The Cake Eating Problem

A simple dynamic programming example about optimal saving

1. The Finite Problem

You have a cake of size W and you have 3 periods to consume it. You need to choose c_1, c_2, and c_3 (consumption in each period) to maximize your discounted sum of utilities $\sum_t \beta^{t-1} u(c_t) = c_1 + \beta c_2 + \beta^2 c_3$ subject to the constraint that $\sum_t c_t = W$ and considering $u(c_t) = \log(c_t)$.

$$\boxed{\begin{aligned} \max_{c_1, c_2, c_3} \quad & \log(c_1) + \beta \log(c_2) + \beta^2 \log(c_3) \\ \text{s/t} \quad & c_1 + c_2 + c_3 = W \end{aligned}}$$

$$\mathcal{L}(c_1, c_2, c_3, \lambda) = \log(c_1) + \beta \log(c_2) + \beta^2 \log(c_3) - \lambda(W - c_1 - c_2 - c_3)$$

No corner solutions b/c $\log(0) = -\text{Inf}$

So $\nabla \mathcal{L} = \vec{0}$:

$$\frac{1}{c_1} = \lambda, \quad \frac{\beta}{c_2} = \lambda, \quad \frac{\beta^2}{c_3} = \lambda, \quad W = c_1 + c_2 + c_3$$

$$c_1 = \frac{1}{\lambda} \qquad c_2 = \frac{\beta}{\lambda} \qquad c_3 = \frac{\beta^2}{\lambda} \qquad W = \frac{1}{\lambda} + \frac{\beta}{\lambda} + \frac{\beta^2}{\lambda}$$

$$c_2 = \beta c_1 \qquad c_3 = \beta^2 c_1 \qquad W = \frac{1 + \beta + \beta^2}{\lambda}$$

$$\lambda = \frac{1 + \beta + \beta^2}{W}$$

$$\boxed{c_1 = \frac{W}{1 + \beta + \beta^2} \qquad c_2 = \frac{\beta W}{1 + \beta + \beta^2} \qquad c_3 = \frac{\beta^2 W}{1 + \beta + \beta^2}}$$

Take $\beta = 0.9$ (β: discount factor, time preference, probability you're alive in the period) and $W = 10$.

$$c_1 = \frac{10}{1 + .9 + .81} = \frac{10}{2.71} \approx 3.69 \qquad c_3 = \frac{8.1}{2.71} \approx 2.99$$

$$c_2 = .9\left(\frac{10}{2.71}\right) = \frac{9}{2.71} \approx 3.32$$

As you care about future periods more ($\beta \uparrow$), $c_2 \uparrow$ relative to c_1 and $c_3 \uparrow$ relative to c_2:

Let $\beta = 0.99$

$$c_1 = \frac{10}{1 + .99 + .99^2} \approx 3.37 \qquad c_2 = .99\, c_1 \approx 3.33$$
$$c_3 = .99^2 c_1 \approx 3.30$$

2. THE INFINITE PROBLEM

Now suppose you could live forever. You still have a cake of size W, $u(c_t) = \log(c_t)$

$$\boxed{\begin{aligned} &\max_{c_t} \sum_{t=1}^{\infty} \beta^{t-1} \log(c_t) \\ &\text{s/t} \sum_{t=1}^{\infty} c_t = W \end{aligned}}$$

$$\mathcal{L} = \log(c_1) + \beta \log(c_2) + \beta^2 \log(c_3) + \ldots - \lambda(c_1 + c_2 + c_3 + \ldots - W)$$

$\nabla \mathcal{L} = \vec{0}$ (corner solutions ruled out because $\log(0) = -\text{Inf}$)

$$\frac{\partial \mathcal{L}}{\partial c_1} = 0 \Rightarrow \frac{1}{c_1} = \lambda$$

$$\frac{\beta}{c_2} = \lambda$$

$$\frac{\beta^2}{c_3} = \lambda$$

$$\ldots$$

So $c_1 = \dfrac{c_2}{\beta} = \dfrac{c_3}{\beta^2}$

and $c_{t+1} = \beta c_t \ \forall t$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow c_1 + c_2 + \ldots = W$$

$$c_1 + \beta c_1 + \beta^2 c_1 + \ldots = W$$

Note: $\beta W = W - c_1$

Solve for W:

$$c_1 = W - \beta W$$

$$W = c_1 / (1-\beta)$$

Let $W = 10$ and $\beta = 0.9$. Then $c_1 = (1-.9)10 = 1$

$$c_2 = .9(1) = .9$$
$$c_3 = .9^2(1) = .81$$
$$c_4 = .9^3(1) = .729$$
$$\ldots$$

If $\beta \uparrow$ to 0.99, will $c_2 \uparrow$ relative to c_1? Yes:

$$c_1 = 10(1-.99) = 0.1$$
$$c_2 = .99(.1) = .099$$
$$c_3 = .99^2(.1) = .09801$$

3: The cake grows

Consider a retiree who stops working with wealth W. They must choose $c_1, c_2, c_3, \ldots$ consumption each period, but this is a twist on the cake eating problem because the more they consume early on, the less they can accumulate in interest. Suppose the interest rate is r and the gross interest rate is $R = 1 + r$.

\$10 invested now at $r = 2\%$ accumulates to $10R = \$10.02$ next period, and $10R^t$ in t periods. Inversely, if you have \$10 in your savings account which you've been accumulating for 3 years, then your initial investment was $IR^3 = 10$, $I = \frac{10}{R^3}$. So the present value of \$10 in 3 years is $10/R^3$.

$$\boxed{\begin{aligned}&\max_{c_1, c_2, c_3} \log(c_1) + \beta \log(c_2) + \beta^2 \log(c_3)\\ &\text{s/t} \quad c_1 + \frac{c_2}{R} + \frac{c_3}{R^2} = W\end{aligned}}$$

$$\mathcal{L} = \log(c_1) + \beta \log(c_2) + \beta^2 \log(c_3)$$
$$- \lambda\left(c_1 + \frac{c_2}{R} + \frac{c_3}{R^2} - W\right)$$

$\nabla \mathcal{L} = \vec{0}$:

$$\frac{1}{c_1} = \lambda \qquad \frac{\beta}{c_2} = \frac{\lambda}{R} \qquad \frac{\beta^2}{c_3} = \frac{\lambda}{R^2}$$

$$c_1 = 1/\lambda \qquad \frac{c_2}{\beta} = \frac{R}{\lambda} \qquad \frac{c_3}{\beta^2} = \frac{R^2}{\lambda}$$

$$c_2 = \beta R c_1 \qquad c_3 = \beta^2 R^2 c_1$$

$$c_1 + \beta c_1 + \beta^2 c_1 = W$$
$$c_1 = \frac{W}{1 + \beta + \beta^2}$$

Let $W = 10, \ \beta = 0.9, \ R = 1.1$

$$c_1 = \frac{10}{1 + .9 + .81} = 2.71$$

$$c_2 = \beta R c_1 = 0.9(1.1)(2.71) = 2.68$$

$c_3 = \beta^2 R^2 c_1 = 0.9^2 (1.1^2)(2.71) = 2.66$

4 : Optimal Saving with ∞

$$\boxed{\begin{aligned} \max\ & \textstyle\sum_{t=1}^{\infty} \beta^{t-1} \log(c_t) \\ s/t\ & \textstyle\sum_{t=1}^{\infty} \frac{1}{R^{t-1}} c_t = W \end{aligned}}$$

$$\mathcal{L} = \log(c_1) + \beta \log(c_2) + \beta^2 \log(c_3) + \dots$$
$$- \lambda \left(c_1 + \frac{c_2}{R} + \frac{c_3}{R^2} + \dots - W \right)$$

$\nabla \mathcal{L} = \vec{0} \Rightarrow$

$$\frac{1}{c_1} = \lambda \qquad c_1 = 1/\lambda$$

$$\frac{\beta}{c_2} = \frac{\lambda}{R} \qquad c_2 = R\beta/\lambda = R\beta c_1$$

$$\frac{\beta^2}{c_3} = \frac{\lambda}{R^2} \qquad c_3 = R^2\beta^2/\lambda = R\beta c_2$$

$$c_{t+1} = R\beta c_t \quad \leftarrow \text{Euler Equation}$$

$$c_1 + \frac{c_2}{R} + \frac{c_3}{R^2} + \dots = W$$

$$c_1 + \beta c_1 + \beta^2 c_1 + \dots = W$$

Note : $W - c_1 = \beta W$

$W(1-\beta) = c_1$

Let $W = 10,\ \beta = 0.9,\ R = 1.1$

$c_1 = 10\,(1 - .9) = 1 \qquad c_2 = 1.1\,(.9)\,(1) = 0.99$

$c_3 = 1.1\,(.9)\,(.99) = 0.9801$

5. Dynamic Programming Approach

Let $V(W)$ be the value of the cake:

$$\boxed{V(W) = \max_{c_t} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \quad s/t \quad \sum_{t=1}^{\infty} c_t = W}$$

V can be defined recursively:

$$V(W) = \max_{c \in [0, W]} \left\{ u(c) + \beta V(W') \right\}$$

$$s/t \quad W' = W - c$$

the value of having W units of cake in a certain period is the utility you get from consuming some of the cake this period plus the discounted value of having some of the cake left over next period.

