ENE 3031
Computer Simulation
Week 4: Hand Simulation
(attributed by Dr. Alexopoulos and Dr. Goldsman)

Chuljin Park
Assistant Professor
Industrial Engineering
Hanyang University

Outline

Goal: Look at some examples of easy problems that we can simulate by hand (or almost by hand).

Monte Carlo Integration

First off, let's integrate

$$I \;=\; \int_a^b f(x)\,dx \;=\; (b-a)\int_0^1 f(a+(b-a)u)\,du,$$

where we get the last equality by substituting $u = (x-a)/(b-a)$.

Of course, we can often do this by analytical methods that we learned back in calculus class, or by numerical methods like the trapezoid rule or something like Gauss-Laguerre integration. But if these methods aren't possible, you can always use MC simulation. . . .

Suppose $U_1, U_2, \ldots$ are iid Unif(0,1), and define

$$I_i \;\equiv\; (b-a)f(a+(b-a)U_i) \quad \text{for } i = 1, 2, \ldots, n.$$

We can use the sample average of the $\bar{I}_n \equiv \frac{1}{n}\sum_{i=1}^n I_i$ as an estimator for I.

By the Law of the Unconscious Statistician, notice that

$$\begin{aligned}
\mathrm{E}[\bar{I}_n] &= (b-a)\mathrm{E}[f(a+(b-a)U_i] \\
&= (b-a)\int_{-\infty}^{\infty} f(a+(b-a)u)g(u)\,du \\
&\quad \text{(where } g(u) \text{ is the Unif(0,1) pdf)} \\
&= (b-a)\int_0^1 f(a+(b-a)u)\,du \;=\; I.
\end{aligned}$$

So $\bar{I}_n$ is unbiased for I. Since it can also be shown that $\text{Var}(\bar{I}_n) = O(1/n)$, the LLN implies $\bar{I}_n \to I$ as $n \to \infty$.

Example: Suppose $I = \int_0^1 \sin(\pi x)\,dx$ (and pretend we don't know the actual answer, $2/\pi \doteq 0.6366$).

Let's take $n = 4$ Unif(0,1) observations:

$$U_1 = 0.79 \quad U_2 = 0.11 \quad U_3 = 0.68 \quad U_4 = 0.31$$

which is close to $2/\pi$! (Actually, we got lucky.) $\square$

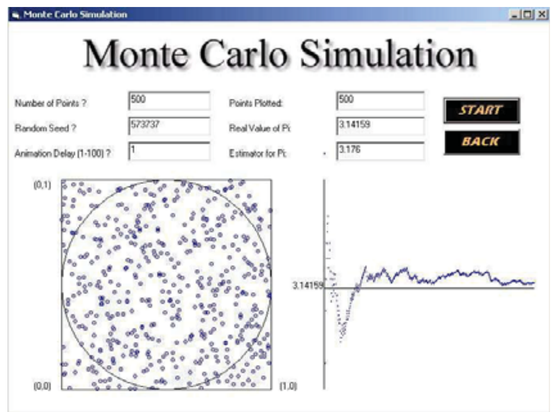
Outline

Making Some π

Consider a unit square (with area one). Inscribe in the square a circle with radius 1/2 (with area $\pi/4$). Suppose we toss darts randomly at the square. The probability that a particular dart will land in the inscribed circle is obviously $\pi/4$ (the ratio of the two areas). We can use this fact to estimate π.

Toss n such darts at the square and calculate the proportion $\hat{p}_n$ that land in the circle. Then an estimate for π is $\hat{\pi}_n = 4\hat{p}_n$, which converges to π as n becomes large by the LLN.

For instance, suppose that we throw $n = 500$ darts at the square and 397 of them land in the circle. Then $\hat{p}_n = 0.794$, and our estimate for π is $\hat{\pi}_n = 3.176$ — not so bad.



How would we actually conduct such an experiment?

To simulate a dart toss, suppose U_1 and U_2 are iid Unif(0,1). Then (U_1, U_2) represents the random position of the dart on the unit square. The dart lands in the circle if

$$\left(U_1 - \frac{1}{2}\right)^2 + \left(U_2 - \frac{1}{2}\right)^2 \le \frac{1}{4}.$$

Generate n such pairs of uniforms and count up how many of them fall in the circle. Then plug into $\hat{\pi}_n$. $\square$

Outline

Single-Server Queue

Customers arrive at a single-server queue with iid interarrival times and iid service times. Customers and must wait in a FIFO line if the server is busy.

