DEFINING AN ORIENTATION VECTOR FIELD FOR A 3D IMAGE USING THE STRUCTURE TENSOR

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Introduction

The structure tensor of a 3D image f can be represented in every point (voxel) (x, y, z) by a 3×3-matrix

$$M_{(x,y,z)} = \begin{pmatrix} \langle f_x, f_x \rangle_{\omega_{(x,y,z)}} & \langle f_x, f_y \rangle_{\omega_{(x,y,z)}} & \langle f_x, f_z \rangle_{\omega_{(x,y,z)}} \\ \langle f_y, f_x \rangle_{\omega_{(x,y,z)}} & \langle f_y, f_y \rangle_{\omega_{(x,y,z)}} & \langle f_y, f_z \rangle_{\omega_{(x,y,z)}} \\ \langle f_z, f_x \rangle_{\omega_{(x,y,z)}} & \langle f_z, f_y \rangle_{\omega_{(x,y,z)}} & \langle f_z, f_z \rangle_{\omega_{(x,y,z)}} \end{pmatrix}$$

where the inner product is defined by

$$\langle f, g \rangle_{\omega_{(x,y,z)}} = \sum_{i,j,k} \omega_{(x,y,z)}(i, j, k) \cdot f(i, j, k) \cdot g(i, j, k)$$

$\omega_{(x,y,z)}$ being a fixed "window function", a distribution of three variables centered in the voxel (x, y, z). As a symmetric matrix, $M_{(x,y,z)}$ is diagonalisable in $\mathbb{R}$, i.e. has real eigenvalues, the sum of their geometric multiplicities being equal to three. We will see further that the eigenvalues are non-negative.

If we want to define an **orientation vector** in the point (x, y, z), we look for the direction along which the derivative *around this point* is minimal. In other words, we look for the eigenvector(s) for the minimal eigenvalue of the matrix $M_{(x,y,z)}$ in this point. It is important to notice that the minimal eigenvalue may have multiplicity two, in which case a plane will be the corresponding eigenspace. Two vectors will then be necessary to define the orientation in this point. In case of a triple eigenvalue, there is a local isotropy in the voxel.

Let us consider first the case where $M_{(x,y,z)}$ is not singular and then the case where $M_{(x,y,z)}$ is singular. In the following text, the subscript (x, y, z) will not be written anymore for the elements $M_{(x,y,z)}$ and $\omega_{(x,y,z)}$, the dependance being evident.

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1. The structure tensor M is not singular

In this case, let us define

$$m = \frac{1}{3}\operatorname{tr}(M) \tag{1}$$

$$q = \frac{1}{2}\det(M - mI) \tag{2}$$

$$p = \frac{1}{6}\sum_{i,j}((M - mI)_{ij})^2 \tag{3}$$

Then we have for M the three following eigenvalues:

$$\lambda_1 = m + 2\sqrt{p}\cos(\phi) \tag{4}$$

$$\lambda_2 = m - \sqrt{p}\cos(\phi) - \sqrt{3p}\sin(\phi) \tag{5}$$

$$\lambda_3 = m - \sqrt{p}\cos(\phi) + \sqrt{3p}\sin(\phi) \tag{6}$$

with $\phi = \frac{1}{3}\arccos(q/p^{3/2}) \in [0, \pi/3]$, following K. Smith's paper[1].

Computing $\lambda_1 - \lambda_2 = \sqrt{3p}\,(\sin(\phi) + \sqrt{3p}\cos(\phi)) > 0$ and $\lambda_3 - \lambda_2 = 2\sqrt{3p}\,(\sin(\phi)) \geq 0$, we know that λ_2 is the minimal eigenvalue. Furthermore, λ_2 has multiplicity 1 except if p is zero or $\sin(\phi)$ is zero.

1.1. Case $p = 0$.

$$p = 0 \Leftrightarrow M - mI = 0 \Leftrightarrow M = mI$$

1.2. Case $\sin(\phi) = 0$.

$$\sin(\phi) = 0 \text{ and } \phi \in [0, \pi/3] \Leftrightarrow \phi = 0 \Leftrightarrow q = p^{3/2}$$

2. The structure tensor M is singular

Because of the definition of M, all eigenvalues are non-negative: let λ be an eigenvalue of M, x be a unitary eigenvector,

$$\lambda = \lambda x^t x = x^t \lambda x = x^t M x = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \langle x_1 f_x + x_2 f_y + x_3 f_z, x_1 f_x + x_2 f_y + x_3 f_z \rangle_\omega$$

$$= \sum_{i,j,k} \omega(i,j,k)((x_1 f_x + x_2 f_y + x_3 f_z)(i,j,k))^2 \geq 0$$

We can conclude that if the matrix M is singular, the minimal eigenvalue is 0. Its multiplicity is three if and only if M is zero. Its multiplicity is two if and only if M is not zero and $\operatorname{tr}(M^2) - \operatorname{tr}(M)^2$ is zero. This can be proven considering the characterestic polynomial of M: the coefficient in λ is $1/2\left(\operatorname{tr}(M^2) - \operatorname{tr}(M)^2\right)$. For λ to have multiplicity two, this coefficient has to be zero.

