HW#5 Solution

#1

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#1

① $E[\bar{X}_n] = E[\frac{1}{n}\sum_{i=1}^{n} X_i]$

$= \frac{1}{n} E[\sum_{i=1}^{n} X_i]$

$= \frac{1}{n}\sum_{i=1}^{n} E[X_i]$

$= E[X_i]$

$\therefore$ $\bar{X}_n$ is an unbiased estimator of $E[X_i]$.

② $E[S_n^2] = E[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2]$

$= E[\frac{1}{n}\sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X}_n + \bar{X}_n^2)]$

$= E[\frac{1}{n}\sum_{i=1}^{n} X_i^2 - 2\frac{\bar{X}_n}{n}\sum_{i=1}^{n} X_i + \frac{1}{n}\cdot n\bar{X}_n^2]$

$= E[\frac{1}{n}\sum_{i=1}^{n} X_i^2 - 2\bar{X}_n^2 + \bar{X}_n^2]$

$= E[X_i^2] - E[\bar{X}_n^2]$

$= \{Var(X_i) - (E[X_i])^2\} - \{Var(\bar{X}_n) - (E[\bar{X}_n])^2\}$ ①

$= Var(X_i) - \cancel{(E[X_i])^2} - \frac{1}{n}Var(X_i) + \cancel{(E[X_i])^2}$

$= Var(X_i) - \frac{1}{n}Var(X_i)$

$= \frac{n-1}{n}Var(X_i) \neq Var(X_i)$

$\therefore$ S_n^2 is <u>not</u> an unbiased estimator of $Var(X_i)$.

#2

Suppose that a random sample of size $n, X_1, X_2, \ldots, X_n$, has been taken and that the observations are assumed to come from a Weibull distribution. The likelihood function derived by using the pdf given by Equation (5.47) can be shown to be

$$L(\alpha, \beta) = \frac{\beta^n}{\alpha^{\beta n}} \left[\prod_{i=1}^{n} X_i^{(\beta-1)} \right] \exp \left[-\sum_{i=1}^{n} \left(\frac{X_i}{\alpha} \right)^{\beta} \right] \tag{9.8}$$

The maximum-likelihood estimates are those values of $\widehat{\alpha}$ and $\widehat{\beta}$ that maximize $L(\alpha, \beta)$ or, equivalently, maximize $\ln L(\alpha, \beta)$, denoted by $l(\alpha, \beta)$. The maximum value of $l(\alpha, \beta)$ is obtained by taking the partial derivatives $\partial l(\alpha, \beta)/\partial\alpha$ and $\partial l(\alpha, \beta)/\partial\beta$, setting each to zero, and solving the resulting equations, which, after substitution, become

$$f(\beta) = 0 \tag{9.9}$$

and

$$\alpha = \left(\frac{1}{n} \sum_{i=1}^{n} X_i^{\beta} \right)^{1/\beta} \tag{9.10}$$

where

$$f(\beta) = \frac{n}{\beta} + \sum_{i=1}^{n} \ln X_i - \frac{n \sum_{i=1}^{n} X_i^{\beta} \ln X_i}{\sum_{i=1}^{n} X_i^{\beta}} \tag{9.11}$$

The maximum-likelihood estimates, $\widehat{\alpha}$ and $\widehat{\beta}$, are the solutions of Equations (9.9) and (9.10). First, $\widehat{\beta}$ is found via the iterative procedure explained below. Then $\widehat{\alpha}$ is found from Equation (9.10), with $\beta = \widehat{\beta}$.

Equation (9.9) is nonlinear, so it is necessary to use a numerical-analysis technique to solve it. In Table 9.3, a suggested iterative method for computing $\widehat{\beta}$ is given as

$$\widehat{\beta}_j = \widehat{\beta}_{j-1} - \frac{f(\widehat{\beta}_{j-1})}{f'(\widehat{\beta}_{j-1})} \tag{9.12}$$

Equation (9.12) employs Newton's method in reaching $\widehat{\beta}$, where $\widehat{\beta}_j$ is the jth iteration, beginning with an initial estimate for $\widehat{\beta}_0$, given in Table 9.3, as follows:

$$\widehat{\beta}_0 = \frac{\bar{X}}{S} \tag{9.13}$$

If the initial estimate, $\widehat{\beta}_0$, is sufficiently close to the solution $\widehat{\beta}$, then $\widehat{\beta}_j$ approaches $\widehat{\beta}$ as $j \longrightarrow \infty$. In Newton's method, $\widehat{\beta}$ is approached through increments of size $f(\widehat{\beta}_{j-1})/f'(\widehat{\beta}_{j-1})$. Equation (9.11)