We will estimate the expected customer waiting time, the expected number of people in the system, and the server utilization (proportion of busy time). First, some notation.

Customer number $=$ i

Interarrival time between customers $i-1$ and i is I_i

Customer i's arrival time is $A_i = \sum_{j=1}^{i} I_j$

Customer i starts service at time $T_i = \max(A_i, D_{i-1})$

Customer i's waiting time is $W_i = T_i - A_i$

Customer i's service time is S_i

Customer i's departure time is $D_i = T_i + S_i$

Example: Suppose we have the following sequence of events...

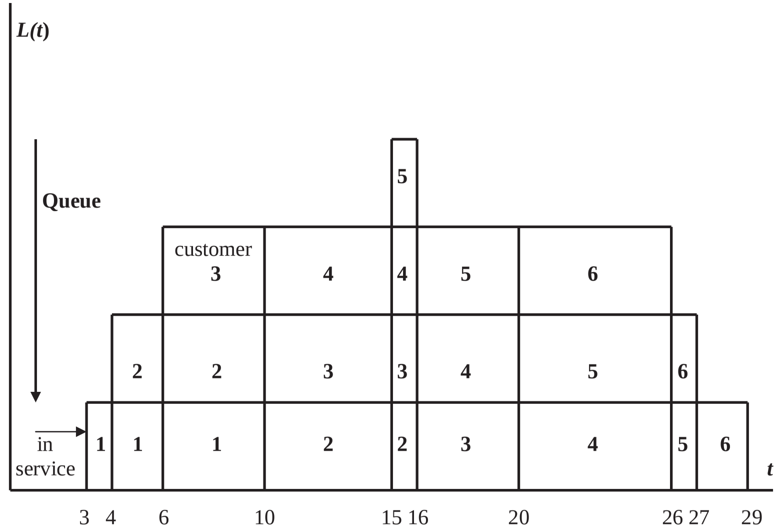
i	I_i	A_i	T_i	W_i	S_i	D_i
1	3				7	
2	1				6	
3	2				4	
4	4				6	
5	5				1	
6	5				2	

The average waiting time for the six customers is obviously

How do we get the average number of people in the system (in line + in service)?

Note that arrivals and departures are the only possible times for the number of people in the system, $L(t)$, to change.

time t	event	$L(t)$
0	simulation begins	0
3	customer 1 arrives	1
4	2 arrives	2
6	3 arrives	3
10	1 departs; 4 arrives	3
15	5 arrives	4
16	2 departs	3
20	3 departs; 6 arrives	3
26	4 departs	2
27	5 departs	1
29	6 departs	0



The average number in the system is ______. □

Another way to get the average number in the system is to calculate

Finally, to obtain the estimated server utilization, we easily see (from the picture) that the proportion of time that the server is busy is ______ □

Example: Same events, but *last*-in-first-out (LIFO) services. . .

i	I_i	A_i	T_i	W_i	S_i	D_i
1	3				7	
2	1				6	
3	2				4	
4	4				6	
5	5				1	
6	5				2	

The average waiting time for the six customers is ___, and the average number of people in the system turns out to be ___, which in this case turn out to better than FIFO.

Outline

1 Monte Carlo Integration

2 Making Some π

3 Single-Server Queue

4 (s, S) Inventory System

5 Simulating Random Variables

(s, S) Inventory System

A store sells a product at \$$d$/unit. Our inventory policy is to have at least s units in stock at the start of each day. If the stock slips to less than s by the end of the day, we place an order with our supplier to push the stock back up to S by the beginning of the next day.

Let I_i denote the inventory at the *end* of day i, and let Z_i denote the order that we place at the end of day i. Clearly,

$$Z_i = \begin{cases} S - I_i & \text{if } I_i < s \\ 0 & \text{otherwise} \end{cases}.$$

If an order is placed to the supplier at the end of day i, it costs the store $K + cZ_i$. It costs \$$h$/unit for the store to hold unsold inventory overnight, and a penalty cost of \$$p$/unit if demand can't be met. No backlogs are allowed. Demand on day i is D_i.

