Operations Management I

Inventory Management (2)

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Inventory Management

Economic Order Quantity (EOQ)

- Introduction
- Assumptions
- Model
- Key Insights

Extensions of EOQ

- EOQ with backorders
- Economic Production Quantity (EPQ)

Hopp and Spearman, 2008, **Factory Physics**, McGraw Hill. **(Chapter 2)**

Krajewski and Ritzman, 2005, **Operations Management**, Prentice Hall. **(Chapter 15)**

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Economic Order Quantity (EOQ)

Introduction

- Harris' model (1913)

Basic trade-off in a metalworking shop that produces copper connectors

✓ Large lots

Setup cost ↓ ← Cost required to ready the shop to produce a connector (manufacturing environment)

Costs in stores of connectors → Inventory holding cost ↑

→ Less frequent changeovers

✓ Small lots

Setup cost ↑

Inventory holding cost ↓

→ Frequent changeovers

→ Setting economic lot size
(for ordering or manufacturing)

In the case that the connectors are purchased (ordering environment)

Ordering cost = cost of placing a purchase order

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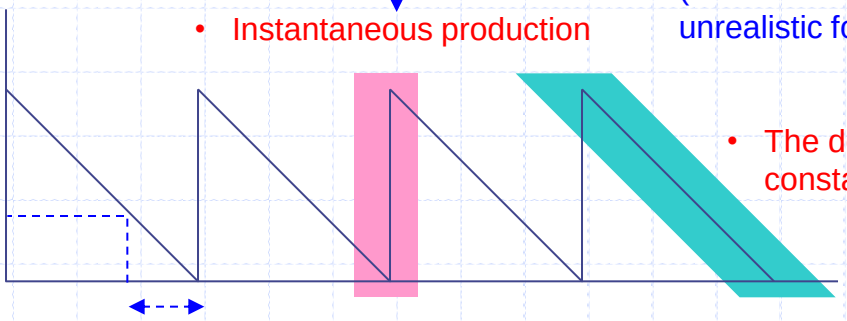
Economic Order Quantity

Assumptions

- Product can be analyzed individually.
  (single product type, no interactions between products)
- There are no constraints on the size of each lot.

No capacity constraint
Simultaneous production of the entire lot

(Suitable for purchasing systems , unrealistic for production system)



- Instantaneous production

Decision variable

- The demand rate is constant and constant with certainty.

e.g., 2000 units / year

Deterministic and static

- Shortages are not allowed.
- There is no uncertainty in lead time or supply

The lead time is constant and known with certainty.

Problem

- Determine the order quantity Q that minimizes the sum of production, setup and inventory holding costs

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D demand rate (units/year)

c unit production (purchase) cost, not counting setup or inventory costs ($/unit)

A fixed setup (ordering) cost to produce (purchase) a lot ($)

h inventory holding cost ($/unit•year)
h = i·c if the holding cost consists entirely of interest on money tied up in inventory, where i is the interest rate.

Q lot size

Economic Order Quantity

Model (1)

- Total cost (per year)

✓ Inventory holding cost (per year)

$\frac{Q}{2} \cdot h$ ← average inventory level / year

✓ Ordering (setup) cost (per year)

$\frac{D}{Q} \cdot A$ ← number of orders per year

e.g., D = 1200, Q = 100
→ frequency = 12 (1200 / 100)

✓ Production or purchase cost (per year) = $c \cdot D$

$$T(Q) = \frac{h \cdot Q}{2} + \frac{A \cdot D}{Q} + c \cdot D$$

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Economic Order Quantity

Model (2)

- Analysis of the total cost function

$$T(Q) = \frac{h \cdot Q}{2} + \frac{A \cdot D}{Q} + c \cdot D$$

Inventory holding cost increase linearly in the lot size Q

Prodution cost is not relevant to the lot size Q.

