Spacetime Algebra

Cocoa

January 27, 2023

Ashoka University

Table of contents

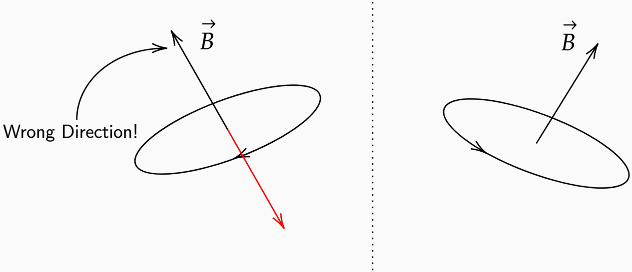
Geometric Algebra

The Physical Problem

Physical quantities such as angular momentum and electromagnetic fields should be invariant under parity inversion.

The Physical Problem

Physical quantities such as angular momentum and electromagnetic fields should be invariant under parity inversion.



The Physical Problem

Quantities defined in terms of the cross product all suffer from this issue, and are thus called *pseudovectors*.

The Physical Problem

Quantities defined in terms of the cross product all suffer from this issue, and are thus called *pseudovectors*.

Thus a mathematical construction is needed that defines these quantities rigorously while following sensible symmetries of physical systems.

The Physical Problem

Quantities defined in terms of the cross product all suffer from this issue, and are thus called *pseudovectors*.

Thus a mathematical construction is needed that defines these quantities rigorously while following sensible symmetries of physical systems.

This is achieved through the introduction of Geometric Algebra also known as Clifford Algebra

Consider the orthonormal basis for the vector space $\mathbb{R}^3$, $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$. Using this we can construct the *inner* and *exterior* products of combinations.

Consider the orthonormal basis for the vector space $\mathbb{R}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Using this we can construct the *inner* and *exterior* products of combinations.

- Inner product maps two vectors to a scalar

$$\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R} \tag{1}$$

Consider the orthonormal basis for the vector space $\mathbb{R}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Using this we can construct the *inner* and *exterior* products of combinations.

- Inner product maps two vectors to a scalar

$$\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R} \tag{1}$$

- Exterior product maps two vectors to an element of a higher rank tensor product space

$$\wedge : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3 \otimes \mathbb{R}^3 \tag{2}$$

Consider the orthonormal basis for the vector space $\mathbb{R}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Using this we can construct the *inner* and *exterior* products of combinations.

- Symmetric Inner product maps two vectors to a scalar

$$\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R} \tag{1}$$

- Exterior product maps two vectors to an element of a higher rank tensor product space

$$\wedge : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3 \otimes \mathbb{R}^3 \tag{2}$$

Consider the orthonormal basis for the vector space $\mathbb{R}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Using this we can construct the *inner* and *exterior* products of combinations.

- Symmetric Inner product maps two vectors to a scalar

$$\cdot : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R} \tag{1}$$

- Antisymmetric Exterior product maps two vectors to an element of a higher rank tensor product space

$$\wedge : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3 \otimes \mathbb{R}^3 \tag{2}$$

For some $a, b \in \mathbb{V}$, the object $a \cdot b$ is a scalar, which we will call grade 0 objects.

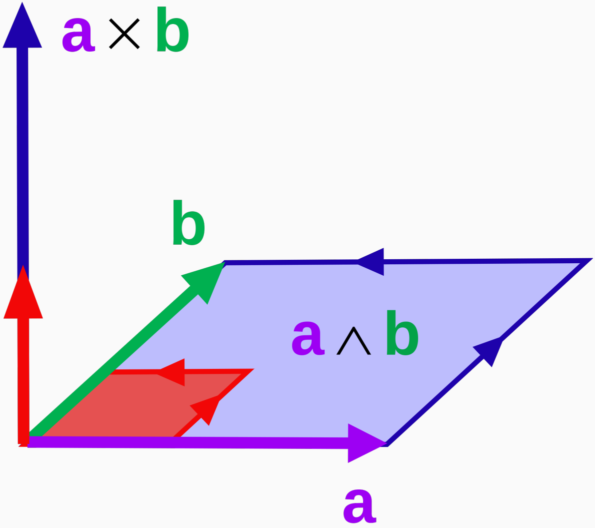
For some $a, b \in \mathbb{V}$, the object $a \cdot b$ is a scalar, which we will call grade 0 objects.

