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Research Article

Topological feature vectors for exploring topological relationships

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Topological relationships between spatial objects such as *overlap*, *disjoint*, and *inside* have for a long time been a focus of research in a number of disciplines like cognitive science, robotics, linguistics, artificial intelligence, and spatial reasoning. In particular as predicates, they support the design of suitable query languages for spatial data retrieval and analysis in spatial database systems and Geographic Information Systems. While conceptual aspects of topological predicates (like their definition and reasoning with them) as well as strategies for avoiding unnecessary or repetitive predicate evaluations (like predicate migration and spatial index structures) have been emphasized, the development of correct and efficient implementation techniques for them has been rather neglected. Recently, the design of topological predicates for different combinations of *complex* spatial data types has led to a large increase of their numbers and accentuated the need for their efficient implementation. The goal of this article is to develop efficient implementation techniques of topological predicates for all combinations of the complex spatial data types *point2D*, *line2D*, and *region2D*, as they have been specified by different authors and in different commercial and public domain software packages. Our solution is a two-phase approach. In the *exploration phase*, for a given scene of two spatial objects, all *topological events* like intersection and meeting situations are recorded in two precisely defined *topological feature vectors* (one for each argument object of a topological predicate) whose specifications are characteristic and unique for each combination of spatial data types. These vectors serve as input for the *evaluation phase* which analyzes the topological events and determines the Boolean result of a topological predicate or the kind of topological predicate. This paper puts an emphasis on the exploration phase and the definition of the topological feature vectors. In addition, it presents a straightforward evaluation method.

Keywords: topological relationship; exploration phase; evaluation phase; exploration algorithm; topological feature vector

1. Introduction

Topological predicates (like *overlap*, *meet*, *inside*) between spatial objects (like points, lines, regions) have always been a main area of research on spatial data handling, reasoning, and query languages in a number of disciplines like artificial intelligence, linguistics, robotics, and cognitive science. They characterize the relative position between two (or more) objects in space, deliberately exclude any

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consideration of quantitative, metric measures like distance or direction measures, and are associated with notions like adjacency, coincidence, connectivity, inclusion, and continuity. In particular, they support the design of suitable query languages for spatial data retrieval and analysis in Geographic Information Systems (GIS) and spatial database systems. The focus of this research has been on the conceptual design of and reasoning with these predicates as well as on strategies for avoiding unnecessary or repetitive predicate evaluations. The two central conceptual approaches, upon which almost all publications in this field have been based and which have produced very similar results, are the 9-*intersection model* (Egenhofer and Herring 1990a) and the *RCC model* (Cui *et al.* 1993). Until recently, topological predicates have only been defined for *simple* spatial objects. In this article, we are especially interested in topological predicates for *complex* spatial objects, as they have been recently specified by Schneider and Behr (2006) on the basis of the 9-intersection model.

In contrast to the large amount of conceptual work, implementation issues for topological predicates have been widely neglected. Since topological predicates are *expensive predicates* that cannot be evaluated in constant time, the strategy in query plans has consequently been to avoid their computation. The extensive work on spatial index structures as a filtering technique in query processing is an important example of this strategy. It aims at identifying a hopefully small collection of candidate pairs of spatial objects that could possibly fulfil the predicate of interest and at excluding a large collection of pairs of spatial objects that definitely cannot satisfy the predicate. The main reason for neglecting the implementation issue of topological predicates in the literature is probably the simplifying view that some plane-sweep (Preparata and Shamos 1985) algorithm is sufficient to implement topological predicates. Certainly, the plane sweep paradigm plays an important role for the implementation of these predicates, but there are at least three aspects that make such an implementation much more challenging. A first aspect refers to the details of the plane sweep itself. Issues are whether each topological predicate necessarily requires an own, tailored plane sweep algorithm, how the plane sweep processes *complex* instead of *simple* spatial objects, whether spatial objects have been preprocessed in the sense that their intersections have been computed in advance (e.g., by employing a *realm-based* approach (Güting and Schneider 1995)), how intersections are handled, and what kind of information the plane sweep must output so that this information can be leveraged for predicate evaluation. A second aspect is that the number of topological predicates increases to a large extent with the transition from simple to complex spatial objects (Schneider and Behr 2006). The two implementation alternatives of a single, specialized algorithm for each predicate or a single algorithm for all predicates with an exhaustive case analysis are error-prone, inappropriate, and thus unacceptable options from a correctness and a performance point of view. A third aspect deals with the kind of query posed. This has impact on the evaluation process. Given two objects A and B of any complex spatial data type *point2D*, *line2D*, or *region2D* (Schneider 1997), we can pose at least two kinds of topological queries: (1) "Do A and B satisfy the topological predicate p?" and (2) "What is the topological predicate p between A and B?". Only query 1 yields a Boolean value, and we call it hence a *verification query*. This kind of query is of large interest for the query processing of spatial joins and spatial selections in spatial databases and GIS. Query 2 returns a predicate (name), and we call it hence a *determination query*. This kind of query is interesting for spatial reasoning and all applications analyzing the topological relationships of spatial objects.

The goal of this (and a follow-up) article is to develop and present efficient implementation strategies for topological predicates between all combinations of the three complex spatial data types *point2D*, *line2D*, and *region2D* within the framework of the spatial algebra SPAL2D. We distinguish two phases of predicate execution: In an *exploration phase*, a plane sweep scans a given configuration of two spatial objects, detects all *topological events* (like intersections), and records them in so-called *topological feature vectors*. These vectors serve as input for the *evaluation phase* which analyzes these topological data and determines the Boolean result of a topological predicate (query 1) or the kind of topological predicate (query 2). This paper emphasizes the exploration phase and in addition presents a straightforward evaluation method. A follow-up article (Praing and Schneider 2008) deals in detail with improved and efficient methods for the evaluation phase. The two-phase approach provides a direct and sound interaction and synergy between conceptual work (9-intersection model) and implementation (algorithmic design).

The special goals of the exploration phase explain the introduction of topological feature vectors. The design of individualized, ad hoc algorithms for each topological predicate of each spatial data type combination turns out to be very cumbersome and error-prone and raises correctness issues. To avoid these problems, we propagate a systematic exploration approach which identifies all topological events that are possible for a particular type combination. These possible topological events are stored in two precisely defined topological feature vectors (one for each argument object of the topological predicate) whose specifications are characteristic and thus unique for each type combination. However, the specification of the topological feature vector of a particular spatial data type is different in different type combinations. The exploration approach leads to very reliable and robust predicate implementations and forms a sound basis for the subsequent evaluation phase. Further goals of the exploration phase are the treatment of not only simple but also complex spatial objects and an integrated handling of general and realm-based spatial objects.

Section 2 discusses related work about spatial data types as well as available design and implementation concepts for topological predicates. In Section 3, we sketch the data structures that we assume for all spatial data types. Section 4 presents some basic algorithmic concepts needed for the exploration algorithms. Section 5 deals with the exploration phase for collecting topological information on the basis of the plane sweep paradigm. It introduces a collection of *exploration algorithms*, one for each type combination. The algorithms extract the needed topological information from a given scene of two spatial objects and determine the topological feature vectors that are specific to each type combination. We also determine the runtime complexity of each algorithm in Big-O notation. Section 6 gives a first, simple evaluation method leveraging topological feature vectors. In Section 7, we provide a brief overview of the implementation environment and present our testing strategy of the topological feature vectors. Since we are not aware of any other, published implementation description of topological predicates to compare with, we cannot and do not provide an empirical performance analysis. Finally, Section 8 draws some conclusions and discusses future work.

2. Related Work

In this section we present related work on spatial objects as the operands of topological predicates (Section 2.1), sketch briefly the essential conceptual models for topological predicates (Section 2.2), and discuss implementation aspects of topological predicates (Section 2.3).

2.1 *Spatial Objects*

In the past, numerous data models and query languages for spatial data have been proposed with the aim of formulating and processing spatial queries in databases and GIS (e.g., Egenhofer (1994a), Güting (1988), Güting and Schneider (1995), Orenstein and Manola (1988), Roussopoulos *et al.* (1988), Schneider (1997)). *Spatial data types* (see Schneider (1997) for a survey) like *point*, *line*, or *region* are the central concept of these approaches. They provide fundamental abstractions for modeling the structure of geometric entities, their relationships, properties, and operations. Topological predicates operate on instances of these data types, called *spatial objects*. Until recently, these predicates have only been defined for simple object structures like single points, continuous lines, and simple regions (Figure 1(a)–(c)). However, it is broad consensus that these simple geometric structures are inadequate abstractions for real spatial applications since they are insufficient to cope with the variety and complexity of geographic reality. Consequently, universal and versatile type specifications are needed for (more) complex spatial objects so that they are usable in many different applications. *Complex points* (Figure 1(d)) allow finite collections of single points as point objects (e.g., to gather the positions of all lighthouses in the U.S.). *Complex lines* (Figure 1(e)) permit arbitrary, finite collections of one-dimensional curves, i.e., spatially embedded networks possibly consisting of several disjoint connected components, as line objects (e.g., to model the ramifications of the Nile Delta). *Complex regions* (Figure 1(f)) model areal regions that may have separations of the exterior (holes) and separations of the interior (multiple components). For example, countries (like Italy) can be made up of multiple components (like the mainland and the offshore islands) and can have holes (like the Vatican). From a formal point of view, spatial data types should be closed under the geometric operations *union*, *intersection*, and *difference*. This is guaranteed for complex but not for simple spatial data types. The formal specification of complex spatial data types given by Schneider and Behr (2006) generalizes the definitions that can be found in the literature and rests on point set theory and point set topology (Gaal 1964).

So far, only a few models have been developed for complex spatial objects. The works by Clementini and Di Felice (1996), Güting and Schneider (1993, 1995), Schneider and Behr (2006), Worboys and Bofakos (1993) are the only formal approaches; they all share the same, main structural features. The OpenGIS Consortium (OGC) has proposed similar geometric structures called *simple features* in their OGC Abstract Specification (OGC Abstract Specification 1999) and in their

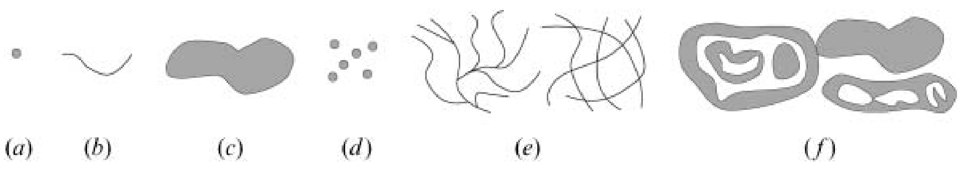


Figure 1. Examples of a simple point object (a), a simple line object (b), a simple region object (c), a complex point object (d), a complex line object (e), and a complex region object (f).

Geography Markup Language (GML) (OGC Geography Markup Language 2001), which is an XML encoding for the transport and storage of geographic information. These geometric structures are described informally and called *MultiPoint*, *MultiLineString*, and *MultiPolygon*. Another similar but also informally described spatial data type specification is provided by ESRI's Spatial Database Engine (ArcSDE) (ESRI Spatial Database Engine (SDE) 1995). Further, database vendors have added spatial extension packages that include spatial data types through extensibility mechanisms to their database systems. Examples are the Informix Geodetic DataBlade (Informix 1997), the Oracle Spatial Cartridge (Oracle 1997), and DB2's Spatial Extender (Davis 1998). All these spatial data type implementations can be used as a basis for the predicate implementation concepts proposed in this article. Our main interest regarding complex spatial data types relates to the development of effective geometric data structures that are able to support the efficient implementation of topological predicates.

2.2 *Conceptual Models for Topological Relationships*

The conceptual study and determination of topological relationships has been an interdisciplinary research topic for a long time. Two fundamental approaches have turned out to be the starting point for a number of extensions and variations; these are the 9-*intersection model* (Egenhofer and Herring 1990a), which is based on point set theory and point set topology, and the *RCC model* (Cui *et al.* 1993), which is based on spatial logic. Despite rather different foundations, both approaches come to very similar results. The implementation strategy described in this article relies heavily on the 9-intersection model. The reason is that we are able to create a direct link between the concepts of this model and our implementation approach, as we will see later. This enables us to prove the correctness of our implementation strategy.

Based on the 9-intersection model, a complete collection of mutually exclusive topological relationships can be determined for each combination of simple and recently also complex spatial data types. The model is based on the nine possible intersections of the boundary (∂A), the interior (A°), and the exterior (A^-) of a spatial object A with the corresponding components of another object B. Each intersection is tested with regard to the topologically invariant criteria of non-emptiness. The topological relationship between two spatial objects A and B can be expressed by evaluating the 3×3-matrix in Figure 2(a). For this matrix, $2^9 = 512$ different configurations are possible from which only a certain subset makes sense depending on the *definition* and *combination* of the types of the spatial objects considered. For each combination of spatial types, this means that each of its predicates is associated with a unique intersection matrix so that all predicates are mutually exclusive and complete with regard to the topologically invariant criteria of emptiness and non-emptiness.

$$\begin{pmatrix} A^\circ \cap B^\circ \neq \emptyset & A^\circ \cap \partial B \neq \emptyset & A^\circ \cap B^- \neq \emptyset \\ \partial A \cap B^\circ \neq \emptyset & \partial A \cap \partial B \neq \emptyset & \partial A \cap B^- \neq \emptyset \\ A^- \cap B^\circ \neq \emptyset & A^- \cap \partial B \neq \emptyset & A^- \cap B^- \neq \emptyset \end{pmatrix}$$

(*a*)

	point	line	region
point	2/5	3/14	3/7
line	3/14	33/82	19/43
region	3/7	19/43	8/33

(*b*)

Figure 2. The 9-intersection matrix (a) and the numbers of topological predicates between two simple/complex spatial objects (b).

