ENE3031 Computer Simulation — Practice Exam (Final)

Fall 2014

Name:

- You will have 1 hours and 50 minutes.
- This exam is closed book and closed notes. Calculators are not allowed. No scrap paper is allowed. Make sure that there is nothing on your desk except pens and erasers.
- If you need extra space, use the back of the page and indicate that you have done so.
- The test must be **stapled** when you turn in your test.
- **Show your work** except problem #1. A correct answer without work will receive zero.
- **We will not select among several answers**. Make sure it is clear what part of your work you want graded. Otherwise, zero point will be given to the problem.

1. (20pts) Questions on Random Number Generation.

   (a) Consider the pseudo-random number generator $X_i = (3X_{i-1} + X_{i-2} + 2)\text{mod}(5)$ with seeds $X_0 = 0$ and $X_1 = 1$. Find X_{25}.

   (b) Consider the PRN generator $X_i = 16807X_{i-1}\text{mod}(2^{31} - 1)$ with seed $X_0 = 1111111$. Find the second PRN, U_2.

   (c) Consider the following n = 30 pseudo-random numbers. (Read from left to right, and then down.)

   0.29 0.37 0.46 0.69 0.90 0.93 0.99 0.86 0.72 0.47
   0.30 0.18 0.29 0.38 0.69 0.76 0.91 0.62 0.41 0.30
   0.11 0.45 0.72 0.88 0.65 0.55 0.31 0.27 0.15 0.92

   Conduct a runs up-and-down test on these numbers and state what the conclusion of the test is. Use $\alpha = 0.10$. (Use $z_{\alpha/2} = z_{0.05} = 1.645$).

   (d) Same question as (c) above, except do a χ^2 goodness-of-fit test when $k = 3$ and $\alpha = 0.10$ (Use one of the tables in the last page).

2. (20pts) Questions on Random Variate Generation.

   (a) If U_1 and U_2 are i.i.d. $Unif(0,1)$, what is the distribution of $-\ln(U_1^2(1-U_2)^2)$?

   (b) Suppose the random variables X_1 and X_2 are i.i.d. with p.d.f. $f(x) = 2x$ for $0 \leq x \leq 1$. Let $Y = \min(X_1, X_2)$. Find the inverse of Y 's c.d.f.

   (c) Consider a Pois(200) random variable N. Using exactly one $Unif(0,1)$ PRN U, show how to generate a realization of N.

   (d) Suppose $U = 0.05$. Use inverse transform to generate a Geometric(0.6) random variate.

3. (30pts) Consider the continuous Pareto distribution, whose p.d.f. is given by

$$f(x) = \theta a^{\theta} x^{-(\theta+1)}; \tag{1}$$

where $x \geq a \geq 0$ and $\theta > 0$. (The constant a is known and θ is unknown.)

(a) Suppose that $X_1, X_2, ...$ are i.i.d. Pareto with the p.d.f. given above. Find the MLE of θ.

(b) Find the c.d.f. $F(x)$ of X_i.

(c) Derive the inverse c.d.f. formula for generating observations from the Pareto distribution. Using $a = 1.5$ and $\theta = 2$, generate Pareto random variates X_1, X_2 for $Unif(0, 1)$ inputs $U_1 = 0.245$ and $U_2 = 0.839$.

(d) Use a χ^2 goodness-of-fit test to test the hypothesis that the Pareto distribution with $a = 1.5$ and $\hat{\theta} = 2$ adequately fits the sampled values whose frequencies are summarized below. Use $\alpha = 0.05$ and assume only $\hat{\theta}$ was estimated from the data. (Use one of the tables in the last page.)

Interval	Observed Frequency
$[1.5, 1.677)$	11
$[1.677, 1.936)$	30
$[1.936, 2.372)$	20
$[2.372, \infty)$	39
	100

4. (30pts) (Questions on Output Analysis)

   (a) Suppose $[0, 1]$ is a 90% confidence interval for the mean μ based on 10 independent replications of size 1000. Now the boss has decided that she wants a 99% CI for 2μ based on those same 10 replications of size 1000. What is it?

