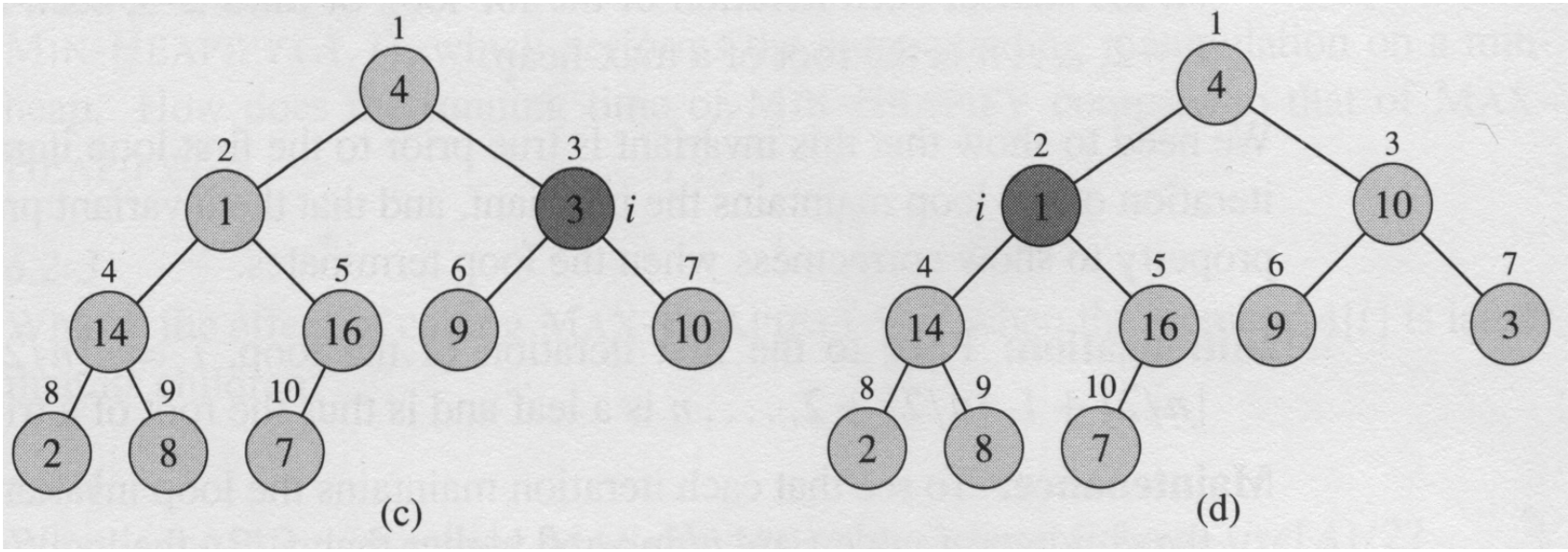
Thinking Assignments:

Why does this loop start from ⌊length(A)/2⌋ ?

Why does the loop not go from 1 to ⌊heap-size(A)/2⌋ ?