Topological relationships have been first investigated for simple spatial objects (Figure 2(b)), i.e., for two simple regions (*disjoint*, *meet*, *overlap*, *equal*, *inside*, *contains*, *covers*, *coveredBy*) (Clementini *et al.* 1993, Egenhofer and Herring 1990a), for two simple lines (Clementini and Di Felice 1998, Egenhofer and Herring 1990b), and for a simple line and a simple region (Egenhofer and Mark 1995). Topological predicates involving simple points are trivial. The implementation concepts in this article can be applied to all these predicate collections.

The two works by Clementini *et al.* (1995) and Egenhofer *et al.* (1994) are the first but restricted attempts to a definition of topological relationships on complex spatial objects. In Clementini *et al.* (1995) the TRCR (Topological Relationships for Composite Regions) model only allows sets of disjoint, simple regions without holes. Topological relationships between these composite regions are defined in an *ad hoc* manner and are not systematically derived from the underlying model. The work by Egenhofer *et al.* (1994) only considers topological relationships of simple regions with holes; multi-part regions are not permitted. A main problem of this approach is its dependance on the number of holes of the operand objects.

In Schneider and Behr (2006), with two precursors in Behr and Schneider (2001) and Schneider and Behr (2005), we have given a thorough, systematic, and complete specification of topological relationships for all combinations of *complex* spatial data types. Details about the determination process and prototypical drawings of spatial scenarios visualizing all topological relationships can be found in these publications. This approach, which is also based on the 9-intersection model, is the basis of our implementation. Figure 2(b) shows the increase of topological predicates for complex objects compared to simple objects and underpins the need for sophisticated and efficient predicate execution techniques.

2.3 *Implementation Aspects of Topological Predicates*

In spatial queries, topological predicates usually appear as filter conditions of spatial selections and spatial joins. At least two main strategies can be distinguished for their processing. Either we attempt to avoid the execution of topological predicates since they are expensive predicates and cannot be executed in constant time, or we design sophisticated implementation methods for them. The latter strategy is always needed while the former strategy is worthwhile to do.

Avoidance strategies for expensive predicate executions can be leveraged at the algebraic level and at the physical level. At the algebraic level, consistency checking procedures examine the collection of topological relationships contained in a query for topological consistency by employing topological reasoning techniques (Rodriguez *et al.* 2003). Optimization minimizes the number of computations needed (Clementini *et al.* 1994) and aims at eliminating topological relationships that are implied uniquely by composition (Egenhofer 1994b). At the physical level, several methods pursue the concept of avoiding unnecessary or repetitive predicate evaluations in query access plans. Examples of these methods are predicate migration (Hellerstein and Stonebraker 1993), predicate placement (Hellerstein 1994), disjunctive query optimization (Claussen *et al.* 2000), and approximation-based evaluation (Brodsky and Wang 1995). Another important method is the deployment of spatial index structures (Gaede and Günther 1998) to identify those candidate pairs of spatial objects that could possibly fulfil the predicate of interest. This is done by a filtering test on the basis of minimum bounding rectangles. In this article, we assume that all these techniques have been applied without success so that the topological predicate execution cannot be avoided.

At some point, avoidance strategies are of no help anymore, and the application of physical predicate execution algorithms is required. So far, there has been no published research on the efficient implementation of topological predicates, their optimization, and their connection to the underlying theory. All three aspects are objectives of our work. We leverage the well known plane sweep paradigm (Berg *et al.* 2000) and go far beyond our own initial attempts in Schneider (2002, 2004). These attempts extend the concept of the eight topological predicates for two simple regions to a concept and implementation of these eight predicates for complex regions.

The spatial data management and analysis packages mentioned in Section 2.1 like the ESRI ArcSDE, the Informix Geodetic DataBlade, the Oracle Spatial Cartridge, and DB2's Spatial Extender offer limited sets of named topological predicates for simple and complex spatial objects. But their definitions are unclear and their underlying algorithms unpublished. The open source JTS Topology Suite (JTS Topology Suite 2007) conforms to the simple features specification (OGC Abstract Specification 1999) of the Open GIS Consortium and implements the aforementioned eight topological predicates for complex spatial objects through *topology graphs*. A topology graph stores topology explicitly and contains labeled nodes and edges corresponding to the endpoints and segments of a spatial object's geometry. For each node and edge of a spatial object, one determines whether it is located in the interior, in the exterior, or on the boundary of another spatial object. Computing the topology graphs and deriving the 9-intersection matrix from them require quadratic time and quadratic space in terms of the nodes and edges of the two operand objects. Our solution requires linearithmic (loglinear) time and linear space due to the use of a plane sweep.

3. Data Structures

Our approach to the verification (query type 1) as well as the determination (query type 2) of a topological relationship requires a two-phase algorithm consisting of an *exploration phase* and an *evaluation phase*. Only the exploration phase makes use of the two spatial input objects. Each input object can be of type *point2D*, *line2D*, or *region2D* as they are provided under different names in many commercial and public domain spatial extension packages (see Section 2.1). In this section, we describe the geometric data structures that we assume and need for our exploration algorithms and that can be found in this or a similar way in current public spatial extension packages.

At the lowest level, we assume a number system Ω ensuring robust geometric computation, e.g., by means of rational numbers allowing value representations of arbitrarily, finite length, or by means of infinite precision numbers. At the next higher level, we introduce some two-dimensional *robust geometric primitives* that we assume to be implemented on the basis of Ω. The primitives serve as elementary building blocks for all higher-level structures and contribute significantly to their behavior, performance, and robustness. All objects of *robust geometric primitive types* are stored in records. The type *poi2D* incorporates all single, two-dimensional points we can represent on the basis of our robust number system. That is, this type is defined as

$$poi2D = \{(x, y) | x, y \in \Omega\} \cup \{\epsilon\}$$

The value ϵ represents *the empty object* and is an element of *all* data types. The empty object is needed to represent the case that an operation yields an "empty" result. Consider the case of a primitive operation that computes the intersection of two segments in a point. If both segments do not intersect at all or they intersect in a common segment, the result is empty and represented by ϵ.

Given two points p, $q \in poi2D$, we assume a predicate "=" ($p = q \Leftrightarrow p.x = q.x \wedge p.y = q.y$) and the lexicographic order relation "<" ($p < q \Leftrightarrow p.x < q.x \vee (p.x = q.x \wedge p.y < q.y)$).

The type *seg2D* includes all straight segments bounded by two endpoints. That is

$$seg2D = \{(p, q) | p, q \in poi2D, p < q\} \cup \{\epsilon\}$$

The order defined on the endpoints normalizes segments and provides for a unique representation. This enables us to speak of a *left end point* and a *right end point* of a segment.

The predicates *on*, *in* : *poi2D* × *seg2D*→*bool* check whether a point is located on a segment including and excluding its endpoints respectively. The predicates *poiIntersect*, *segIntersect* : *seg2D* × *seg2D*→*bool* test whether two segments intersect in a point or a segment respectively. The predicates *collinear*, *equal*, *disjoint*, *meet* : *seg2D* × *seg2D*→*bool* determine whether two segments lie on the same infinite line, are identical, do not share any point, and touch each other in exactly one common endpoint respectively. The function *len* : *seg2D*→*real* computes the length of a segment. The type *real* is our own approximation type for the real numbers and implemented on the basis of Ω. The operation *poiIntersection* : *seg2D* × *seg2D*→*poi2D* returns the intersection point of two segments.

The type *mbb2D* comprises all minimum bounding boxes, i.e., axis-parallel rectangles. It is defined as

$$mbb2D = \{(p, q) | p, q \in poi2D, p.x < q.x, p.y < q.y\} \cup \{\epsilon\}$$

Here, the predicate *disjoint* : *mbb2D* × *mbb2D*→*bool* checks whether two minimum bounding boxes are disjoint; otherwise, they interfere with each other.

At the next higher level, we assume the *geometric component data type halfsegment2D* that introduces halfsegments as the basic implementation components of objects of the spatial data types *line2D* and *region2D*. A *halfsegment*, which is stored in a record, is a hybrid between a point and a segment. That is, it has features of both geometric structures; each feature can be inquired on demand. We define the set of all halfsegments as the component data type

$$halfsegment2D = \{(s, d) | s \in seg2D - \{\epsilon\}, d \in bool\}$$

For a halfsegment $h = (s, d)$, the Boolean flag d emphasizes one of the segment's end points, which is called the *dominating point* of h. If $d = true$ ($d = false$), the left (right) end point of s is the dominating point of h, and h is called *left* (*right*) *halfsegment*. Hence, each segment s is mapped to two halfsegments (s, *true*) and (s, *false*). Let *dp* be the function which yields the dominating point of a halfsegment.

The representation of *line2D* and *region2D* objects requires an order relation on halfsegments. For two distinct halfsegments h_1 and h_2 with a common end point p, let α be the enclosed angle such that $0° < \alpha \leqslant 180°$. Let a predicate *rot* be defined as follows: *rot*(h_1, h_2) is *true* if, and only if, h_1 can be rotated around p through α to overlap h_2 in counterclockwise direction. This enables us now to define a complete order on halfsegments. For two halfsegments $h_1=(s_1, d_1)$ and $h_2=(s_2, d_2)$ we obtain:

$$h_1 < h_2 \Leftrightarrow dp(h_1) < dp(h_2) \vee \tag{1}$$

$$(dp(h_1) = dp(h_2) \wedge ((\neg d_1 \wedge d_2) \vee \tag{2a}$$

$$(d_1 = d_2 \wedge rot(h_1, h_2)) \vee \tag{2b}$$

$$(d_1 = d_2 \wedge collinear(s_1, s_2) \wedge len(s_1) < len(s_2)))) \tag{3}$$

Examples of the order relation on halfsegments are given in Figure 3. Case 1 is exclusively based on the (x, y)-lexicographical order on dominating points. In the other cases the dominating points of h_1 and h_2 coincide. Case 2a deals with the situation that h_1 is a right halfsegment and h_2 is a left halfsegment. Case 2b handles the situation that h_1 and h_2 are either both left halfsegments or both right halfsegments so that the angle criterion is applied. Finally, case 3 treats the situation that h_1 and h_2 are collinear. Two halfsegments $h_1=(s_1, d_1)$ and $h_2=(s_2, d_2)$ are equal if, and only if, $s_1=s_2$ and $d_1=d_2$.

We will also need an order relation between a point $v \in poi2D$ and a halfsegment $h \in halfsegment2D$. We define $v<h \Leftrightarrow v<dp(h)$ and $v=h \Leftrightarrow v=dp(h)$. This shows the hybrid nature of halfsegments having point and segment features.

At the highest level, we have the three *complex spatial data types point2D*, *line2D*, and *region2D* (see Section 2.1 for their intuitive description and Schneider and Behr (2006) for their formal definition). They are the input data types for topological predicates and are essentially represented as *ordered sequences of elements* of variable length. We here ignore additionally stored information about spatial objects since it is not needed for our purposes. The type of the elements is *poi2D* for *point2D* objects, *halfsegment2D* for *line2D* objects, and *attributed halfsegments* (see below) for *region2D* objects. Ordered sequences are selected as representation structures, since they directly and efficiently support parallel traversals (Section 4.1) and the plane sweep paradigm (see Section 4.3).

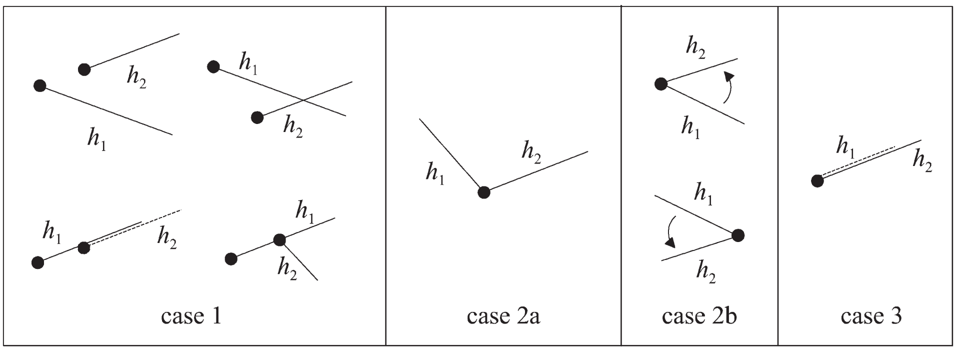


Figure 3. Examples of the order relation on halfsegments: $h_1<h_2$

We now have a closer look at the type definitions. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We define the type *point2D* as

$$point2D = \{\langle p_1, \ldots, p_n\rangle \mid n \in \mathbb{N}_0, \forall 1 \le i \le n : p_i \in poi2D - \{\epsilon\}, \forall 1 \le i \le n : p_i < p_{i+1}\}$$

Since $n=0$ is allowed, the empty sequence <> represents the empty *point2D* object.

The spatial data type *line2D* is defined as

$$\begin{aligned} line2D = \{\langle h_1, \ldots, h_{2n}\rangle \mid & \text{(i)} \quad n \in \mathrm{N}_0 \\ & \text{(ii)} \quad \forall 1 \le i \le 2n : h_i \in halfsegment2D \\ & \text{(iii)} \quad \forall h_i = (s_i, d_i) \in \{h_1, \ldots, h_{2n}\} \exists h_j = (s_j, d_j) \\ & \qquad \in \{h_1, \ldots, h_{2n}\}, 1 \le i < j \le 2n : equal(s_i, s_j) \wedge d_i = \neg d_j \\ & \text{(iv)} \quad \forall 1 \le i < 2n : h_i < h_{i+1} \\ & \text{(v)} \quad \forall h_i = (s_i, d_i), h_j = (s_j, d_j) \in \{h_1, \ldots, h_{2n}\}, i \ne j : \\ & \qquad equal(s_i, s_j) \vee disjoint(s_i, s_j) \vee meet(s_i, s_j)\} \end{aligned}$$

The value n is equal to the number of segments of a *line2D* object. Since each segment is represented by a left and a right halfsegment (condition (iii)), a *line2D* object has $2n$ halfsegments. Since $n=0$ is allowed (condition (i)), the empty sequence <> represents the empty *line2D* object. Condition (iv) expresses that a *line2D* object is given as an *ordered halfsegment sequence*. Condition (v) requires that the segments of two distinct halfsegments are either equal (this only holds for the left and right halfsegments of a segment), disjoint, or meet.