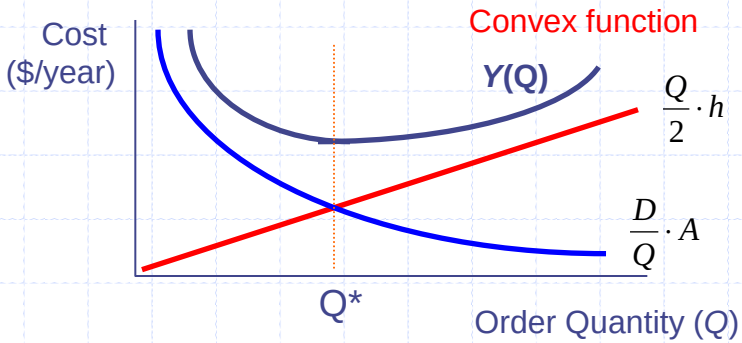
$$Y(Q) = \frac{h \cdot Q}{2} + \frac{A \cdot D}{Q}$$

Setup cost diminishes quickly in Q.

✓ Y(Q) is a convex function.

$$\frac{d}{dQ} Y(Q) = \frac{h}{2} - \frac{A \cdot D}{Q^2}$$

$$\frac{d^2}{dQ^2} Y(Q) = 2\frac{A \cdot D}{Q^3} > 0 \text{ for all } Q > 0$$



Total cost Y(Q) is minimized by some lot size Q.
(There exists a unique point Q* that minimizes the total cost function.)

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Economic Order Quantity

Model (3)

- Calculating the EOQ (Q^*)

$$\frac{d}{dQ}Y(Q) = \frac{h}{2} - \frac{A \cdot D}{Q^2} = 0$$

$$\Rightarrow \quad Q^2 = \frac{2AD}{h} \quad \longleftarrow \quad \frac{\$\cdot(\text{unit} / \text{year})}{\$ / \text{unit} \times \text{year}} = \frac{\$\cdot\text{unit}^2\cdot\text{year}}{\$\cdot\text{year}} = \text{units}^2$$

$$\Rightarrow \quad \boxed{Q^* = \sqrt{\frac{2AD}{h}}} \quad \longleftarrow$$

Economic lot size Q^* turns out to occur precisely at the value of Q for which the holding cost and setup cost are exactly balanced.

$$\frac{h \cdot Q}{2} = \frac{A \cdot D}{Q}$$

- Total cost (at EOQ)

$$Y^* = Y(Q^*) = \frac{hQ^*}{2} + \frac{AD}{Q^*} = \frac{h\sqrt{2AD/h}}{2} + \frac{AD}{\sqrt{2AD/h}}$$

$$= \sqrt{\frac{ADh}{2}} + \sqrt{\frac{ADh}{2}} = \sqrt{2ADh}$$

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Economic Order Quantity

Model (4)

- Time betwen two orders (T)

$$T^* = \frac{Q^*}{D} = \frac{\sqrt{2AD/h}}{D} = \sqrt{\frac{2A}{hD}}$$

- Frequency of orders

$$F^* = \frac{1}{T^*} = \sqrt{\frac{hD}{2A}}$$

- Reorder point (r) under given lead time L (L < T)

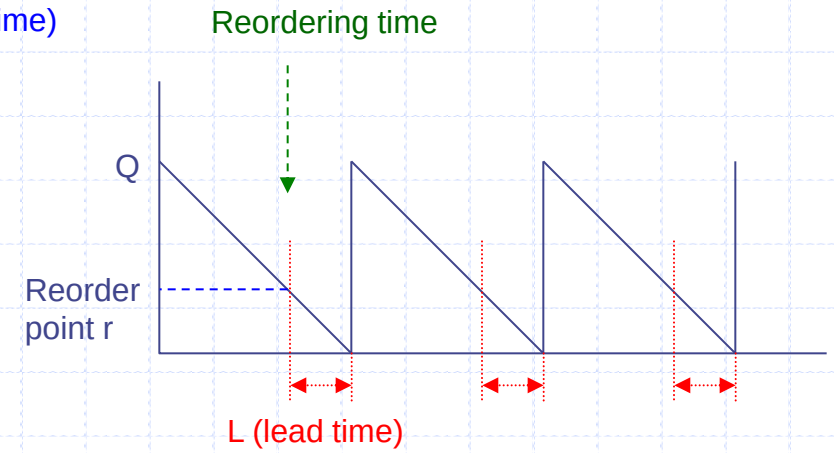
✓ L years $r = D \cdot L$ (demand during lead time)

