Spacetime Algebra

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January 26, 2023

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Geometric Algebra

The Physical Problem

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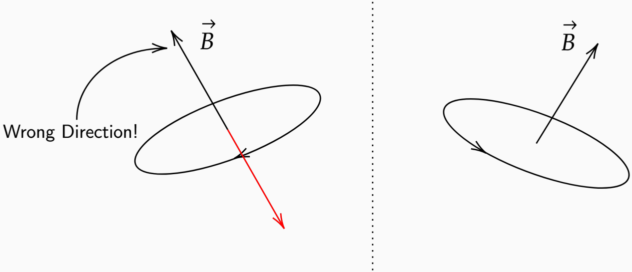

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Thus a mathematical construction is needed that defines these quantities rigorously while following sensible symmetries of physical systems.

This is achieved through the introduction of Geometric Algebra also known as Clifford Algebra

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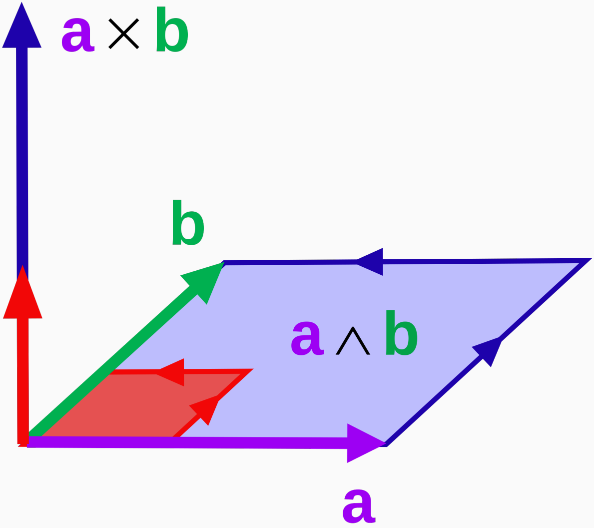
The object $a \wedge b$ is a *bivector*, and may be considered as an oriented plane segment. We will call these grade 2 elements.


$\mathbf{a} \times \mathbf{b}$
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$\mathbf{a}$

Using a combination of these we introduce a vector multiplication operation known as the *geometric product* of vectors. For some $a, b \in \mathbb{V}$, the geometric product of these vectors is

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Grade	Basis
0	1
1	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
2	$\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3$
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These form a *tensor basis* for the Clifford algebra $Cl(\mathbb{R}^3)$ also known as The Pauli Algebra.

The Pauli Algebra

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Consider the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ and take their geometric product with the grade 3 trivector $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ -

$$\begin{aligned} \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 &= -\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_3 \\ &= -(\mathbf{e}_1 \cdot \mathbf{e}_1)(\mathbf{e}_2 \cdot \mathbf{e}_2)\mathbf{e}_3 \\ &= -\mathbf{e}_3 \\ &= -(\mathbf{e}_1 \times \mathbf{e}_2) \end{aligned}$$

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This pattern follows for all combinations, and thus we define the cross product as

$$a \times b \equiv -i(a \wedge b), \tag{4}$$

where $i \equiv \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the highest grade basis element.

Hodge Duality

Multiplication by the highest grade element in an n dimensional Clifford algebra induces a duality operation which maps grade k elements to grade $n - k$ elements.

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Vectors in a Clifford algebra have a multiplicative inverse defined as

$$a^{-1} = \frac{a}{aa}. \tag{5}$$

The Gradient

We can define a reciprocal basis given $\{\mathsf{e}_i\}$ as $\mathsf{e}^i = (\mathsf{e}_i)^{-1}$. The gradient is defined as

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For a vector $\mathbf{a}$, the gradient can be decomposed as

$$\boldsymbol{\nabla}\mathbf{a} = \boldsymbol{\nabla} \cdot \mathbf{a} + \boldsymbol{\nabla} \wedge \mathbf{a},$$

where $\boldsymbol{\nabla} \cdot \mathbf{a}$ is the *divergence* and $\boldsymbol{\nabla} \wedge \mathbf{a}$ is the *curl*.

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Take a set of orthonormal basis elements $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ of an inner product space $\mathbb{V}_4$. The metric signature is given by

$$\gamma_0 \cdot \gamma_0 = 1 \quad \text{and} \quad \gamma_k \cdot \gamma_j = -\delta_{kj}$$

Vectors that square to positive one are called *timelike* and negative one are *spacelike*.

Spacetime Splits

The Dirac algebra is a subalgebra of the Pauli algebra. This can be seen by constructing elements of the Dirac algebra that square to positive one, which will form the basis of the Pauli subalgebra.

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Thus timelike bivectors of the Dirac algebra are the vectors of the Pauli algebra.

From this, we can see that for bivectors of the Pauli algebra we have

$$\begin{aligned}
\mathbf{e}_k \wedge \mathbf{e}_j &= \gamma_k\gamma_0 \wedge \gamma_j\gamma_0 \\
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Thus, spacelike bivectors of the Dirac algebra form the bivectors of the Pauli algebra.

The scalars map to scalars and pseudoscalars map to pseudoscalars. Thus, by choosing a timelike vector, we can split all even graded elements of the Dirac algebra into the Pauli algebra.

Spacetime Splits

Vectors can also be decomposed into the Pauli algebra under multiplication by γ_0. We have for a Dirac vector p

$$p\gamma_0 = p \cdot \gamma_0 + p \wedge \gamma_0$$

We define $p_0 \equiv p \cdot \gamma_0$ and $\mathbf{p} = p \wedge \gamma_0$.

The Dirac Gradient

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The $\Box$ operator is itself a vector. Thus it can be split into a scalar and a Pauli vector by the choice of a timelike vector γ_0.

$$\begin{aligned}\gamma_0 \Box &= \gamma_0 \cdot \Box + \gamma_0 \wedge \Box \\ &= \partial_0 + \boldsymbol{\nabla}\end{aligned}$$

The Curvature Tensor

Maxwell’s Equations