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[1]Reference: Oliver K. Smith: Eigenvalues of a symmetric 3×3 matrix. Communications of the ACM Volume 4 (Issue 4): page 168 (1961)

Eigenvectors

Let λ be an eigenvalue for M. The vector x is an eigenvector for λ if and only if x is orthogonal to the image of $M - \lambda I$. It can be easily proved noting that if x is an eigenvector for λ, $Mx = \lambda x$. Let us take a $w \in \mathrm{Im}(M - \lambda I)$, then there exists a $w' \in \mathbb{R}^3$ with $w = (M - \lambda I)w'$. Then, using the symmetry of the matrix M and the symmetry of the scalar product, $\langle v, w\rangle = \langle v, (M - \lambda I)w'\rangle = \langle v, Mw'\rangle - \langle v, w'\rangle = \langle Mv, w'\rangle - \langle \lambda v, w'\rangle = 0$. The other direction is similar.

It is why, if $(M - \lambda I)e_1$ is non zero, we can take $(M - \lambda I)e_1 \times (M - \lambda I)e_2$ or $(M - \lambda I)e_1 \times (M - \lambda I)e_3$ as an eigenvector for the eigenvalue λ of multiplicity one. If λ has multiplicity 2 and $(M - \lambda I)e_1$ is non zero, we can take $(M - \lambda I)e_1 \times e_1$ or $(M - \lambda I)e_1 \times (e_1 + e_2 + e_3)$ as one of the eigenvectors.

My algorithm

We compute M, $\det(M)$, m, q, p and ϕ like in the former definitions. We then compute

$$\lambda_{min} = \begin{cases} m - \sqrt{p}\cos(\phi) - \sqrt{3p}\sin(\phi) & \text{if} \quad p \neq 0 \\ 0 & \text{if} \quad p = 0 \end{cases}$$

The former discussion would suggest the form of a tree, testing first if $\det(M)$ is zero. If $\det(M)$ is zero, we will have to test then if $\mathrm{tr}(M^2) - \mathrm{tr}(M)^2$ if zero and eventually, if it is the case, if M is zero. In each of the defined subcases, we have another multiplicity for the minimal eigenvalue 0 and we are able to tell what the corresponding eigenvector(s) is (are). If $\det(M)$ is not zero, test if p is zero, if yes test if $q = p^{3/2}$. In each of the defined subcases, we have another multiplicity for the minimal eigenvalue (which is non zero) and we are able to tell what the corresponding eigenvector(s) is (are).

The cases M singular and M non singular, separated because Smith's formula applies only to non singular matrices, can also be reunited: instead of following this subcase construction, we **order** the columns of $M - \lambda_{min} I$ to have the zero ones at the end. We then compute the candidate eigenvectors

$$\begin{aligned} c_1 &= \frac{(M - \lambda_{min} I)e_1}{\|(M - \lambda_{min} I)e_1\|} \times \frac{(M - \lambda_{min} I)e_2}{\|(M - \lambda_{min} I)e_2\|} \\ c_2 &= \frac{(M - \lambda_{min} I)e_1}{\|(M - \lambda_{min} I)e_1\|} \times \frac{(M - \lambda_{min} I)e_3}{\|(M - \lambda_{min} I)e_3\|} \\ c_3 &= \frac{(M - \lambda_{min} I)e_1}{\|(M - \lambda_{min} I)e_1\|} \times \frac{e_1 + e_2 + e_3}{\sqrt{3}} \\ c_4 &= \frac{(M - \lambda_{min} I)e_1}{\|(M - \lambda_{min} I)e_1\|} \times e_1 \end{aligned}$$

My first orientation vector v_1 will be the first of these vectors having a non-zero norm. If the multiplicity is two, the second vector will be

$$v_2 = \frac{(M - \lambda_{min} I)e_1}{\|(M - \lambda_{min} I)e_1\|} \times v_1.$$

The former differentiation of all different subcases confirms the validity of this computation:

2.1. λ_{min} **has multiplicity one.** This happens in cases:

(1) $\det(M) = 0$, $\operatorname{tr}(M^2) - \operatorname{tr}(M)^2 \neq 0$
(2) $\det(M) \neq 0$, $p \neq 0$ and $q \neq p^{3/2}$

In both cases, c_1 or c_2 will give the orientation vector v_1.

2.2. λ_{min} **has multiplicity two.** This happens in cases:

(1) $\det(M) = 0$, $\operatorname{tr}(M^2) - \operatorname{tr}(M)^2 = 0$ and $M \neq 0$
(2) $\det(M) \neq 0$, $p \neq 0$ and $q = p^{3/2}$

In both cases, c_3 or c_4 will give the first orientation vector. The second one is given by the former formula.

2.3. λ_{min} **has multiplicity three.** This happens in cases:

(1) $M = 0$
(2) $p = 0$

In both cases, there is no orientation in the point and we have $v_1 = v_2 = 0$.

Note that the computation of c_1, c_2, c_3 and c_4 can be vectorized, as well as the final formula for v_1 and v_2, if we use logical matrices for the norm of the c_i being zero. In `MATLAB`, the gain in comparison of a triple `for`-loop with `if`-tests inside will be important!