✓ L months

$$r = D \cdot \frac{L}{12}$$ ← month → year

✓ L weeks

$$r = D \cdot \frac{L}{52}$$ ← week → year



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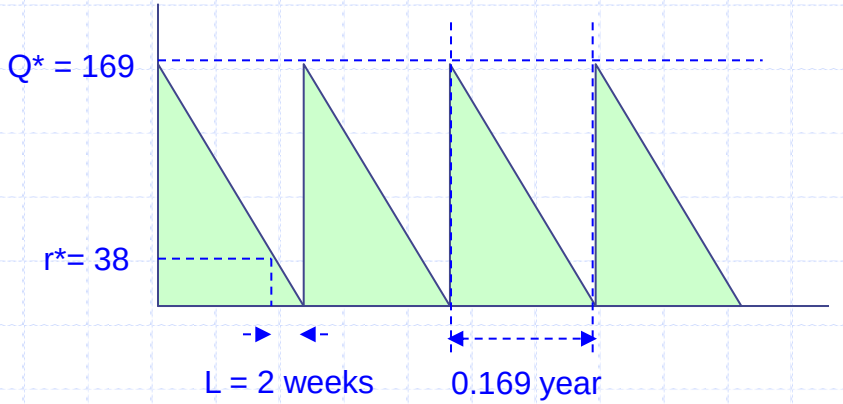

Economic Order Quantity

Model (5)

- Example
  - ✓ Data
    - ➢ Demand rate (D) = 1000 units /year
    - ➢ Production (purchase) cost (c) = 250 $
    - ➢ Setup (ordering) cost (A) = 500 $
    - ➢ Inventory holding cost (h) = 250·(0.1) + 10 = 35 ($/year·unit)
    - ✓ Interest rate (i) = 10%
    - ✓ Cost required to store a product = 10 ($/year·unit)
  - ✓ Solutions
    - ➢ Economic order quanity

$$Q^* = \sqrt{\frac{2AD}{h}} = \sqrt{\frac{2(500)(1000)}{35}} = \sqrt{28571} \approx 169$$

- ➢ Time betwen two orders

$$T^* = \frac{Q^*}{D} = \frac{169}{1000} = 0.169\,(\text{year})$$

- ➢ Frequency of orders

$$F^* = \frac{1}{T^*} = \frac{D}{Q^*} = \frac{1000}{169} = 5.91\,(\text{about 6 times per year})$$

- ➢ Reorder point (Lead time = 2 weeks)

$$r^* = 1000 \cdot \left(\frac{2}{52}\right) \cong 38$$

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Economic Order Quantity

Key Insights (1)

- Fundamental insights
  - ✓ Setup cost ∝ lot size ◄---- $\sqrt{A}\uparrow \rightarrow Q^*\uparrow$
  - ✓ Demand rate ∝ lot size ◄---- $\sqrt{D}\uparrow \rightarrow Q^*\uparrow$
  - ✓ Inventory holding costs ∝ 1 / lot size ◄---- $\sqrt{h}\uparrow \rightarrow Q^*\downarrow$

◄---- $Q^* = \sqrt{\frac{2AD}{h}}$

✓ Setup cost ↑, Inventory holding costs ↑ → total cost ↑

$$Y^* = Y(Q^*) = \sqrt{2ADh}$$

◄---- e.g., Toyota production system

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Economic Order Quantity

Key Insights (2)

- Other insights

✓ There is a close relation between lot size and inventory.

Lot size (Q*) ↑ ---→
- Average amount of inventory (Q*/2) ↑
- Number of lots (number of setups) per year (D/Q*) ↓

✓ Holding and setup costs are fairly insensitive to lot size.

