Elementary Graph Algorithms

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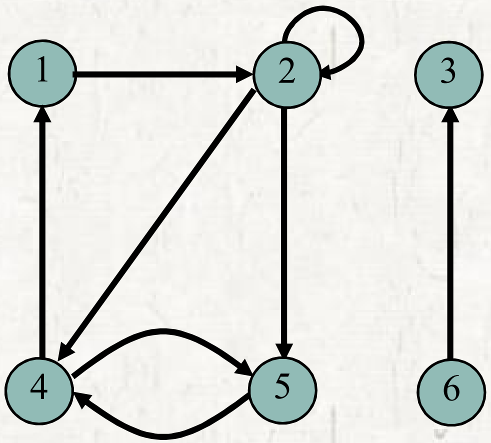
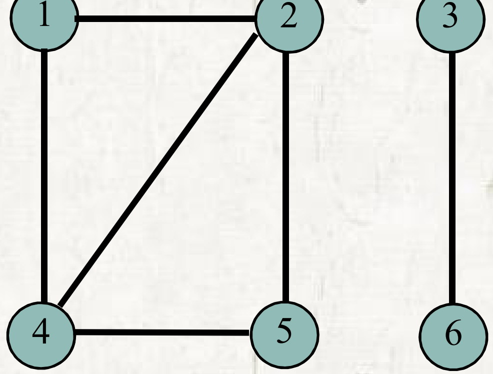
  - Topological sort

Graph basics

- A ***graph G*** is a pair (***V***, ***E***)
  where ***V*** is a ***vertex*** set and ***E*** is an ***edge*** set.
- A vertex (node) is a stand-alone object.
  - Represented by a circle.
- An edge **(**link**)** is an object connecting two vertices.
  - Represented by either an arrow or a line.



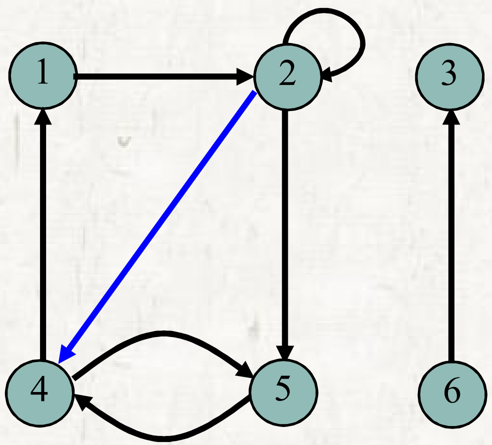
A directed graph



An undirected graph

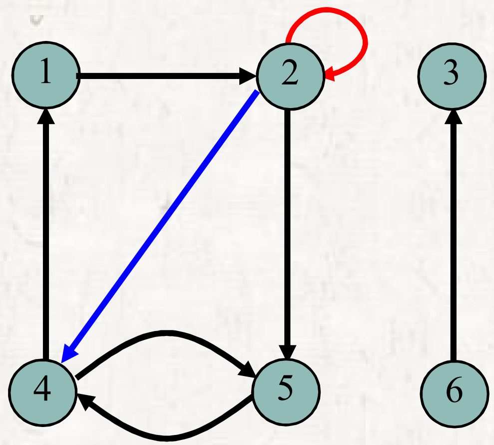
Graph basics

- A ***directed graph*** (or ***digraph***) is a graph with ***directed edges***.
  - Edges have directions so they are represented by arrows.
  - Each edge ***leaves*** a vertex and ***enters*** a vertex.
    - The blue edge leaves vertex 2 and enters vertex 4.



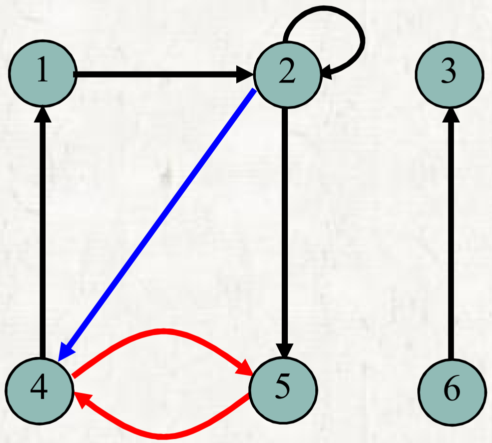
Graph basics

- An edge leaving a vertex *u* and entering a vertex *v* is said it is ***incident from*** *u* and ***incident to*** *v*.
  - The blue edge is incident from vertex 2 and to vertex 4.
- In a digraph, ***self-loops*** (edges from a vertex to itself) are possible.
  - The red edge is a self-loop.



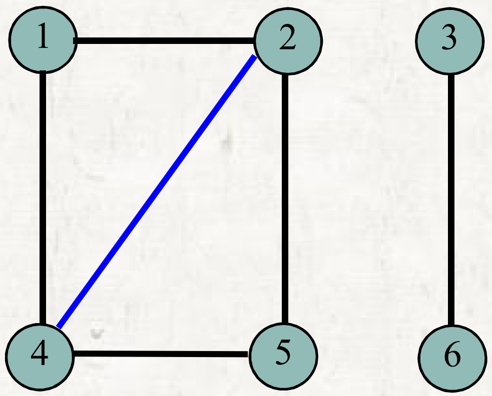
Graph basics

- Normally, each vertex is identified by a number or a name.
  - $V = \{1, 2, 3, 4, 5, 6\}$
- Each edge is identified by the ***ordered pair of vertices*** it leaves and enters.
  - $E = \{(1,2), (2,2), (2,4), (2,5), (4,1), (4,5), (5,4), (6,3)\}$
- In a digraph, there are at most 2 edges between two vertices.



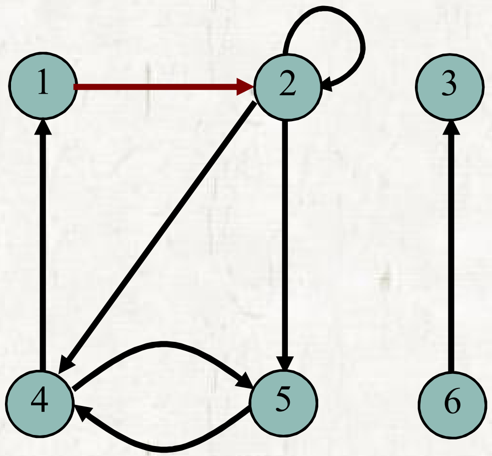
Graph basics

- An ***undirected graph*** is a graph with ***undirected edges***.
  - Edges have no directions so they are represented by lines.
  - Self-loops are forbidden.
  - Edge (u,v) is the same as edge (v,u).
    - (2,4) = (4,2)
    - The blue edge is ***incident on*** vertices 2 and 4.



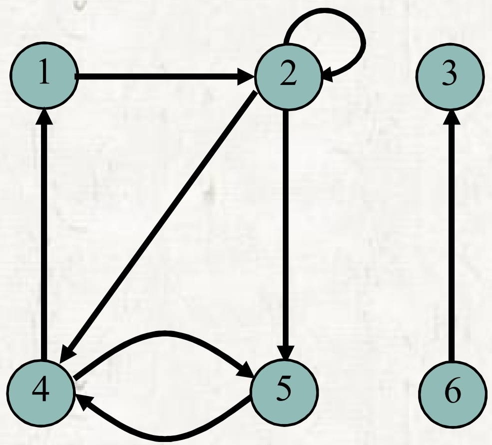
Graph basics

- ***Adjacency***
  - If (u,v) is an edge, vertex v is ***adjacent*** to vertex u.
  - In an undirected graph, adjacency relation is symmetric.
    - If u is adjacent to v, v is adjacent to u.
  - In a directed graph, it is not symmetric.
    - Vertex 2 is adjacent to 1.
    - But vertex 1 is not adjacent to 2.



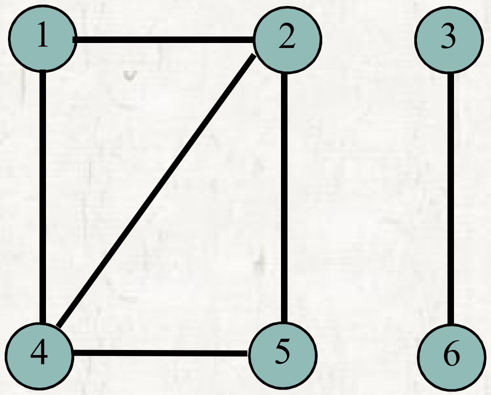
Graph basics

- ***Degree***
  - The ***out-degree*** of a vertex is the number of edges leaving it.
    - The out-degree of vertex 2 is 3.
  - The ***in-degree*** of a vertex is the number of edges entering it.
    - The in-degree of vertext 2 is 2.
  - ***degree*** **= out-degree + in-degree.**



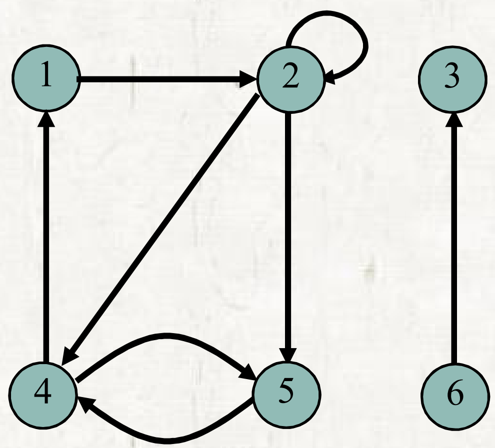
Graph basics

- ***Degree***
  - In an undirected graph,
    - The out-degree and the in-degree are not defined.
    - Only the degree of a vertex is defined.
  - The degree of vertex 2 is 3.



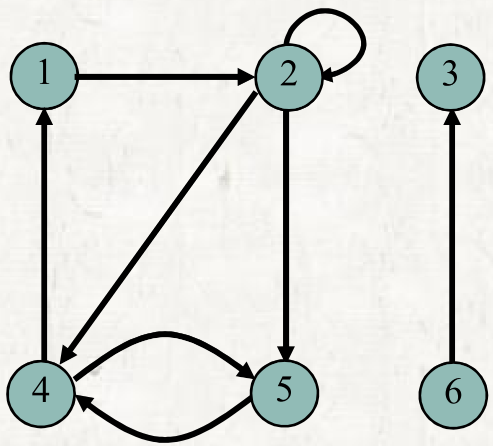
Graph basics

- ***Path***
  - A path from vertex u to vertex v is a sequence of vertices $<v_0, v_1, v_2, \ldots, v_k>$ where
    - $v_0 = u$, $v_k = v$, and
    - every vertex v_{i+1} $(0 \le i \le k\text{-}1)$ is adjacent to v_i.
      - There is an edge (v_i, v_{i+1}) for all i.
    - <1, 2, 4, 5> is a path.
    - <1, 2, 4, 1, 2> is a path.
    - <1, 2, 4, 2> is not a path.



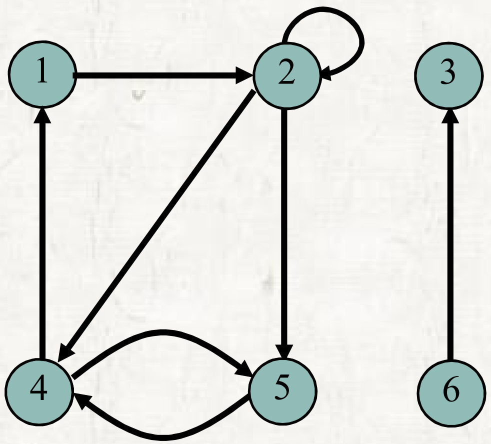
Graph basics

- ***Path***
  - The ***length*** of a path is the number of edges in the path.
    - The length of a path <1, 2, 4, 5> is 3.
    - If there is a path from vertex *u* to vertex *v*, *v* is called ***reachable*** from *u*.
      - Vertex 5 is reachable from vertex 1.
      - Vertex 3 is not reachable from vertex 1.



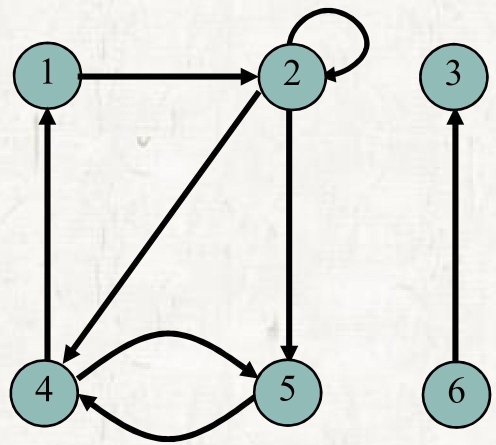
Graph basics

- ***Simple path***
  - A path is simple if all vertices in the path are distinct.
  - A path <1, 2, 4, 5> is a simple path.
  - A path <1, 2, 4, 1, 2> is not a simple path.



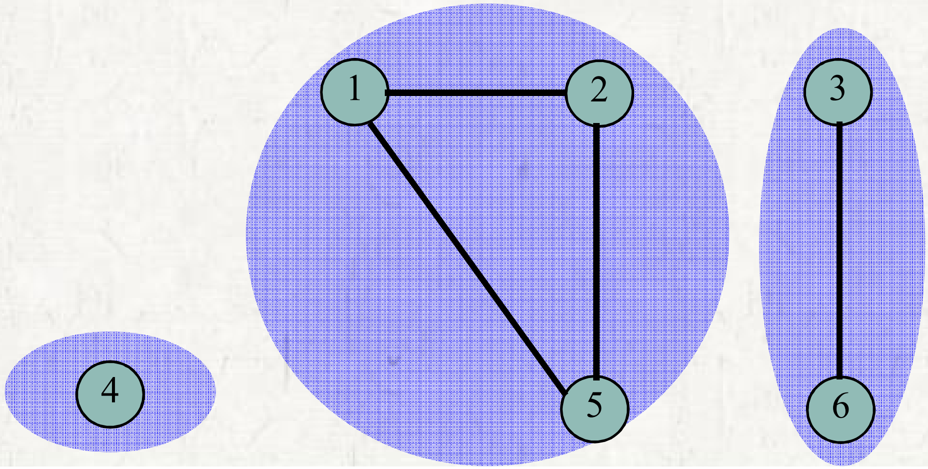
Graph basics

- ***Cycle* and *simple cycle***
  - A path <v_0, v_1, v_2,…, v_k> is a cycle if $v_0 = v_k$
  - A cycle <v_0, v_1, v_2,…, v_k> is simple if v_1, v_2,…, v_k are distinct.
  - A path <1, 2, 4, 5, 4, 1> is a cycle but it is not a simple cycle.
  - A path <1, 2, 4, 1> is a simple cycle.