   (b) Consider a stationary stochastic process $X_1, X_2, ...,$ with covariance function $R_k = \mathsf{Cov}(X_1, X1 + k) = 3 - k$ for $k = 0, 1, 2, 3$, and $R_k = 0$ for $k \geq 4$. Find $\mathsf{Var}(\overline{X}_4)$.

   (c) Consider the following (approximately normal) average waiting times from 4 independent replications of a complicated queueing network. Suppose that each output is based on the average of 500 waiting times:

   $$30 \quad 40 \quad 10 \quad 50$$

   Use the method of independent replications to calculate a two-sided 90% confidence interval for the mean μ.

   (d) Consider the following 10 snowfall totals in Buffalo, NY over consecutive years:

   $$130 \quad 94 \quad 125 \quad 112 \quad 150 \quad 123 \quad 141 \quad 133 \quad 128 \quad 152$$

   Use the method of batch means to calculate a two-sided 90% confidence interval for the mean μ. In particular, use two batches of size 5.

Some Formula

•
$$F(x) = \begin{cases} \int_{-\infty}^{x} f(t)\,dt, & \text{if } X \text{ is continuous} \\ \sum_{\{y|y\le x\}} f(y) = \sum_{\{y|y\le x\}} \Pr(X=y), & \text{if } X \text{ is discrete} \end{cases}.$$

•
$$\mu \equiv \mathsf{E}[X] \equiv \begin{cases} \sum_x xp(x) & \text{if } X \text{ is discrete} \\ \int_{\Re} xf(x)\,dx & \text{if } X \text{ is cts} \end{cases}$$

$$\mathsf{E}[g(X)] \equiv \begin{cases} \sum_x g(x)p(x) & \text{if } X \text{ is discrete} \\ \int_{\Re} g(x)f(x)\,dx & \text{if } X \text{ is cts} \end{cases}$$

$$\mathsf{E}[g(X,Y)] \equiv \begin{cases} \sum_y \sum_x g(x,y)p(x,y) & \text{if } X \text{ is discrete} \\ \int_{\Re}\int_{\Re} g(x,y)f(x,y)\,dxdy & \text{if } X \text{ is cts} \end{cases}$$

• $\mathsf{Var}[X] \equiv \mathsf{E}[(X-\mu)^2] = \mathsf{E}[(X-\mathsf{E}[X])^2] = \mathsf{E}[X^2] - (\mathsf{E}[X])^2$

• $p_X(x) = \sum_y p(x,y); p_Y(y) = \sum_x p(x,y); f_X(x) = \int_y f(x,y)dy; f_Y(y) = \int_x f(x,y)dx.$

•
$$F(x,y) = P(X \le x, Y \le y) = \begin{cases} \sum\sum_{s\le x, t\le y} f(s,t) & \text{discrete} \\ \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt & \text{continuous} \end{cases}$$

•
$$\mathsf{E}[Y|X=x] \equiv \begin{cases} \sum_y yf(y|x) & \text{discrete} \\ \int_{\Re} yf(y|x)\,dy & \text{continuous} \end{cases}$$

•
$$\mathsf{Cov}(X,Y) \equiv \sigma_{XY} \equiv \mathsf{E}[(X-\mathsf{E}[X])(Y-\mathsf{E}[Y])] = \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y].$$

•
$$\rho = \mathrm{Corr}(X,Y) \equiv \frac{\mathsf{Cov}(X,Y)}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y}.$$

• Mgf: $M_X(t) = \mathsf{E}[e^{tX}]$ and $\mathsf{E}[X^k] = \frac{d^k}{dt^k}\mathsf{E}[e^{tX}]|_{t=0}$.