Sensitivity of the cost to lot size
= total cost at some other arbitrary lot size / optimal total cost

$$\frac{Y(Q')}{Y^*} = \frac{hQ'/2 + AD/Q'}{\sqrt{2ADh}}$$

Optimal total cost (at EOQ)

$$= \frac{Q'}{2}\sqrt{\frac{h}{2AD}} + \frac{1}{Q'}\sqrt{\frac{AD}{2h}} = \frac{Q'}{2}\cdot\frac{1}{Q^*} + \frac{1}{2Q'}\cdot Q^* = \frac{1}{2}\left[\frac{Q'}{Q^*} + \frac{Q^*}{Q'}\right]$$

Total cost at some other arbitrary lot size Q' instead of Q^* ($Q' > Q^*$ or $Q' < Q^*$)

$$Y(Q') = \frac{hQ'}{2} + \frac{AD}{Q'}$$

e.g., $Q' = 2Q^*$

$$\frac{Y(Q')}{Y^*} = \frac{1}{2}\left[\frac{Q'}{Q^*} + \frac{Q^*}{Q'}\right] = \frac{1}{2}\left[\frac{2Q^*}{Q^*} + \frac{Q^*}{2Q^*}\right] = 1.25$$

Two times in lot size results in 25% increase in total cost .

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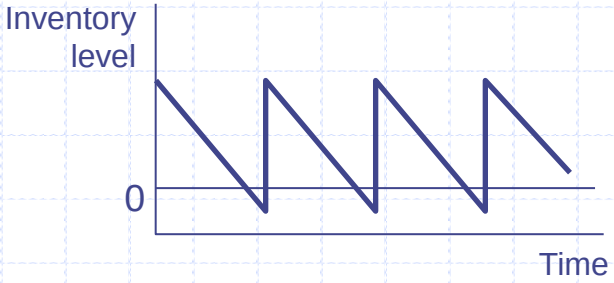
Extensions of EOQ

Overview

- Relaxation of the basic assumtions of the EOQ model



✓ EOQ with backorders

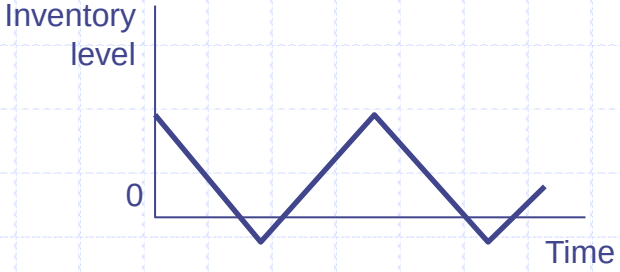
Backorder is allowed. ◄---- Orders that are not filled immediately, but have to wait until stock is available.

✓ Economic Production Quantity (EPQ)

Production (replenishment) is not instantaneous
(infinite production capacity → finite production capacity)

✓ Others

- EPQ with backorders
- EOQ with quantity discounts
- Multple items, etc.



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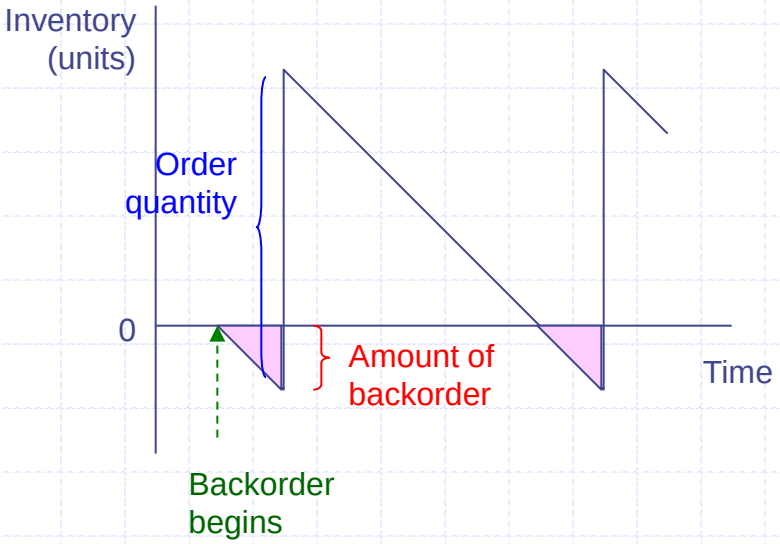
Extensions of EOQ