The object a is a vector, which may be considered geometrically as an oriented line segment. We will call these grade 1 elements.

For some $a, b \in \mathbb{V}$, the object $a \cdot b$ is a scalar, which we will call grade 0 objects.

The object a is a vector, which may be considered geometrically as an oriented line segment. We will call these grade 1 elements.

The object $a \wedge b$ is a *bivector*, and may be considered as an oriented plane segment. We will call these grade 2 elements.


$\mathbf{a} \times \mathbf{b}$
$\mathbf{b}$
$\mathbf{a} \wedge \mathbf{b}$
$\mathbf{a}$

Using a combination of these we introduce a vector multiplication operation known as the *geometric product* of vectors. For some $a, b \in \mathbb{V}$, the geometric product of these vectors is

$$ab = a \cdot b + a \wedge b \tag{3}$$

Using a combination of these we introduce a vector multiplication operation known as the *geometric product* of vectors. For some $a, b \in \mathbb{V}$, the geometric product of these vectors is

$$ab = a \cdot b + a \wedge b \tag{3}$$

This new construction does not return an element of the vector space, thus a new mathematical object is needed based on the elements of $\mathbb{V}$.

Using a combination of these we introduce a vector multiplication operation known as the *geometric product* of vectors. For some $a, b \in \mathbb{V}$, the geometric product of these vectors is

$$ab = a \cdot b + a \wedge b \tag{3}$$

This new construction does not return an element of the vector space, thus a new mathematical object is needed based on the elements of $\mathbb{V}$. This object is the *Clifford algebra of* $\mathbb{V}$ denoted by $Cl(\mathbb{R}^3)$.

To construct a basis for the Clifford algebra, we construct a basis of elements of every grade.

To construct a basis for the Clifford algebra, we construct a basis of elements of every grade.

Exterior products are antisymmetric $\implies a \wedge b = 0$ if a and b are linearly dependent. Thus the only possible basis elements are

Grade	Basis
0	1
1	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
2	$\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3$
3	$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$

To construct a basis for the Clifford algebra, we construct a basis of elements of every grade.

Exterior products are antisymmetric $\implies a \wedge b = 0$ if a and b are linearly dependent. Thus the only possible basis elements are

Grade	Basis
0	1
1	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
2	$\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3$
3	$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$

These form a *tensor basis* for the Clifford algebra $Cl(\mathbb{R}^3)$ also known as The Pauli Algebra.

The issues with the magnetic field and angular momentum arise due to their definition relying on the cross product. Only in 3 dimensions one finds a unique line orthogonal to a plane.

The issues with the magnetic field and angular momentum arise due to their definition relying on the cross product. Only in 3 dimensions one finds a unique line orthogonal to a plane. Thus a generalization of the cross product is required.

The issues with the magnetic field and angular momentum arise due to their definition relying on the cross product. Only in 3 dimensions one finds a unique line orthogonal to a plane. Thus a generalization of the cross product is required.

Consider the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ and take their geometric product with the grade 3 trivector $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ -

$$\begin{aligned} \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 &= -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_3 \\ &= -(\mathbf{e}_1 \cdot \mathbf{e}_1)(\mathbf{e}_2 \cdot \mathbf{e}_2)\mathbf{e}_3 \\ &= -\mathbf{e}_3 \\ &= -(\mathbf{e}_1 \times \mathbf{e}_2) \end{aligned}$$

The issues with the magnetic field and angular momentum arise due to their definition relying on the cross product. Only in 3 dimensions one finds a unique line orthogonal to a plane. Thus a generalization of the cross product is required.