• **Central Limit Theorem:** Suppose $X_1, \ldots, X_n$ are i.i.d. with $\mathsf{E}[X_i] = \mu$ and $\mathsf{Var}(X_i) = \sigma^2$. Then as $n \to \infty$,

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{\mathcal{D}} \mathsf{Nor}(0,1),$$

where "$\xrightarrow{\mathcal{D}}$" means that the c.d.f. $\to$ the $\mathsf{Nor}(0,1)$ c.d.f.

• **Normal Approximation to Binomial** If $X \sim$Bin(n,p) and n is large, then

$$\frac{Y - \mathsf{E}[Y]}{\sqrt{\mathsf{Var}(Y)}} \approx \mathsf{Nor}(0,1).$$

• **Bernoulli(p) Distribution: success or failure**

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } q \end{cases}$$

$\mathsf{E}[X] = p$, $\mathsf{Var}(X) = pq$, and $M_X(t) = pe^t + q$.
• **Binomial(n, p) Distribution: number of successes out of n trials**
$\text{Bin}(n, p)$ is the sum of n i.i.d. $\text{Bern}(p)$.
$\Pr(Y = k) = \binom{n}{k} p^k q^{n-k}, n = 0, 1, \ldots,\ \mathsf{E}[Y] = np,\ \mathsf{Var}[Y] = npq.$
• **Geometric(p): number of trials until the 1st success**
$\Pr(Z = k) = q^{k-1}p$, for $k = 1, 2, \ldots,\ M_Z(t) = \frac{pe^t}{1-qe^t}$, for $t < \ell\text{n}(1/q)$
$\mathsf{E}[Z] = \frac{1}{p}$, and $\mathsf{Var}(Z) = \frac{q}{p^2}$.
• **Poisson with rate λ: number of events in a fixed interval**
$\Pr(X = k) = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, 2, \ldots$ and $\mathsf{E}[X] = \mathsf{Var}[X] = \lambda$.
• **Uniform(a, b) Distribution**

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mathsf{E}[X] = \frac{a+b}{2} \qquad \mathsf{Var}(X) = \frac{(a-b)^2}{12}.$$

• **Exponential with rate λ: lifetime with constant failure rate**

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = 1 - e^{-\lambda x} \quad \mathsf{E}[X] = 1/\lambda, \quad \mathsf{Var}(X) = 1/\lambda^2.$$

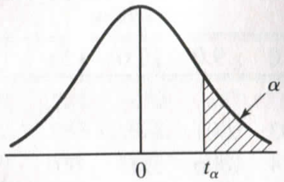
• **Standard Normal** Z The p.d.f. of the $\mathsf{Nor}(0, 1)$ is

$$\phi(z) \equiv \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad z \in \Re.$$

$$\Pr\{Z \le z\} \equiv \Phi(z) \equiv \int_{-\infty}^{z} \phi(t)\, dt, \quad z \in \Re.$$

$$X \sim \mathsf{Nor}(\mu, \sigma^2) \Rightarrow Z \equiv \frac{X - \mu}{\sigma} \sim \mathsf{Nor}(0, 1).$$

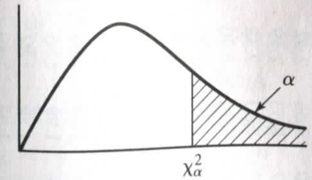
A.5 Percentage Points of The Student's t Distribution with ν Degrees of Freedom



