GABRIEL-ULMER DUALITY

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Abstract. A short note presenting Gabriel Ulmer duality.

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1. Introduction

Gabriel-Ulmer duality is a syntax-semantics-type duality establishing that a locally κ-presentable category is uniquely determined by its set of κ-presentables. In this note we give a very concise introduction to this theorem. We may start by a very brief recall of what a locally presentable category is.

Definition 1.1 (Locally $\boldsymbol{\kappa}$-presentable category)**.** A locally κ-presentable category is a cocomplete category with a strong generator made by κ-presentable objects.

The most known reference on the theory of locally presentable category is [AR94]. The theory of locally presentable categories has nowadays gained a primary position in categorical algebra, due to its connections with categorical model theory, homotopy theory, and universal algebra.

Example 1.2. Most of cocomple category of the *working mathematician* are locally presentable, as shown by the following list of examples.

(1) Set is locally finitely presentable.
(2) Grp is locally finitely presentable.
(3) Mod(R) is locally presentable.

Remark 1.3. Be careful, some very important categories are not locally presentable, that's the case of topological spaces. The previous list of examples and the last counterexample should suggest what characterizes locally presentable categories, these are essentially categories of models of essentially algebraic theories. This is much more then a motto and appears as [AR94][Thm. 3.36].

Remark 1.4. Let $T : \mathbf{Set}^{\mathsf{C}^\circ} \to \mathbf{Set}^{\mathsf{C}^\circ}$ be an idempotent monad preserving κ-directed colimits for some regular κ, then its category of algebras is a locally presentable category. Every locally presentable category can be described in this way.

Example 1.5. Let C be a cocomplete category and $U : \mathsf{C} \rightleftarrows \mathbf{Set} : F$ be an adjunction where the right adjoint U is conservative and preserve κ-directed colimits, then C is locally κ-presentable. For every locally κ-presentable category there exists such an adjunction where U is $\kappa'(\geq \kappa)$ accessible.

2. Gabriel-Ulmer duality

Theorem 2.1 (Gabriel-Ulmer duality)**.** *There is a biequivalence of 2-categories*

$$\mathrm{Ind}_\kappa : \mathrm{Rex}_\kappa^{\mathrm{coop}} \leftrightarrows \mathrm{Pres}_\kappa : \mathrm{Pres}_\kappa$$

where:

- $\mathrm{Rex}_\kappa^{\mathrm{coop}}$ *is the 2-category of small categories with κ-small colimits, where 1-cells are the κ-small colimit preserving functors and 2-cells are the natural transformations.*
- Pres_κ *is the 2-category of locally κ-presentable categories with κ-accessible right adjoints.*

A classical reference for the proof is [CV02, Th. 3.1], a more recent one that sets the result in the enriched context is [LP09], an even more recent one that studies this duality from the point of view of formal category theory is [DL18, Th. 4.12].

Remark 2.2. Some readers might be familiar with a different presentation of Gabriel-Ulmer duality involving, instead of Rex_κ, the 2-category Lex_κ of small categories with κ-small limits, where 1-cells are the κ-small limit preserving functors and 2-cells are the natural transformations. Both versions of the duality are related as follows. We have a (pseudo)commutative diagram of biequivalences

$$\begin{tikzcd}
\mathrm{Rex}_\kappa^{\mathrm{coop}} \arrow[rr, bend left, "\mathrm{Ind}_\kappa"] \arrow[dr, bend right, "\circ"'] & & \mathrm{Pres}_\kappa \arrow[ll, bend left, "\mathrm{Pres}_\kappa"] \arrow[dl, bend right, "\mathrm{Pres}_\kappa^\circ"'] \\
& \mathrm{Lex}_\kappa^{\mathrm{op}}, \arrow[ul, bend right, "\circ"'] \arrow[ur, bend right, "\mathrm{Lex}_\kappa"'] &
\end{tikzcd}$$

where $\circ$ denotes the biequivalence provided by taking the opposite category and Lex_κ is a short notation for the pseudofunctor $\mathrm{Lex}_\kappa(-, \mathbf{Set})$. In particular this means that

$$\mathrm{Ind}_\kappa \cong \mathrm{Lex}_\kappa((-)^\circ, \mathbf{Set}).$$

2.1. **The 2-functor Pres_κ.** The 2-functor Pres_κ acts as follows:

0-cells It maps C to C_κ, the full subcategory of κ-presentable objects. Observe that this category is (essentially) small.