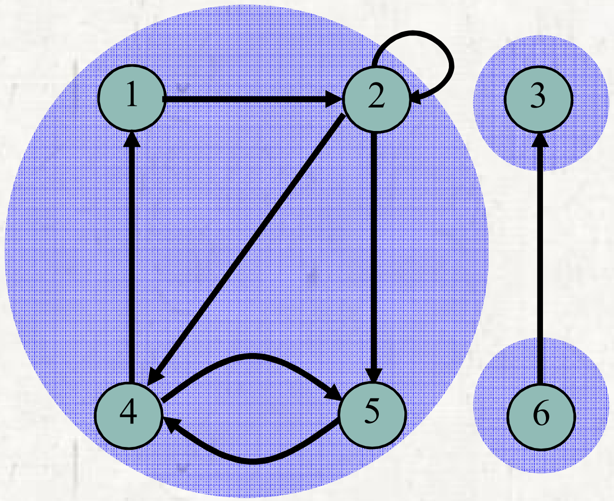
Graph basics

- ***An acyclic graph***
  - A graph without cycles
- ***A connected graph***
  - An undirected graph is ***connected*** if every pair of vertices is connected by a path.
- ***Connected components***
  - Maximally connected subsets of vertices of an undirected graph.



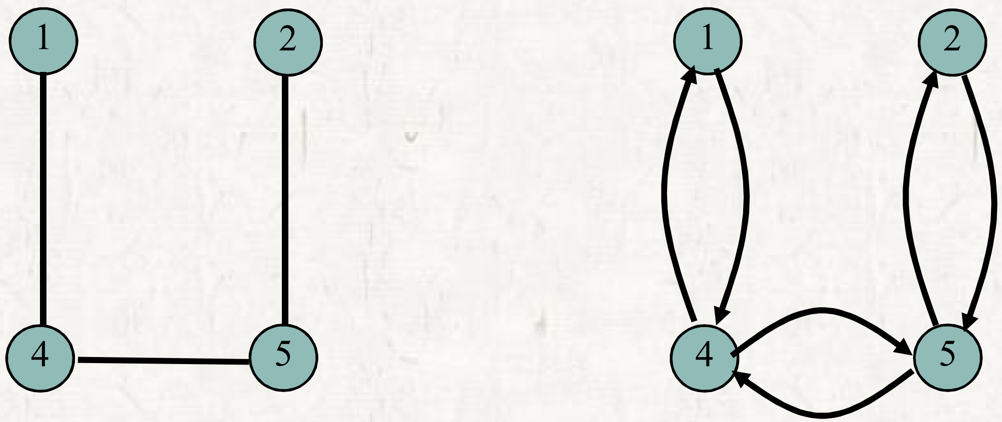
Graph basics

- ***Strongly connected***
  - A directed graph is ***strongly connected*** if every pair of vertices is reachable from each other.
- ***Strongly connected components***
  - Maximally strongly connected subsets of vertices in a directed graph.



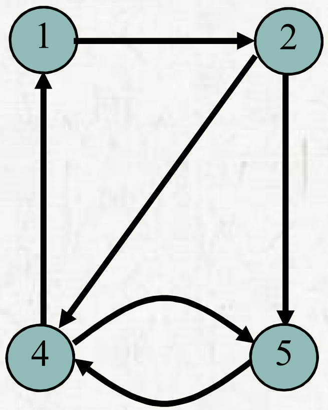
Graph basics

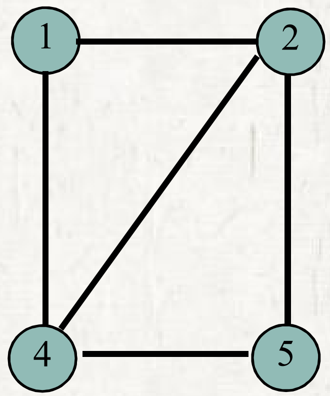
- ***Directed version*** of an undirected graph
  - Replace each undirected edge (u,v) by two directed edges (u,v) and (v,u).



Graph basics

- ***Undirected version*** of a directed graph
  - Replace each directed edge (u,v) by an undirected edge (u,v)





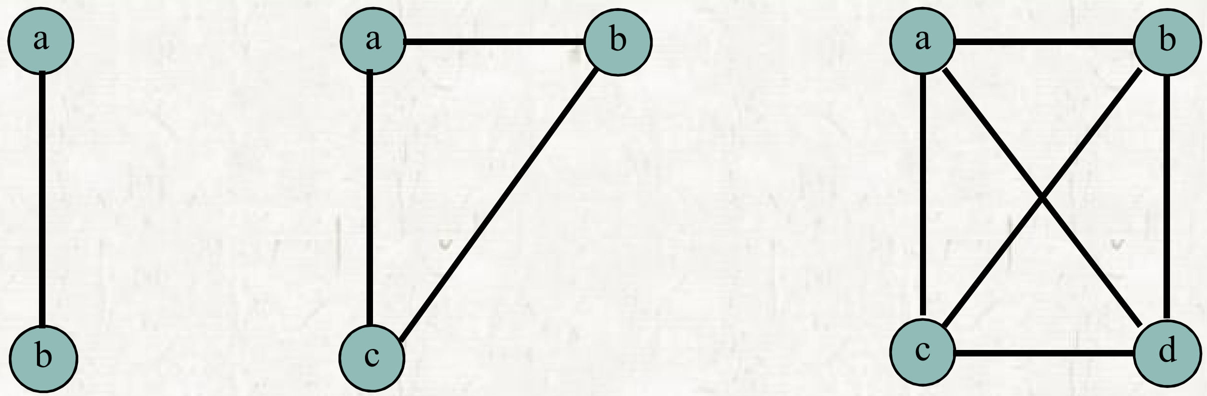
Graph basics

- Undirected graph G → directed ver. G' → undirected ver. G''
  - Are G and G'' the same?

- Directed graph G → undirected ver. G' → directed ver. G''
  - Are G and G'' the same?

Graph basics

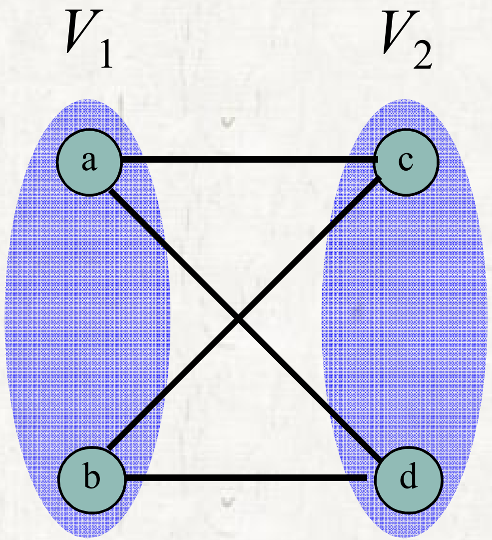
- ***A complete graph***
  - An undirected graph in which every pair of vertices is adjacent.



  - The number of edges with n vertices?

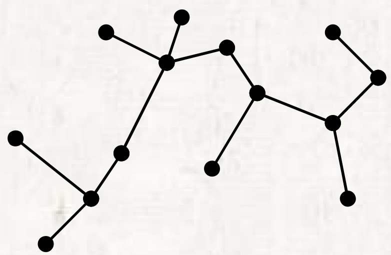
Graph basics

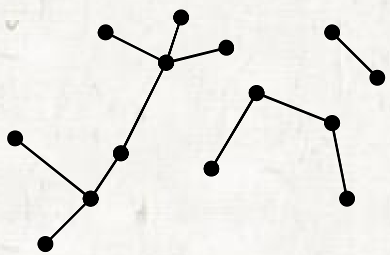
- ***A bipartite graph***
  - An undirected graph $G = (V,E)$ in which
    V can be partitioned into two sets V_1 and V_2 such that
    for each edge (u,v), either $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$.

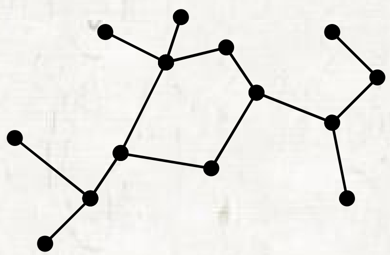


Graph basics

- ***Forest***
  - An acyclic, undirected graph
- ***Tree***
  - A connected forest
  - A connected, acyclic, undirected graph

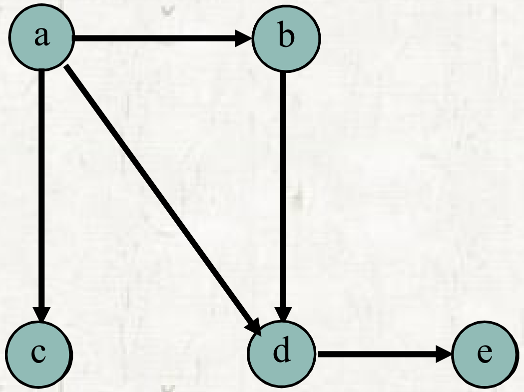






Graph basics

- ***Dag***
  - A directed acyclic graph

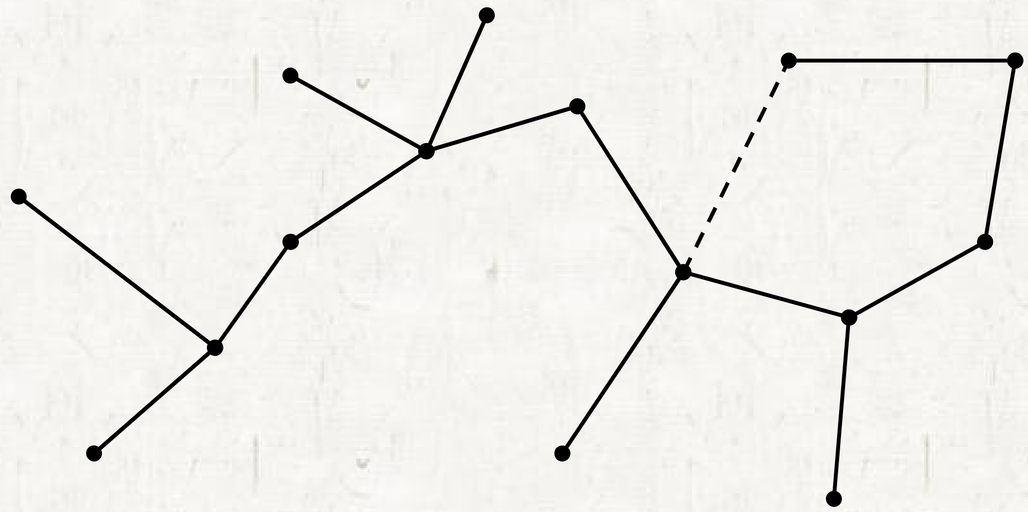


- ***Handshaking lemma***
  - If $G = (V, E)$ is an undirected graph

$$\sum_{v \in V} \text{degree}(v) = 2|E|$$

Graph basics

- **Tree: connected, acyclic, and undirected graph**
  - Any two vertices are connected by a unique simple path.
  - If any edge is removed, the resulting graph is disconnected.
  - If any edge is added, the resulting graph contains a cycle.
  - |E| = |V| - 1



Graph basics

- ***G* is a tree.**

  = G is a connected, acyclic, and undirected graph

  = In G, any two vertices are connected by a unique simple path.

  = G is connected, and if any edge is removed, the resulting graph is disconnected.

  = G is connected, $|E| = |V| - 1$.

  = G is acyclic, $|E| = |V| - 1$.

  = G is acyclic, but if any edge is added, the resulting graph contains a cycle.

Graph basics

- **The number of edges**
  - Directed graph
    - $|E| \le |V|^2$
  - Undirected graph
    - $|E| \le |V| (|V|-1) / 2$

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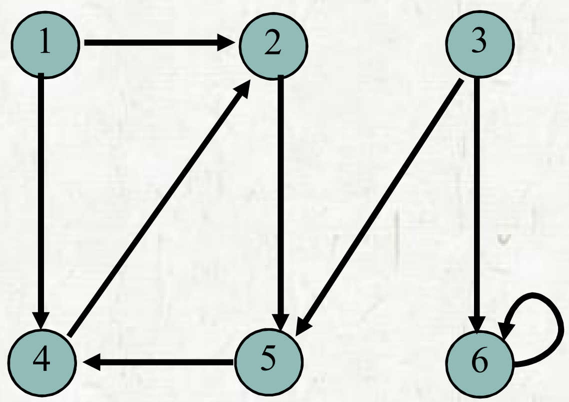
Graph representation

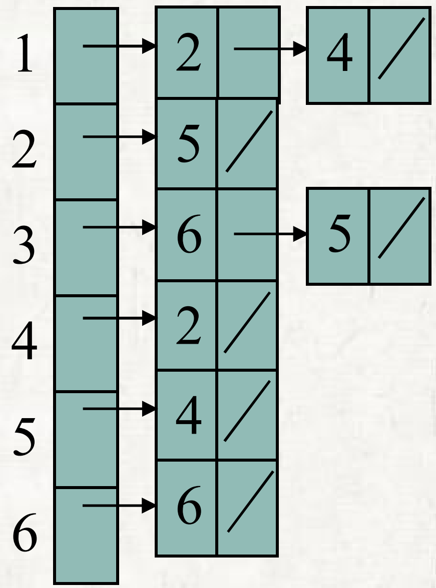
- **Representations of graphs**
  - Adjacency-list representation
  - Adjacency-matrix representation

Graph representation

- **Adjacency-list representation**
  - An array of $|V|$ lists, one for each vertex.
  - For vertex u, its adjacency list contains all vertices adjacent to u.