The internal representation of *region2D* objects is similar to that of *line2D* objects. But for each halfsegment, we also need the information whether the interior of the region is above/left or below/right of it. We denote such a halfsegment as *attributed halfsegment*. Each element of the sequence consists of a halfsegment that is augmented by a Boolean flag *ia* (for "interior above"). If *ia*=*true* holds, the interior of the region is above or, for vertical segments, left of the segment. The type *region2D* is defined as

$$\begin{aligned} region2D = \{\langle (h_1, ia_1), \ldots, (h_{2n}, ia_{2n})\rangle \mid & \\ & \text{(i)} \quad n \in \mathrm{N}_0 \\ & \text{(ii)} \quad \forall 1 \le i \le 2n : h_i \in halfsegment2D, ia_i \in bool \\ & \text{(iii)} \quad \forall h_i = (s_i, d_i) \in \{h_1, \ldots, h_{2n}\} \\ & \qquad \exists h_j = (s_j, d_j) \in \{h_1, \ldots, h_{2n}\}, 1 \le i < j \le 2n : \\ & \qquad s_i = s_j \wedge d_i = \neg d_j \wedge ia_i = ia_j \\ & \text{(iv)} \quad \forall 1 \le i < 2n : h_i < h_{i+1} \\ & \text{(v)} \quad \text{"additional topological constraints"}\} \end{aligned}$$

Condition (v) refers to some additional topological constraints that all components (faces) of a region must be edge-disjoint from each other and that for each face its holes must be located inside its outer polygon, be edge-disjoint to

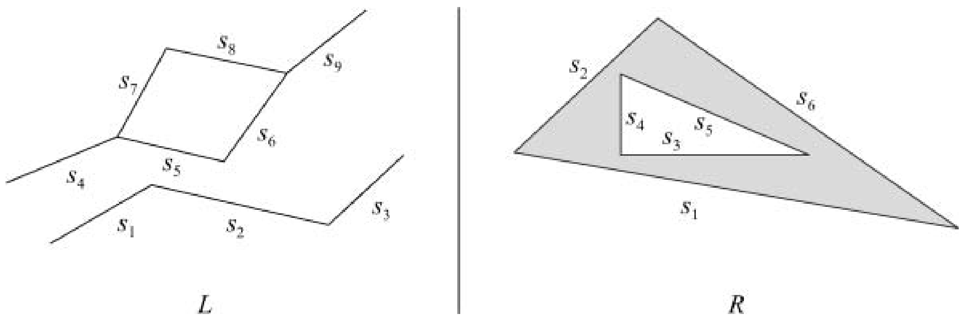


Figure 4. A *line2D* object L and a *region2D* object R

the outer polygon, and be edge-disjoint among each other. We mention these constraints for reasons of completeness and assume their satisfaction.

As an example, Figure 4 shows a *line2D* object L (with two components (blocks)) and a *region2D* object R (with a single face containing a hole). Both objects are annotated with segment names s_i. We determine the halfsegment sequences of L and R and let $h_i^l = (s_i, \mathit{true})$ and $h_i^r = (s_i, \mathit{false})$ denote the left halfsegment and right halfsegment of a segment s_i respectively. For L we obtain the ordered halfsegment sequence

$$L = \langle h_4^l, h_1^l, h_4^r, h_5^l, h_7^l, h_1^r, h_2^l, h_7^r, h_8^l, h_5^r, h_6^l, h_8^r, h_6^r, h_9^l, h_2^r, h_3^l, h_9^r, h_3^r \rangle$$

For R we obtain the following ordered sequence of attributed halfsegments ($t \equiv \mathit{true}, f \equiv \mathit{false}$):

$$R = \langle (h_1^l, t), (h_2^l, f), (h_3^l, f), (h_4^l, t), (h_4^r, t), (h_5^l, t), (h_2^r, f), (h_6^l, f), (h_5^r, t), \\ (h_5^r, t), (h_3^r, f), (h_6^r, f), (h_1^r, t) \rangle$$

Since inserting a halfsegment at an arbitrary position needs $O(n)$ time, in our implementation we use an AVL-tree embedded into an array whose elements are linked in halfsegment order. An insertion then requires $O(\log n)$ time.

If we take into account that the segments of a *line2D* object as well as the segments of a *region2D* object are not allowed to intersect each other or touch each other in their interiors according to their type specifications, the definition of the order relation on halfsegments seems to be too intricate. If, in Figure 3, we take away all subcases of case 1 except for the upper left subcase as well as case 3, the restricted order relation can already be leveraged for complex lines and complex regions. In case that all spatial objects of an application space are defined over the same *realm*[1] (Güting and Schneider 1993, Schneider 1997), the restricted order relation can also be applied for a parallel traversal of the sequences of two (or more) *realm-based line2D* or *region2D* objects. Only in the general case of intersecting spatial objects, the full order relation on halfsegments is needed for a parallel traversal of the objects' halfsegment sequences.

[1] A *realm* provides a discrete geometric basis for the construction of spatial objects and consists of a finite set of points and *non-intersecting* segments, called *realm objects*. That is, a segment inserted into the realm is intersected and split according to a special strategy with all realm objects. All spatial objects like complex points, complex lines, and complex regions are then defined in terms of these realm objects. Hence, all spatial objects defined over the same realm become aquainted with each other beforehand.

4. Basic Algorithmic Concepts

In this section, we describe three algorithmic concepts that serve as the foundation of the *exploration algorithms* in Section 5. These concepts are the *parallel object traversal* (Section 4.1), *overlap numbers* (Section 4.2), and the *plane sweep* paradigm (Section 4.3). Parallel object traversal and overlap numbers are employed during a plane sweep. We will not describe these three concepts in full detail here, since they are well known methods in Computational Geometry (Berg *et al.* 2000) and spatial databases (Güting *et al.* 1995). However, we will focus on the specialties of these concepts in our setting, including some improvements compared to standard plane sweep implementations. These comprise a smoothly integrated handling of general (i.e., intersecting) and realm-based (i.e., non-intersecting) pairs of spatial objects. An objective of this section is also to introduce a number of auxiliary operations and predicates that make the description of the exploration algorithms later much easier and more comprehensible.

4.1 *Parallel Object Traversal*

For a plane sweep, the representation elements (points or segments) of the spatial operand objects have usually to be merged together and sorted afterwards according to some order relation (e.g., the order on x-coordinates). This initial merging and sorting is rather expensive and requires $O(n \log n)$ time, if n is the number of representation elements of both operand objects. Our approach avoids this initial sorting, since the representation elements of *point2D*, *line2D*, and *region2D* objects are already stored in the order we need (point order or halfsegment order). We also do not have to merge the object representations, since we can deploy a *parallel object traversal* that allows us to traverse the point or halfsegment sequences of both operand objects in parallel. Hence, by employing a cursor on both sequences, it is sufficient to check the point or halfsegment at the current cursor positions of both sequences and to take the lower one with respect to the point order or halfsegment order for further computation.

If the operand objects have already been intersected with each other, like in the realm case (Güting and Schneider 1993), the parallel object traversal has only to operate on two *static* point or halfsegment sequences. But in the general case, intersections between both objects can exist and are detected during the plane sweep. A purely static sequence structure is insufficient in this case, since detected intersections have to be stored and handled later during the plane sweep. In order to avoid a change of the original object representations, which would be very expensive and only temporarily needed, each object is associated with an additional and temporary *dynamic* sequence, which stores newly detected points or halfsegments of interest. Hence, our parallel object traversal has to handle a static and a dynamic sequence part for each operand object and thus four instead of two point or halfsegment sequences. It returns the smallest point or halfsegment from the four current cursor positions. We will give an example of the parallel object traversal when we discuss our plane sweep approach in Section 4.3.

To simplify the description of this parallel scan, two operations are provided. Let $O_1 \in \alpha$ and $O_2 \in \beta$ with $\alpha, \beta \in \{point2D, line2D, region2D\}$. The operation *select_first*(O_1, O_2, *object*, *status*) selects the first point or halfsegment of each of the operand objects O_1 and O_2 and positions a logical pointer on both of them. The parameter *object* with a possible value out of the set {*none*, *first*, *second*, *both*} indicates which of the two object representations contains the smaller point or halfsegment. If the value of *object* is *none*, no point or halfsegment is selected, since O_1 and O_2 are empty. If the value is *first* (*second*), the smaller point or halfsegment belongs to O_1 (O_2). If it is *both*, the first point or halfsegment of O_1 and O_2 are identical. The parameter *status* with a possible value out of the set {*end_of_none*, *end_of_first*, *end_of_second*, *end_of_both*} describes the state of both object representations. If the value of *status* is *end_of_none*, both objects still have points or halfsegments. If it is *end_of_first* (*end_of_second*), O_1 (O_2) is exhausted. If it is *end_of_both*, both object representations are exhausted.

The operation *select_next*(O_1, O_2, *object*, *status*), which has the same parameters as *select_first*, searches for the next smallest point or halfsegment of O_1 and O_2. Two points (halfsegments) are compared with respect to the lexicographic (halfsegment) order. For the comparison between a point and a halfsegment, the dominating point of the halfsegment and hence the lexicographic order is used. If before this operation *object* was equal to *both*, *select_next* moves forward the logical pointers of both sequences; otherwise, if *object* was equal to *first* (*second*), it only moves forward the logical pointer of the first (second) sequence. In contrast to the first operation, which only has to consider the static sequence part of an object, this operation also has to check the dynamic sequence part of each object. Both operations together allow one to scan in linear time two object representations like one ordered sequence.

4.2 *Overlap Numbers*

The concept of *overlap numbers* is exclusively needed for the computation of the topological relationships between two *region2D* objects. The reason is that we have to find out the degree of overlapping of region parts. Points in the plane that are shared by both *region2D* objects obtain the overlap number 2. Points that are shared only by one of the objects obtain the overlap number 1. All other points are outside of both objects and obtain the overlap number 0. In our implementation, since a segment s_h of an attributed halfsegment (h, ia_h)=((s_h, d_h), ia_h) of a region object separates space into two parts, an interior and an exterior one, during a plane sweep each such segment is associated with a *segment class* which is a pair (m/n) of overlap numbers, a lower (or right) one m and an upper (or left) one n ($m, n \in \mathrm{N}_0$). The lower (upper) overlap number indicates the number of overlapping *region2D* objects below (above) the segment. In this way, we obtain a *segment classification* of two *region2D* objects and speak about (m/n)-segments. Obviously, $0 \leqslant m, n \leqslant 2$ holds. Of the nine possible combinations only seven describe valid segment classes. This is because a (0/0)-segment contradicts the definition of a complex *region2D* object, since then at least one of both regions would have two holes or an outer cycle and a hole with a common border segment. Similarly, (2/2)-segments cannot exist, since then at least one of the two regions would have a segment which is common to two outer cycles of the object. Hence, possible (m/n)-segments are (0/1)-, (0/2)-, (1/0)-, (1/1)-, (1/2)-, (2/0)-, and (2/1)-segments. Figure 5 gives an example.

4.3 *Plane Sweep*

The plane sweep technique (Berg *et al.* 2000, Preparata and Shamos 1985) is a well known algorithmic scheme in Computational Geometry. Its central idea is to reduce

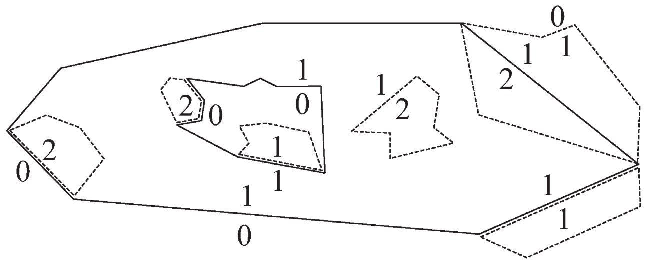


Figure 5. Example of the segment classification of two *region2D* objects

a two-dimensional geometric problem to a simpler one-dimensional geometric problem. A vertical *sweep line* traversing the plane from left to right stops at special *event points* which are stored in a queue called *event point schedule*. The event point schedule must allow one to insert new event points discovered during processing; these are normally the initially unknown intersections of line segments. The state of the intersection of the sweep line with the geometric structure being swept at the current sweep line position is recorded in vertical order in a data structure called *sweep line status*. Whenever the sweep line reaches an event point, the sweep line status is updated. Event points which are passed by the sweep line are removed from the event point schedule. Note that, in general, an efficient and fully dynamic data structure is needed to represent the event point schedule and that, in many plane-sweep algorithms, an initial sorting step is needed to produce the sequence of event points in (x, y)-lexicographical order.

In our case, the event points are either the points of the static point sequences of *point2D* objects or the (attributed) halfsegments of the static halfsegment sequences of *line2D* (*region2D*) objects. This especially holds and is sufficient for the realm case. In addition, in the general case, new event points are determined during the plane sweep as intersections of line segments; they are stored as points or halfsegments in the dynamic sequence parts of the operand objects and are needed only temporarily for the plane sweep. As we have seen in Section 4.1, the concepts of point order, halfsegment order, and parallel object traversal avoid an expensive initial sorting at the beginning of the plane sweep. We use the operation *get_event* to provide the element to which the logical pointer of a point or halfsegment sequence is currently pointing. The Boolean predicate *look_ahead* tests whether the dominating points of a given halfsegment and the next halfsegment after the logical pointer of a given halfsegment sequence are equal.