1-cells It maps the adjunction in $L : \mathsf{C} \leftrightarrows \mathsf{D} : R$ to the restriction of L to C_κ. Observe that the left adjoint of an κ-accessible functor preserves κ-presentable objects and thus this restriction actually lands in D_κ. Observe that the inclusion $\mathsf{C}_\kappa \hookrightarrow \mathsf{C}$ creates κ-small colimits, thus they are preserved by the restriction of L, since L is a left adjoint.

2.2. **The pseudofunctor Lex_κ.** The pseudofunctor Lex_κ acts as follows:

0-cells It maps C to $\mathrm{Lex}_\kappa(\mathsf{C}, \mathbf{Set})$. Recall that by *$\kappa$-left exact* we mean *preserving κ-small limits*.

1-cells It maps $f : \mathsf{C} \to \mathsf{D}$ to the adjunction

$$L(f) : \mathrm{Lex}_\kappa(\mathsf{C}, \mathbf{Set}) \leftrightarrows \mathrm{Lex}_\kappa(\mathsf{D}, \mathbf{Set}) : R(f).$$

The map $R(f)$ can be described in many fashions. One possible description is to define $R(f)$ as the restriction to $\mathrm{Lex}_\kappa(\mathsf{D}, \mathbf{Set})$ of the restriction of scalars functor $f^* : \mathrm{Cat}(\mathsf{D}, \mathbf{Set}) \to \mathrm{Cat}(\mathsf{C}, \mathbf{Set})$, which is easily seen to take values in $\mathrm{Lex}_\kappa(\mathsf{C}, \mathbf{Set})$ because f preserves κ-small limits. $R(f)$ has a left adjoint because of the Special adjoint functor theorem.

Remark 2.3. In many occassions one may want to work with the Rex-presentation of the duality. As already pointed out in Remark 2.2, the two categories Rex and Lex are closely related, in fact the passage to the opposite category provides a biequivalence

$$^\circ : \mathrm{Rex}^{\mathrm{co}} \leftrightarrows \mathrm{Lex} : {}^\circ.$$

Observe that the passage to the conjugate category $\mathrm{Rex}^{\mathrm{co}}$ is needed just because $\mathrm{Cat}(\mathsf{C}^\circ, \mathsf{D}^\circ) \cong \mathrm{Cat}(\mathsf{C}, \mathsf{D})^\circ$.

Remark 2.4. Observe that the notation Ind_κ for the functor $\mathrm{Lex}_\kappa(\mathsf{C}^\circ, \mathbf{Set})$ is a suitable choice due to the fact that $\mathrm{Lex}_\kappa(\mathsf{C}^\circ, \mathbf{Set})$ corresponds to the free completion under κ-directed colimits of C, in the same sense as $\mathrm{Cat}(\mathsf{C}^\circ, \mathbf{Set})$ is the free completion under all small colimits of C, see for example [AR94, Prop 1.46].

Remark 2.5. It might be useful to give a different presentation of Theorem 2.1, and more precisely of the action of Ind on 1-cells. As first, we set the notation that we will use in the section.

$$\begin{tikzcd}
\mathsf{A} \arrow[r, "f"] \arrow[d, "\alpha_{\mathsf{A}}"'] & \mathsf{B} \arrow[d, "\alpha_{\mathsf{B}}"] \\
\mathrm{Ind}_k \mathsf{A} \arrow[r, shift left, "S(f)"] & \mathrm{Ind}_k \mathsf{B} \arrow[l, shift left, "Z(f)"]
\end{tikzcd}$$