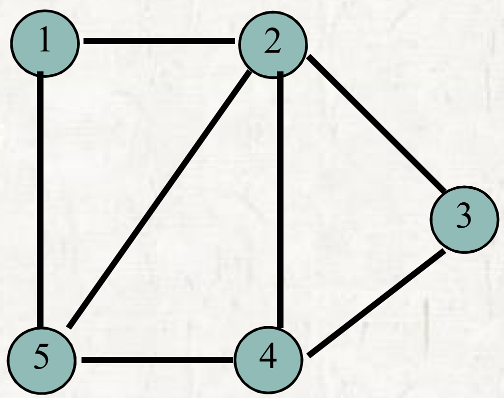


A directed graph

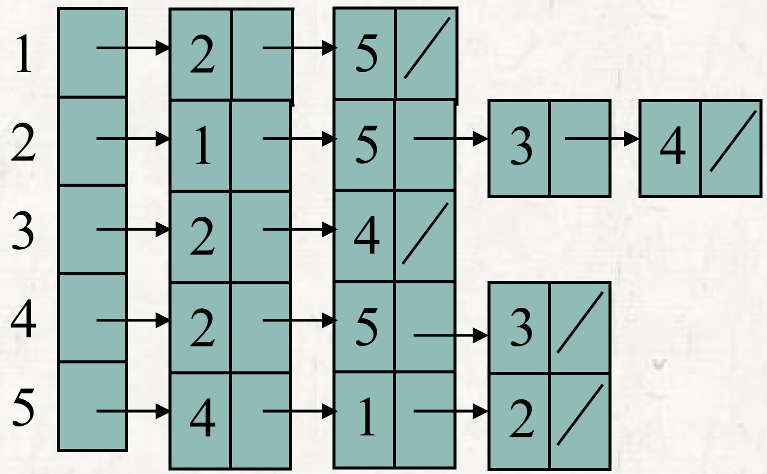


Graph representation

- **Adjacency-list representation**
  - For an undirected graph, its directed version is stored.



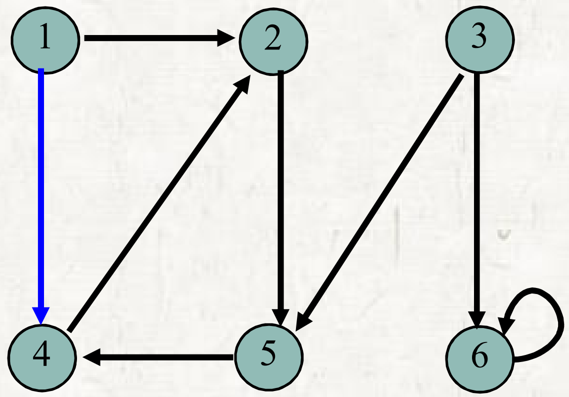
An undirected graph



  - $\Theta(V + E)$ space

Graph representation

- **Adjacency-matrix representation**
  - $|V| \times |V|$ matrix: $\Theta(V^2)$ space
  - Entry (i,j) is 1 if there is an edge and 0 otherwise.

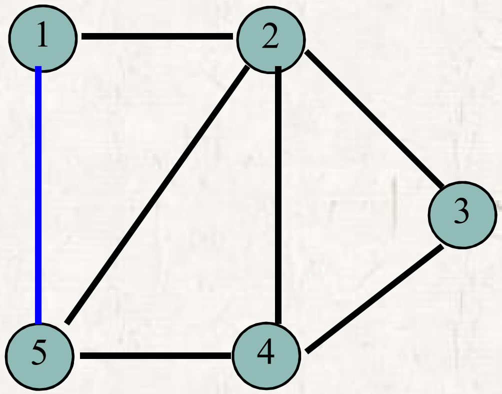


A directed graph

	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

Graph representation

- **Adjacency-matrix representation**
  - $|V| \times |V|$ matrix
  - Entry (i,j) is 1 if there is an edge and 0 otherwise.

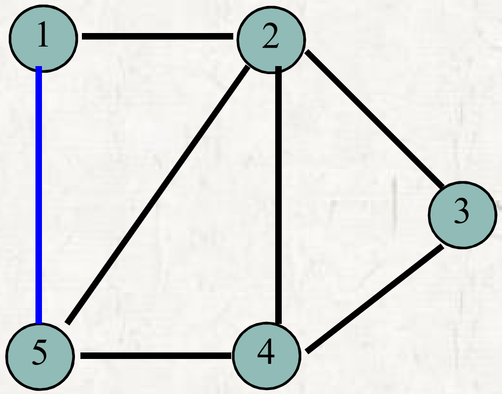


An undirected graph

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

Graph representation

- **Adjacency-matrix representation**
  - For an undirected graph, there is a symmetry along the main diagonal of its adjacency matrix.
  - Storing the lower matrix is enough.



An undirected graph

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

Graph representation

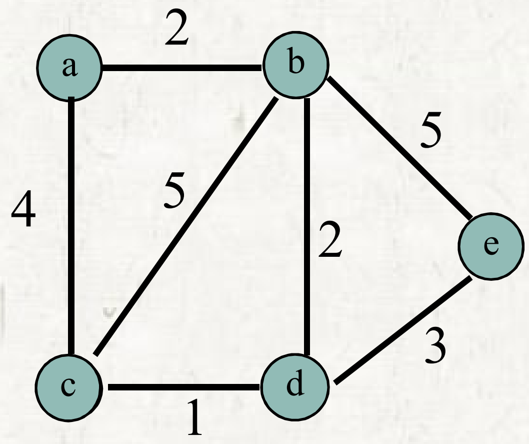
- **Comparison of adjacency list an adjacency matrix**
  - Storage
    - If G is sparse, adjacency list is better.
      - because $|E| < |V|^2$.
    - If G is dense, adjacency matrix is better.
      - because adjacency matrix uses only one bit for an entry.
  - Edge present test: does an edge (i,j) exist?
    - Adjacency matrix: $\Theta(1)$ time.
    - Adjacency list: $O(V)$ time.

Graph representation

- **Comparison of adjacency list and adjacency matrix**
  - Listing or visiting all edges
    - Adjacency matrix: $\Theta(V^2)$ time.
    - Adjacency list: $\Theta(V + E)$ time.

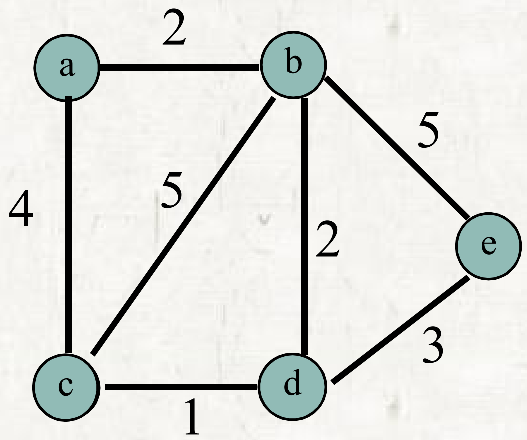
Graph representation

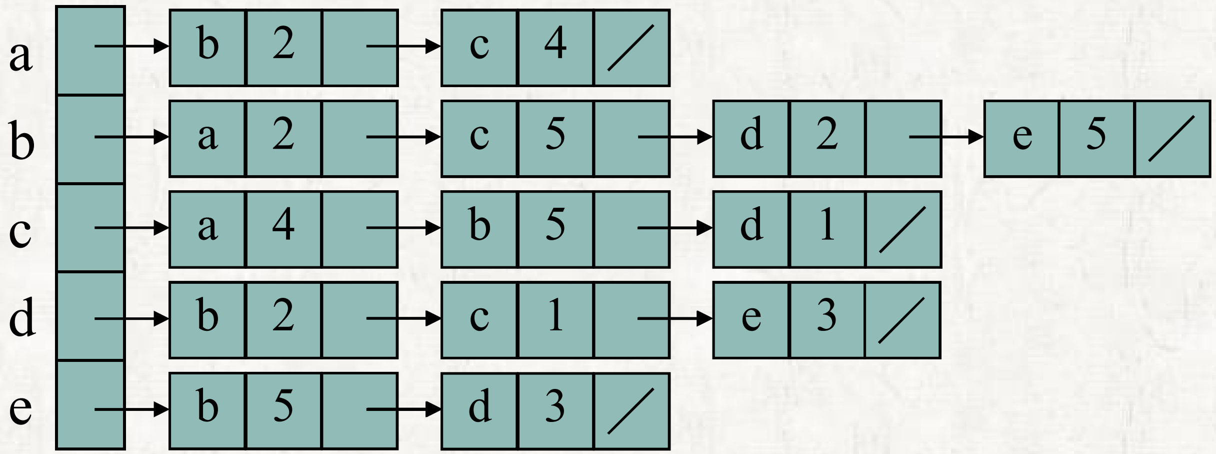
- **Weighted graph**
  - Edges have weights.



Graph representation

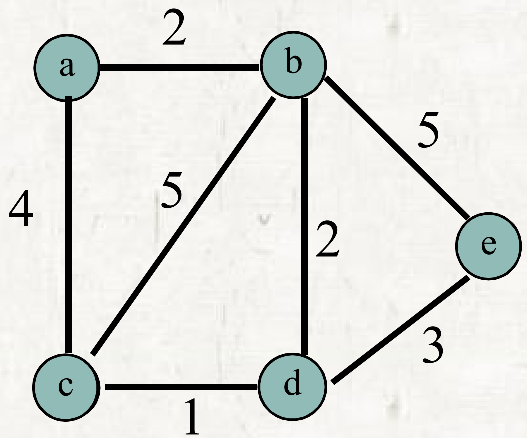
- **Weighted graph representation**
  - adjacency list





Graph representation

- **Weighted graph representation**
  - adjacency matrix
    - $\Theta(V^2)$ space



	a	b	c	d	e
a	0	2	4	0	0
b	2	0	5	2	5
c	4	5	0	1	0
d	0	2	1	0	3
e	0	5	0	3	0

Graph representation

- **Transpose of a matrix**
  - The ***transpose*** of a matrix $A = (a_{ij})$ is
  - $A^T = (a_{ij}^T)$ where $a_{ij}^T = a_{ji}$
  - An undirected graph is its own transpose: $A = A^T$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Self-study

- **Exercise 22.1-3**
  - The transpose of a directed graph
- **Exercise 22.1-4**
  - Removing duplicate edges in a multigraph in $O(V+E)$ time.
- **Exercise 22.1-6**
  - Universal sink detection in $O(V)$ time.

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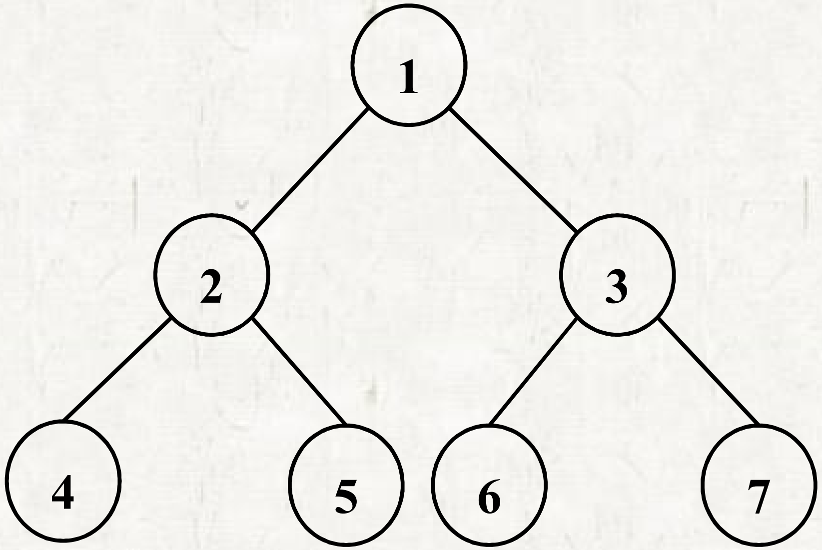
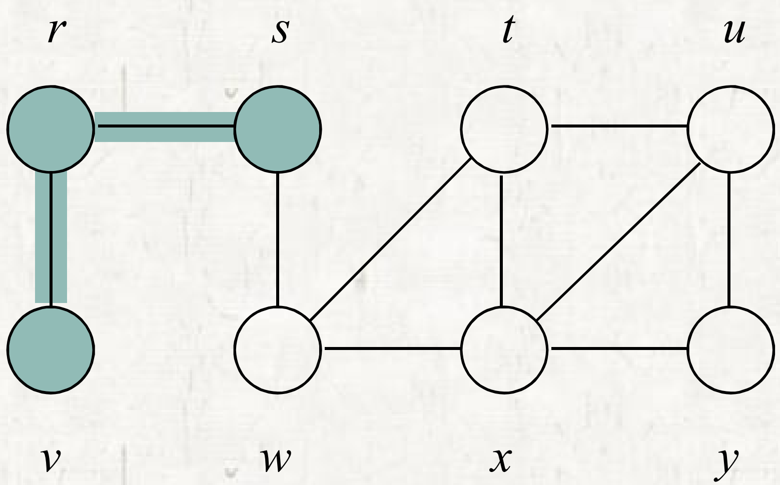
Searching a tree



- Breadth-first search
- Depth-first search

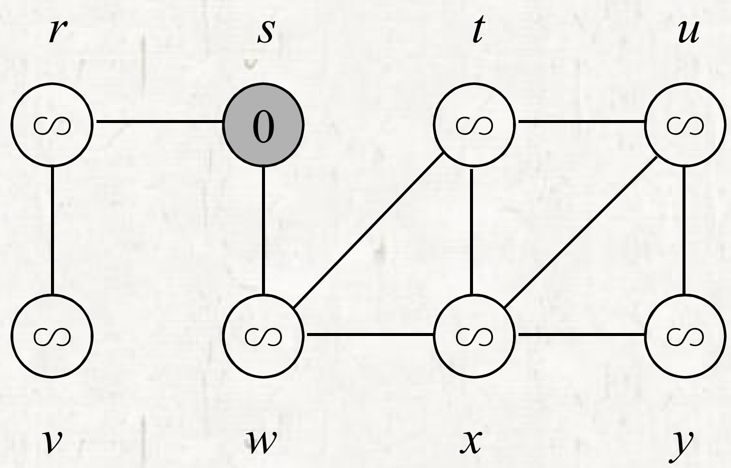
Breadth-first search

- ***Distance***
  - Distance from u to v
    - The number of edges in the shortest path from u to v.
    - The distance from s to v is 2.



