Hints on Interval Halving

The Guessing Game

- Rules:
  - You are trying to determine $\lfloor s \rfloor$ for some **s**ecret non-negative ***`real`*** number s
  - You know s is in the half-open-half-closed interval $[lowEnough, tooHigh)$ where the end-points are ***`integer`***s
  - You may ask only one kind of question about an ***`integer`*** **g**uess g: is it true that $s < g$?

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This notation, pronounced ***floor of s***, means the greatest integer less than or equal to s; for non-negative s, it is also known as the ***integer part of s***. (Example: ⌊4.47⌋ = 4)

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The end-points have meaningful names:
`lowEnough` is ***low enough to be*** $\lfloor s \rfloor$,
and
`tooHigh` is ***too high to be*** $\lfloor s \rfloor$.

- Rules:
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    non-negative ***`rea`*** ber `s`
  - You know `s` is in the f-open-half-closed interval `[lowEnough, tooHigh)` where the end-points are ***`integer`***s
  - You may ask only one kind of question about an ***`integer`*** guess `g`: is it true that `s < g`?

Approach to the Guessing Game

- As long as the interval `[lowEnough, tooHigh)` contains more than one integer (i.e., `tooHigh - lowEnough > 1`), repeat:
  - Guess the floor of the midpoint of the interval as `g`, asking whether `s < g`
  - Depending on the answer to this question, replace either `lowEnough` or `tooHigh` with `g`
- When `tooHigh - lowEnough = 1`, there is only one possible answer: `lowEnough`

Approach

The term ***interval halving*** for this algorithm (also called ***bisection*** or ***binary search***) comes from the fact that each iteration eliminates half the previous interval.

- As long as `tooHigh)`
  (i.e., `tooHi`
  repeat:
  - Guess the floor of the midpoint of the interval as `g`, asking whether `s < g`
  - Depending on the answer to this question, replace either `lowEnough` or `tooHigh` with `g`
- When `tooHigh – lowEnough = 1`, there is only one possible answer: `lowEnough`

The Root-Guessing Game

- Rules:
  - You are trying to determine $\lfloor n^{1/r} \rfloor$ for given non-negative **`integer`**s n and r (and you can't compute $n^{1/r}$ directly, so $n^{1/r}$ is just like the secret **`real`** number s)
  - You know $n^{1/r}$ is in the half-open-half-closed interval *`[lowEnough, tooHigh)`* where the end-points are **`integer`**s
  - You may ask only one kind of question about an **`integer`** guess g: is it true that $n^{1/r} < g$?

Approach to the Root-Guessing Game

- As long as the interval `[lowEnough, tooHigh)` contains more than one integer (i.e., `tooHigh – lowEnough > 1`), repeat:
  - Guess the floor of the midpoint of the interval as g, asking whether $n^{1/r} < g$
  - Depending on the answer to this question, replace either `lowEnough` or `tooHigh` with `g`
- When `tooHigh – lowEnough = 1`, there is only one possible answer: `lowEnough`

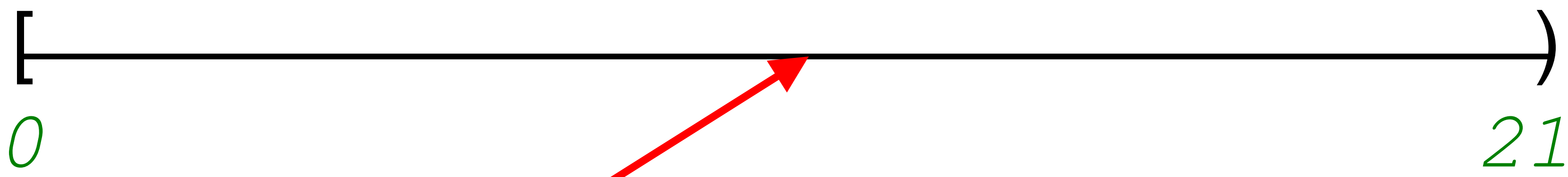
How Can This Algorithm Work?