EOQ with Backorders (1)

- Situation
  - ✓ Backorder is allowed.
  - ✓ Other assumptions are the same as those of EOQ.
- Problem

  Determine the order quantity and backorder level that minimize the sum of setup, production, inventory holding and backorder costs



Zero backorder cost
----- No inventory (all backordered)

Infinite backorder cost
----- No backorder

Appropriate backorder cost
----- Backorder is economical.

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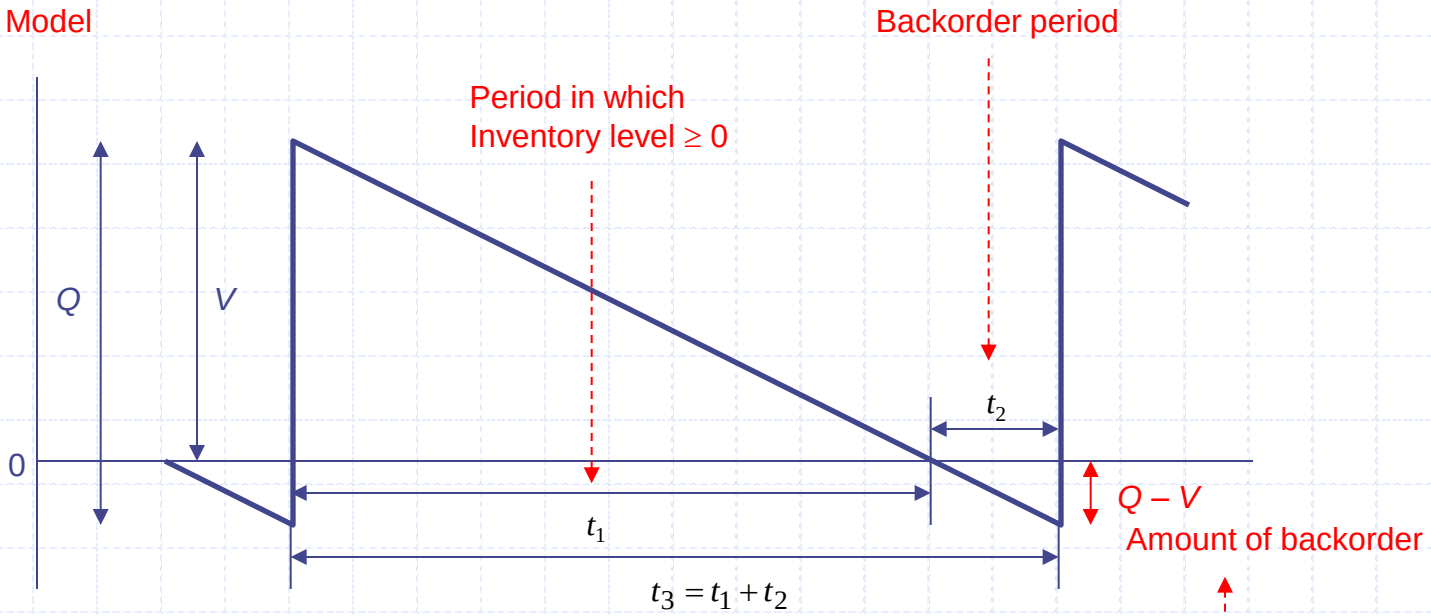
D	demand rate (units/year)
c	unit production (purchase) cost, not counting setup or inventory costs ($/unit)
A	fixed setup (ordering) cost to produce (purchase) a lot ($)
h	inventory holding cost ($/unit•year)
π	backorder cost ($/unit•year)

Extensions of EOQ

EOQ with Backorders (2)

