Intro to the Characteristic Equation of a 2nd-Order Linear System

Consider a linear, second-order, unforced, undamped system, which can be represented with the following vector ODE:

$$M\ddot{X} + KX = 0, \qquad X(t) = \begin{bmatrix} x_1(t) \\ ... \\ x_n(t) \end{bmatrix}$$

Since the system is undamped and unforced, we can expect to find solutions of the form:

$$X_0 \cos(\omega_n t)$$

Where X_0 is a vector constant called the **mode shape** and ω_n is a scalar constant called the **natural frequency**. Assuming that $X(t)$ has this form, then it follows that $\ddot{X}(t)$ is given by:

$$X(t) = X_0 \cos(\omega_n t) \;\; \rightarrow \;\; \ddot{X}(t) = -\gamma_n X_0 \cos(\omega_n t)$$

where $\gamma_n = \omega_n^2$. Let's find the possible values for X_0 and ω_n. To do this, we plug $X(t)$ and $\ddot{X}(t)$ back into our differential equation:

$$-\gamma_n M X_0 \cos(\omega_n t) + K X_0 \cos(\omega_n t) = 0 \;\; \rightarrow \;\; (K - \gamma_n M) X_0 \cos(\omega_n t) = 0$$

Since $\cos(\omega_n t)$ can be nonzero, and $(K - \gamma_n M) X_0$ is a constant, it follows that:

$$(K - \gamma_n M) X_0 = 0$$

Assuming that the solution is nontrivial (i.e. that $X_0 \neq 0$), we conclude that the determinant of $K - \gamma_n M$ must be zero:

$$\det(K - \gamma_n M) = 0$$

We call this our **characteristic equation**, and $\det(K - \gamma_n M)$ is called the **characteristic polynomial**. Note that there can (and usually are) multiple values of $\gamma_n = \omega_n^2$ that satisfy our characteristic equation, with each mode shape X_0 corresponding to a specific natural frequency ω_n. Once we have found a natural frequency ω_i, we can find the corresponding mode shape(s) X_i by evaluating $K - \omega_i^2 M$, and then solving the following equation for X_i:

$$\left(K - \omega_i^2 M\right) X_i = 0$$

Since the above equation is linear w/respect to X_i, it follows that:

1. If X_i is a mode shape with natural frequency ω_i, then αX_i is also a mode shape with natural frequency ω_i (for any constant scalar α):
$$\left(K - \omega_i^2 M\right) X_i = 0$$
$$\left(K - \omega_i^2 M\right)(\alpha X_i) = \alpha\left(\left(K - \omega_i^2 M\right) X_i\right) = \alpha(0) = 0$$

2. If X_1 and X_2 are mode shapes with the same natural frequency ω_i, then any linear combination of X_1 and X_2 is also a mode shape with natural frequency ω_i:
$$\left(K - \omega_i^2 M\right) X_1 = 0, \qquad \left(K - \omega_i^2 M\right) X_2 = 0$$
$$\left(K - \omega_i^2 M\right)(\alpha X_1 + \beta X_2) = \left(K - \omega_i^2 M\right)(\alpha X_1) + \left(K - \omega_i^2 M\right)(\beta X_2) = 0 + 0 = 0$$

Note that, at least for this particular linear ODE (and assuming that M and K are both symmetric and that M is also positive definite), the multiplicity of each root γ_i of our characteristic polynomial (the algebraic multiplicity) will equal the number of linearly independent mode shapes with that particular natural frequency (the geometric multiplicity).

If we were to have instead assumed that the solution had one of the following forms:

$$X(t) = X_0 \sin(\omega_n t), \qquad X(t) = X_0 \cos(\omega_n t + \phi), \qquad X(t) = X_0 \left(a \cos(\omega_n t) + b \sin(\omega_n t)\right)$$

we would end up with the same exact characteristic equation as before:

$$\det\left(K - \gamma_n M\right) = 0$$

Thus, if $X_0 \cos\left(\omega_n t\right)$ is a solution to our original linear ODE, then

$$X(t) = X_0 \sin\left(\omega_n t\right), \quad X(t) = X_0 \cos\left(\omega_n t + \phi\right), \quad X(t) = X_0\left(a\cos\left(\omega_n t\right) + b\sin\left(\omega_n t\right)\right)$$

are also solutions.

Linear ODE's, Superposition, and the Initial Value Problem (IVP)

Suppose we have found two different solutions $X_1(t)$, $X_2(t)$ to our ODE. By linearity, it follows that any linear combination of $X_1(t)$ and $X_2(t)$ is also a solution:

$$M\ddot{X}_1 + KX_1 = 0, \quad M\ddot{X}_2 + KX_2 = 0, \quad X^*(t) = \alpha X_1(t) + \beta X_2(t)$$

$$M\ddot{X}^* + KX^* = M\left(\alpha\ddot{X}_1 + \beta\ddot{X}_2\right) + K\left(\alpha X_1 + \beta X_2\right) = \alpha\left(M\ddot{X}_1 + KX_1\right) + \beta\left(M\ddot{X}_2 + KX_2\right) = \alpha(0) + \beta(0) = 0$$

Thus, once we have found our natural frequencies ω_i, and mode shapes X_i, we can construct a general solution of the form:

$$X(t) = \sum_i X_i\left(a_i\cos\left(\omega_i t\right) + \frac{b_i}{\omega_i}\sin\left(\omega_i t\right)\right)$$

which has the derivative

$$\dot{X}(t) = \sum_i X_i\left(-a_i\omega_i\sin\left(\omega_i t\right) + b_i\cos\left(\omega_i t\right)\right)$$

Let's solve the initial value problem: given initial conditions $X(0)$ and $\dot{X}(0)$, find the values of the coefficients a_i and b_i. To do this, we first evaluate our expression for $X(t)$ and $\dot{X}(t)$ for $t = 0$:

$$X(0) = \sum_i X_i\left(a_i\cos(0) + \frac{b_i}{\omega_i}\sin(0)\right) = \sum_i a_i X_i$$

$$\dot{X}(0) = \sum_i X_i\left(-a_i\omega_i\sin(0) + b_i\cos(0)\right) = \sum_i b_i X_i$$

Let's define the **modal matrix** $\Phi = [X_1, ..., X_n]$. It follows that:

$$X(0) = \sum_i a_i X_i = \Phi A, \quad A = \begin{bmatrix} a_1 \\ ... \\ a_n \end{bmatrix}$$

$$\dot{X}(0) = \sum_i b_i X_i = \Phi B, \quad B = \begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix}$$

We can now solve for the coefficient vectors A and B:

$$A = \Phi^{-1}X(0), \quad B = \Phi^{-1}\dot{X}(0)$$