Breadth-first search

- **Breadth-first search**
  - Given a graph $G = (V, E)$ and a ***source*** vertex s, it explores the edges of G to "discover" every reachable vertex from s.
  - It discovers vertices in the increasing order of distance from the source. It first discovers all vertices at distance 1, then 2, and etc.



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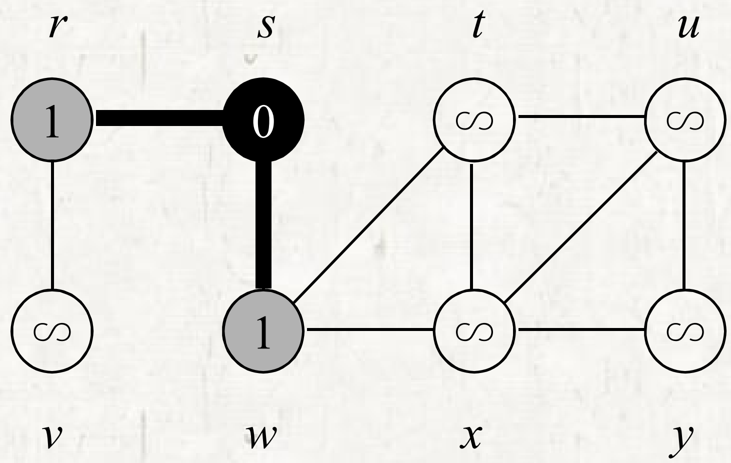

Breadth-first search

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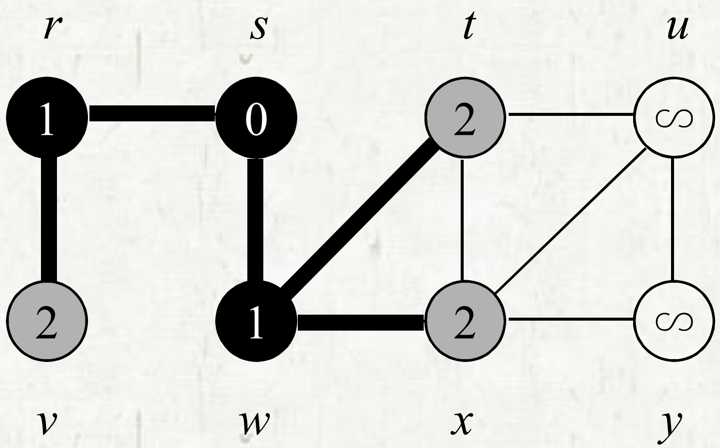
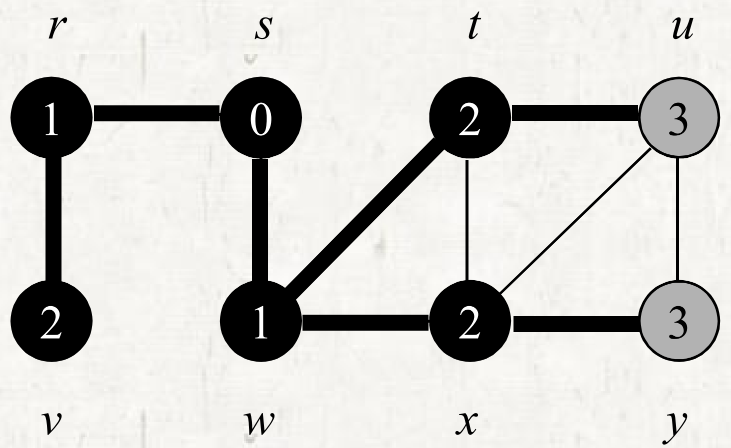


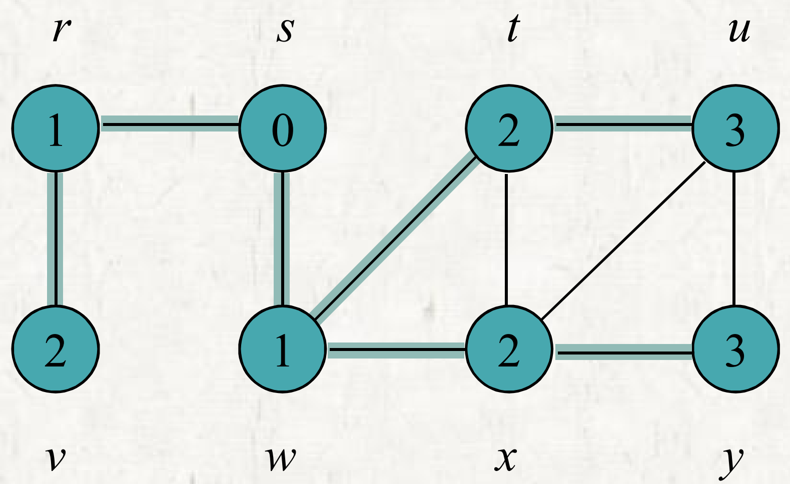
Breadth-first search

- **Breadth-first search**
  - Given a graph $G = (V, E)$ and a ***source*** vertex s, it explores the edges of G to "discover" every reachable vertex from s.
  - It discovers vertices in the increasing order of distance from the source. It first discovers all vertices at distance 1, then 2, and etc.



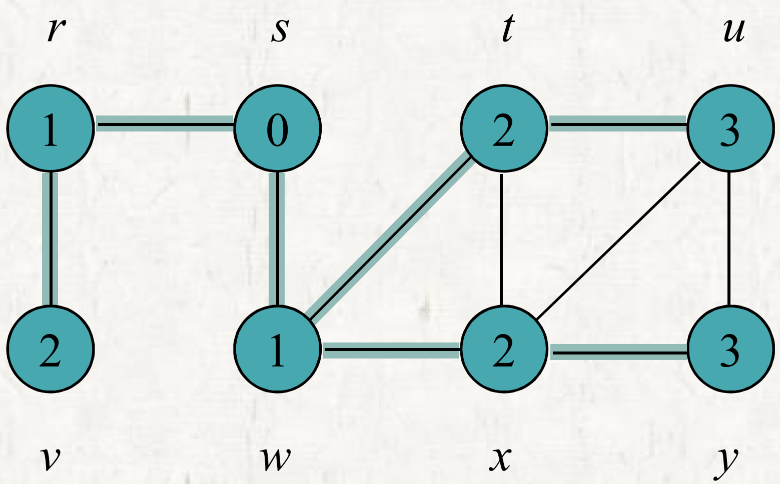
Breadth-first search

- **Breadth-first search**
  - It also computes
    - the distance of vertices from the source: $u.d = 3$
    - the predecessor of vertices: $u.\pi = t$



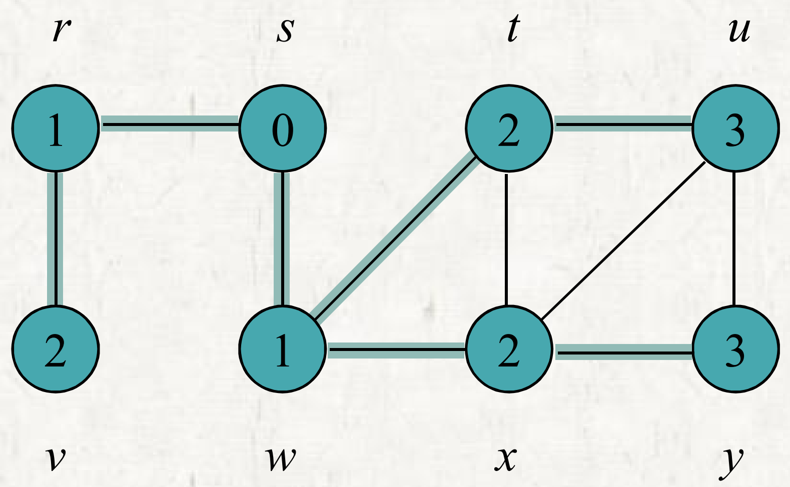
Breadth-first search

- The ***predecessor subgraph*** of G as $G_\pi = (V_\pi, E_\pi)$,
  - $V_\pi = \{v \in V: v.\pi \neq \text{NIL}\} \cup \{s\}$
  - $E_\pi = \{(v.\pi, v) : v \in V_\pi - \{s\}\}$.



Breadth-first search

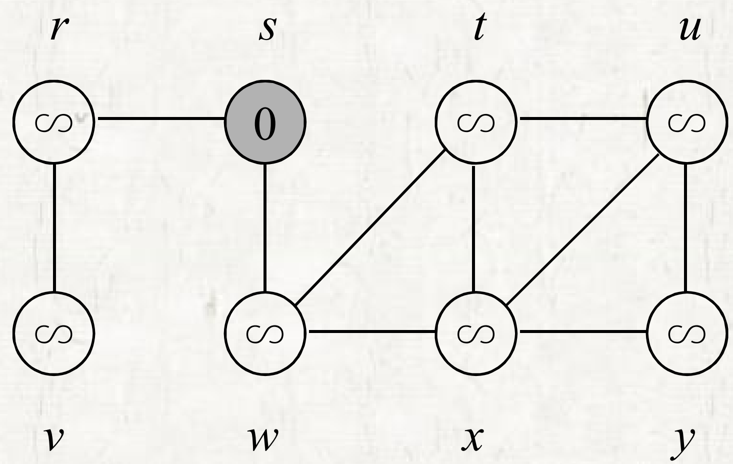
- The predecessor subgraph G_π is a ***breadth-first tree.***
  - since it is connected and $|E_\pi| = |V_\pi|$ -1.
  - The edges in E_π are called ***tree edges***.

