Spacetime Algebra

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The Physical Problem

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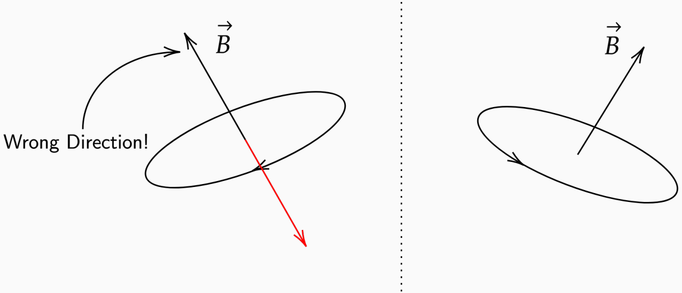

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This is achieved through the introduction of Geometric Algebra also known as Clifford Algebra

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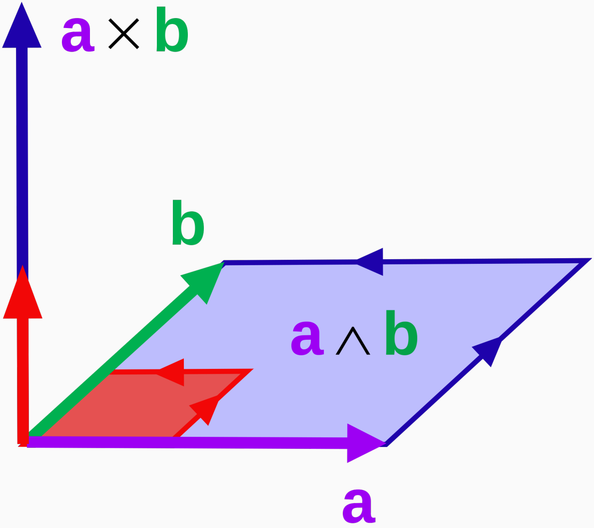
The object $a \wedge b$ is a *bivector*, and may be considered as an oriented plane segment. We will call these grade 2 elements.


$\mathbf{a} \times \mathbf{b}$
$\mathbf{b}$
$\mathbf{a} \wedge \mathbf{b}$
$\mathbf{a}$

Using a combination of these we introduce a vector multiplication operation known as the *geometric product* of vectors. For some $a, b \in \mathbb{V}$, the geometric product of these vectors is

$$ab = a \cdot b + a \wedge b \tag{3}$$

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Geometric Algebra

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Grade	Basis
0	1
1	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
2	$\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3$
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These form a *tensor basis* for the Clifford algebra $Cl(\mathbb{R}^3)$ also known as The Pauli Algebra.

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Consider the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ and take their geometric product with the grade 3 trivector $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ -

$$\begin{aligned}
\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 &= -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_3 \\
&= -(\mathbf{e}_1 \cdot \mathbf{e}_1)(\mathbf{e}_2 \cdot \mathbf{e}_2)\mathbf{e}_3 \\
&= -\mathbf{e}_3 \\
&= -(\mathbf{e}_1 \times \mathbf{e}_2)
\end{aligned}$$

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This pattern follows for all combinations, and thus we define the cross product as

$$a \times b \equiv -i(a \wedge b), \tag{4}$$

where $i \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the highest grade basis element.

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Vectors in a Clifford algebra have a multiplicative inverse defined as

$$a^{-1} = \frac{a}{aa}. \tag{5}$$

The Gradient

We can define a reciprocal basis given $\{\mathsf{e}_i\}$ as $\mathsf{e}^i = (\mathsf{e}_i)^{-1}$. The gradient is defined as

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For a vector a, the gradient can be decomposed as

$$\boldsymbol{\nabla}\mathsf{a} = \boldsymbol{\nabla} \cdot \mathsf{a} + \boldsymbol{\nabla} \wedge \mathsf{a},$$

where $\boldsymbol{\nabla} \cdot \mathsf{a}$ is the *divergence* and $\boldsymbol{\nabla} \wedge \mathsf{a}$ is the *curl*.