Several operations are needed for managing the sweep line status. The operation *new_sweep* creates a new, empty sweep line status. If a left (right) halfsegment of a *line2D* or *region2D* object is reached during a plane-sweep, the operation *add_left* (*del_right*) stores (removes) its segment component into (from) the segment sequence of the sweep line status. The predicate *coincident* checks whether the just inserted segment partially coincides with a segment of the other object in the sweep line status. The operation *set_attr* (*get_attr*) sets (gets) an attribute for (from) a segment in the sweep line status. This attribute can be either a Boolean value indicating whether the interior of the region is above the segment or not (the "Interior Above" flag), or it can be an assigned segment classification. The operation *get_pred_attr* yields the attribute from the predecessor of a segment in the sweep line status. The operation *pred_exists* (*common_point_exists*) checks whether for a segment in the sweep line status a predecessor according to the vertical *y*-order (a neighbored segment of the *other* object with a common end point) exists. The operation *pred_of_p* searches the nearest segment below a given point in the sweep line status. The predicate *current_exists* tests whether such a segment exists. The predicate *poi_on_seg* (*poi_in_seg*) checks whether a given point lies *on* (*in*) any segment of the sweep line status.

Intersections of line segments stemming from two *lines2D* objects, two *region2D* objects, or a *line2D* object and a *region2D* object are of special interest, since they indicate topological changes. If two segments of two *line2D* objects intersect, this can, e.g., indicate a proper intersection, or a meeting situation between both segments, or an overlapping of both segments. If a segment of a *line2D* object intersects a segment of a *region2D* object, the former segment can, e.g., "enter" the region, "leave" the region, or overlap with the latter segment. Overlap numbers can be employed here to determine entering and leaving situations. If segments of two *region2D* objects intersect, this can, e.g., indicate that they share a common area and/or a common boundary. In this case, intersecting line segments have especially an effect on the overlap numbers of the segments of both *region2D* objects. In Section 4.2 we have tacitly assumed that any two segments from both *region2D* objects are either disjoint, or equal, or meet solely in a common end point. Only if these topological constraints are satisfied, we can use the concepts of overlap numbers and segment classes for a plane sweep. But the general case in particular allows intersections. Figure 6 shows the problem of segment classes for two intersecting segments. The segment class of s_1 [s_2] left of the intersection point is (0/1) [(1/2)]. The segment class of s_1 [s_2] right of the intersection point is (1/2) [(0/1)]. That is, after the intersection point, seen from left to right, s_1 and s_2 exchange their segment classes. The reason is that the topology of both segments changes. Whereas, to the left of the intersection, s_1 (s_2) is outside (inside) the region to which s_2 (s_1) belongs, to the right of the intersection, s_1 (s_2) is inside (outside) the region to which s_2 (s_1) belongs.

In order to be able to make the needed topological detections and to enable the use of overlap numbers for two general regions, in case that two segments from two different regions intersect, partially coincide, or touch each other within the interior of a segment, we pursue a splitting strategy that is executed during the plane sweep "on the fly". If segments intersect, they are temporarily split at their common intersection point so that each of them is replaced by two segments (i.e., four halfsegments) (Figure 7a). If two segments partially coincide, they are split each time the endpoint of one segment lies inside the interior of the other segment. Depending on the topological situations, which can be described by Allen's thirteen basic relations on intervals (Allen 1983), each of the two segments either remains unchanged or is replaced by up to three segments (i.e., six halfsegments). From the thirteen possible relations, eight relations (four pairs of symmetric relations) are of interest here (Figure 7b). If an endpoint of one segment touches the interior of the other segment, the latter segment is split and replaced by two segments (i.e., four

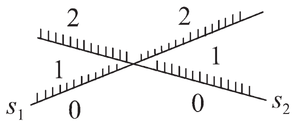


Figure 6. Changing overlap numbers after an intersection.

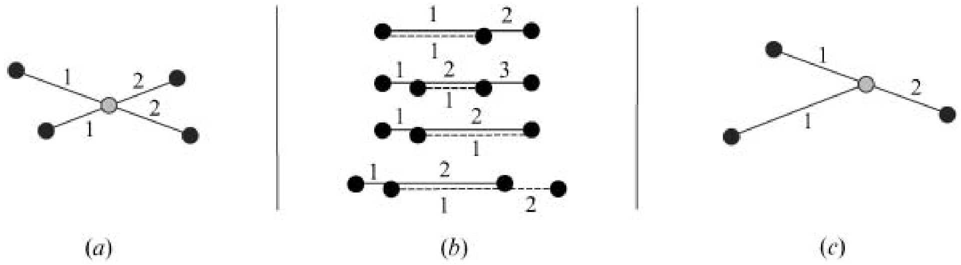


Figure 7. Splitting of two intersecting segments (a), two partially coinciding segments (without symmetric counterparts) (b), and a segment whose interior is touched by another segment (c). Digits indicate part numbers of segments after splitting.

halfsegments) (Figure 7c). This splitting strategy is numerically stable and thus feasible from an implementation standpoint since we assume numerically robust geometric computation that ensures topological consistency of intersection operations. Intersecting and touching points can then be *exactly* computed, lead to representable points, and are thus precisely located on the intersecting or touching segments.

However, as indicated before, the splitting of segments entails some algorithmic effort. On the one hand, we want to keep the halfsegment sequences of the *line2D* and *region2D* objects unchanged, since their update is expensive and only temporarily needed for the plane sweep. On the other hand, the splitting of halfsegments has an effect on these sequences. As a compromise, for each *line2D* or *region2D* object, we maintain its "static" representation, and the halfsegments obtained by the splitting process are stored in an additional "dynamic" halfsegment sequence. The dynamic part is also organized as an AVL tree which is embedded in an array and whose elements are linked in sequence order. Assuming that k splitting points are detected during the plane sweep, we need $O(k)$ additional space, and to insert them requires $O(k\log k)$ time. After the plane sweep, this additional space is released.

To illustrate the splitting process in more detail, we consider two *region2D* objects R_1 and R_2. In general, we have to deal with the three cases in Figure 7. We first consider the case that the plane sweep detects an intersection. This leads to a situation like in Figure 8a. The two static and the two dynamic halfsegment sequences of R_1 and R_2 are shown in Table 1. Together they form the *event point schedule* of the plane sweep and are processed by a parallel object traversal. Before the current position of the sweep line (indicated by the vertical dashed line in Figure 8), the parallel object traversal has already processed the attributed halfsegments $\left(h^l_{s_1}, t\right)$, $\left(h^l_{s_2}, f\right)$, $\left(h^l_{v_1}, t\right)$, and $\left(h^l_{v_2}, f\right)$ in this order. At the current position of the sweep line, the parallel object traversal encounters the halfsegments $\left(h^l_{s_3}, t\right)$ and $\left(h^l_{s_4}, f\right)$. For each left halfsegment visited, the corresponding segment is inserted into the sweep line status according to the y-coordinate of its dominating point and checked for intersections with its direct upper and lower neighbors. In our example, the insertion of s_4 leads to an intersection with its upper neighbor v_1. This requires *segment splitting*; we split v_1 into the two segments $v_{1,1}$ and $v_{1,2}$ and s_4 into the two segments $s_{4,1}$ and $s_{4,2}$. In the sweep line status, we have to replace v_1 by $v_{1,1}$ and s_4 by $s_{4,1}$ (Figure 8b). The new halfsegments $\left(h^r_{s_{4,1}}, f\right)$, $\left(h^l_{s_{4,2}}, f\right)$, and $\left(h^r_{s_{4,2}}, f\right)$

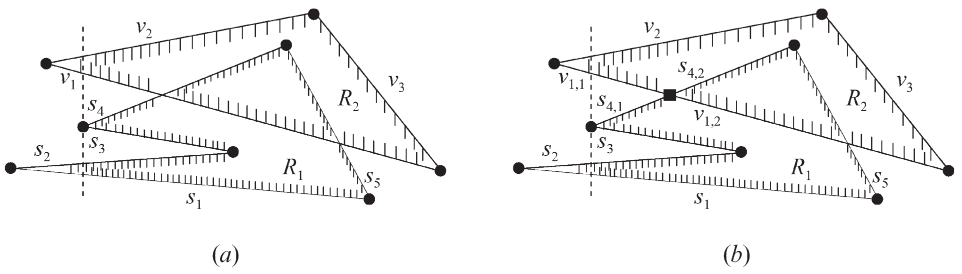


Figure 8. Sweep line status before the splitting (s_4 to be inserted) (a) and after the splitting (b). The vertical dashed line indicates the current position of the sweep line.

are inserted into the dynamic halfsegment sequence of R_1. Into the dynamic halfsegment sequence of R_2, we insert the halfsegments $\left(h^r_{v_{1,1}}, t\right)$, $\left(h^l_{v_{1,2}}, t\right)$, and $\left(h^r_{v_{1,2}}, t\right)$. We need not store the two halfsegments $\left(h^l_{s_{4,1}}, f\right)$ and $\left(h^l_{v_{1,1}}, t\right)$ since they refer to the "past" and have already been processed.

On purpose we have accepted a little inconsistency in this procedure, which can fortunately be easily controlled. Since, for the duration of the plane sweep, s_4 (v_1) has been replaced by $s_{4,1}$ ($v_{1,1}$) and $s_{4,2}$ ($v_{1,2}$), the problem is that the static sequence part of R_1 (R_2) still includes the now invalid halfsegment $\left(h^r_{s_4}, f\right)$ $\left(\left(h^r_{v_1}, t\right)\right)$, which we may not delete (see Figure 8b). However, this is not a problem due to the following observation. If we find a right halfsegment in the dynamic sequence part of a *region2D* object, we know that it stems from splitting a longer, collinear, right halfsegment that is stored in the static sequence part of this object, has the same right end point, and has to be skipped during the parallel object traversal.

For the second and third case in Figure 7, the procedure is the same but more splits can occur. In case of overlapping, collinear segments, we obtain up to six new halfsegments. In case of a touching situation, the segment whose interior is touched is split.

5. The Exploration Phase for Collecting Topological Information

For a given scene of two spatial objects, the goal of the exploration phase is to discover appropriate topological information that is characteristic and unique for

Table 1. Static and dynamic halfsegment sequences of the regions R_1 and R_2 in Figure 8.

R_1 *dynamic* sequence part	$\left(h^r_{s_{4,1}}, f\right)$	$\left(h^l_{s_{4,2}}, f\right)$	$\left(h^r_{s_{4,2}}, f\right)$		
R_1 *static* sequence part	$\left(h^l_{s_1}, t\right)$	$\left(h^l_{s_2}, f\right)$	$\left(h^l_{s_3}, t\right)$	$\left(h^l_{s_4}, f\right)$	$\left(h^r_{s_3}, t\right)$
	$\left(h^r_{s_2}, f\right)$	$\left(h^r_{s_4}, f\right)$	$\left(h^l_{s_5}, f\right)$	$\left(h^r_{s_5}, f\right)$	$\left(h^r_{s_1}, t\right)$
R_2 *static* sequence part	$\left(h^l_{v_1}, t\right)$	$\left(h^l_{v_2}, f\right)$	$\left(h^r_{v_2}, f\right)$	$\left(h^l_{v_3}, f\right)$	$\left(h^r_{v_3}, f\right)$
	$\left(h^r_{v_1}, t\right)$				
R_2 *dynamic* sequence part	$\left(h^r_{v_{1,1}}, t\right)$	$\left(h^l_{v_{1,2}}, t\right)$	$\left(h^r_{v_{1,2}}, t\right)$		

this scene and that is suitable both for verification queries (query type 1) and determination queries (query type 2). Our approach is to scan such a scene from left to right by a plane sweep and to collect topological data during this traversal that later in the evaluation phase helps us confirm, deny, or derive the topological relationship between both objects. From both phases, the exploration phase is the computationally expensive one since topological information has to be explicitly derived by geometric computation.

Our research shows that it is unfavorable to aim at designing a *universal* exploration algorithm that covers all combinations of spatial data types. This has three main reasons. First, each of the data types *point2D*, *line2D*, and *region2D* has very type-specific, well known properties that are different from each other (like different dimensionality). Second, for each combination of spatial data types, the topological information we have to collect is very specific and especially different from all other type combinations. Third, the topological information we collect about each spatial data type is different in different type combinations. Therefore, using the basic algorithmic concepts of Section 4, in this section, we present exploration algorithms for all combinations of complex spatial data types. Between two objects of types *point2D*, *line2D*, or *region2D*, we have to distinguish six different cases, if we assume that the first operand has an equal or lower dimension than the second operand[2]. All algorithms except for the *point2D*/*point2D* case require the plane sweep technique.

Depending on the types of spatial objects involved, a boolean vector v_F consisting of a special set of "topological flags" is assigned to *each* object *F*. We call it a *topological feature vector*. Its flags are all initialized to *false*. Once certain topological information about an object has been discovered, the corresponding flag of its topological feature vector is set to *true*. Topological flags represent topological facts of interest. We obtain them by analyzing the topological intersections between the exterior, interior and boundary of a complex spatial object with the corresponding components of another complex spatial object according to the 9-intersection matrix (Figure 2(a)). That is, the concept is to map the matrix elements of the 9-intersection matrix, which are predicates, into a topological feature vector and to eliminate redundancy given by symmetric matrix elements. For all type combinations, we aim at minimizing the number of topological flags of both spatial argument objects. In symmetric cases, only the first object gets the flag. The topological feature vectors are later used in the evaluation phase for predicate matching. Hence, the selection of topological flags is highly motivated by the requirements of the evaluation phase (Section 6, Praing and Schneider (2008)).

Let $P(F)$ be the *set* of all points of a *point2D* object F, $H(F)$ be the *set* of all (attributed) halfsegments [including those resulting from our splitting strategy] of a *line2D* (*region2D*) object F, and $B(F)$ be the set of all boundary points of a *line2D* object F. For $f \in H(F)$, let $f.s$ denote its segment component, and, if F is a *region2D* object, let $f.ia$ denote its attribute component. The definitions in the next subsections make use of the operations on robust geometric primitives and halfsegments (Section 3).