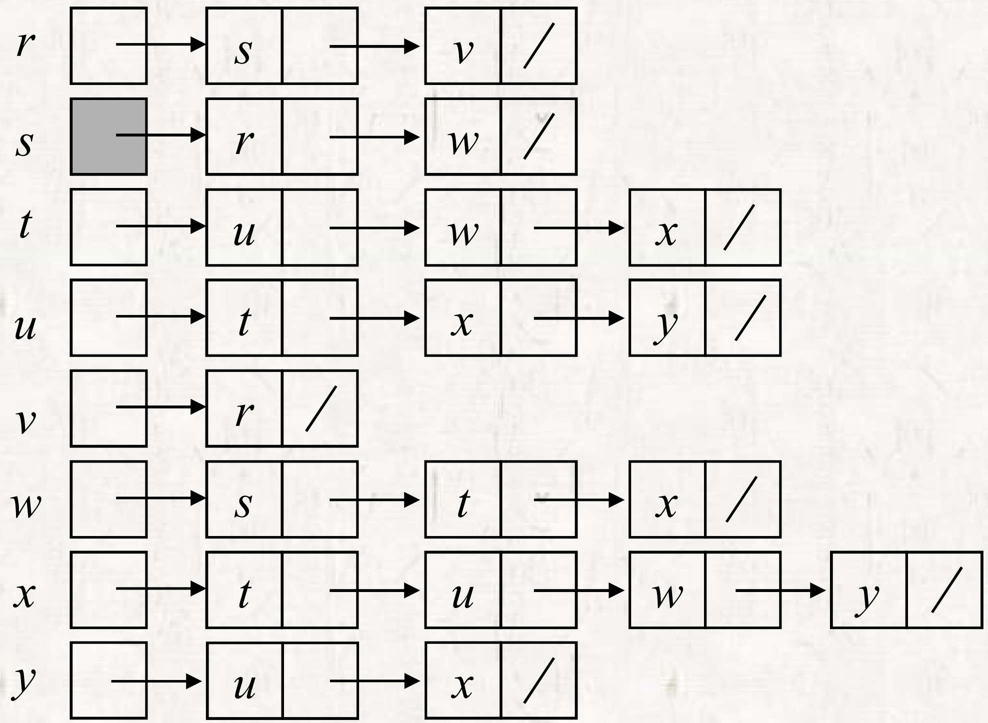


Breadth-first search

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
        u.color = WHITE
        u.d = ∞
        u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = Ø
ENQUEUE(Q, s)
while Q ≠ Ø
        u = DEQUEUE(Q)
        for each v ∈ G.Adj[u]
                if v.color == WHITE
                        v.color = GRAY
                        v.d = u.d + 1
                        v.π = u
                        ENQUEUE(Q, v)
        u.color = BLACK
```

Breadth-first search

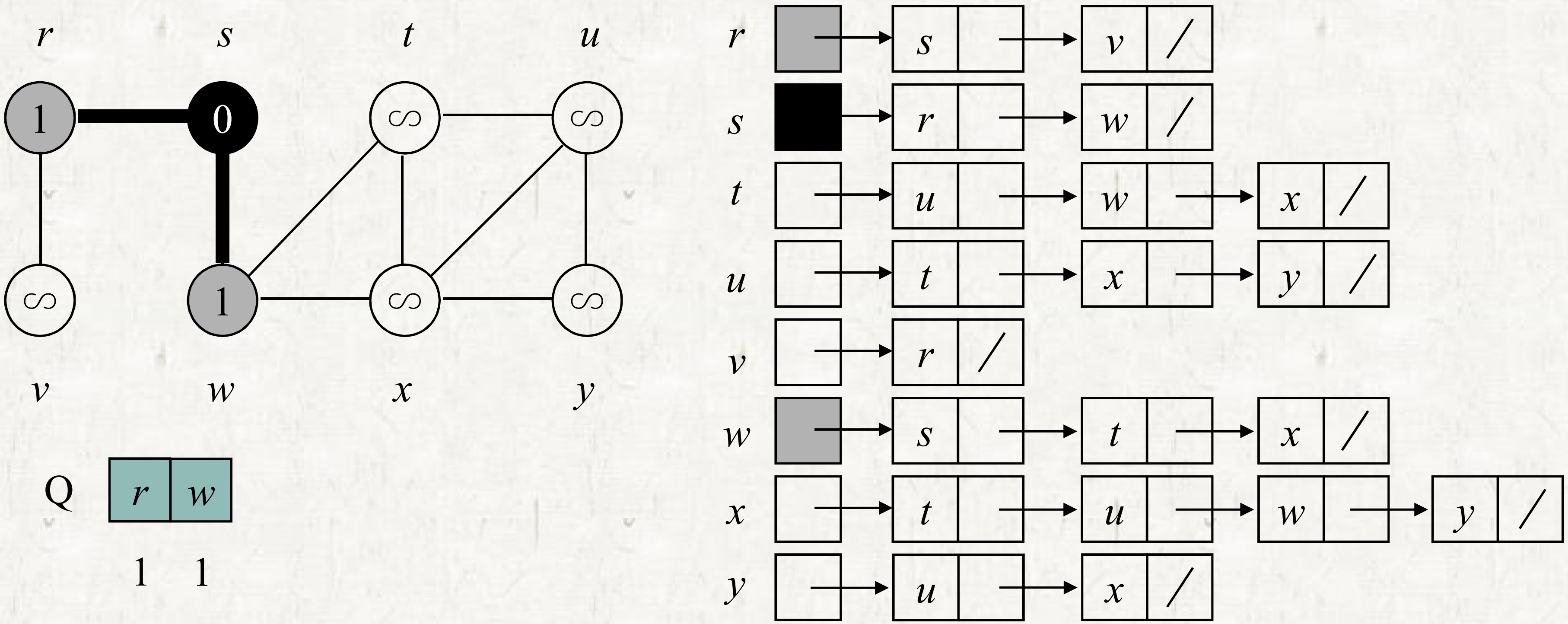




Q s

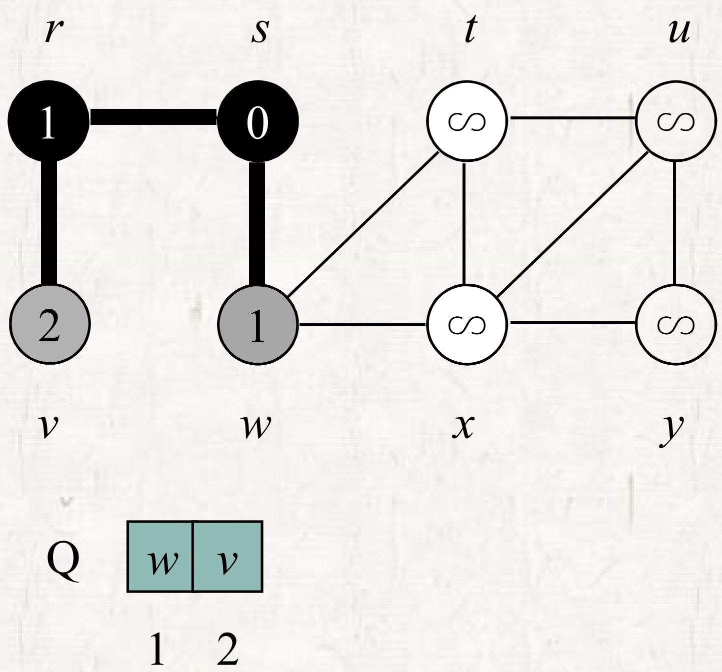
0

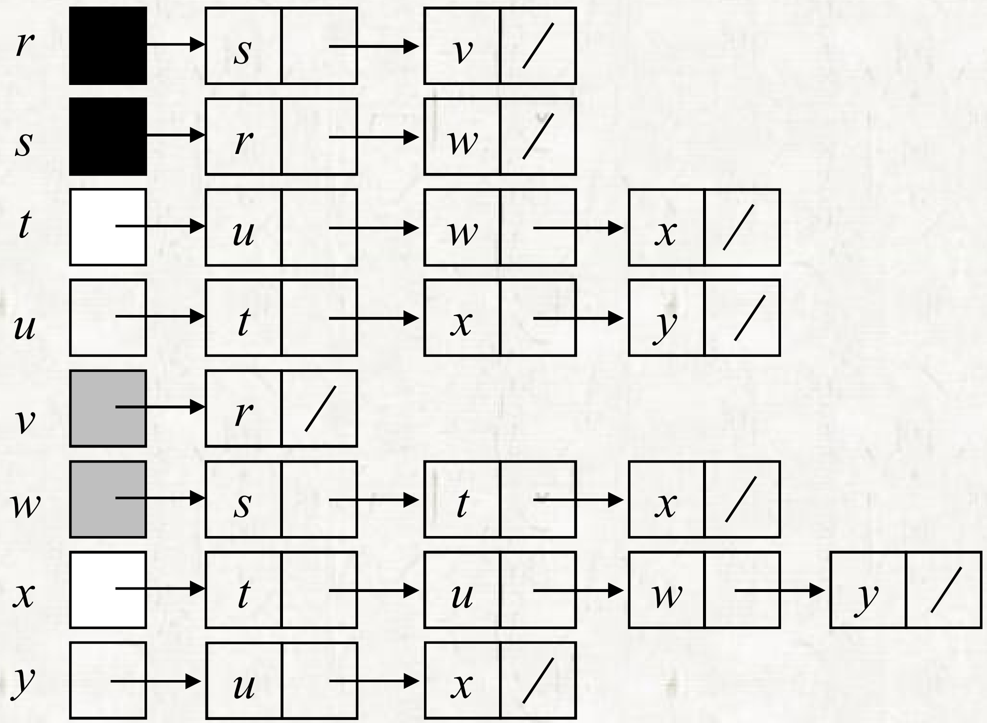
Breadth-first search



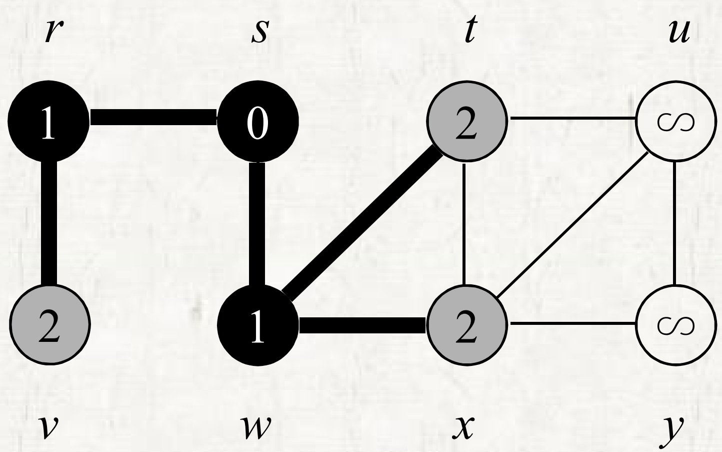
- white: not discovered (not entered the Q)
- gray: discovered (in the Q)
- black: finished (out of the Q)

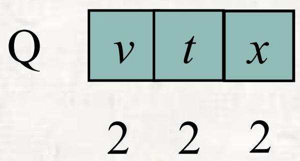
Breadth-first search

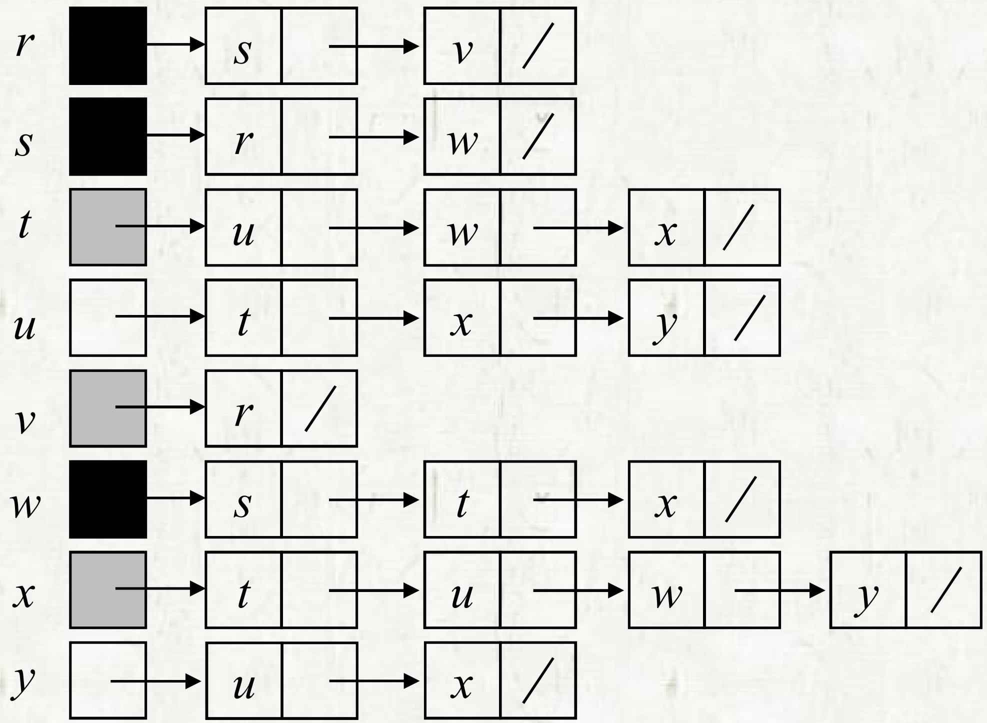




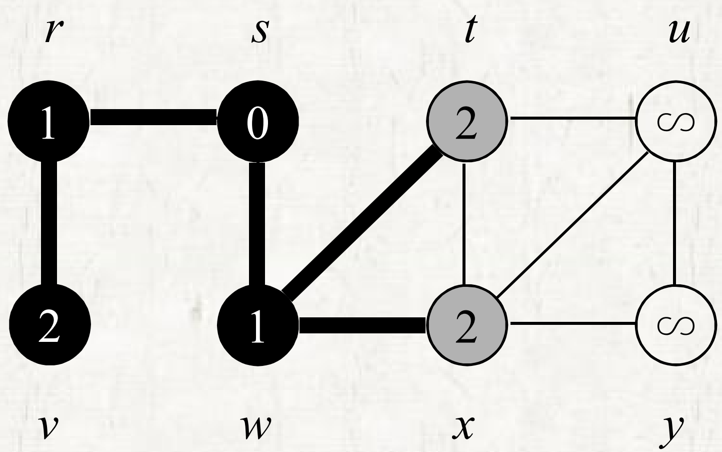