Build-Max-Heap

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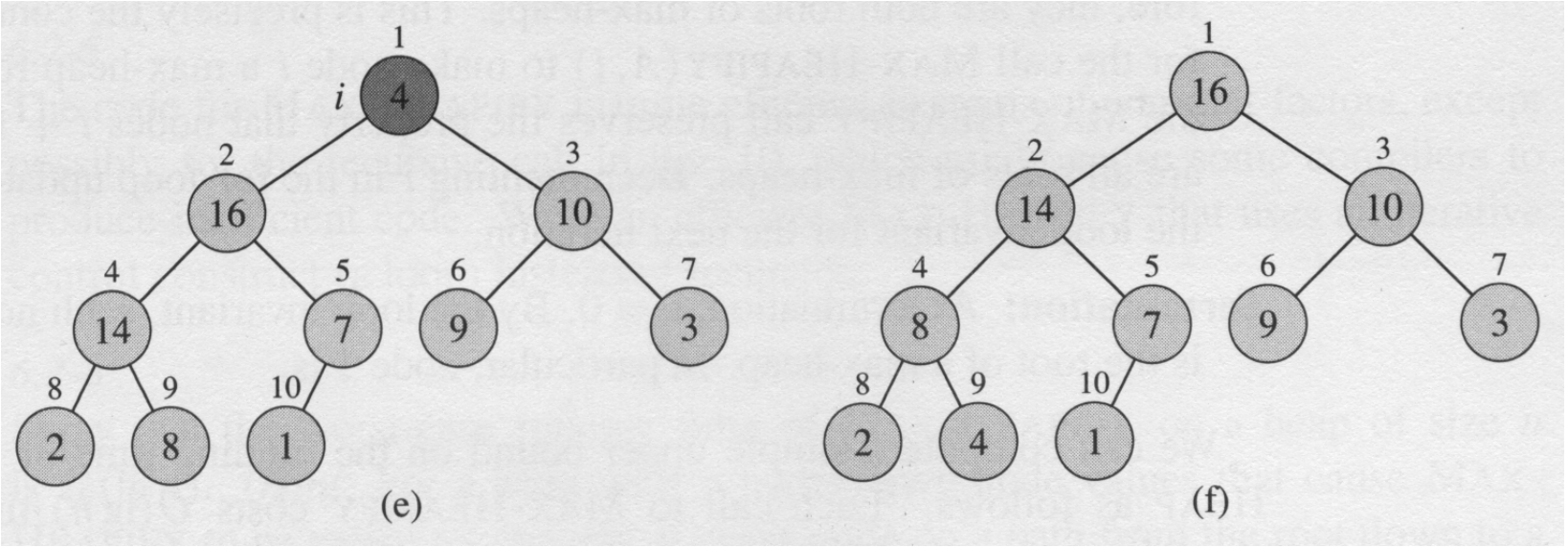
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What is Build-Max-Heap's complexity?

O(nlgn) is a good estimate – why?

But a tighter upper bound can be found: Build-Max-Heap is actually O(n) – linear!

Optional Reading Assignment: If you want to know why, read text p. 159

The Heapsort algorithm

```
Heapsort(A[1...n]: array of number)
1  Build-Max-Heap(A)
2  for i = A.length down to 2
3      swap A[1] and A[i]
4          A.heap-size = A.heap-size -1
5.         Max-Heapify(A,1)
```