The Dirac Algebra

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Take a set of orthonormal basis elements $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ of an inner product space $\mathbb{V}_4$. The metric signature is given by

$$\gamma_0 \cdot \gamma_0 = 1 \quad \text{and} \quad \gamma_k \cdot \gamma_j = -\delta_{kj}$$

Vectors that square to positive one are called *timelike* and negative one are *spacelike*.

Spacetime Splits

The Dirac algebra is a subalgebra of the Pauli algebra. This can be seen by constructing elements of the Dirac algebra that square to positive one, which will form the basis of the Pauli subalgebra.

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Thus timelike bivectors of the Dirac algebra are the vectors of the Pauli algebra.

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From this, we can see that for bivectors of the Pauli algebra we have

$$\begin{aligned} \mathsf{e}_k \wedge \mathsf{e}_j &= \gamma_k \gamma_0 \wedge \gamma_j \gamma_0 \\ &= -\gamma_k \gamma_j \gamma_0^2 \\ &= -\gamma_k \gamma_j \end{aligned}$$

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Thus, spacelike bivectors of the Dirac algebra form the bivectors of the Pauli algebra.

The scalars map to scalars and pseudoscalars map to pseudoscalars. Thus, by choosing a timelike vector, we can split all even graded elements of the Dirac algebra into the Pauli algebra.

Spacetime Splits

Vectors can also be decomposed into the Pauli algebra under multiplication by γ_0. We have for a Dirac vector p

$$p\gamma_0 = p \cdot \gamma_0 + p \wedge \gamma_0$$

We define $p_0 \equiv p \cdot \gamma_0$ and $\mathbf{p} = p \wedge \gamma_0$.

The Dirac Gradient

To finish the Dirac algebra, the gradient operator is defined similar to the Pauli gradient operator as

$$\Box \equiv \gamma^\mu \partial_\mu \tag{6}$$

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The $\Box$ operator is itself a vector. Thus it can be split into a scalar and a Pauli vector by the choice of a timelike vector γ_0.

$$\begin{aligned} \gamma_0 \Box &= \gamma_0 \cdot \Box + \gamma_0 \wedge \Box \\ &= \partial_0 + \boldsymbol{\nabla} \end{aligned}$$

Geometric Differential Calculus

A set of four linearly independent vector fields at every point in spacetime define a frame at every point $\{\gamma_\mu\}$. This is called a *frame field*.

The derivative in the direction of one of these vectors is denoted by $\Box_k$ and follows the following axioms.

1. $\Box_\mu$ maps scalars to scalars $\Box_\mu \phi = \partial_\mu \phi$
2. $\Box_\mu$ is a linear combination of partial derivatives.

The Gradient, Revisited ii

3. $\Box_\mu$ maps vectors to vectors. Any vector can be written as a linear combination of vectors a basis, and thus we have

$$\Box_\mu \gamma_\nu = -L^\alpha_{\mu\nu} \gamma_\alpha,$$

   where $L^\alpha_{\mu\nu}$ are the connection coefficients.

4. $\Box_\mu$ obeys the Leibnitz rule

$$\Box_\mu (A + B) = (\Box_\mu A) B + A (\Box_\mu B)$$

5. $\Box_\mu$ is a linear operator.
6. $\Box_\mu$ transforms as a vector field.

The Curvature Tensor

This derivative behaves as the covarient derivative on vector fields. Given a field $a = a^\mu \gamma_\mu$

$$
\begin{aligned}
\square_\nu(a) &= (\square_\nu a^\alpha)\gamma_\alpha + a^\mu(\square_\nu \gamma_\mu) \\
&= \partial_\nu a^\alpha \gamma_\alpha - a^\mu L^{\alpha}_{\nu\mu}\gamma_\alpha \\
&= (\partial_\nu a^\alpha - a^\mu L^{\alpha}_{\nu\mu})\gamma_\alpha.
\end{aligned}
$$