Breadth-first search



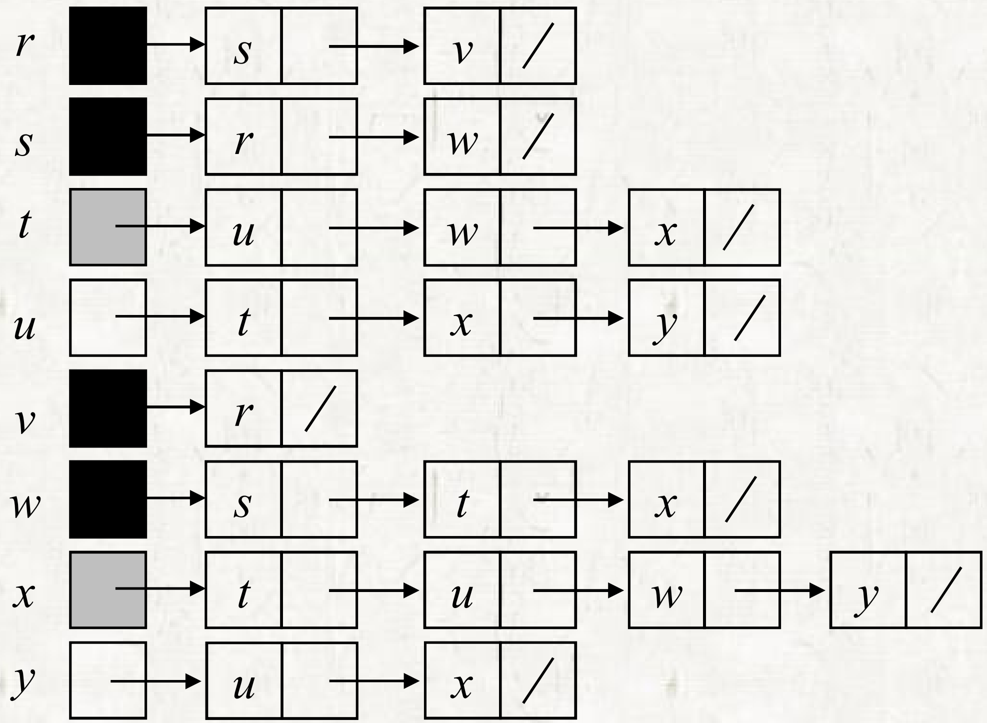




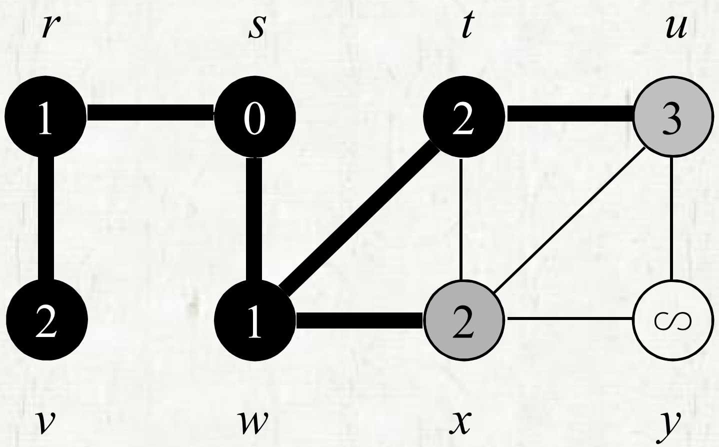
Breadth-first search

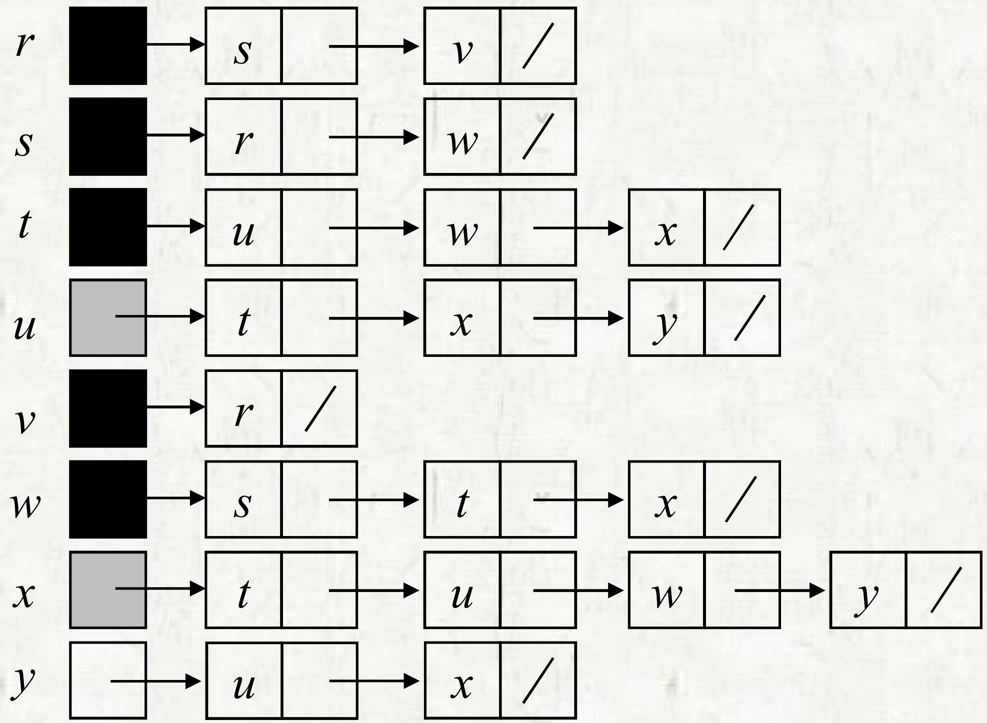


Q	t	x
	2	2



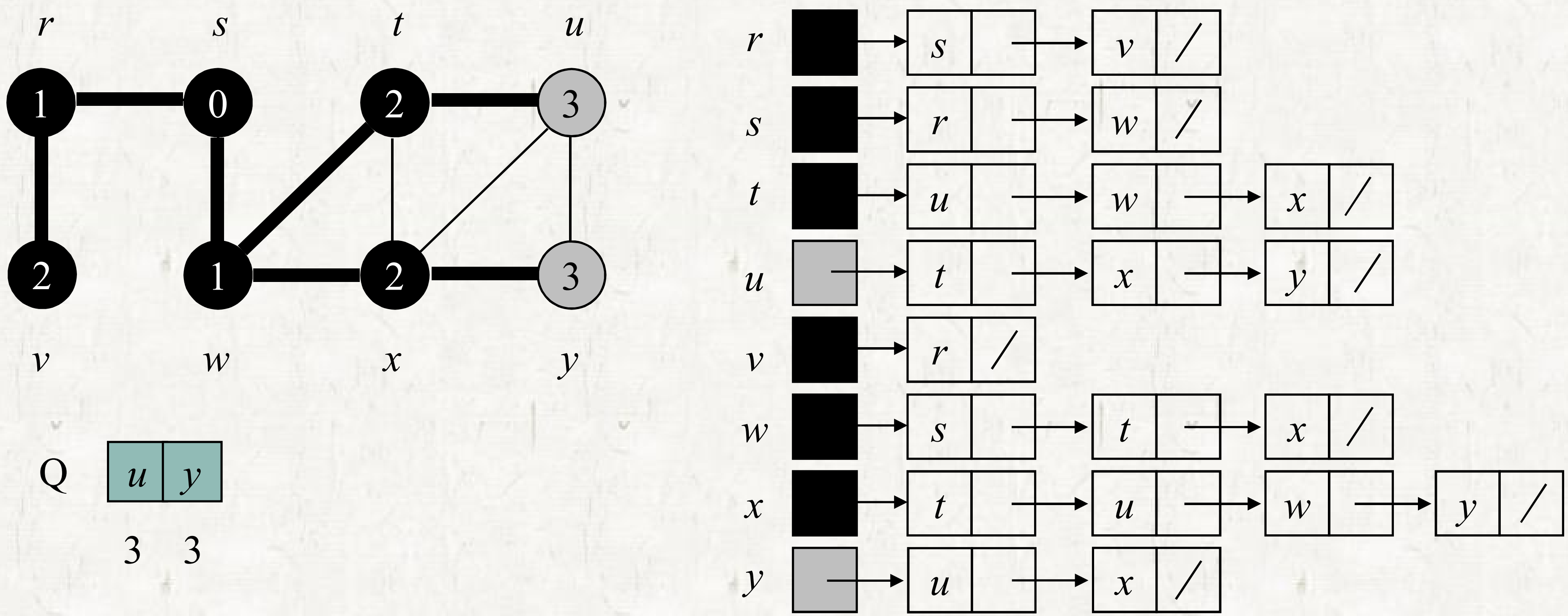
Breadth-first search



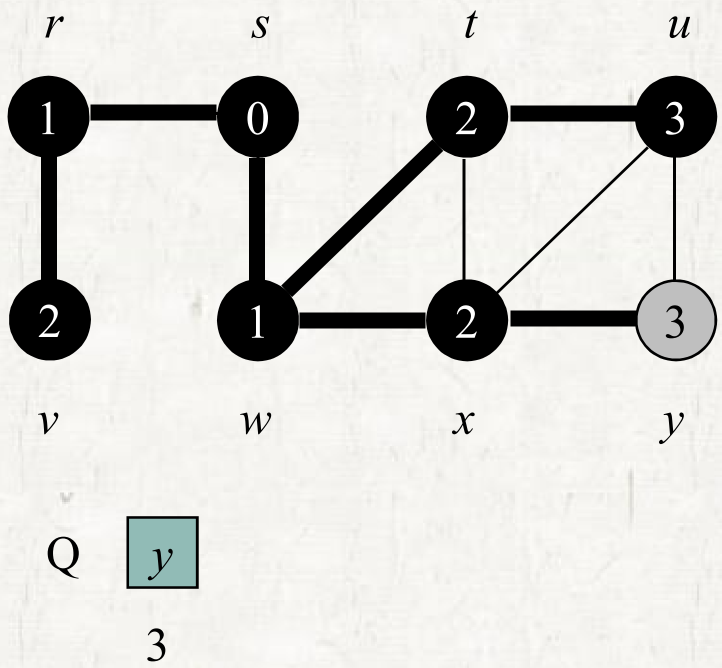


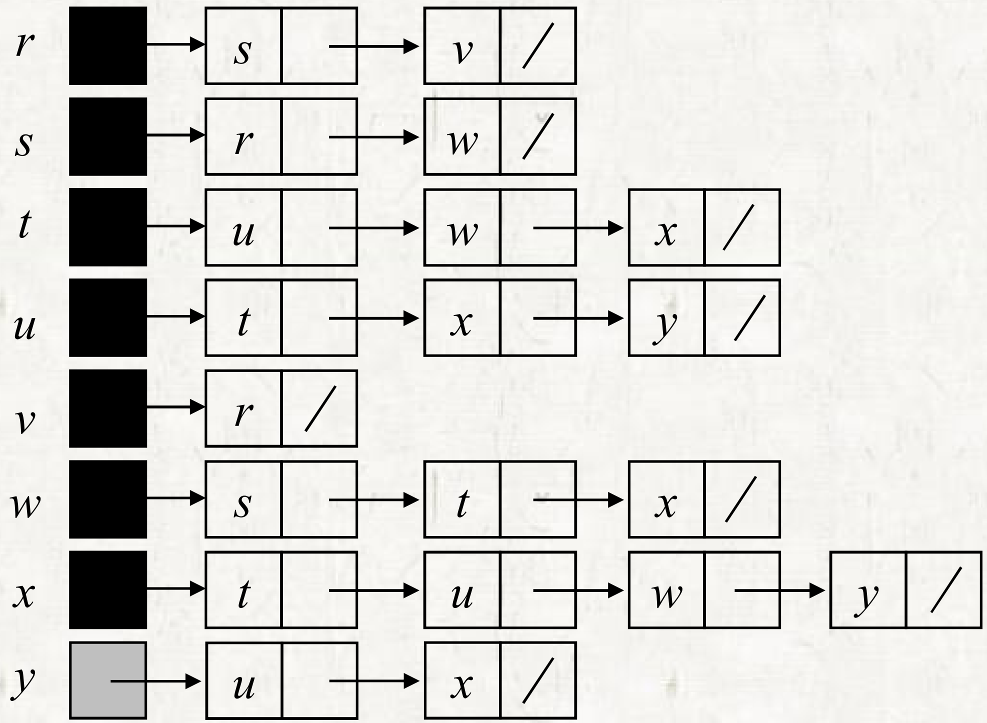
Q	x	u
	2	3

Breadth-first search

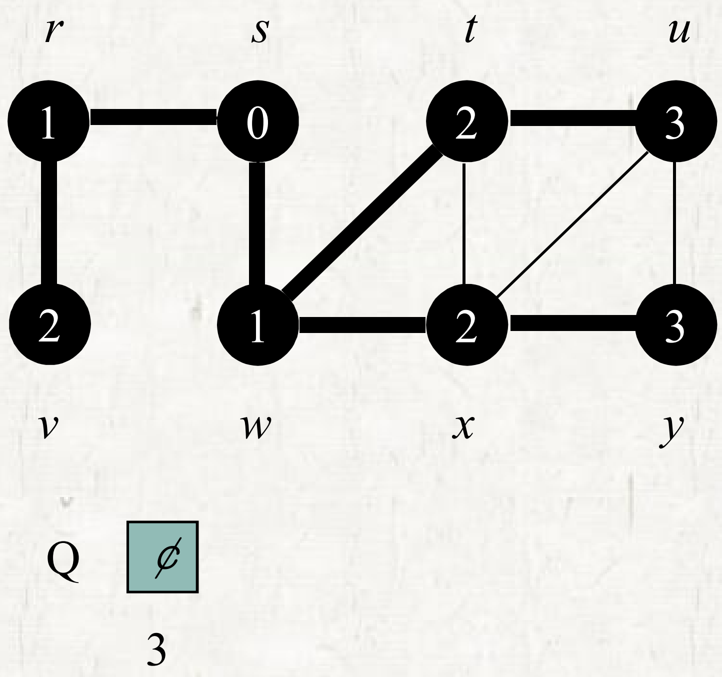


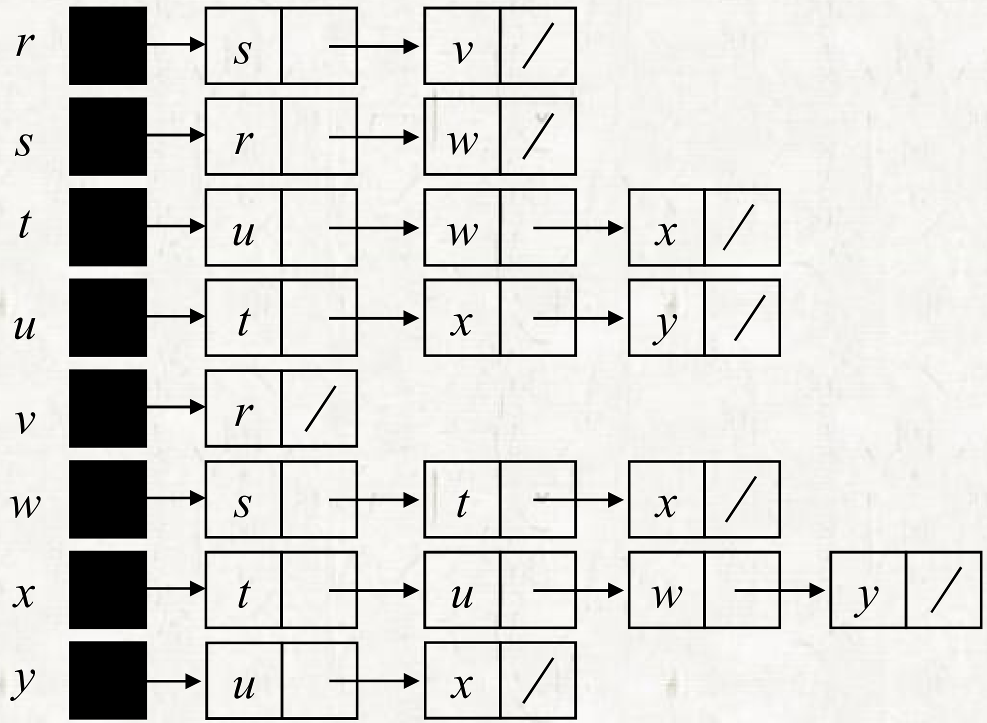
Breadth-first search





Breadth-first search





Breadth-first search

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
        u.color = WHITE
        u.d  = ∞
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s.color = GRAY
s.d = 0
s.π = NIL
Q = Ø
ENQUEUE(Q, s)
while Q ≠ Ø
        u = DEQUEUE(Q)
        for each v ∈ G.Adj[u]
                if  v.color == WHITE
                        v.color = GRAY
                        v.d = u.d + 1
                        v.π = u
                        ENQUEUE(Q, v)
        u.color = BLACK
```