[2] If, in the determination query case, a predicate $p(A, B)$ has to be processed, for which the dimension of object A is higher than the dimension of object B, we process the converse predicate $p^{conv}(B, A)$ where p^{conv} has the transpose of the 9-intersection matrix (see Figure 2(a)) of p.

5.1 *The Exploration Algorithm for the point2D/point2D Case*

The first and simplest case considers the exploration of topological information for two *point2D* objects F and G. Here, the topological facts of interest are whether (i) both objects have a point in common and (ii) F (G) contains a point that is not part of G (F). Hence, both topological feature vectors v_F and v_G get the flag *poi_disjoint*. But only v_F in addition gets the flag *poi_shared* since the sharing of a point is symmetric. We obtain (the symbol "$:\Leftrightarrow$" means "equivalent by definition"):

Definition 1 Let F, $G \in point2D$, and let v_F and v_G be their topological feature vectors. Then

(i) $v_F[poi\_shared] \quad :\Leftrightarrow \exists f \in P(F) \exists g \in P(G) : f = g$
(ii) $v_F[poi\_disjoint] \quad :\Leftrightarrow \exists f \in P(F) \forall g \in P(G) : f \neq g$
(iii) $v_G[poi\_disjoint] \quad :\Leftrightarrow \exists g \in P(G) \forall f \in P(F) : f \neq g$

For the computation of the topological feature vectors, a plane sweep is not needed; a parallel traversal suffices, as the algorithm in Figure 9 shows. The while-loop terminates if either the end of one of the objects has been reached or all topological flags have been set to *true* (lines 7 and 8). In the worst case, the loop has to be traversed $l+m$ times where l (m) is the number of points of the first (second) *point2D* object. Since the body of the while-loop requires constant time, the overall time complexity is $O(l+m)$.

5.2 *The Exploration Algorithm for the point2D/line2D Case*

In case of a *point2D* object F and a *line2D* object G, at the lowest level of detail, we are interested in the possible relative positions between the individual points of F and the halfsegments of G. This requires a precise understanding of the definition of the boundary of a *line2D* object, as it has been given in Schneider and Behr (2006). It follows from this definition that each boundary point of G is an endpoint of a (half)segment of G and that this does not necessarily hold vice versa, as Figure 10(a) indicates. The black segment endpoints belong to the boundary of G, since exactly one segment emanates from each of them. Intuitively, they "bound" G. In contrast, the grey segment endpoints belong to the interior of G, since several segments emanate from each of them. Intuitively, they are "connector points" between different segments of G.

The following argumentation leads to the needed topological flags for F and G. Seen from the perspective of F, we can distinguish three cases since the boundary of F is empty (Schneider and Behr 2006) and the interior of F can interact with the exterior, interior, or boundary of G. First, (the interior of) a point f of F can be disjoint from G (flag *poi_disjoint*). Second, a point f can lie in the interior of a

```
01 algorithm ExplorePoint2DPoint2D
02 input:  point2D objects F and G, topological feature
03         vectors $v_F$ and $v_G$ initialized with false
04 output: updated vectors $v_F$ and $v_G$
05 begin
06   select_first(F, G, object, status);
07   while status = end_of_none and not ($v_F$[poi_disjoint]
08         and $v_G$[poi_disjoint] and $v_F$[poi_shared]) do
09     if object = first then $v_F$[poi_disjoint] := true
10     else if object = second then $v_G$[poi_disjoint] := true
11     else /* object = both */
12       $v_F$[poi_shared] := true;
13     endif
14     select_next(F, G, object, status);
15   endwhile;
16   if status = end_of_first then $v_G$[poi_disjoint] := true
17   else if status = end_of_second then
18     $v_F$[poi_disjoint] := true
19   endif
20 end ExplorePoint2DPoint2D
```

Figure 9. Algorithm for computing the topological feature vectors for two *point2D* objects

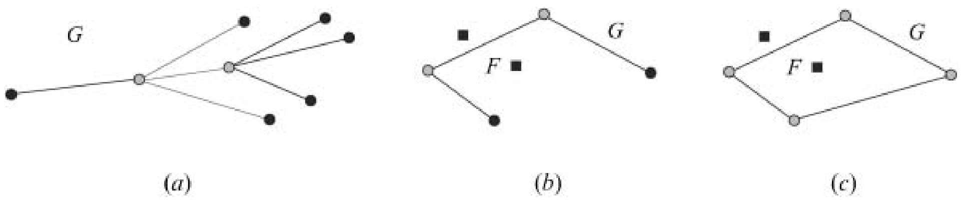


Figure 10. Boundary points (in black) and connector points (in grey) of a *line2D* object (a), a scenario where a boundary point of a *line2D* object exists that is unequal to all points of a *point2D* object (b), a scenario where this is not the case (c).

segment of *G* (flag *poi_on_interior*). This includes an endpoint of such a segment, if the endpoint is a connector point of *G*. Third, a point *f* can be equal to a boundary point of *G* (flag *poi_on_bound*). Seen from the perspective of *G*, we can distinguish four cases since the boundary and the interior of *G* can interact with the interior and exterior of *F*. First, *G* can contain a boundary point that is unequal to all points in *F* (flag *bound_poi_disjoint*). Second, *G* can have a boundary point that is equal to a point in *F*. But the flag *poi_on_bound* already takes care of this situation. Third, the interior of a segment of *G* (including connector points) can comprehend a point of *F*. This situation is already covered by the flag *poi_on_interior*. Fourth, the interior of a segment of *G* can be part of the exterior of *F*. This is always true since a segment of *G* represents an infinite point set that cannot be covered by the finite number of points in *F*. Hence, we need not handle this as a special situation. Formally, we define the semantics of the topological flags as follows:

Definition 2 Let $F \in point2D$, $G \in line2D$, and v_F and v_G be their topological feature vectors. Then

(i) $v_F[poi\_disjoint] \quad :\Leftrightarrow \exists f \in P(F) \forall g \in H(G) : \neg on(f, g.s)$

(ii) $v_F[poi\_on\_interior] \quad :\Leftrightarrow \exists f \in P(F) \exists g \in H(G) \forall b \in B(G) : on(f, g.s) \wedge f \neq b$

(iii) $v_F[poi\_on\_bound] \quad :\Leftrightarrow \exists f \in P(F) \exists g \in B(G) : f = g$

(iv) $v_G[bound\_poi\_disjoint] \quad :\Leftrightarrow \exists g \in B(G) \forall f \in P(F) : f \neq g$

Our algorithm for computing the topological information for this case is shown in Figure 11. The while-loop is executed until the end of the *line2D* object (line 9) and as long as not all topological flags have been set to *true* (lines 10 to 11). The operations *select_first* and *select_next* compare a point and a halfsegment according to the order relation defined in Section 3 in order to determine the next element(s) to be processed. If only a point has to be processed (line 12), we know that it does not coincide with an endpoint of a segment of *G* and hence not with a boundary point of *G*. But we have to check whether the point lies in the interior of a segment in the sweep line status structure *S*. This is done by the search operation *poi_in_seg* on *S* (line 13). If this is not the case, the point must be located outside the segment (line 14). If only a halfsegment *h* has to be processed (line 15), its segment component is inserted into (deleted from) *S* if *h* is a left (right) halfsegment (line 17). We also have to test if the dominating point of *h*, say *v*, is a boundary point of *G*. This is the case if *v* is unequal to the previous dominating point stored in the variable *last_dp* (line 18) and if the operation *look_ahead* finds out that *v* does also not coincide with the dominating point of the next halfsegment (lines 19 to 20). In case that a point *v* of *F* is equal to a dominating point of a halfsegment *h* in *G* (line 23), we know that *v* has never been visited before and that it is an end point of the segment component of *h*.

```
01 algorithm ExplorePoint2DPoint2D
02 input:   point2D objects F and line2D object G,
03          topological feature vectors $v_F$ and $v_G$
04          initialized with false
05 output: updated vectors $v_F$ and $v_G$
06 begin
07   S := new_sweep(); last_dp := $\epsilon$;
08   select_first(F, G, object, status);
09   while status $\neq$ end_of_second and status $\neq$ end_of_both and
10         not ($v_F$[poi_disjoint] and $v_F$[poi_on_interior] and
11         $v_F$[poi_on_bound]) and $v_G$[bound_poi_disjoint] do
12     if object = first then p := get_event(F);
13       if poi_in_seg(S, p) then $v_F$[poi_on_interior] := true
14       else $v_F$[poi_disjoint] := true endif
15     else if object = second then
16       h := get_event(G); /* h = (s, d) */
17       if d then add_left(S, s) else del_right(S, s) endif;
18       if dp(h) $\neq$ last_dp then last_dp := dp(h);
19         if not look_ahead(h, G) then
20           $v_G$[bound_poi_disjoint] := true
21         endif
22       endif
23     else /* object = both */
24       h := get_event(G); /* h = (s, d) */
25       if d then add_left(S, s) else del_right(S, s) endif;
26       last_dp := dp(h);
27       if look_ahead(h, G) then
28         $v_F$[poi_on_interior] := true
29       else $v_F$[poi_on_bound] := true endif
30     endif
31     select_next(F, G, object, status);
32 endwhile;
33 if status = end_of_second then
34   $v_F$[poi_disjoint] := true
35   endif
36 end ExplorePoint2DLine2D.
```

Figure 11. Algorithm for computing the topological feature vectors for a *point2D* object and a *line2D* object

Besides the update of S (line 25), it remains to decide whether v is an interior point (line 28) or a boundary point (line 29) of h. For this, we look ahead (line 27) to see whether the next halfsegment's dominating point is equal to v or not.

If l is the number of points of F and m is the number of halfsegments of G, the while-loop is executed at most $l+m$ times. The insertion of a left halfsegment into and the removal of a right halfsegment from the sweep line status needs $O(\log m)$ time. The check whether a point lies within or outside a segment (predicate *poi_in_seg*) also requires $O(\log m)$ time. Altogether, the worst time complexity is $O((l+m)\log m)$. The while-loop has to be executed at least m times for processing the entire *line2D* object in order to find out if a boundary point exists that is unequal to all points of the *point2D* object (Figures 10(b) and (c)).

5.3 *The Exploration Algorithm for the point2D/region2D Case*

In case of a *point2D* object F and a *region2D* object G, the situation is simpler than in the previous case. Seen from the perspective of F, we can again distinguish three cases between (the interior of) a point of F and the exterior, interior, or boundary of G. First, a point of F lies either inside G (flag *poi_inside*), on the boundary of G (flag *poi_on_bound*), or outside of region G (flag *poi_outside*). Seen from the perspective of G, we can distinguish four cases between the boundary and the interior of G with the interior and exterior of F. The intersection of the boundary (interior) of G with the interior of F implies that a point of F is located on the boundary (inside) of G. This situation is already covered by the flag *poi_on_bound* (*poi_inside*). The intersection of the boundary (interior) of G with the exterior of F is always true, since F as a finite point set cannot cover G's boundary segments (interior) representing an infinite point set. More formally, we define the semantics of the topological flags as follows (we assume *poiInRegion* to be a predicate which checks whether a point lies inside a *region2D* object):

Definition 3 Let $F \in point2D$, $G \in region2D$, and v_F and v_G be their topological feature vectors. Then

(i) $v_F[poi\_inside] \quad :\Leftrightarrow \exists f \in P(E) : poiInRegion(f, G)$

(ii) $v_F[poi\_on\_bound] :\Leftrightarrow \exists f \in P(F) \exists g \in H(G) : on(f, g.s)$

(iii) $v_F[poi\_outside] \quad :\Leftrightarrow \exists f \in P(F) \forall g \in H(G) : \neg poiInRegion(f, G) \wedge \neg on(f, g.s)$

```
01 algorithm ExplorePoint2DRegion2D
02 input:  point2D object F and region2D object G,
03         topological feature vectors $v_F$ and $v_G$
04         initialized with false
05 output: updated vectors $v_F$ and $v_G$
06 begin
07   S := new_sweep();
08   select_first(F, G, object, status);
09   while status = end_of_none and not ($v_F$[poi_inside]
10         and $v_F$[poi_on_bound] and $v_F$[poi_outside]) do
11     if object = first then p := get_event(F);
12       if poi_on_seg(S, p) then $v_F$[poi_on_bound] := true
13       else pred_of_p(S, p);
14         if current_exists(S) then ia := get_attr(S);
15           if ia then $v_F$[poi_inside] := true
16           else $v_F$[poi_outside] := true endif
17         else $v_F$[poi_outside] := true
18         endif
19       endif
20     else h := get_event(G); ia := get_attr(G); /* h = (s, d) */
21       if d then add_left(S, s); set_attr(S, ia)
22       else del_right(S, s) endif;
23       if object = both then $v_F$[poi_on_bound] := true endif
24     endif
25     select_next(F, G, object, status);
26   endwhile;
27   if status = end_of_second then
28     $v_F$[poi_outside] := true
29   endif
30 end ExplorePoint2DRegion2D.
```

Figure 12. Algorithm for computing the topological feature vectors for a *point2D* object and a *region2D* object

We see that v_G is not needed. The algorithm for this case is shown in Figure 12. The while-loop is executed as long as none of the two objects has been processed (line 9) and as long as not all topological flags have been set to *true* (lines 9 to 10). If only a point has to be processed (line 11), we must check its location. The first case is that it lies on a boundary segment; this is checked by the sweep line status predicate *poi_on_seg* (line 12). Otherwise, it must be located inside or outside of G. We use the operation *pred_of_p* (line 13) to determine the nearest segment in the sweep line status whose intersection point with the sweep line has a lower y-coordinate than the y-coordinate of the point. The predicate *current_exists* checks whether such a segment exists (line 14). If this is not the case, the point must be outside of G (line 17). Otherwise, we ask for the information whether the interior of G is above the segment (line 14). We can then derive whether the point is inside or outside the region (lines 15 to 16). If only a halfsegment h has to be processed or a point v of F is equal to a dominating point of a halfsegment h in G (line 20), h's segment component is inserted into (deleted from) S if h is a left (right) halfsegment (line 20 to 21). In case of a left halfsegment, in addition, the information whether the interior of G is above the segment is stored in the sweep line status (line 21). If v and the dominating point of h coincide, we know that the point is on the boundary of G (line 23).