- The problem seems to be that, without *already knowing* the secret number $n^{1/r}$, you cannot directly answer the question: is it true that $n^{1/r} < g$?
- Observe: answering whether $n^{1/r} < g$ is the same as answering whether $n < g^r$
  - In other words, if you can compute g^r, then you can guess $n^{1/r}$ using the same approach as you used to guess the secret number s

Example: Find $\lfloor 20^{1/2} \rfloor$

- What is the actual answer?
  - Since the 2nd (i.e., square) root of 20 is about 4.47, we have $\lfloor 20^{1/2} \rfloor = 4$
  - Let's see how this can be determined by interval halving
- We need a starting interval known to contain $20^{1/2}$
  - 0 is low enough to be the answer
  - $20 + 1 = 21$ is too high to be the answer

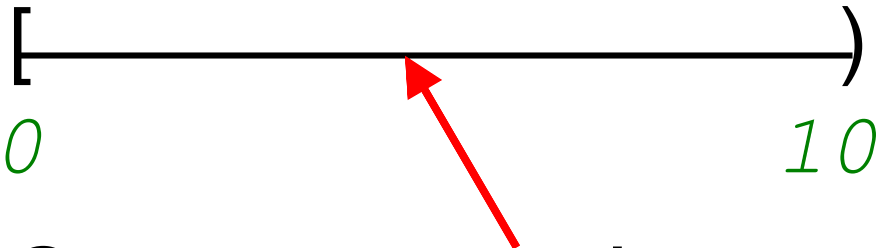
Example: Find $\lfloor 20^{1/2} \rfloor$



Guess $g = 10$. Is $20 < 10^2$? Yes.

- So, 10 is too high to be $\lfloor 20^{1/2} \rfloor$
- In other words, there is no point in ever guessing 10 or higher

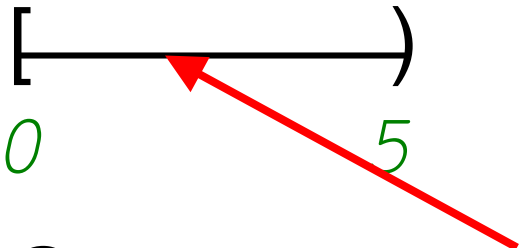
Example: Find $\lfloor 20^{1/2} \rfloor$



Guess $g = 5$. Is $20 < 5^2$? Yes.

- So, 5 is too high to be $\lfloor 20^{1/2} \rfloor$
- In other words, there is no point in ever guessing 5 or higher

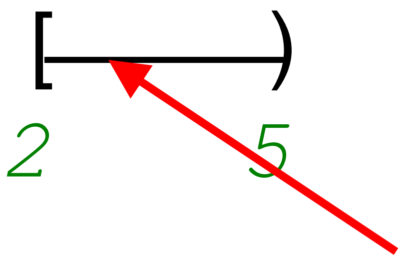
Example: Find $\lfloor 20^{1/2} \rfloor$



Guess $g = 2$. Is $20 < 2^2$? No.

- So, 2 is low enough to be $\lfloor 20^{1/2} \rfloor$
- In other words, there is no point in ever guessing lower than 2

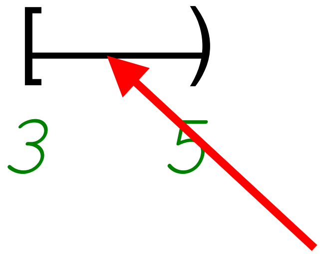
Example: Find $\lfloor 20^{1/2} \rfloor$



Guess $g = 3$. Is $20 < 3^2$? No.

- So, 3 is low enough to be $\lfloor 20^{1/2} \rfloor$
- In other words, there is no point in ever guessing lower than 3

Example: Find $\lfloor 20^{1/2} \rfloor$



Guess $g = 4$. Is $20 < 4^2$? No.

- So, 4 is low enough to be $\lfloor 20^{1/2} \rfloor$
- In other words, there is no point in ever guessing lower than 4

Example: Find $\lfloor 20^{1/2} \rfloor$

$[\!\!\longrightarrow\!)$

$4 \quad 5$

We now know that $4 \leq 20^{1/2} < 5$ so the answer must be $\lfloor 20^{1/2} \rfloor = 4$.