How much money does the store make on day i?

$$\begin{aligned}
&\text{Total} \\
&= \text{Sales} - \text{Ordering Cost} - \text{Holding Cost} - \text{Penalty Cost} \\
&= d \min(D_i, \text{inventory at beginning of day } i) \\
&\quad - \begin{cases} K + cZ_i & \text{if } I_i < s \\ 0 & \text{otherwise} \end{cases} \\
&\quad - hI_i - p \max(0, D_i - \text{inventory at beginning of day } i) \\
&= d \min(D_i, I_{i-1} + Z_{i-1}) \\
&\quad - \begin{cases} K + cZ_i & \text{if } I_i < s \\ 0 & \text{otherwise} \end{cases} \\
&\quad - hI_i - p \max(0, D_i - (I_{i-1} + Z_{i-1})).
\end{aligned}$$

Example: Suppose

$$d = 10, \quad s = 3, \quad S = 10, \quad K = 2, \quad c = 4, \quad h = 1, \quad p = 2.$$

Consider the following sequence of demands:

$$D_1 = 5, \quad D_2 = 2, \quad D_3 = 8, \quad D_4 = 6, \quad D_5 = 2, \quad D_6 = 1.$$

Suppose that we start out with an initial stock of $I_0 + Z_0 = 10$.

Day i	begin stock	D_i	I_i	Z_i	sales rev	order cost	hold cost	penalty cost	TOTAL rev
1	10	5							
2		2							
3		8							
4		6							
5		2							
6		1							

Outline

Simulating Random Variables

Example (Discrete Uniform): Consider a D.U. on $\{1, 2, \ldots, n\}$, i.e., $X = i$ with probability $1/n$ for $i = 1, 2, \ldots, n$. (Think of this as an n-sided dice toss for you Dungeons and Dragons fans.)

If $U \sim \text{Unif}(0,1)$, we can obtain a D.U. random variate simply by setting $X = \lceil nU \rceil$, where $\lceil \cdot \rceil$ is the "ceiling" (or "round up") function.

For example, if $n = 10$ and we sample a Unif(0,1) random variable $U = 0.73$, then $X =$. $\square$

Example (Another Discrete Random Variable):

$$P(X = x) = \begin{cases} 0.25 & \text{if } x = -2 \\ 0.10 & \text{if } x = 3 \\ 0.65 & \text{if } x = 4.2 \\ 0 & \text{otherwise} \end{cases}$$

Can't use a die toss to simulate this random variable. Instead, use what's called the *inverse transform method*.

x	$P(X = x)$	$P(X \le x)$	Unif(0,1)'s
-2	0.25		
3	0.10		
4.2	0.65		

Sample $U \sim \text{Unif}(0,1)$. Choose the corresponding x-value, i.e., $X = F^{-1}(U)$. For example, $U = 0.46$ means that $\square$

Now we'll use the *inverse transform method* to generate a continuous random variable. Recall. . .

Theorem: If X is a continuous random variable with cdf $F(x)$, then the random variable $F(X) \sim \text{Unif}(0,1)$.

This suggests a way to generate realizations of the RV X. Simply set $F(X) = U \sim \text{Unif}(0,1)$ and solve for $X = F^{-1}(U)$.

Old Example: Suppose $X \sim \text{Exp}(\lambda)$. Then $F(x) = 1 - e^{-\lambda x}$ for $x > 0$. Set $F(X) = 1 - e^{-\lambda X} = U$. Solve for X,

$$X = \frac{-1}{\lambda} \ell n(1 - U) \sim \text{Exp}(\lambda). \quad \square$$

Example (Generating Uniforms): All of the above RV generation examples relied on our ability to generate a Unif(0,1) RV. For now, let's assume that we can generate numbers that are "practically" iid Unif(0,1).

If you don't like programming, you can use Excel function `RAND()` or something similar to generate Unif(0,1)'s.

Here's an algorithm to generate *pseudo-random numbers (PRN's)*, i.e., a series $R_1, R_2, \dots$ of *deterministic* numbers that *appear* to be iid Unif(0,1). Pick a *seed* integer X_0, and calculate

$$X_i \;=\; 16807 X_{i-1} \mathrm{mod}(2^{31}-1), \quad i = 1, 2, \dots.$$

Then set $R_i = X_i/(2^{31}-1)$, $i = 1, 2, \dots$.

Next Class

- Random Number Generation