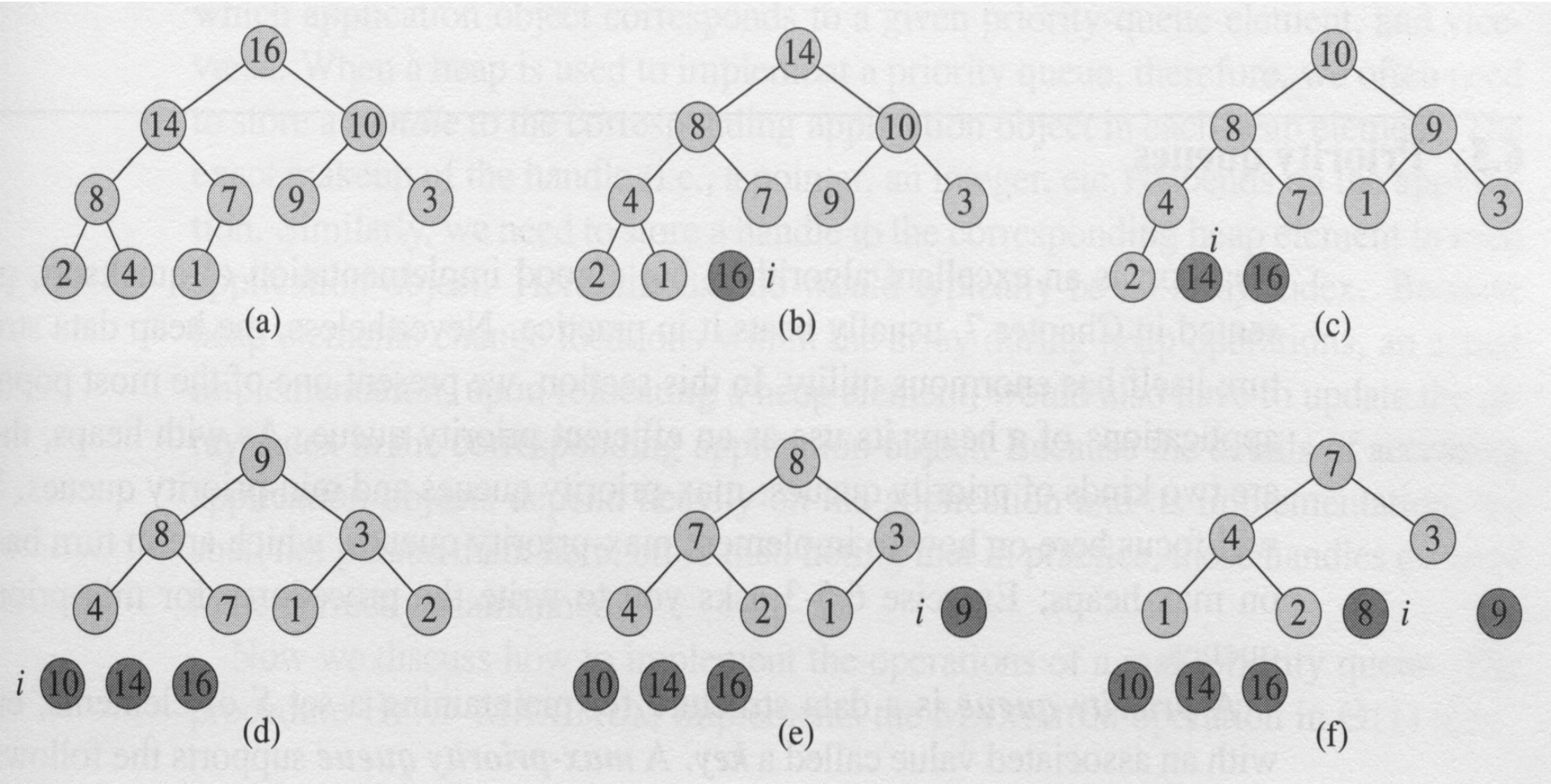
Thinking Assignments:

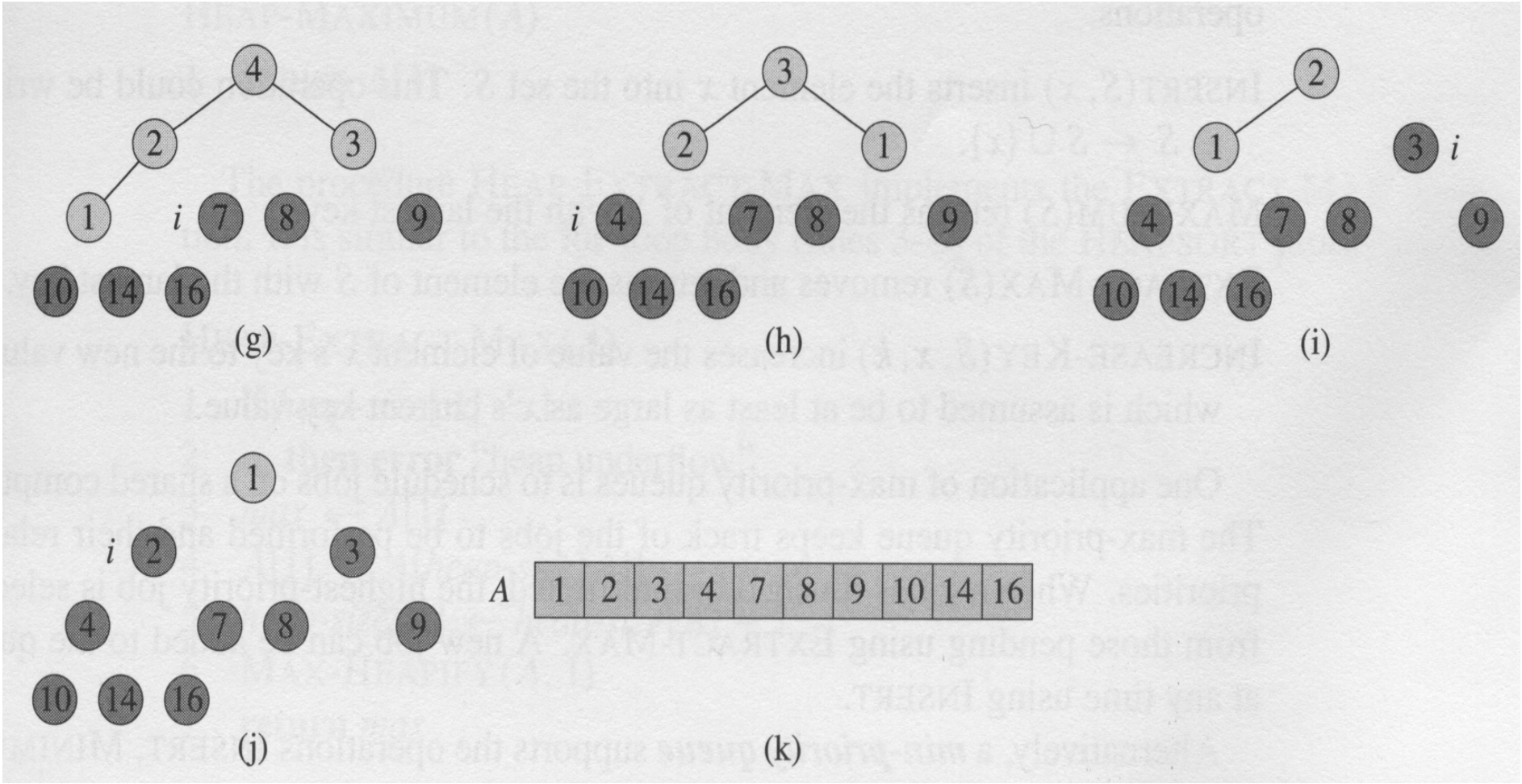
Does this algorithm sort in ascending order?

Why is its worst-case complexity O(nlgn)?

What is its best case complexity? For what kind of input?

The operation of Heapsort





Complexity: O(n logn)

Priority Queues

The heap data structure is not only used in sorting but also in priority queues. A **priority queue** is a data structure that maintain a set S of elements, each with an associated value call a **key**. Priority queues has many applications. A **max (or min) priority queue** should at least support the following operations:

- **Maximum/Minimum** (S)
- **Extract-Max/Min** (S)
- **Decrease-Key** (S, x, k)
- **Increase-Key** (S, x, k)
- **Insert** (S, x)