The adjunction $S(f) \dashv Z(f)$ is $\mathrm{Ind}(f)$. Observe that looking at the definition of the 2-functor Pres, exposed in 2.1, we have that $S(f)\alpha_{\mathsf{A}} = \alpha_{\mathsf{B}} f$. Observe that since $S(f)$ is cocontinous and α_{A} is dense (i.e. $\mathrm{Lan}_{\alpha_{\mathsf{A}}}(\alpha_{\mathsf{A}}) = \mathrm{id}_{\mathrm{Ind}_\kappa \mathsf{A}}$), $S(f)$ must coincide with $\mathrm{Lan}_{\alpha_{\mathsf{A}}}(\alpha_{\mathsf{B}} f)$, in fact

$$\begin{aligned}
S(f) &\cong S(f) \circ \mathrm{id}_{\mathrm{Ind}_\kappa \mathsf{A}} \\
&\cong S(f) \circ \mathrm{Lan}_{\alpha_{\mathsf{A}}}(\alpha_{\mathsf{A}}) \\
&\cong \mathrm{Lan}_{\alpha_{\mathsf{A}}}(S(f)\alpha_{\mathsf{A}}) \\
&\cong \mathrm{Lan}_{\alpha_{\mathsf{A}}}(\alpha_{\mathsf{B}} f).
\end{aligned}$$

A quite formal argument that can be found in [DL18][2.16] proves that in this case the right adjoint $Z(f)$ has to coincide with $\mathrm{Lan}_{\alpha_{\mathsf{B}} f}(\alpha_{\mathsf{A}})$. In this proof we intensely used the fact that a cocontinous functor preserves Kan extensions. This will be a key tool later.

Remark 2.6. Observe that $S(f)$ has to preserve κ-presentable objects, because $S(f)\alpha_{\mathsf{A}} = \alpha_{\mathsf{B}} f$. Since $S(f)$ is also cocontinous, it is κ-accessible functor that preserve κ-presentable objects. The literature calls these functors strongly κ-accessible.

Remark 2.7. In [MP87] the authors observed that the typical paradigm of Stone-like dualities can be used to re-enact Gabriel-Ulmer duality as follows.

$$\begin{aligned}
\mathrm{Pres}_\kappa(\mathsf{C})^\circ &\cong \mathrm{Lex}_\kappa(\mathbb{1}, \mathrm{Pres}_\kappa(\mathsf{C})^\circ) \\
&\cong \mathrm{Lex}_\kappa^{\mathrm{op}}(\mathrm{Pres}_\kappa(\mathsf{C})^\circ, \mathbb{1}) \\
&\overset{\mathrm{GU}}{\cong} \mathrm{Pres}_\kappa(\mathsf{C}, \mathrm{Lex}_\kappa(\mathbb{1}, \mathbf{Set})) \\
&\cong \mathrm{Pres}_\kappa(\mathsf{C}, \mathbf{Set}).
\end{aligned}$$

This very short chain of isomorphisms proves that the 2-functor $\mathrm{Pres}_\kappa^\circ$ is in fact represented in the 2-category Pres_κ by the object $\mathbf{Set}$.

2.3. **Syntax-Semantics.** Gabriel-Ulmer duality is usually referred to as a duality of syntax-semantics kind. This is easily illustrated from the perspective of cartesian logic, also known as finite limit logic, the study of which was introduced in the works by Freyd [Fre02], Isbell [Isb72] and Coste [Cos76]. A category with finite limits C can be seen as a cartesian theory and a functor $M : \mathsf{C} \to \mathbf{Set}$ preserving finite limits can be seen as a model of the theory C in $\mathbf{Set}$. The category $\mathrm{Lex}(\mathsf{C}, \mathbf{Set})$ can be thus identified with the category of models of C. From this perspective, Gabriel-Ulmer duality establishes a reconstruction result: the category of models of a cartesian theory (the semantics) fully determines the theory (the syntax). Observe that this same observation applies equally to infinitary cartesian logic. Accordingly, throughout the whole paper we will refer to the 2-categories Lex_κ and Rex_κ as the syntactic side of the duality, while Ind_κ will be referred to as the semantic part.

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