Breadth-first search

- **Running time**
  - **Initialization**: $\Theta(V)$
  - **Exploring the graph**: $O(V + E)$
    - A vertex is examined at most once.
    - An edge is explored at most twice.
  - **Overall:** $O(V + E)$

Self-study

- **Exercise 22.2-4 (22.2-3 in the 2^{nd} ed.)**
  - The running time of BFS
    with adjacency matrix representation.

- **Exercise 22.2-6 (22.2-5 in the 2^{nd} ed.)**
  - Impossible breadth-first trees.

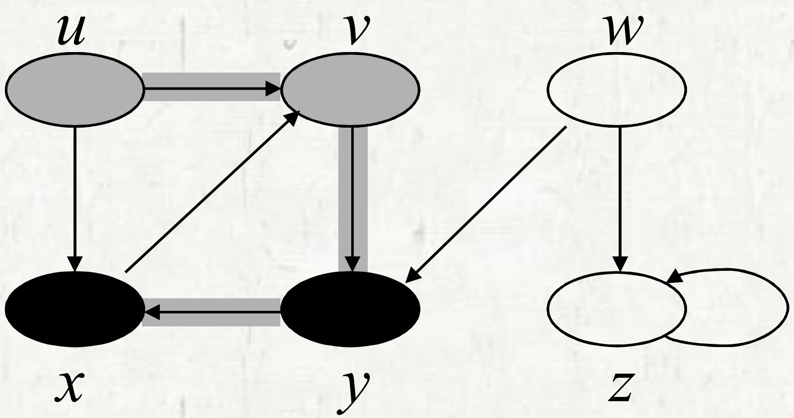
- **Exercise 22.2-7 (22.2-6 in the 2^{nd} ed.)**
  - Rivalry

Contents

- ***Graphs***
  - *Graphs basics*
  - *Graph representation*
- ***Searching a graph***
  - *Breadth-first search*
  - Depth-first search
- **Applications of depth-first search**
  - Topological sort

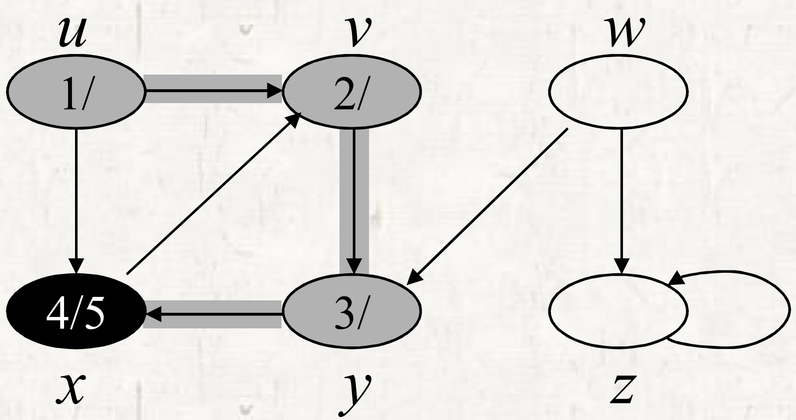
Depth-first search

- ***Colors of vertices***
  - Each vertex is initially ***white***. (not discovered)
  - The vertex is ***grayed*** when it is ***discovered.***
  - The vertex is ***blackened*** when it is ***finished***, that is, when its adjacency list has been examined completely.

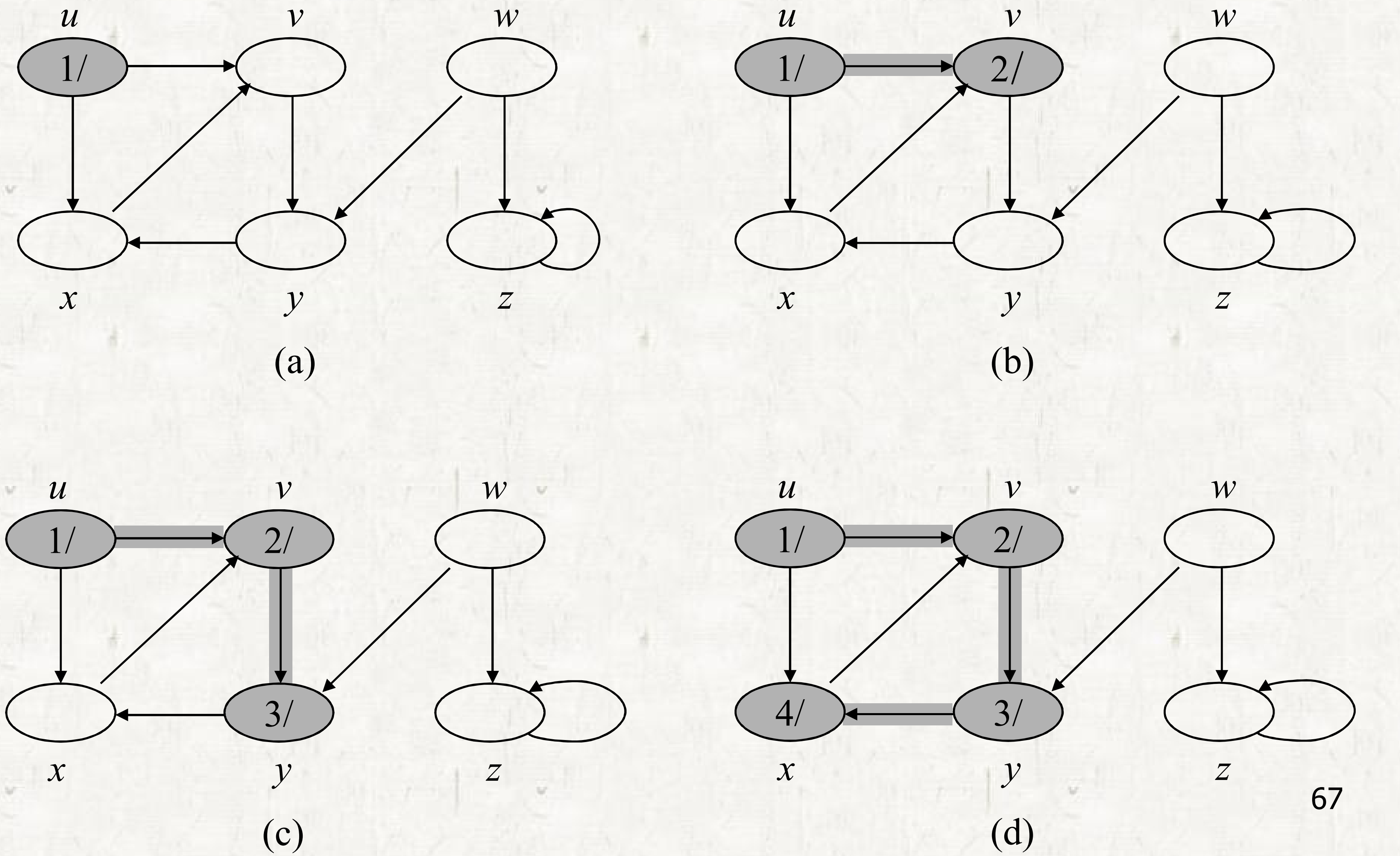


Depth-first search

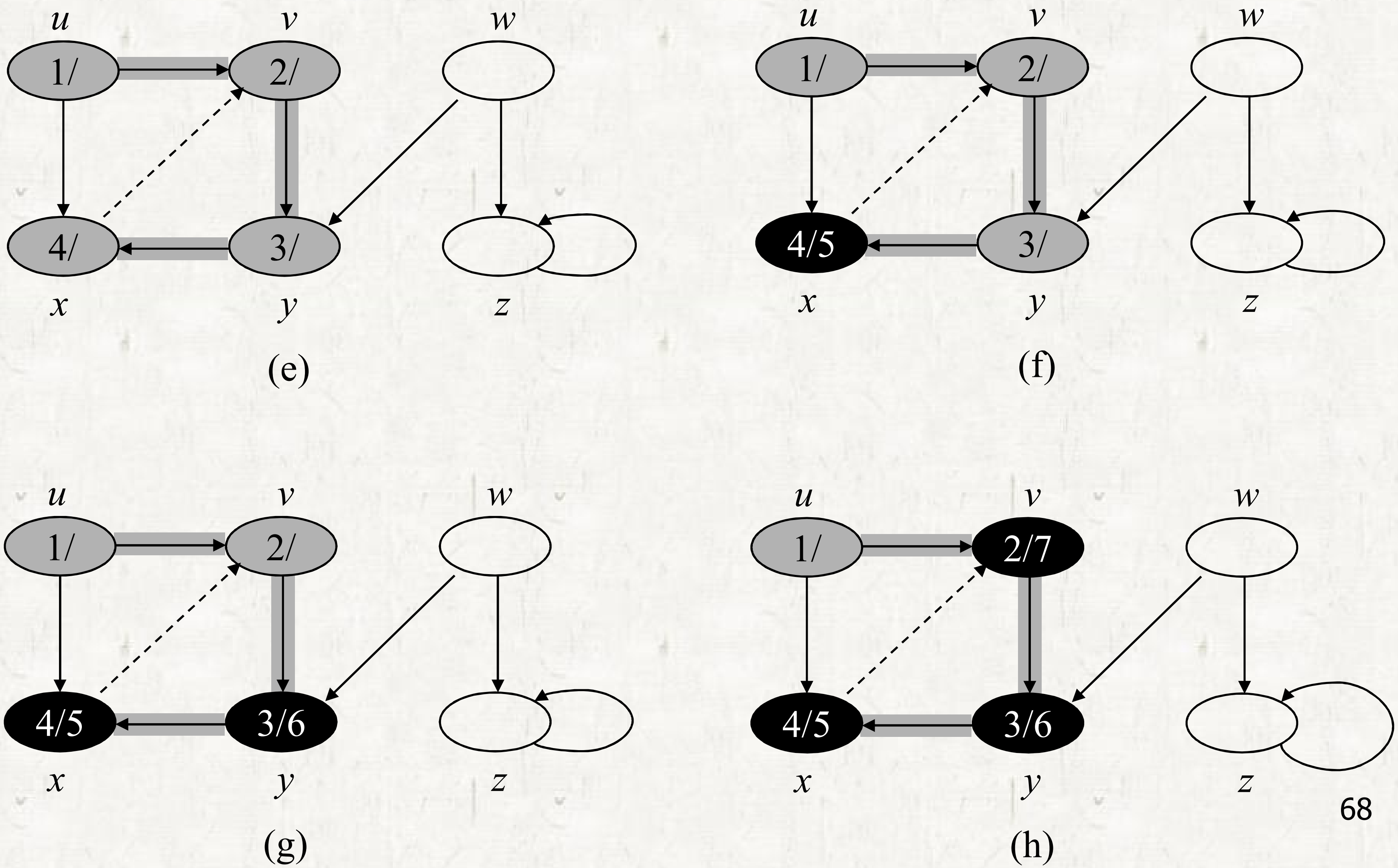
- ***Timestamps***
  - Each vertex *v* has two timestamps.
    - *v.d*: ***discovery time*** (when *v* is grayed)
    - *v.f*: ***finishing time*** (when *v* is blacken)