This pattern follows for all combinations, and thus we define the cross product as

$$a \times b \equiv -i(a \wedge b), \tag{4}$$

where $i \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the highest grade basis element.

Hodge Duality

Hodge Duality

Multiplication by the highest grade element in an n dimensional Clifford algebra induces a duality operation which maps grade k elements to grade $n-k$ elements.

Hodge Duality

Multiplication by the highest grade element in an n dimensional Clifford algebra induces a duality operation which maps grade k elements to grade $n - k$ elements.

In the 3 dimensional Pauli algebra, the cross product mappings are mappings from grade 2 elements ($a \wedge b$) to grade $3 - 2 = 1$ elements which are the orthogonal vectors produced by the cross product.

Hodge Duality

Multiplication by the highest grade element in an n dimensional Clifford algebra induces a duality operation which maps grade k elements to grade $n-k$ elements.

In the 3 dimensional Pauli algebra, the cross product mappings are mappings from grade 2 elements ($a \wedge b$) to grade $3-2=1$ elements which are the orthogonal vectors produced by the cross product.

Vectors in a Clifford algebra have a multiplicative inverse defined as

$$a^{-1} = \frac{a}{aa}. \tag{5}$$

The Gradient

We can define a reciprocal basis given $\{\mathsf{e}_i\}$ as $\mathsf{e}^i = (\mathsf{e}_i)^{-1}$. The gradient is defined as

$$\boldsymbol{\nabla} = \mathsf{e}^i \partial_i$$

We can define a reciprocal basis given $\{\mathbf{e}_i\}$ as $\mathbf{e}^i = (\mathbf{e}_i)^{-1}$. The gradient is defined as

$$\boldsymbol{\nabla} = \mathbf{e}^i \partial_i$$

For a vector $\mathbf{a}$, the gradient can be decomposed as

$$\boldsymbol{\nabla}\mathbf{a} = \boldsymbol{\nabla} \cdot \mathbf{a} + \boldsymbol{\nabla} \wedge \mathbf{a},$$

where $\boldsymbol{\nabla} \cdot \mathbf{a}$ is the *divergence* and $\boldsymbol{\nabla} \wedge \mathbf{a}$ is the *curl*.

The Dirac Algebra

The Pauli algebra is the algebra of 3 dimensional Equclidean space. The Dirac algebra is the algebra of 4 dimensional Minkowski spacetime.

The Dirac Algebra

The Pauli algebra is the algebra of 3 dimensional Equclidean space. The Dirac algebra is the algebra of 4 dimensional Minkowski spacetime.

Take a set of orthonormal basis elements $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ of an inner product space $\mathbb{V}_4$. The metric signature is given by

$$\gamma_0 \cdot \gamma_0 = 1 \quad \text{and} \quad \gamma_k \cdot \gamma_j = -\delta_{kj}$$

Vectors that square to positive one are called *timelike* and negative one are *spacelike*.

Spacetime Splits

The Dirac algebra is a subalgebra of the Pauli algebra. This can be seen by constructing elements of the Dirac algebra that square to positive one, which will form the basis of the Pauli subalgebra.

The Dirac algebra is a subalgebra of the Pauli algebra. This can be seen by constructing elements of the Dirac algebra that square to positive one, which will form the basis of the Pauli subalgebra.

Consider the elements $\gamma_k\gamma_0$ and take their square.

$$\begin{aligned}\gamma_k\gamma_0\gamma_k\gamma_0 &= -\gamma_k^2\gamma_0^2 \\ &= -(-1)(1) = 1 = \mathsf{e}_k\end{aligned}$$

Spacetime Splits

The Dirac algebra is a subalgebra of the Pauli algebra. This can be seen by constructing elements of the Dirac algebra that square to positive one, which will form the basis of the Pauli subalgebra.