- Model



Decision variables

Q	Order quantity
V	Maximum inventory level

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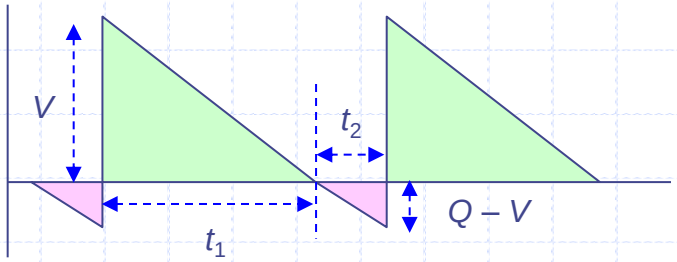
Extensions of EOQ

EOQ with Backorders (3)



- Total cost (per cycle)
  - ✓ Ordering (setup) cost = A
  - ✓ Production (purchase) cost = $c \cdot Q$
  - ✓ Inventory holding cost = $h \cdot \left[\frac{1}{2} \cdot V \cdot t_1\right] = \frac{hV^2}{2D}$ ⟵ $t_1 = \frac{V}{D}$

average Inventory level per cycle

  - ✓ Backorder cost = $\pi \cdot \left[\frac{1}{2} \cdot (Q-V) \cdot t_2\right] = \frac{\pi(Q-V)^2}{2D}$ ⟵ $t_2 = \frac{(Q-V)}{D}$

average backorder level per cycle

→ Total cost (per year)

$$Y(Q,V) = \left[A + c \cdot Q + \frac{hV^2}{2D} + \frac{\pi(Q-V)^2}{2D}\right] \cdot \frac{D}{Q}$$

Number of orders (setups) per year

$$= \frac{AD}{Q} + c \cdot D + \frac{hV^2}{2Q} + \frac{\pi(Q-V)^2}{2Q}$$

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Extensions of EOQ

EOQ with Backorders (4)

- EOQ with backorders (Q*)
- Optimal maximum inventory level (V*)

HW (a) Derive Q* and V*

As $\pi \to \infty$ $\quad Q^* \to \sqrt{\frac{2AD}{h}}$ (EOQ)

Inifinte backorder cost

⇢ No backorder

As $\pi \to 0$ $\quad V^* \to 0$

Zero backorder cost

⇢ No inventory (all backordered)

- Reorder point (r)

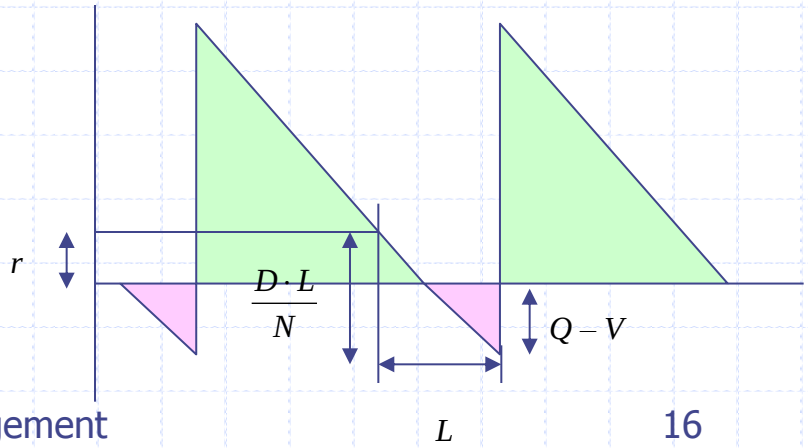
Given lead time L (days)

$$r^* = \frac{D \cdot L}{N} - (Q^* - V^*)$$

$(Q^* - V^*)$: amount of backorder

N: number of working days (weeks or months) per year

r* can be negative.