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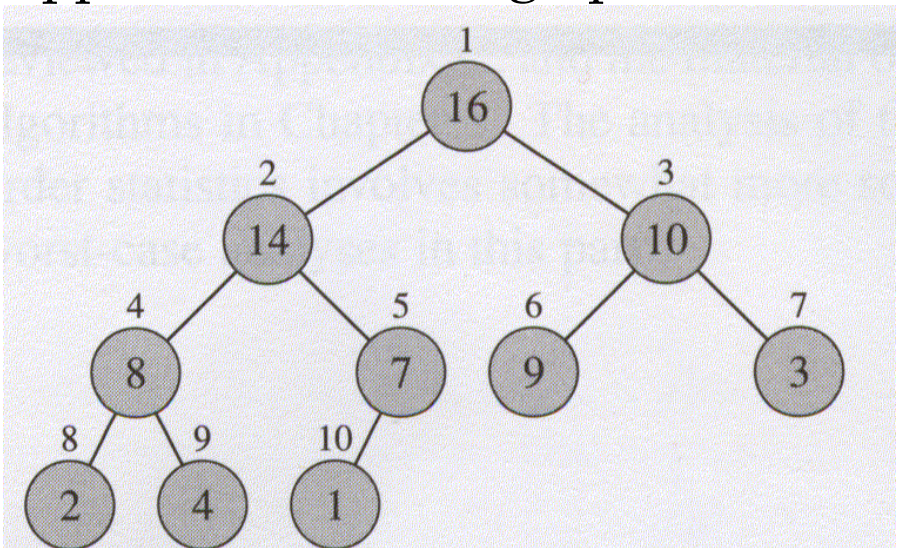

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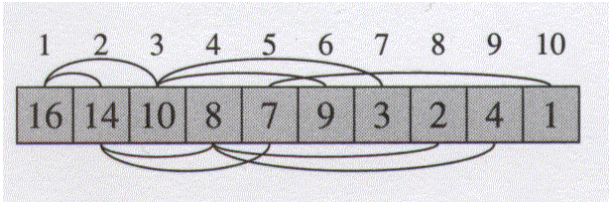
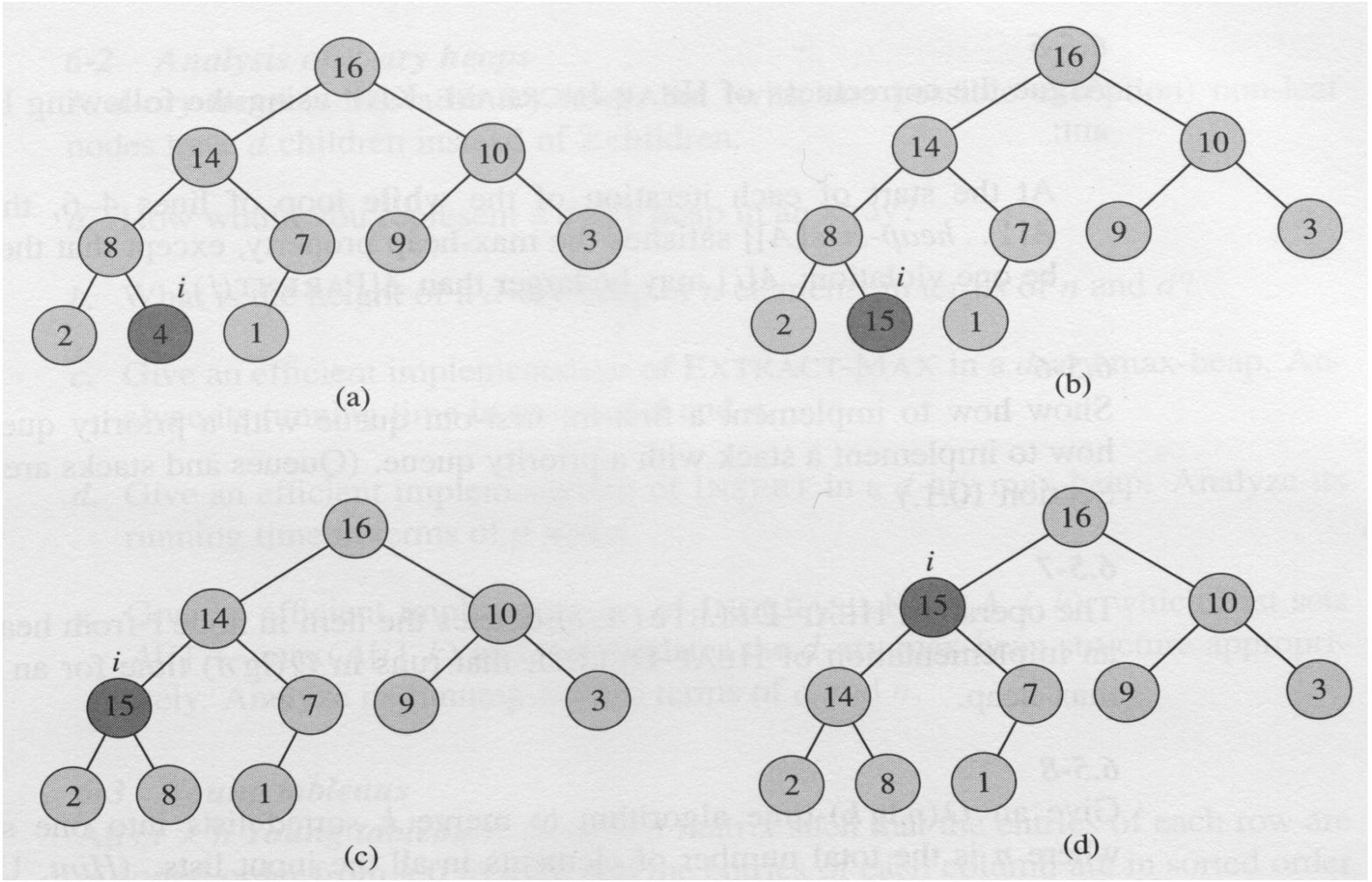