If l is the number of points of F and m is the number of halfsegments of G, the while-loop is executed at most $l+m$ times. Each of the sweep line status operations *add_left*, *del_right*, *poi_on_seg*, *pred_of_p*, *current_exists*, *get_attr*, and *set_attr* needs $O(\log m)$ time. The total worst time complexity is $O((l+m)\log m)$.

5.4 *The Exploration Algorithm for the line2D/line2D Case*

We now consider the exploration algorithm for two *line2D* objects F and G. Seen from the perspective of F, we can differentiate six cases between the interior and boundary of F and the interior, boundary, and exterior of G. First, the interiors of two segments of F and G can partially or completely coincide (flag *seg_shared*). Second, if a segment of F does not partially or completely coincide with any segment of G, we register this in the flag *seg_unshared*. Third, we set the flag *interior_poi_shared* if two segments intersect in a single point that does not belong to the boundaries of F or G. Fourth, a boundary endpoint of a segment of F can be located in the interior of a segment (including connector points) of G (flag *bound_on_interior*). Fifth, both objects F and G can share a boundary point (flag *bound_shared*). Sixth, if a boundary endpoint of a segment of F lies outside of all segments of G, we set the flag *bound_disjoint*. Seen from the perspective of G, we can identify the same cases. But due to the symmetry of three of the six topological cases, we do not need all flags for *G*. For example, if a segment of *F* partially coincides with a segment of *G*, this also holds vice versa. Hence, it is sufficient to introduce the flags *seg_unshared*, *bound_on_interior*, and *bound_disjoint* for *G*. We define the semantics of the topological flags as follows:

Definition 4 Let $F, G \in line2D$, and let v_F and v_G be their topological feature vectors. Then

(i) $v_F[seg\_shared]$ $:\Leftrightarrow \exists f \in H(F) \exists\, g \in H(G) : segIntersect(f.s,\ g.s)$

(ii) $v_F[interior\_poi\_shared]$ $:\Leftrightarrow \exists f \in H(F) \exists g \in H(G) \forall p \in B(F) \cup B(G) :$ $poiIntersect(f.s,\ g.s) \wedge poiIntersection(f.s,\ g.s) \neq p$

(iii) $v_F[seg\_unshared]$ $:\Leftrightarrow \exists f \in H(F) \forall g \in H(G) : \neg segIntersect(f.s,\ g.s)$

(iv) $v_F[bound\_on\_interior]$ $:\Leftrightarrow \exists f \in H(F) \exists g. \in H(G) \exists p \in B(F) \backslash B(G) :$ $poiIntersection(f.s,\ g.s) = p$

(v) $v_F[bound\_shared]$ $:\Leftrightarrow \exists p \in B(F) \exists q \in B(G) : p = q$

(vi) $v_F[bound\_disjoint]$ $:\Leftrightarrow \exists p \in B(F) \forall g \in H(G) : \neg on(p,\ g.s)$

(vii) $v_G[seg\_unshared]$ $:\Leftrightarrow \exists g \in H(G) \forall f \in H(F) : \neg segIntersect(f.s,\ g.s)$

(viii) $v_G[bound\_on\_interior]$ $:\Leftrightarrow \exists f \in H(F) \exists g \in H(G) \exists p \in B(G) \backslash B(F) :$ $poiIntersection(f.s,\ g.s) = p$

(ix) $v_G[bound\_disjoint]$ $:\Leftrightarrow \exists q \in B(G) \forall f \in H(F) : \neg on(q,\ f.s)$

The exploration algorithm for this case is given in Figure 13. The while-loop is executed until both objects have been processed (line 9) and as long as not all topological flags have been set to *true* (lines 9 to 13). If a single left (right) halfsegment of *F* (line 14) has to be processed (the same for *G* (line 36)), we insert it into (delete it from) the sweep line status (lines 15 and 16). The deletion of a single right halfsegment further indicates that it is not shared by *G* (line 16). If the current dominating point, say *v*, is unequal to the previous dominating point of *F* (line 17) and if the operation look_ahead finds out that *v* is also unequal to the dominating point of the next halfsegment of *F* (line 18), *v* must be a boundary point of *F* (line 19). In this case, we perform three checks. First, if *v* coincides with the current boundary point in *G*, both objects share a part of their boundary (lines 20 to 21). Second, otherwise, if *v* is equal to the current dominating point, say *w*, in *G*, *w* must be an interior point of *G*, and the boundary of *F* and the interior of *G* share a point (lines 22 to 23). Third, otherwise, if *v* is different from the dominating point of the next halfsegment in *G*, *F* contains a boundary point that is disjoint from *G* (lines 24 to 25). If *v* has not been identified as a boundary point in the previous step (line 29), it must be an interior point of *F*. In this case, we check whether it coincides with the current boundary point in *G* (lines 30 to 31) or whether it is also an interior point in *G* (lines 32 to 33). If a halfsegment belongs to both objects (line 38), we can conclude that it is shared by them (line 39). Depending on whether it is a left or right halfsegment, it is inserted into or deleted from the sweepline status (line 40). Lines 41 to 45 (46 to 50) test whether the dominating point *v* of the halfsegment is a boundary point of *F* (*G*). Afterwards, we check whether *v* is a boundary point of both objects (lines 51 to 52). If this is not the case, we examine whether one of them is a boundary point and the other one is an interior point (lines 54 to 59). Lines 62 to 66 handle the case that exactly one of the two halfsegment sequences is exhausted.

Let *l* (*m*) be the number of halfsegments of *F* (*G*). Segments of both objects can intersect or partially coincide (Figure 7), and we handle these topological situations with the splitting strategy described in Section 4.3. If, due to splitting, *k* is the total

```
01 algorithm ExploreLine2DRegion2D
02 input:  line2D objects F and G, topological feature
03            vectors $v_F$ and $v_G$ initialized with false
04 output: updated vectors $v_F$ and $v_G$
05 begin
06    S := new_sweep(); last_dp_in_F := $\epsilon$; last_dp_in_G := $\epsilon$;
07    last_bound_in_F := $\epsilon$; last_bound_in_G := $\epsilon$;
08    select_first(F, G, object, status);
09    while status $\neq$ end_of_both and not ($v_F$[seg_shared]
10          and $v_F$[interior_poi_shared] and $v_F$[seg_unshared]
11          and $v_F$[bound_on_interior] and $v_F$[bound_shared]
12          and $v_F$[bound_disjoint] and $v_G$[bound_disjoint]
13          and $v_G$[bound_on_interior] and $v_G$[seg_unshared]) do
14       if object = first then h := get_event(F); /* h = (s, d) */
15          if d then add_left(S, s)
16          else del_right(S, s); $v_F$[seg_unshared] := true endif;
17          if dp(h) $\neq$ last_dp_in_F then last_dp_in_F := dp(h);
18          if not look_ahead(h, F) then
19             last_bound_in_F := dp(h);
20             if last_bound_in_F = last_bound_in_G then
21                $v_F$[bound_shared] := true
22             else if last_bound_in_F = last_dp_in_G then
23                $v_F$[bound_on_interior] := true
24             else if not look_ahead(h, G) then
25                $v_F$[bound_disjoint] := true
26             endif
27          endif
28       endif;
29       if dp(h) $\neq$ last_bound_in_F then
30          if dp(h) = last_bound_in_G then
31             $v_G$[bound_on_interior] := true
32          else if dp(h) = last_dp_in_G then
33             $v_F$[interior_poi_shared] := true
34          endif
35       endif
36       else if object = second then h := get_event(G);
37          .../* like lines 15 to 35 with F and G swapped */
38       else /* object = both */
39          h := get_event(F); $v_F$[seg_shared] := true;
40          if d then add_left(S, s) else del_right(S, s) endif;
41          if dp(h) $\neq$ last_dp_in_F then last_dp_in_F := dp(h);
42             if not look_ahead(h, F) then
43                last_bound_in_F := dp(h)
44             endif
45          endif;
46          if dp(h) $\neq$ last_dp_in_G then last_dp_in_G := dp(h);
47             if not look_ahead(h, G) then
48                last_bound_in_G := dp(h)
49             endif
50          endif;
51          if last_bound_in_F = last_bound_in_G then
52             $v_F$[bound_shared] := true
53          else
54             if last_bound_in_F = last_dp_in_G then
55                $v_F$[bound_on_interior] := true
56             endif;
57             if last_bound_in_G = last_dp_in_F then
58                $v_G$[bound_on_interior] := true
59             endif
60          endif
61       endif;
62       if status = end_of_first then
63          $v_G$[seg_unshared] := true;
64       else if status = end_of_second then
65          $v_F$[seg_unshared] := true;
66       endif
67       select_next(F, G, object, status);
68    endwhile;
69 end ExploreLine2DLine2D.
```

Figure 13. Algorithm for computing the topological feature vectors for two *line2D* objects

number of additional halfsegments stored in the dynamic halfsegment sequences of both objects, the while-loop is executed at most $l+m+k$ times. The only operations needed on the sweep line status are *add_left* and *del_right* for inserting and deleting halfsegments; they require $O(\log(l+m+k))$ time each. No special predicates have to be deployed for discovering topological information. Due to the splitting strategy, all dominating end points either are already endpoints of existing segments or become endpoints of newly created segments. The operation *look_ahead* needs constant time. In total, the algorithm requires $O((l+m+k)\log(l+m+k))$ time and $O(l+m+k)$ space.

5.5 *The Exploration Algorithm for the line2D|region2D Case*

Next, we describe the exploration algorithm for a *line2D* object *F* and a *region2D* object *G*. Seen from the perspective of *F*, we can distinguish six cases between the interior and boundary of *F* and the interior, boundary, and exterior of *G*. First, the intersection of the interiors of *F* and *G* means that a segment of *F* lies in *G* (flag *seg_inside*). Second, the interior of a segment of *F* intersects with a boundary segment of *G* if either both segments partially or fully coincide (flag *seg_shared*), or if they properly intersect in a single point (flag *poi_shared*). Third, the interior of a segment of *F* intersects with the exterior of *G* if the segment is disjoint from *G* (flag *seg_outside*). Fourth, a boundary point of *F* intersects the interior of *G* if the boundary point lies inside of *G* (flag *bound_inside*). Fifth, if it lies on the boundary of *G*, we set the flag *bound_shared*. Sixth, if it lies outside of *G*, we set the flag *bound_disjoint*.

Seen from the perspective of *G*, we can differentiate the same six cases as before and obtain most of the topological flags as before. First, if the interiors of *G* and *F* intersect, a segment of *F* must partially or totally lie in *G* (already covered by flag *seg_inside*). Second, if the interior of G and the boundary of F intersect, the boundary point of a segment of F must be located in G (already covered by flag *bound_inside*). Third, the case that the interior of G intersects the exterior of F is always true due to the different dimensionality of both objects; hence, we do not need a flag. Fourth, if the boundary of G intersects the interior of F, a segment of F must partially or fully coincide with a boundary segment of G (already covered by flag *seg_shared*). Fifth, if the boundary of G intersects the boundary of F, a boundary point of a segment of F must lie on a boundary segment of G (already covered by flag *bound_shared*). Sixth, if the boundary of G intersects the exterior of F, a boundary segment of G must be disjoint from F (new flag *seg_unshared*). More formally, we define the semantics of the topological flags as follows:

Definition 5 Let $F \in line2D$, $G \in region2D$, and v_F and v_G be their topological feature vectors. Then

(i) $v_F[seg\_inside] :\Leftrightarrow \exists f \in H(F) \forall g \in H(G) : \neg segIntersect(f.s, g.s) \wedge segInRegion(f.s, G)$

(ii) $v_F[seg\_shared] :\Leftrightarrow \exists f \in H(F) \exists g \in H(G) : segIntersect(f.s, g.s)$

(iii) $v_F[seg\_outside] :\Leftrightarrow \exists f \in H(F) \forall g \in H(G) : \neg segIntersect(f.s, g.s) \wedge \neg segInRegion(f.s, G)$

(iv) $v_F[poi\_shared] :\Leftrightarrow \exists f \in H(F) \exists g \in H(G) : poiIntersect(f.s, g.s) \wedge poiIntersection(f.s, g.s) \notin B(F)$

(v) $v_F[bound\_inside] :\Leftrightarrow \exists f \in H(F) : poiInRegion(dp(f), G) \wedge dp(f) \in B(F)$

(vi) $v_F[bound\_shared] :\Leftrightarrow \exists f \in H(F) \exists g \in H(G) : poiIntersect(f.s, g.s) \wedge poiIntersection(f.s, g.s) \in B(F)$

(vii) $v_F[bound\_disjoint] :\Leftrightarrow \exists f \in H(F) \forall g \in H(G) : \neg poiInRegion(dp(f), G) \wedge dp(f) \in B(F) \wedge \neg on(dp(f), g.s)$

(viii) $v_G[seg\_unshared] :\Leftrightarrow \exists g \in H(G) \forall f \in H(F) : \neg segIntersect(f.s, g.s)$

The operation *segInRegion* is assumed to check whether a segment is located inside a region; it is an imaginary predicate and not implemented as a robust geometric primitive.