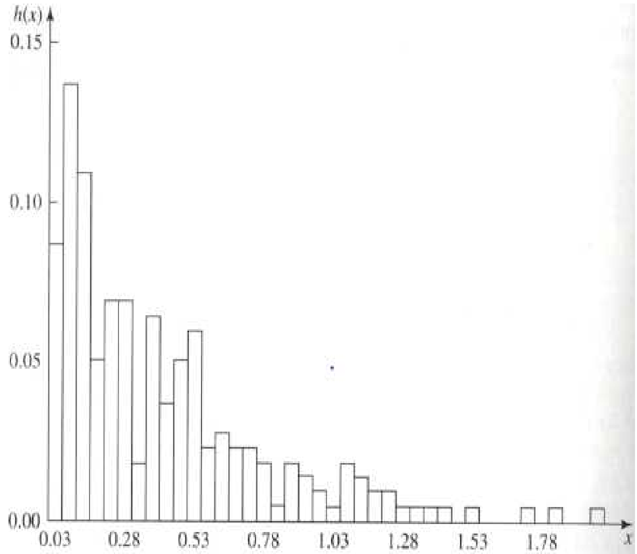
#3

j	$\widehat{\beta}_j$	$\sum_{i=1}^{20} X_i^{\widehat{\beta}_j}$	$\sum_{i=1}^{20} X_i^{\widehat{\beta}}j \ln X_i$	$\sum_{i=1}^{20} X_i^{\widehat{\beta}}j (\ln X_i)^2$	$f(\widehat{\beta}_j)$	$f'(\widehat{\beta}_j)$	$\widehat{\beta}_{j+1}$
0	2.539	1359.088	2442.221	4488.722	1.473	-4.577	2.861
1	2.861	2432.557	4425.376	8208.658	.141	-3.742	2.899
2	2.899	2605.816	4746.920	8813.966	.002	-3.660	2.899
3	2.899	2607.844	4750.684	8821.054	.000	-3.699	2.899

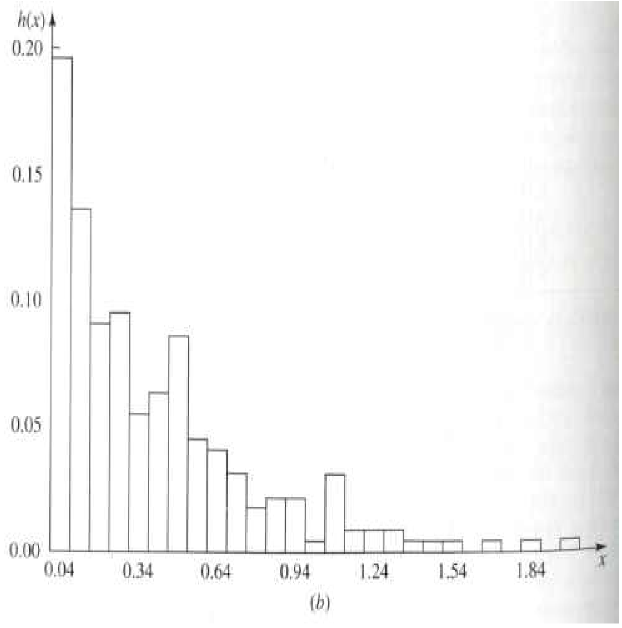
$\widehat{\beta} = 2.899$

$\widehat{\alpha} = 5.366$

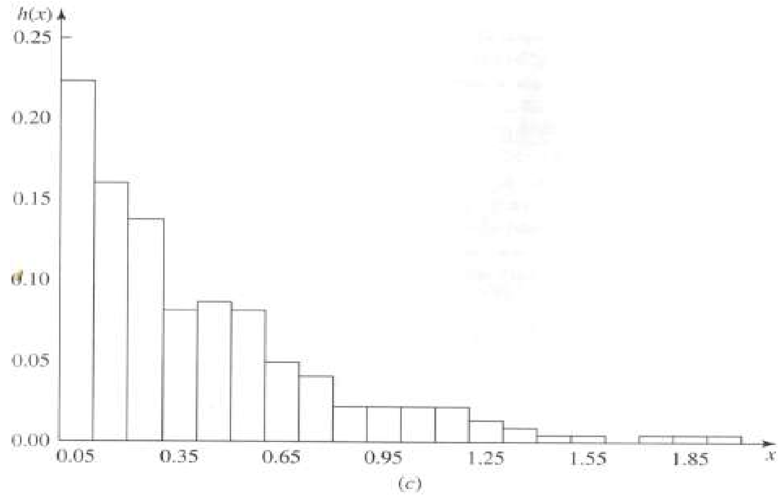
#4



(a)



(b)



Histograms of the interarrival-time data in Table 6.7: (a)$\triangle b$=0.005 (b)$\triangle b$=0.075 (c) $\triangle b$=0.100

