Weak saturation and weak amalgamation property

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- finitely saturated;
- finitely homogeneous;
- countable;
- unique up to isomorphism between structures having these properties above.

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Let $\mathcal{K}$ be a λ-accessible category such that:

- has directed colimits;
- has the amalgamation property;
- has the joint embedding property;

then there is an object K which is:

- λ-saturated;
- λ-homogeneous;
- if it is λ^+ presentable, then it is unique up to isomorphism.

Definition

A category $\mathcal{K}$ has the weak amalgamation property if, for any object A there is an arrow $A \xrightarrow{e} B$ such that any span like the following

$$\begin{tikzcd}
& C & \\
& B \arrow[u, "f"] \arrow[r, "g"'] & D \\
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can be completed to a square such that the diagram below is commutative.

$$\begin{tikzcd}
C \arrow[r] & E \\
A \arrow[u] \arrow[r] & D \arrow[u]
\end{tikzcd}$$

We call such an arrow $A \xrightarrow{e} B$ *amalgamable*.

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To each Fraïssé class $\mathcal{K}$ one can associate the class $\mathcal{K}_p$ of all systems, $S = (A, \psi : B \to C)$ where A, B, C are structures in $\mathcal{K}$, B and C are substructures of A. Moreover, ψ is an isomorphism.

Theorem (Kechris-Rosendal)

The following are equivalent:

- $\mathcal{K}_p$ has the weak amalgamation property and the joint embedding property;
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Examples

Many class of graphs.

A concrete example

The class of all finite cycle-free graphs in which no two vertices of degree > 2 are adjacent.

Definition

An object K is weakly λ-saturated when for any arrow $A \to K$ where A is λ-presentable there exists $A \to B$, with B λ-presentable such that for any prolongation $A \to B \to C$ where C is λ -presentable there is an arrow $C \to K$ making the obvious diagram commutative.

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then there is a weakly finitely saturated object which is also weakly finitely homogeneous and unique up to isomorphism.

Definition

We say that a category $\mathcal{K}$ with the weak amalgamation property satisfies the smallness condition if, given a λ-presentable object A and an amalgamable arrow $A \to M$, there exists a λ-presentable object B and arrows $A \to B, B \to M$ such that $A \to B$ is amalgamable and the diagram below commutes.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[dl] \\ M & \end{tikzcd}$$

Theorem

Let $\mathcal{K}$ be a λ-accessible category with the weak amalgamation property and directed colimits, satisfying the smallness condition, then any object K has a map $K \to M$ where M is weakly λ-saturated.

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Theorem

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