Note: When the cake increases in size by 1 unit, the lifetime utility of the agent ↑ by λ units. So $V'(W) = \lambda$.

WTS $c_1 = W(1-\beta)$ and $c_{t+1} = \beta c_t \quad \forall t > 1$

Solving for V using the method of undetermined coefficients: Guess that the value function has the form $V(W) = A + B\ln(w)$ and find A and B:

$$A + B\ln(w) = \max_{c} \left\{ \ln(c) + \beta\left(A + B\ln(w - c)\right) \right\}$$

$$\text{FOC:} \quad \frac{1}{c} = \frac{\beta B}{w - c}$$

$$w - c = \beta B c$$

$$w = c(1 + \beta B)$$

$$c = \frac{W}{1+\beta B}$$

$$\text{So } W' = W - c = \frac{W(\cancel{1}+\beta B)}{1+\beta B} - \frac{\cancel{W}}{1+\beta B}$$

$$W' = \frac{W\beta B}{1+\beta B}$$

$$A + B\ln(w) = \ln\left(\frac{W}{1+\beta B}\right) + \beta\left(A + B\ln\left(\frac{W\beta B}{1+\beta B}\right)\right)$$

$$= \underline{\quad} \ln(W) + \underline{\quad}$$

$$1 + \beta B$$

$$-\ln(1+\beta B) + \underline{\beta A} +$$
$$\beta B \ln(\beta B) - \beta B \ln(1 + \beta B)$$

$$B = 1 + \beta B$$
$$B(1-\beta) = 1$$
$$B = 1/1-\beta$$

$$\beta B = \beta/1-\beta$$

$$1 + \beta B = \frac{1-\beta}{1-\beta} + \frac{\beta}{1-\beta} = 1/1-\beta$$

$$A - \beta A = \ln\left(\frac{1}{1+\beta B} \cdot (\beta B)^{\beta B} \cdot \frac{1}{(1+\beta B)^{\beta B}}\right)$$

$$A = \frac{1}{1-\beta} \ln\left((1-\beta)\left(\frac{\beta}{1-\beta}\right)^{\beta/1-\beta} \left(\frac{1}{1-\beta}\right)^{-\beta/1-\beta}\right)$$

$$= \frac{1}{1-\beta} \ln\left(\beta^{\beta/1-\beta} (1-\beta)^{1 - \cancel{\frac{\beta}{1-\beta}} + \cancel{\frac{\beta}{1-\beta}}}\right)$$

$$= \frac{1}{1-\beta} \ln\left(\beta^{\beta/1-\beta}(1-\beta)\right)$$

$$= \frac{\beta}{(1-\beta)^2}\ln(\beta) + \frac{1}{(1-\beta)}\ln(1-\beta)$$

$$\boxed{V(W) = \frac{1}{1-\beta}\left(\frac{\beta}{1-\beta}\ln(\beta) + \ln(1-\beta)\right) + \frac{1}{1-\beta}\ln W}$$

$$c = \frac{W}{1+\beta B} \Rightarrow \quad c = \frac{W}{1+\beta\left(\frac{1}{1-\beta}\right)} = \frac{W}{\frac{1-\cancel{\beta}}{1-\beta} + \frac{\cancel{\beta}}{1-\beta}}$$

$$= W(1-\beta)$$

And $c_t = W_t(1-\beta)$ so

since $c_0 = W(1-\beta)$

then $c_1 = (W - W(1-\beta))(1-\beta)$

$$\Rightarrow c_1 = W(1-\beta) - W(1-\beta)^2$$
$$= c_0 - (1-\beta)c_0$$
$$= c_0(\cancel{1} - \cancel{1} + \beta)$$
$$= \beta c_0$$

Or $c_{t+1} = \beta c_t$ ■