The exploration algorithm for this case is given in Figure 14. The while-loop is executed until at least the first object has been processed (line 10) and as long as not all topological flags have been set to *true* (lines 11 to 14). In case that we only encounter a halfsegment h of F (line 15), we insert its segment component s into the sweep line status if it is a left halfsegment (line 16). If it is a right halfsegment, we find out whether h is located inside or outside of G (lines 18 to 23). We know that it cannot coincide with a boundary segment of G, since this is another case. The predicate *pred_exists* checks whether s has a predecessor in the sweep line status (line 18); it ignores segments in the sweep line status that stem from F. If this is not the case (line 22), s must lie outside of G. Otherwise, we check the upper overlap number of s's predecessor (line 19). The overlap number 1 indicates that s lies inside G (line 20); otherwise, it is outside of G (line 21). After this check, we remove s from the sweep line status (line 23). Next we test whether the dominating point of h is a boundary point of F (line 26) by using the predicate *look_ahead*. If this is the case, we determine whether this point is shared by G (lines 28 to 30) or whether this point is located inside or outside of G (lines 31 to 38). Last, if the dominating point turns out not to be a boundary point of F, we check whether it is an interior point that shares a boundary point with G (lines 40 to 43). In case that we only obtain a halfsegment h of G (line 44), we insert its segment component s into the sweep line

```
01 algorithm ExploreLine2DRegion2D
02 input:   line2D object F and region2D object G,
03          topological feature vectors $v_F$ and $v_G$
04          initialized with false
05 output:  updated vectors $v_F$ and $v_G$
06 begin
07   S := new_sweep(); last_dp_in_F := $\epsilon$; last_dp_in_G := $\epsilon$;
08   last_bound_in_F := $\epsilon$;
09   select_first(F, G, object, status);
10   while status $\neq$ end_of_first and status $\neq$ end_of_both
11         and not ($v_F$[seg_inside] and $v_F$[seg_shared]
12         and $v_F$[seg_outside] and $v_F$[poi_shared]
13         and $v_F$[bound_inside] and $v_F$[bound_shared]
14         and $v_G$[bound_disjoint] and $v_G$[seg_unshared]) do
15     if object = first then h := get_event(F); /* h = (s, d) */
16       if d then add_left(S, s)
17       else
18         if pred_exists(S, s) then
19           ($m_p$/$n_p$) := get_pred_attr(S, s);
20           if $n_p$ = 1 then $v_F$[seg_inside] := true
21           else $v_F$[seg_outside] := true endif
22         else $v_F$[seg_outside] := true endif;
23         del_right(S, s);
24       endif;
25       if dp(h) $\neq$ last_dp_in_F then last_dp_in_F := dp(h);
26         if not look_ahead(h, F) then
27           last_bound_in_F := dp(h);
28           if last_bound_in_F = last_dp_in_G
29             or look_ahead(h, G) then
30             $v_F$[bound_shared] := true
31           else
32             if pred_exists(S, s) then
33               ($m_p$/$n_p$) := get_pred_attr(S, s);
34               if $n_p$ = 1 then $v_F$[bound_inside] := true
35               else $v_F$[bound_outside] := true endif
36             else $v_F$[bound_outside] := true endif
37           endif
38         endif
39       endif;
40       if dp(h) $\neq$ last_bound_in_F and
41         (dp(h) = last_dp_in_G or look_ahead(h, G)) then
42         $v_F$[poi_shared] := true
43       endif
44     else if object = second then
45       h := get_event(G); ia := get_attr(G);
46       if d then add_left(S, s); set_attr(S, ia)
47       else del_right(S, s); $v_G$[seg_unshared] := true endif;
48       if dp(h) $\neq$ last_dp_in_G then
49         last_dp_in_G := dp(h) endif;
50     else /* object = both */ $v_F$[seg_shared] := true;
51       h := get_event(G); ia := get_attr(G);
52       if d then add_left(S, s); set_attr(S, ia)
53       else del_right(S, s) endif;
54       if dp(h) $\neq$ last_dp_in_F then last_dp_in_F := dp(h);
55         if not look_ahead(h, F) then
56           $v_F$[bound_shared] := true
57         else $v_F$[poi_shared] := true endif
58       endif;
59       if dp(h) $\neq$ last_dp_in_G then
60         last_dp_in_G := dp(h) endif;
61     endif;
62     if status = end_of_second then
63       $v_F$[seg_outside] := true endif;
64     select_next(F, G, object, status);
65   endwhile;
66   if status = end_of_first then
67     $v_G$[seg_unshared] := true endif
68 end ExploreLine2DRegion2D.
```

Figure 14. Algorithm for computing the topological feature vectors for a *line2D* object and a *region2D* object

status and attach the Boolean flag *ia* indicating whether the interior of G is above s or not (line 46). Otherwise, we delete a right halfsegment h from the sweep line status and know that it is not shared by F (line 47). In case that both F and G share a halfsegment, we know that they also share their segment components (line 50). The sweep line status is then modified depending on the status of h (lines 52 to 53). If we encounter a new dominating point of F, we have to check whether F shares a boundary point (lines 55 to 56) or an interior point (line 57) with the boundary of G. If the halfsegment sequence of G should be exhausted (line 62), we know that F must have a segment whose interior is outside of G (line 63). If after the while-loop only F is exhausted but not G (line 66), G must have a boundary segment that is disjoint from F (line 67).

Let l be the number of halfsegments of F, m be the number of attributed halfsegments of G, and k be the total number of new halfsegments created due to our splitting strategy. The while-loop is then executed at most $l+m+k$ times. All operations needed on the sweep line status require $O(\log(l+m+k))$ time each. Due to the splitting strategy, all dominating end points are already endpoints of existing segments or become endpoints of newly created segments. The operation *look_ahead* needs constant time. In total, the algorithm requires $O((l+m+k)\log(l+m+k))$ time and $O(l+m+k)$ space.

5.6 *The Exploration Algorithm for the region2D/region2D Case*

The exploration algorithm for the *region2D/region2D* case is quite different from the preceding five cases, since it has to take into account the areal extent of both objects. The indices of the vector fields, with one exception described below, are not flags as before but segment classes. The fields of the vectors again contain Boolean values that are initialized with *false*. The main goal of the exploration algorithm is to determine the existing segment classes in each *region2D* object. Hence, the topological feature vector for each object is a *segment classification vector*. Each vector contains a field for the segment classes (0/1), (1/0), (0/2), (2/0), (1/2), (2/1), and (1/1). The following definition makes a connection between representational concepts and point set topological concepts as it is later needed in the evaluation phase. For a segment $s=(p, q) \in seg2D$, the function *pts* yields the infinite point set of s as $pts(s)=\{r \in \mathbb{R}^2 | r=p+\lambda(q-p),\ \lambda \in \mathbb{R},\ 0 \leqslant \lambda \leqslant 1\}$. Further, for $F \in region2D$, we define $\partial F = \bigcup_{f \in H(F)} pts(f.s)$, $F^{\circ} = \{p \in \mathbb{R}^2 | poiInRegion(p, F)\}$, and $F^{-} = \mathbb{R}^2 - F - F^{\circ}$. We can now define the semantics of this vector as follows:

Definition 6 Let $F, G \in region2D$ and v_F be the segment classification vector of F. Then

(i) $v_F[(0/1)] \quad :\Leftrightarrow \exists f \in H(F) : f.ia \wedge pts(f.s) \subset G^{-}$

(ii) $v_F[(1/0)] \quad :\Leftrightarrow \exists f \in H(F) : \neg f.ia \wedge pts(f.s) \subset G^{-}$

(iii) $v_F[(1/2)] \quad :\Leftrightarrow \exists f \in H(F) : f.ia \wedge pts(f.s) \subset G^{\circ}$

(iv) $v_F[(2/1)] \quad :\Leftrightarrow \exists f \in H(F) : \neg f.ia \wedge pts(f.s) \subset G^{\circ}$

(v) $v_F[(0/2)] \quad :\Leftrightarrow \exists f \in H(F) \exists g \in H(G) : f.s=g.s \wedge f.ia \wedge g.ia$

(vi) $v_F[(2/0)] \quad :\Leftrightarrow \exists f \in H(F) \exists g \in H(G) : f.s=g.s \wedge \neg f.ia \wedge \neg g.ia$

(vii) $v_F[(1/1)] \quad :\Leftrightarrow \exists f \in H(F) \exists g \in H(G) : f.s=g.s \wedge ((f.ia \wedge \neg g.ia) \vee (\neg f.ia \wedge g.ia))$

(viii) $v_F[bound\_poi\_shared] \quad :\Leftrightarrow \exists f \in H(F) \exists g \in H(G) : f.s \neq g.s \wedge dp(f)=dp(g)$

The segment classification vector v_G of G includes the cases (i) to (iv) with F and G swapped; we omit the flags for the cases (v) to (viii) due to their symmetry (or equivalence) to flags of F. The flag *bound_poi_shared* indicates whether any two unequal boundary segments of both objects share a common point. Before the splitting, such a point may have been a segment endpoint or a proper segment intersection point for each object. The determination of the boolean value of this flag also includes the treatment of a special case illustrated in Figure 15. If two regions F and G meet in a point like in the example, such a topological meeting situation cannot be detected by a usual plane sweep. The reason is that the plane sweep forgets completely about the already visited segments (right halfsegments) left of the sweep line. In our example, after s_1 and s_2 have been removed from the sweep line status, any information about them is lost. When s_3 is inserted into the status sweep line, its meeting with s_2 cannot be detected. Our solution is to look ahead in object G for a next halfsegment with the same dominating point before s_2 is removed from the sweep line status.

The segment classification is computed by the algorithm in Figure 16. The while-loop is executed as long as none of the two objects have been processed (line 8) and as long as not all topological flags have been set to *true* (lines 8 to 12). Then,

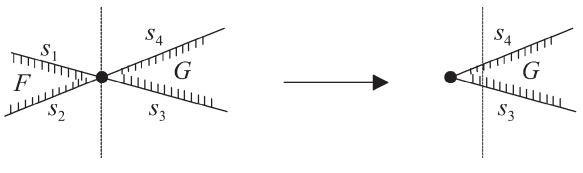


Figure 15. Special case of the plane sweep.

```
01 algorithm ExploreRegion2DRegion2D
02 input:   region2D objects F and G, topological feature
03          vectors $v_F$ and $v_G$ initialized with false
04 output:  updated vectors $v_F$ and $v_G$
05 begin
06   S := new_sweep(); last_dp_in_F := ϵ; last_dp_in_G := ϵ;
07   select_first(F, G, object, status);
08   while status = end_of_none and not ($v_F$[(0/1)] and $v_F$[(1/0)]
09         and $v_F$[(1/2)] and $v_F$[(2/1)] and $v_F$[(0/2)] and
10         $v_F$[(2/0)] and $v_F$[(1/1)] and $v_F$[bound_poi_shared]
11         and $v_G$[(0/1)] and $v_G$[(1/0)] and $v_G$[(1/2)] and
12          $v_G$[(2/1)]) do
13     if object = first then /* h = (s, d) */
14       h := get_event(F); last_dp_in_F := dp(h)
15     else if object = second then
16       h := get_event(G); last_dp_in_G := dp(h)
17     else /* object = both */
18       h := get_event(F);
19        last_dp_in_F := dp(h); last_dp_in_G := dp(h)
20     endif;
21     if last_dp_in_F = last_dp_in_G
22       or last_dp_in_F = look_ahead(h, G)
23       or last_dp_in_G = look_ahead(h, F) then
24        $v_F$[bound_poi_shared] := true
25     endif;
26     if d = right then
27       {($m_s$/$n_s$)} := get_attr(S);
28       if object = first then $v_F$[($m_s$/$n_s$)] := true
29       else if object = second then $v_G$[($m_s$/$n_s$)] := true
30       else if object = both then
31         $v_F$[($m_s$/$n_s$)] := true; $v_G$[($m_s$/$n_s$)] := true
32       endif;
33       del_right(S, s)
34     else add_left(S, s);
35       if coincident(S, s) then object := both endif;
36       if not pred_exists(S, s) then ($m_p$/$n_p$) := (*/0)
37       else {($m_p$/$n_p$)} := get_pred_attr(S) endif;
38       $m_s$ := $n_p$; $n_s$ := $n_p$;
39       if object = first or object = both then
40         if get_attr(F) then $n_s$ := $n_s$ + 1
41         else $n_s$ := $n_s$ −1 endif
42       endif;
43       if object = second or object = both then
44         if get_attr(G) then $n_s$ := $n_s$ + 1
45         else $n_s$ := $n_s$ −1 endif
46       endif;
47       set_attr(S, {($m_s$/$n_s$)});
48     endif;
49     select_next(F, G, object, status);
50   endwhile;
51   if status = end_of_first then
52     $v_G$[(0/1)] := true; $v_G$[(1/0)] := true
53   else if status = end_of_second then
54     $v_F$[(0/1)] := true; $v_F$[(1/0)] := true
55   endif
56 end ExploreRegion2DRegion2D.
```

Figure 16. Algorithm for computing the topological feature vectors for two *region2D* objects

according to the halfsegment order, the next halfsegment h is obtained, which belongs to one or both objects, and the variables for the last considered dominating points in F and/or G are updated (lines 13 to 20). Next, we check for a possible common boundary point in F and G (lines 21 to 25). This is the case if the last dominating points of F and G are equal, or the last dominating point in F (G) coincides with the next dominating point in G (F). The latter algorithmic step, in particular, helps us solve the special situation in Figure 15. If h is a right halfsegment (line 26), we update the topological feature vectors of F and/or G correspondingly (lines 27 to 32) and remove its segment component s from the sweep line status (line 33). In case that h is a left halfsegment, we insert its segment component s into the sweep line status (line 34) according to the y-order of its dominating point and the y-coordinates of the intersection points of the current sweep line with the segments momentarily in the sweep line status. If h's segment component s either belongs to F or to G, but not to both objects, and partially coincides with a segment from the other object in the sweep lines status, our splitting strategy is applied. Its effect is that the segment we currently consider suddenly belongs to *both* objects. Therefore, we modify the *object* variable in line 35 correspondingly. Next, we compute the segment class of s. For this purpose, we determine the lower and upper overlap numbers m_p and n_p of the predecessor p of s (lines 36 to 37). If there is no predecessor, m_p gets the 'undefined' value '*'. The segment classification of the predecessor p is important since the lower overlap number m_s of s is assigned the upper overlap number n_p of p, and the upper overlap number n_s of s is assigned its incremented or decremented lower overlap number, depending on whether the Boolean flag 'Interior Above' obtained by the predicate *get_attr* is *true* or *false* respectively (lines 39 to 46). The newly computed segment classification is then attached to s (line 47). The possible 19 segment class constellations between two consecutive segments in the sweep line status are shown in Table 2. The table shows which segment classes (m_s/n_s) a new segment s just inserted into the sweep line status can get, given a certain segment class (m_p/n_p) of a predecessor

Table 2. Possible segment class constellations between two consecutive segments in the sweep line status.

n_s	1	2	1	2	1	2	0	1	2	0	1	2	0	1	2	0	1	0	1
m_s	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2
n_p	*	*	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2
m_p	*	*	1	1	2	2	0	0	0	1	1	1	2	2	2	0	0	1	1

segment *p*. The first two columns show the special case that at the beginning the sweep line status is empty and the first segment is inserted. This segment can either be shared by both region objects ((0/2)-segment) or stems from one of them ((0/1)-segment). In all these cases (except the first two cases), $n_p = m_s$ must hold.