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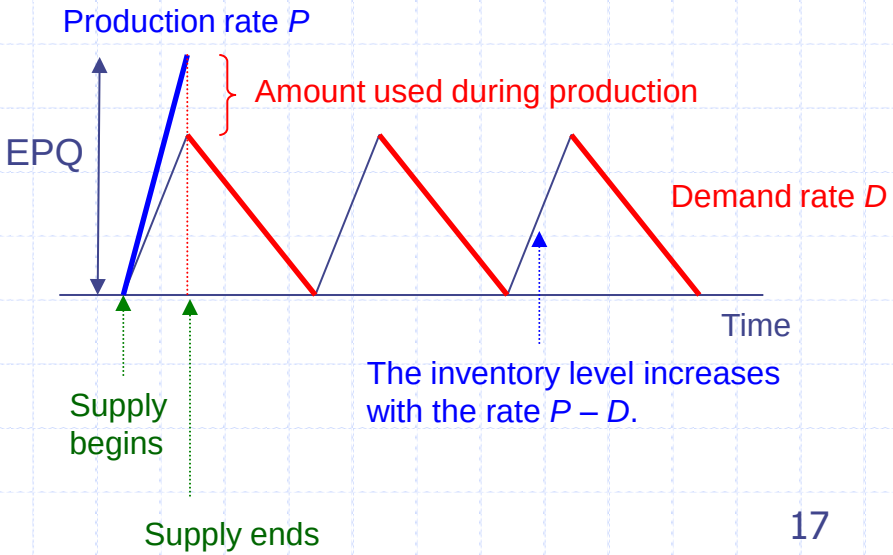
Extensions of EOQ

Economic Production Quantity (EPQ) (1)

- Situation
  - ✓ Production and demand occur simultaneously.
    (Basic assumption: production rate $P \geq$ demand rate D)
    - ⇠ Relaxation of the assumtion that replenishment (or production) is instantaneous
  - ✓ Other assumptions are the same as those of EOQ.
- Problem

  Determine the order quantity that minimizes the sum of setup, production and inventory holding costs



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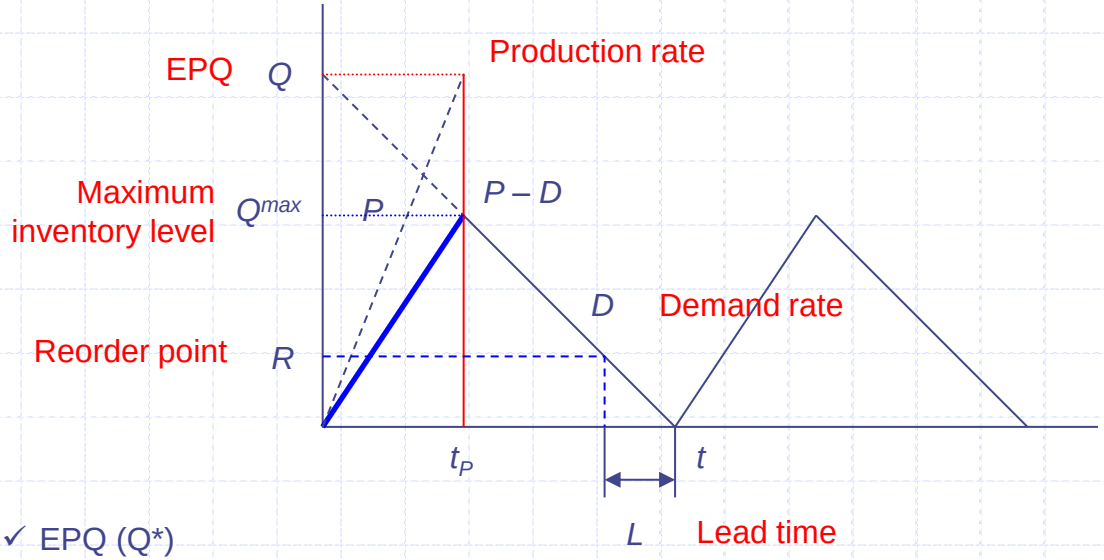
Extensions of EOQ

Economic Production Quantity (EPQ) (2)

- Model
  - ✓ Total cost (per day) Y(Q)



  - ✓ EPQ (Q*)

HW (b) Derive Y(Q) and Q* for EPQ model.