And the curvature tensor has it's usual definition

$$
L^{\mu}_{\alpha\beta\sigma} = \partial_\alpha L^{\mu}_{\beta\sigma} - \partial_\beta L^{\mu}_{\alpha\sigma} + L^{\mu}_{\beta}\rho L^{\rho}_{\alpha\sigma} - L^{\mu}_{\alpha\rho} L^{\rho}_{\beta\sigma}. \tag{7}
$$

The electromagnetic field can now be redefined in terms of the Pauli algebra. The Riemann-Silberstein vector is defined as

$$\mathsf{F} = \mathsf{E} + i\mathsf{B}. \tag{8}$$

The Faraday Multivector

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Since every Pauli multivector is a Dirac multivector, we have

$$E = \mathsf{E} = E^i \mathbf{e}_i = E^i \gamma_i \gamma_0 \tag{9}$$

$$B = \mathsf{B} = B^i \mathbf{e}_i = B^i \gamma_i \gamma_0 \tag{10}$$

Which gives us a Dirac multivector known as the Faraday multivector

$$F = E + iB \tag{11}$$

The Faraday Multivector

The electromagnetic tensor is a separate object from the Faraday Multivector. The tensor can be defined in terms of the multivector however they are separate entities.

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For any two Dirac vectors, the electromagnetic tensor $\underline{F}(a, b)$ can be defined is the contraction of the Faraday multivector with these vectors

$$\underline{F}(a, b) = a \cdot F \cdot b \tag{12}$$

The Faraday Multivector

We can decompose F as a linear combination of bivectors to have

$$F = \frac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu$$

And then the decomposition retrieves the required component

$$\gamma_\alpha \cdot F \cdot \gamma_\beta = \frac{1}{2} F^{\mu\nu} \gamma_\alpha \cdot \gamma_\mu \wedge \gamma_\nu \cdot \gamma_\beta = \frac{1}{2} F^{\alpha\beta}$$

Charged Current Density

The charged current density is a Dirac multivector J which can be decomposed into a scalar and a Pauli vector by multiplication with γ_0

$$\begin{aligned} J &= J\gamma_0\gamma_0 \\ &= (J \cdot \gamma_0 + J \wedge \gamma_0)\gamma_0 \\ &= (\rho + \mathrm{J})\gamma_0 \\ &= \gamma_0(\rho - \mathrm{J}) \end{aligned}$$

Maxwell's Equation

Maxwell's equation in terms of the Faraday multivector is then given as

$$\Box F = J \tag{13}$$

These can be decomposed into the four familiar equations by left multiplication with γ_0 and noting that $\gamma_0 \cdot \Box = \partial_0$ and $\gamma_0 \wedge \Box = \boldsymbol{\nabla}$.

$$\begin{aligned}(\partial_0 + \boldsymbol{\nabla})\,(\mathsf{E} + i\mathsf{B}) &= \rho - \mathsf{J} \\ \partial_0 \mathsf{E} + \boldsymbol{\nabla}\mathsf{E} + i(\partial_0 \mathsf{B} + \boldsymbol{\nabla}\mathsf{B}) &= \rho - \mathsf{J}\end{aligned}$$

Maxwell's Equation

Expanding the geometric products as a dot and wedge product we get

$$\partial_0 \mathsf{E} + \boldsymbol{\nabla} \cdot \mathsf{E} + \boldsymbol{\nabla} \wedge \mathsf{E} + i\partial_0 \mathsf{B} + i\boldsymbol{\nabla} \cdot \mathsf{B} + i\boldsymbol{\nabla} \wedge \mathsf{B} = \rho - \mathsf{J}$$

Collecting scalar, vector, bivector, trivector and pseudoscalar terms we get

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathsf{E} &= \rho \\ \partial_0 \mathsf{E} + i\boldsymbol{\nabla} \wedge \mathsf{B} &= -\mathsf{J} \\ i\partial_0 \mathsf{B} + \boldsymbol{\nabla} \wedge \mathsf{E} &= 0 \\ i\boldsymbol{\nabla} \cdot \mathsf{B} &= 0 \end{aligned}$$

Questions?