Let l be the number of attributed halfsegments of *F*, m be the number of attributed halfsegments of *G*, and k be the total number of new halfsegments created due to our splitting strategy. The while-loop is then executed at most $l+m+k$ times. All operations needed on the sweep line status require at most $O(\log(l+m+k))$ time each. The operations on the halfsegment sequences of *F* and *G* need constant time. In total, the algorithm requires $O((l+m+k)\log(l+m+k))$ time and $O(l+m+k)$ space.

6. Direct Predicate Characterization as an Evaluation Method

In the previous section we have developed sophisticated and efficient algorithms for the computation of the topological feature vectors v_F and v_G of two complex spatial objects $F, G \in \{point2D, line2D, region2D\}$. The vectors v_F and v_G contain specific topological feature flags for each type combination. The flags capture all topological situations between *F* and *G* and are different for different type combinations. Consequently, each type combination has led to a different exploration algorithm.

In this section, we present a method for the *evaluation phase* whose objective is to uniquely characterize the topological relationship of two given spatial objects of any type combination. Our general evaluation strategy is to leverage the information kept in the topological feature vectors and accommodate the objects' mutual topological features with an existing topological predicate for both predicate verification and predicate determination. The method provides a *direct predicate characterization* of all *s* topological predicates of each type combination (see Figure 2(b) for the different values of *s*) that is based on the feature values of v_F and v_G of the two spatial argument objects *F* and *G*. For example, for the *line2D*/*line2D* case, we have to determine which topological flags of v_F and v_G must be turned on and which flags must be turned off so that a given topological predicate (verification query) or a predicate to be found (determination query) is fulfilled. For the *region2D*/*region2D* case, the central question is to which segment classes the segments of both objects must belong so that a given topological predicate or a predicate to be found is satisfied. The direct predicate characterization gives an answer for each individual predicate of each individual type combination. This means that we obtain 184 individual predicate characterizations without converse predicates and 248 individual predicate characterizations with converse predicates (see Figure 2(b)). In general, each characterization is a Boolean expression in conjunctive normal form and expressed in terms of the topological feature vectors v_F and v_G.

We give two examples of direct predicate characterizations. As a first example, we consider the topological predicate number 8 (*meet*) between two *line2D* objects *F* and G (Figure 17(a) and Schneider and Behr (2006)) and see how the flags of the topological feature vectors (Definition 4) are used.

$$\begin{aligned} p_8(F, G) :\Leftrightarrow & \neg v_F[\mathit{seg\_shared}] \wedge \neg v_F[\mathit{interior\_poi\_shared}] \wedge v_F[\mathit{seg\_unshared}] \wedge \\ & \neg v_F[\mathit{bound\_on\_interior}] \wedge v_F[\mathit{bound\_shared}] \wedge v_F[\mathit{bound\_disjoint}] \wedge \\ & v_G[\mathit{seg\_unshared}] \wedge \neg v_G[\mathit{bound\_on\_interior}] \wedge v_G[\mathit{bound\_disjoint}] \end{aligned}$$

If we take into account the semantics of the topological feature flags, the right side of the equivalence means that both objects may only and must share boundary parts. More precisely and by considering the matrix in Figure 17(a), intersections between both interiors ($\neg v_F$[*seg_shared*], $\neg v_F$[*interior_poi_shared*]) as well as between the boundary of one object and the interior of the other object ($\neg v_F$[*bound_on_interior*], $\neg v_G$[*bound_on_interior*]) are not allowed; besides intersections between both boundaries (v_F[*bound_shared*), each component of one object must interact with the exterior of the other object (v_F[*seg_unshared*], v_G[*seg_unshared*], v_F[*bound_disjoint*], v_G[*bound_disjoint*]).

Next, we view the topological predicate number 7 (*inside*) between two *region2D* objects F and G (Figure 17(b) and Schneider and Behr (2006)) and see how the segment classes kept in the topological feature vectors (Definition 6) are used.

$$\begin{aligned} p_7(F, G) :\Leftrightarrow & \neg v_F[(0/1)] \wedge \neg v_F[(1/0)] \wedge \neg v_F[(0/2)] \wedge \neg v_F[(2/0)] \wedge \neg v_F[(1/1)] \wedge \\ & \neg v_F[\mathit{bound\_poi\_shared}] \wedge (v_F[(1/2)] \vee v_F[(2/1)]) \wedge \\ & \neg v_G[(1/2)] \wedge \neg v_G[(2/1)] \wedge (v_G[(0/1)] \vee v_G[(1/0)]) \end{aligned}$$

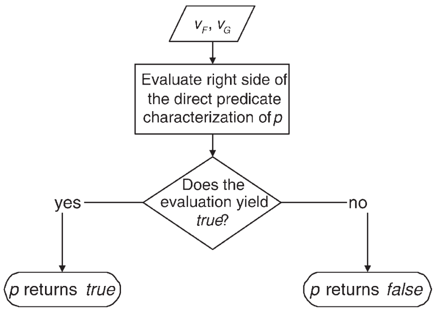
For the *inside* predicate, the segments of F must be located inside of G since the interior and boundary of F must be located in the interior of G; hence they must all have the segment classes (1/2) or (2/1). This "for all" quantification is tested by checking whether v_F[(1/2)] or v_F[(2/1)] are *true* and whether *all* other vector fields are *false*. The fact that all other vector fields are *false* means that the interior and boundary of F do not interact with the boundary and exterior of G. That is, the segments of G must be situated outside of F, and thus they *all* must have the segment classes (0/1) or (1/0); other segment classes are forbidden for G. Further, we must ensure that no segment of F shares a common point with any segment of G ($\neg v_F$[*bound_poi_shared*]).

The predicate characterizations can be read in both directions. If we are interested in *predicate verification*, i.e., in evaluating a specific topological predicate, we look

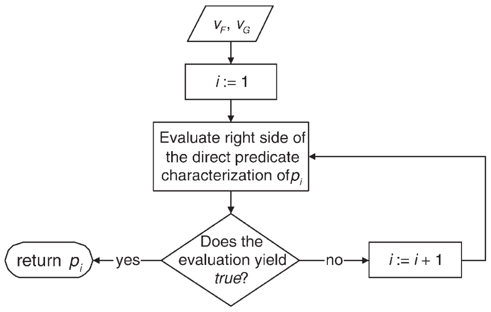
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

(a) (b)

Figure 17. The 9-intersection matrix number 8 for the predicate *meet* between two *line2D* objects (a) and the 9-intersection matrix number 7 for the predicate *inside* between two *region2D* objects (b)


v_F, v_G
Evaluate right side of the direct predicate characterization of *p*
Does the evaluation yield *true*?
yes
no
p returns *true*
p returns *false*

(*a*)


v_F, v_G
$i := 1$
Evaluate right side of the direct predicate characterization of p_i
Does the evaluation yield *true*?
return p_i
yes
no
$i := i + 1$

(*b*)

Figure 18. Predicate verification (a) and predicate determination (b) based on the direct predicate characterization method

from left to right and check the respective right side of the predicate's direct characterization (Figure 18(a)). This corresponds to an explicit implementation of each individual predicate. If we are interested in *predicate determination*, i.e., in deriving the topological relationship from a given spatial configuration of two spatial objects, we have to look from right to left. That is, consecutively we evaluate the right sides of the predicate characterizations by applying them to the given topological feature vectors v_F and v_G. For the characterization that matches we look on its left side to obtain the name or number of the predicate (Figure 18(b)).

7. Implementation and Testing of the Approach

To verify the feasibility, practicality, and correctness of the concepts presented, we have implemented and tested our approach in our own algebra package for handling two-dimensional spatial data, called *Spatial Algebra 2D* (*SPAL2D*). This algebra, or type system, includes the implementation of all six exploration algorithms from Section 5. Each such algorithm is preceded with a minimum bounding box intersection test as a filtering technique. Robustness of geometric computation is ensured by a software library called *RATIO* that permits the representation of and calculation with rational numbers of arbitrary, finite length. The implementation makes use of the complex spatial data types *point2D*, *line2D*, and *region2D*, as they have been described in Sections 2.1 and 3 and as they are available in several spatial extension packages. Hence, an implementation of the exploration algorithms in these extension packages should be straightforward.

A universal interface method *TopPredExploration* is provided to explore the topological events of two interacting spatial objects. This interface is overloaded to accept two spatial objects of any type combination as input. The output consists of two topological feature vectors which hold the topological information for both argument objects. A universal interface method *TopDirectPredCharacterization* then takes the two topological vectors as input and evaluates or determines the topological predicate. This leads to 184 individual predicate implementations excluding converse predicates and 248 individual predicate implementations including converse predicates.

Our implementation incorporating the data structures for the spatial data types and the exploration algorithms underwent various tests in order to verify the correctness of the concepts. We use a mixture of *black-box*[3] and *white-box*[4] testing techniques known as *gray-box*[5] testing. The black-box testing part arranges for well defined input and output objects. Each input object is a correct element of one of our spatial data types. Each output is guaranteed to be a topological feature vector and is tailored to the respective type combination. This enables us to test the functional behavior of our implementation by designing a collection of test cases to cover all type combinations of spatial objects as input and all possible values (*true* and *false*) of all topological feature flags as output. The white-box testing part considers every single execution path and guarantees that each statement is executed at least once. This ensures that all special cases that are specified and handled by the algorithms are properly tested. Our collection of test cases consists of 184 different scenes corresponding to the total number of topological predicates between spatial objects. They have been successfully tested and indicate the correctness of our concepts and the ability of our algorithms to correctly discover the needed topological information from any given scene.

A special test case generation technique has been leveraged to check the functionality of the exploration algorithms and the correctness of the resulting values of the topological feature vectors. The vector values have to be independent of the location of the two spatial objects involved. In order to check this, this technique is able to generate arbitrarily many different orientations of a topologically identical scene of two spatial objects with respect to the same sweep line and coordinate system. The idea is to rotate such a scene by a random and (in our case) rational angle around a central reference point. Special test cases including vertical segments or a meeting situation like in Figure 15 are considered too. For this rotation scenario, we especially take care of maintaining the robustness of geometric computation and preserving the topological consistency of a scene. We ensure that segment end points are rotated to end points with rational coordinates (and not floating point coordinates) by involving the Newton Method for approximating the results of numerical computations by rational numbers from our special rational number system RATIO. Intersecting segments are broken up into four segments in advance.

Starting with a set of 184 explicitly constructed base cases (one for each topological predicate) which cover the special test cases if possible, we have generated at least 20,000 test cases for the topological predicates of each of the six type combinations by our random scene rotation technique. In total, more than 120,000 test cases have been successfully generated, tested, and checked in the exploration phase. All test cases have been checked for predicate verification and predicate determination.

8. Conclusions and Future Work

While, at the core of GIS, from a conceptual perspective, topological predicates have been investigated to a large extent, the design of efficient implementation

[3] Black box testing takes an external perspective of the test object to derive test cases and does not consider the internal structure of the test object. The test designer selects valid and invalid input, determines the correct output, and checks the correctness of the functional behavior of the test object.

[4] White box testing takes an internal perspective of the test object and designs test cases based on its internal structure. The tester chooses test case inputs to exercise all paths through the software and determines the appropriate outputs. It requires programming skills to identify all paths through the software.

[5] Gray box testing keeps the advantages of black-box and white-box testing and is not impeded by the limitations of each single one.

methods for them has been widely neglected. Especially due to the introduction of complex spatial data types, the resulting increase of topological predicates between them asks for efficient and correct implementation concepts. We propose a two-phase approach which consists of an exploration phase and an evaluation phase and which can be applied to both predicate verification and predicate determination. The goal of the exploration phase is to traverse a given scene of two objects in space, collect any topological information of importance, and store it in topological feature vectors. The goal of the evaluation phase is to interpret the gained topological information and match it against the topological predicates. In this article, we have focused on the exploration phase and in detail developed exploration algorithms for all combinations of spatial data types. Our approach can be implemented on any available commercial or public spatial extension package provided by GIS or database vendors. We ourselves have implemented the algorithms as part of our SPAL2D software library which is a sophisticated spatial type system currently under development and determined for an integration into extensible GIS and spatial databases.

The direct predicate characterization demonstrates how we can leverage the concept of topological feature vectors for the evaluation and determination of topological predicates. However, this evaluation method has three main drawbacks. First, the method depends on the number of topological predicates. That is, each of the 184 (248) topological predicates between complex spatial objects requires an own specification and thus own implementation. Second, in the worst case, all direct predicate characterizations with respect to a particular type combination have to be checked for predicate determination. Third, the direct predicate characterization is error-prone. It is difficult to ensure that each predicate characterization is correct and unique and that all predicate characterizations together are mutually exclusive and cover all topological relationships. From this standpoint, this solution is an *ad hoc* approach. In Praing and Schneider (2008), we present sophisticated and systematic evaluation methods that are robust, correct, independent of the number of topological predicates of a particular type combination, and have a formal foundation.

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