Consider the elements $\gamma_k\gamma_0$ and take their square.

$$\begin{aligned}\gamma_k\gamma_0\gamma_k\gamma_0 &= -\gamma_k^2\gamma_0^2 \\ &= -(-1)(1) = 1 = \mathrm{e}_k\end{aligned}$$

Thus timelike bivectors of the Dirac algebra are the vectors of the Pauli algebra.

Spacetime Splits

From this, we can see that for bivectors of the Pauli algebra we have

$$\begin{aligned} \mathbf{e}_k \wedge \mathbf{e}_j &= \gamma_k \gamma_0 \wedge \gamma_j \gamma_0 \\ &= -\gamma_k \gamma_j \gamma_0^2 \\ &= -\gamma_k \gamma_j \end{aligned}$$

Spacetime Splits

From this, we can see that for bivectors of the Pauli algebra we have

$$\begin{aligned}
\mathbf{e}_k \wedge \mathbf{e}_j &= \gamma_k\gamma_0 \wedge \gamma_j\gamma_0 \\
&= -\gamma_k\gamma_j\gamma_0^2 \\
&= -\gamma_k\gamma_j
\end{aligned}$$

Thus, spacelike bivectors of the Dirac algebra form the bivectors of the Pauli algebra.

From this, we can see that for bivectors of the Pauli algebra we have

$$\begin{aligned} \mathbf{e}_k \wedge \mathbf{e}_j &= \gamma_k\gamma_0 \wedge \gamma_j\gamma_0 \\ &= -\gamma_k\gamma_j\gamma_0^2 \\ &= -\gamma_k\gamma_j \end{aligned}$$

Thus, spacelike bivectors of the Dirac algebra form the bivectors of the Pauli algebra.

The scalars map to scalars and pseudoscalars map to pseudoscalars. Thus, by choosing a timelike vector, we can split all even graded elements of the Dirac algebra into the Pauli algebra.

Vectors can also be decomposed into the Pauli algebra under multiplication by γ_0. We have for a Dirac vector p

$$p\gamma_0 = p \cdot \gamma_0 + p \wedge \gamma_0$$

We define $p_0 \equiv p \cdot \gamma_0$ and $\mathbf{p} = p \wedge \gamma_0$.

The Dirac Gradient

To finish the Dirac algebra, the gradient operator is defined similar to the Pauli gradient operator as

$$\Box \equiv \gamma^\mu \partial_\mu \tag{6}$$

To finish the Dirac algebra, the gradient operator is defined similar to the Pauli gradient operator as

$$\Box \equiv \gamma^{\mu} \partial_{\mu} \tag{6}$$

The $\Box$ operator is itself a vector. Thus it can be split into a scalar and a Pauli vector by the choice of a timelike vector γ_0.

$$\begin{aligned} \gamma_0 \Box &= \gamma_0 \cdot \Box + \gamma_0 \wedge \Box \\ &= \partial_0 + \boldsymbol{\nabla} \end{aligned}$$

The Curvature Tensor

A set of four linearly independent vector fields at every point in spacetime define a frame at every point $\{\gamma_\mu\}$. This is called a *frame field*.

The derivative in the direction of one of these vectors is denoted by $\square_k$ and follows the following axioms.

1. $\square_\mu$ maps scalars to scalars $\square_\mu \phi = \partial_\mu \phi$
2. $\square_\mu$ is a linear combination of partial derivatives.

3. $\Box_\mu$ maps vectors to vectors. Any vector can be written as a linear combination of vectors a basis, and thus we have

$$\Box_\mu \gamma_\nu = -L^\alpha_{\mu\nu} \gamma_\alpha,$$

where $L^\alpha_{\mu\nu}$ are the connection coefficients.

4. $\Box_\mu$ obeys the Leibnitz rule

$$\Box_\mu (A + B) = (\Box_\mu A) B + A (\Box_\mu B)$$

5. $\Box_\mu$ is a linear operator.
6. $\Box_\mu$ transforms as a vector field.

Maxwell's Equations