EXAMPLE 6.6. For the exponential distribution, $\theta = \beta$ $(\beta > 0)$ and $f_\beta(x) = (1/\beta)e^{-x/\beta}$ for $x \geq 0$. The likelihood function is

$$L(\beta) = \left(\frac{1}{\beta}e^{-X_1/\beta}\right)\left(\frac{1}{\beta}e^{-X_2/\beta}\right)\cdots\left(\frac{1}{\beta}e^{-X_n/\beta}\right)$$
$$= \beta^{-n}\exp\left(-\frac{1}{\beta}\sum_{i=1}^{n} X_i\right)$$

and we seek the value of β that maximizes $L(\beta)$ over all $\beta > 0$. This task is more easily accomplished if, instead of working directly with $L(\beta)$, we work with its logarithm. Thus, we define the *log-likelihood function* as

$$l(\beta) = \ln L(\beta) = -n \ln \beta - \frac{1}{\beta}\sum_{i=1}^{n} X_i$$

Since the logarithm function is strictly increasing, maximizing $L(\beta)$ is equivalent to maximizing $l(\beta)$, which is much easier; that is, $\hat{\beta}$ maximizes $L(\beta)$ if and only if $\hat{\beta}$ maximizes $l(\beta)$. Standard differential calculus can be used to maximize $l(\beta)$ by setting its derivative to zero and solving for β. That is,

$$\frac{dl}{d\beta} = \frac{-n}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} X_i$$

which equals zero if and only if $\beta = \sum_{i=1}^{n} X_i/n = \overline{X}(n)$. To make sure that $\beta = \overline{X}(n)$ is a maximizer of $l(\beta)$ (as opposed to a minimizer or an inflection point), a sufficient (but not necessary) condition is that $d^2l/d\beta^2$, evaluated at $\beta = \overline{X}(n)$, be negative. But

$$\frac{d^2l}{d\beta^2} = \frac{n}{\beta^2} - \frac{2}{\beta^3}\sum_{i=1}^{n} X_i$$

which is easily seen to be negative when $\beta = \overline{X}(n)$ since the X_i's are positive. Thus, the MLE of β is $\hat{\beta} = \overline{X}(n)$. Notice that the MLE is quite natural here, since β is the mean of the hypothesized distribution and the MLE is the *sample* mean. For the data of Example 6.4, $\hat{\beta} = \overline{X}(219) = 0.399$.

EXAMPLE 6.15. We now use a chi-square test to compare the $n = 219$ interarrival times of Table 6.7 with the fitted exponential distribution having distribution function $\hat{F}(x) = 1 - e^{-x/0.399}$ for $x \geq 0$. If we form, say, $k = 20$ intervals with $p_j = 1/k = 0.05$ for $j = 1, 2, \ldots, 20$, then $np_j = (219)(0.05) = 10.950$, so that this satisfies the guidelines that the intervals be chosen with equal p_j's and $np_j \geq 5$. In this case, it is easy to find the a_j's, since $\hat{F}$ can be inverted. That is, we set $a_0 = 0$ and $a_{20} = \infty$, and

TABLE 6.12
A chi-square goodness-of-fit test for the interarrival-time data

j	Interval	N_j	np_j	$\frac{(N_j - np_j)^2}{np_j}$
1	[0, 0.020)	8	10.950	0.795
2	[0.020, 0.042)	11	10.950	0.000
3	[0.042, 0.065)	14	10.950	0.850
4	[0.065, 0.089)	14	10.950	0.850
5	[0.089, 0.115)	16	10.950	2.329
6	[0.115, 0.142)	10	10.950	0.082
7	[0.142, 0.172)	7	10.950	1.425
8	[0.172, 0.204)	5	10.950	3.233
9	[0.204, 0.239)	13	10.950	0.384
10	[0.239, 0.277)	12	10.950	0.101
11	[0.277, 0.319)	7	10.950	1.425
12	[0.319, 0.366)	7	10.950	1.425
13	[0.366, 0.419)	12	10.950	0.101
14	[0.419, 0.480)	10	10.950	0.082
15	[0.480, 0.553)	20	10.950	7.480
16	[0.553, 0.642)	9	10.950	0.347
17	[0.642, 0.757)	11	10.950	0.000
18	[0.757, 0.919)	9	10.950	0.347
19	[0.919, 1.195)	14	10.950	0.850
20	[1.195, ∞)	10	10.950	0.082
				$\chi^2 = 22.188$

for $j = 1, 2, \ldots, 19$ we want a_j to satisfy $\hat{F}(a_j) = j/20$; this is equivalent to setting $a_j = -0.399 \ln(1 - j/20)$ for $j = 1, 2, \ldots, 19$ since $a_j = \hat{F}^{-1}(j/20)$. (For continuous distributions such as the normal, gamma, and beta, the inverse of the distribution function does not have a simple closed form. In these cases, however, F^{-1} can be evaluated by numerical methods; consult the references given in Table 6.11.) The computations for the test are given in Table 6.12, and the value of the test statistic is $\chi^2 = 22.188$. Referring to Table T.2, we see that $\chi^2_{19,0.90} = 27.204$, which is not exceeded by χ^2, so we would not reject H_0 at the $\alpha = 0.10$ level. (Note that we would also not reject H_0 for certain larger values of α such as 0.25.) Thus, this test gives us no reason to conclude that our data are poorly fitted by the expo(0.399) distribution.