Max-Heap-Increase-Key



Priority Queues

The heap data structure is not only used in sorting but also in priority queues. A **priority queue** is a data structure that maintain a set S of elements, each with an associated value call a **key**. Priority queues has many applications. A **max (or min) priority queue** should at least support the following operations:

- **Maximum/Minimum** (S) — O(1)
- **Extract-Max/Min** (S) — O(log n)
- **Increase-Key** (S, x, k) — O(log n)
- **Insert** (S, x) — O(log n)

Two Heap Operations

Heap-Maximum(A[1...n]: array of number**)**

```
1 return A[1]
```

Heap-Extract-Max(A[1...n]: array of number**)**

```
1  if A.heap-size < 1
2     then error "heap underflow"
3  max = A[1]
4  A[1] =A[A.heap-size]
5  A.heap-size = A.heap-size - 1
6  Max-Heapify (A, 1)
7.    return max
```

Thinking Assignment: Write Heap-Extract-Min

Max-Heap-Increase-Key(A[1...n]: array of number, i: int 1≤i≤n, key)

```
1  if key < A[i]
2    then error "new key is smaller than current key"
3  A[i] = key
4  while i > 1 and A[Parent(i)] < A[i]   {Parent(i) returns ⌊i/2⌋ }
5     swap A[i] and A[Parent(i)]
6.     i = Parent(i)
```

Thinking Assignments:

What is this algorithm's complexity?

Write Max-Heap-Decrease-Key(A, i, key)

Write Min-Heap-Increase-Key(A, i, key)

Write Min-Heap-Decrease-Key(A, i, key)

Max-Heap-Insert(A[1...n]: array of number, key)

1 A.heap-size = A.heap-size + 1

2 A[A.heap-size] = -∞

3. Max-Heap-Increase-Key (A, A.heap-size, key)

Thinking Assignment: Write Min-Heap-Insert(A, key)

Algorithm Design Thinking Assignments

- Min-Heapify
- Build-Min-Heap
- Heapsort procedure using a Min heap
- Heap-Extract-Min
- Heap-Minimum
- Min-Heap-Decrease-Key
- Min-Heap-Insert
- Max-Heap-Decrease-Key
- Min-Heap-Increase-Key

Ch. 6 Reading Assignments

All of the chapter (make sure you understand the Loop Invariant correctness proof of BUILD-MAX-HEAP on p. 157)

Ch. 6 Thinking Assignments

Exercises: 6.1-1:6.1-7, 6.2-1:6.2-6, 6.3-1:6.3-2, 6.4-1:6.4-4, 6.5-1:6.5-8