ν	$t_{0.005}$	$t_{0.01}$	$t_{0.025}$	$t_{0.05}$	$t_{0.10}$
1	63.66	31.82	12.71	6.31	3.08
2	9.92	6.92	4.30	2.92	1.89
3	5.84	4.54	3.18	2.35	1.64
4	4.60	3.75	2.78	2.13	1.53
5	4.03	3.36	2.57	2.02	1.48
6	3.71	3.14	2.45	1.94	1.44
7	3.50	3.00	2.36	1.90	1.42
8	3.36	2.90	2.31	1.86	1.40
9	3.25	2.82	2.26	1.83	1.38
10	3.17	2.76	2.23	1.81	1.37
11	3.11	2.72	2.20	1.80	1.36
12	3.06	2.68	2.18	1.78	1.36
13	3.01	2.65	2.16	1.77	1.35
14	2.98	2.62	2.14	1.76	1.34
15	2.95	2.60	2.13	1.75	1.34
16	2.92	2.58	2.12	1.75	1.34
17	2.90	2.57	2.11	1.74	1.33
18	2.88	2.55	2.10	1.73	1.33
19	2.86	2.54	2.09	1.73	1.33
20	2.84	2.53	2.09	1.72	1.32
21	2.83	2.52	2.08	1.72	1.32
22	2.82	2.51	2.07	1.72	1.32
23	2.81	2.50	2.07	1.71	1.32
24	2.80	2.49	2.06	1.71	1.32
25	2.79	2.48	2.06	1.71	1.32
26	2.78	2.48	2.06	1.71	1.32
27	2.77	2.47	2.05	1.70	1.31
28	2.76	2.47	2.05	1.70	1.31
29	2.76	2.46	2.04	1.70	1.31
30	2.75	2.46	2.04	1.70	1.31
40	2.70	2.42	2.02	1.68	1.30
60	2.66	2.39	2.00	1.67	1.30
120	2.62	2.36	1.98	1.66	1.29
∞	2.58	2.33	1.96	1.645	1.28

Source: Robert E. Shannon, *Systems Simulation: The Art and Science*, © 1975, p. 372. Reprinted by permission of Prentice Hall, Upper Saddle River, NJ.

Table A.6 Percentage Points of The Chi-Square Distribution with ν Degrees of Freed



ν	$\chi^2_{0.005}$	$\chi^2_{0.01}$	$\chi^2_{0.025}$	$\chi^2_{0.05}$	$\chi^2_{0.10}$
1	7.88	6.63	5.02	3.84	2.71
2	10.60	9.21	7.38	5.99	4.61
3	12.84	11.34	9.35	7.81	6.25
4	14.96	13.28	11.14	9.49	7.78
5	16.7	15.1	12.8	11.1	9.2
6	18.5	16.8	14.4	12.6	10.6
7	20.3	18.5	16.0	14.1	12.0
8	22.0	20.1	17.5	15.5	13.4
9	23.6	21.7	19.0	16.9	14.7
10	25.2	23.2	20.5	18.3	16.0
11	26.8	24.7	21.9	19.7	17.3
12	28.3	26.2	23.3	21.0	18.5
13	29.8	27.7	24.7	22.4	19.8
14	31.3	29.1	26.1	23.7	21.1
15	32.8	30.6	27.5	25.0	22.3
16	34.3	32.0	28.8	26.3	23.5
17	35.7	33.4	30.2	27.6	24.8
18	37.2	34.8	31.5	28.9	26.0
19	38.6	36.2	32.9	30.1	27.2
20	40.0	37.6	34.2	31.4	28.4
21	41.4	38.9	35.5	32.7	29.6
22	42.8	40.3	36.8	33.9	30.8
23	44.2	41.6	38.1	35.2	32.0
24	45.6	43.0	39.4	36.4	33.2
25	49.6	44.3	40.6	37.7	34.4
26	48.3	45.6	41.9	38.9	35.6
27	49.6	47.0	43.2	40.1	36.7
28	51.0	48.3	44.5	41.3	37.9
29	52.3	49.6	45.7	42.6	39.1
30	53.7	50.9	47.0	43.8	40.3
40	66.8	63.7	59.3	55.8	51.8
50	79.5	76.2	71.4	67.5	63.2
60	92.0	88.4	83.3	79.1	74.4
70	104.2	100.4	95.0	90.5	85.5
80	116.3	112.3	106.6	101.9	96.6
90	128.3	124.1	118.1	113.1	107.6
100	140.2	135.8	129.6	124.3	118.5

Source: Robert E. Shannon, *Systems Simulation: The Art and Science*, © 1975, p. 372. Reprinted by permission of Prentice Hall, Upper Saddle River, NJ.