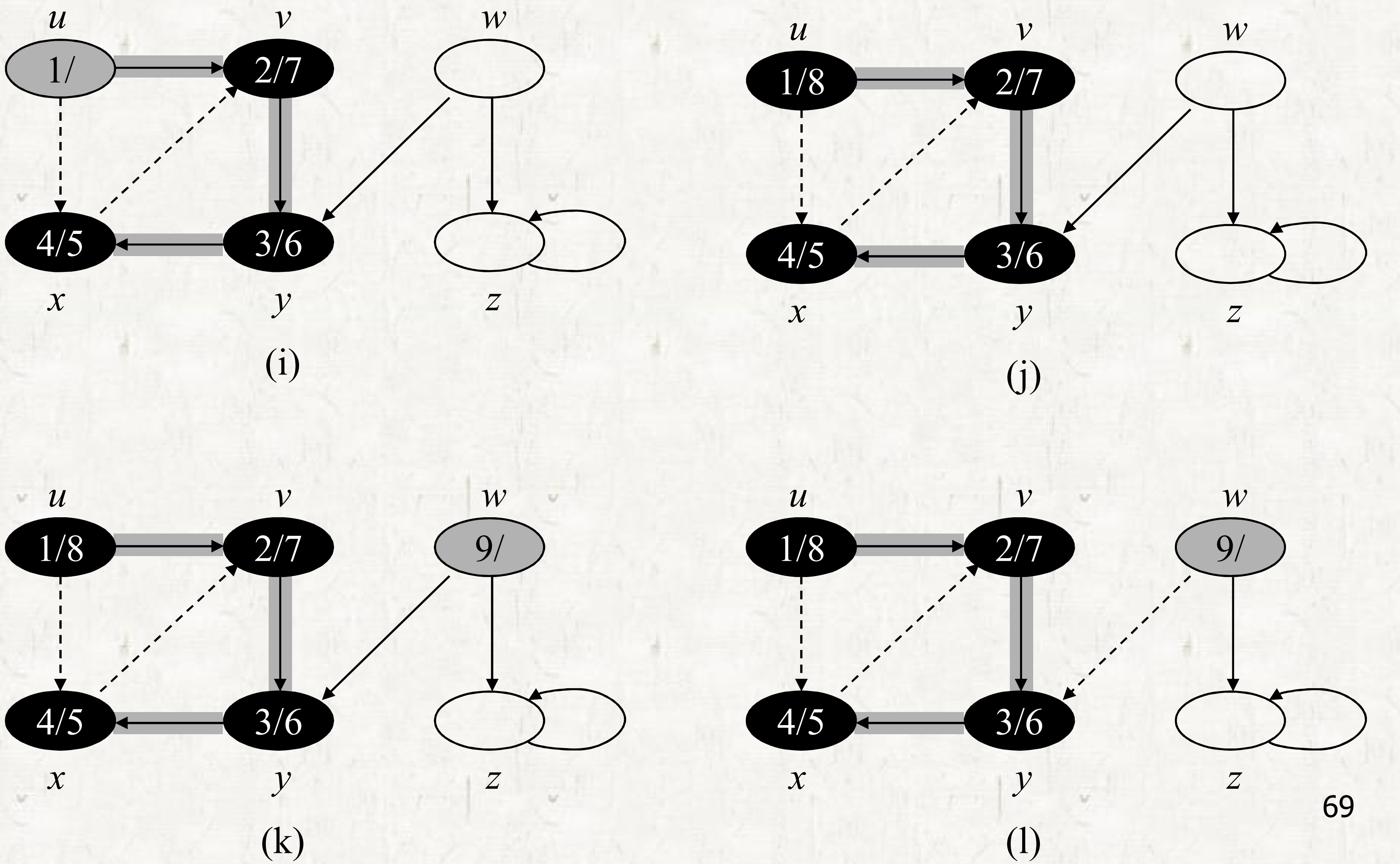
Depth-first search



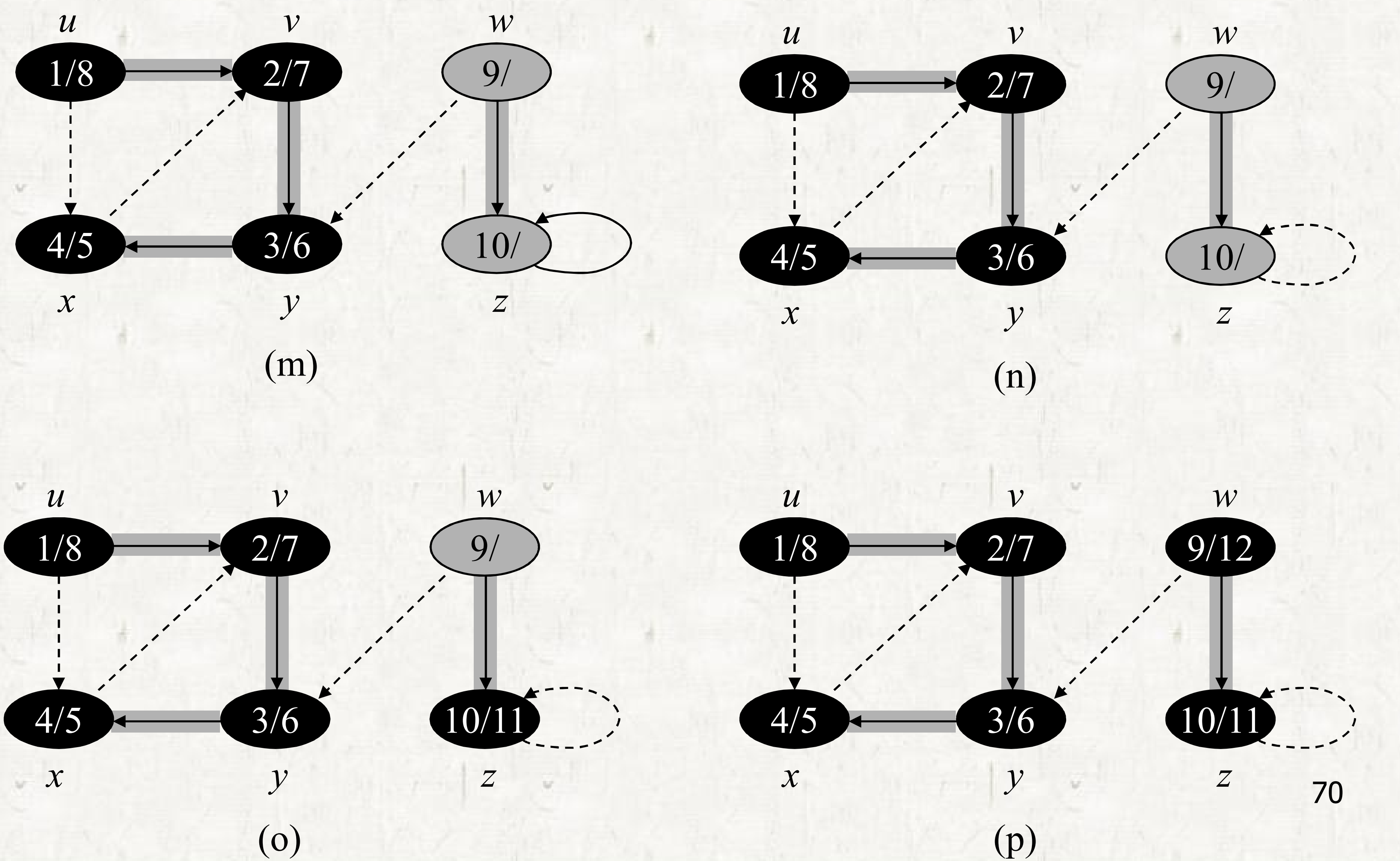
Depth-first search



Depth-first search

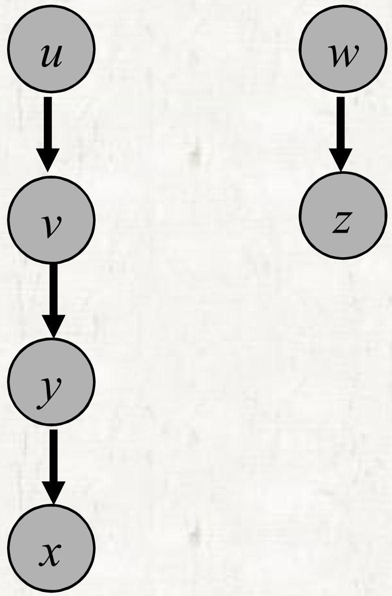


Depth-first search



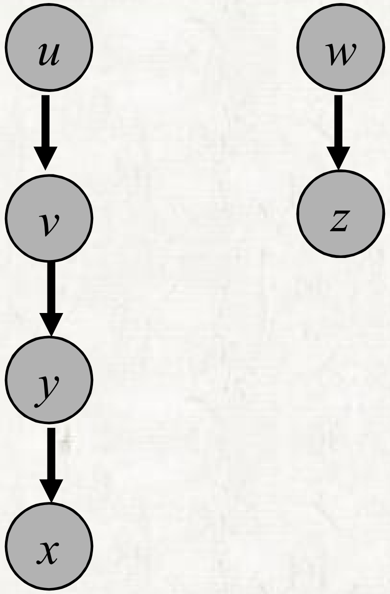
Depth-first search

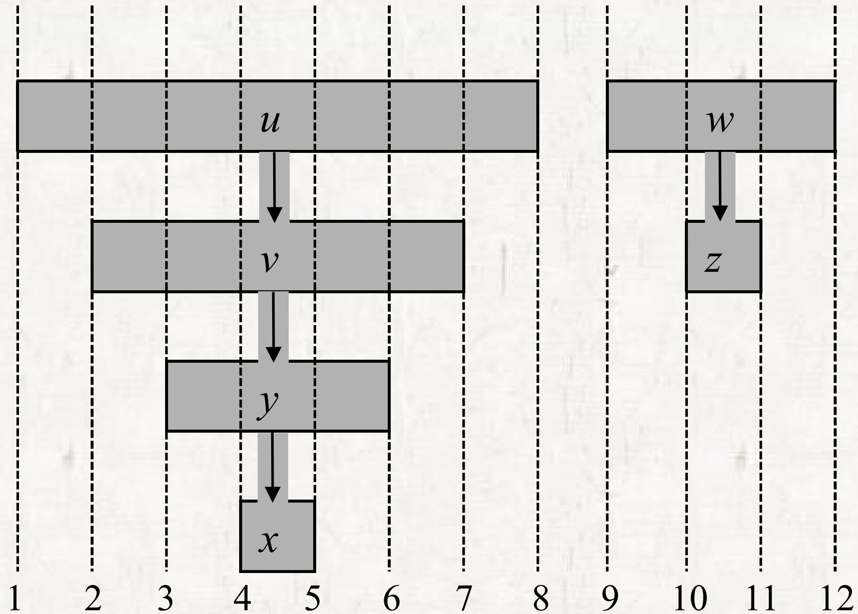
- The ***predecessor subgraph*** is a ***depth-first forest***.



Depth-first search

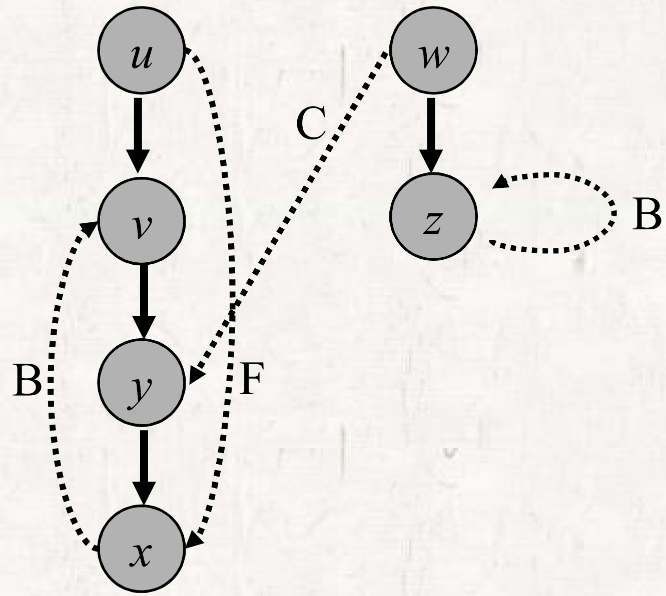
- ***Parenthesis theorem (for gray interval)***
  - ***Inclusion***: The ancestor's includes the descendants'.
  - ***Disjoint***: Otherwise.





Depth-first search

- **Classification of edges**
  - ***Tree edges***
  - ***Back edges***
  - ***Forward edges***
  - ***Cross edges***



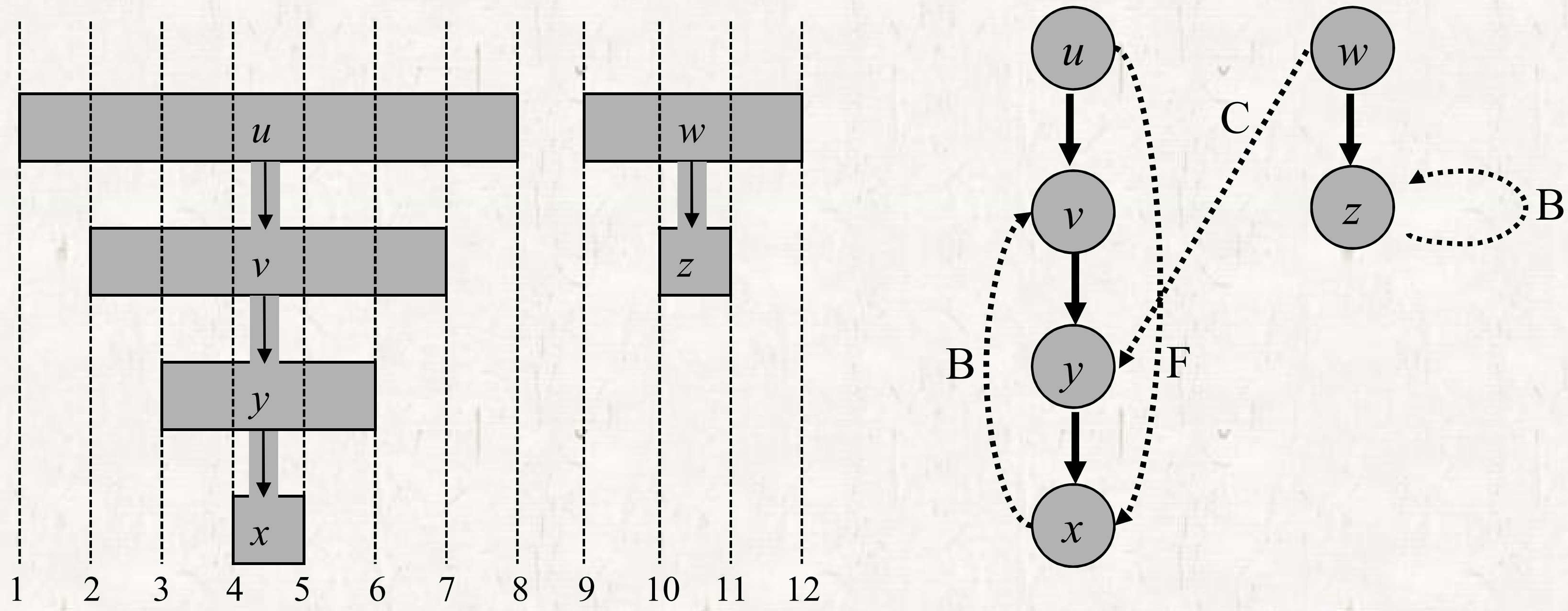
Depth-first search

- **Classification of edges**
  - ***Tree edges*:** Edges in the depth-first forest.
  - ***Back edges***: Those edges (u, v) connecting a vertex u to an ancestor v in a depth-first tree. Self-loops are considered to be back edges.
  - ***Forward edges***: Those edges (u, v) connecting a vertex u to a descendant v in a depth-first tree.
  - ***Cross edges***: All other edges. They can go between vertices in the same depth-first tree, as long as one vertex is not an ancestor of the other, or they can go between vertices in different depth-first trees.

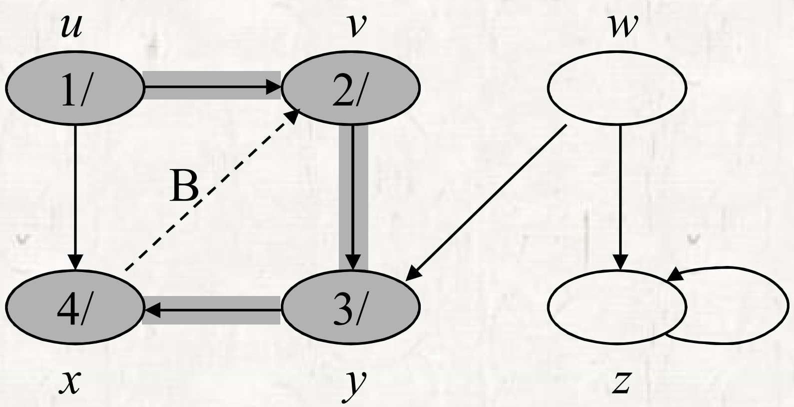
Depth-first search

- **Classification by the DFS algorithm**
  - Each edge (u, v) can be classified by the color of the vertex v that is reached when the edge is first explored:
    - ***white*** indicates a tree edge,
    - ***gray*** indicates a back edge, and
    - ***black*** indicates a forward or cross edge.
  - Forward and cross edges are classified by ***the inclusion of gray intervals of u and v***.

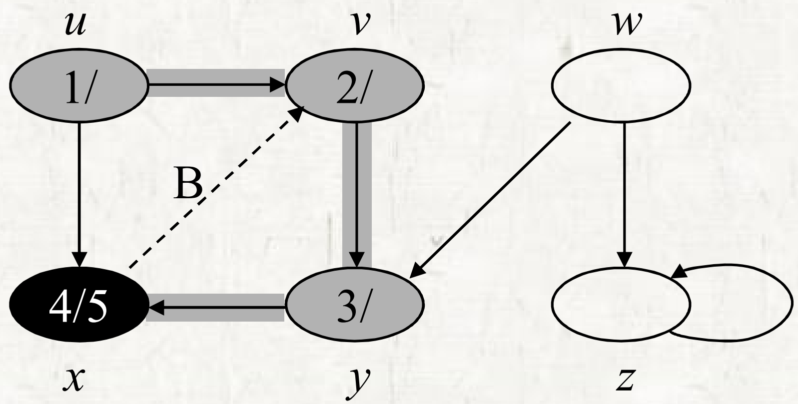
Depth-first search



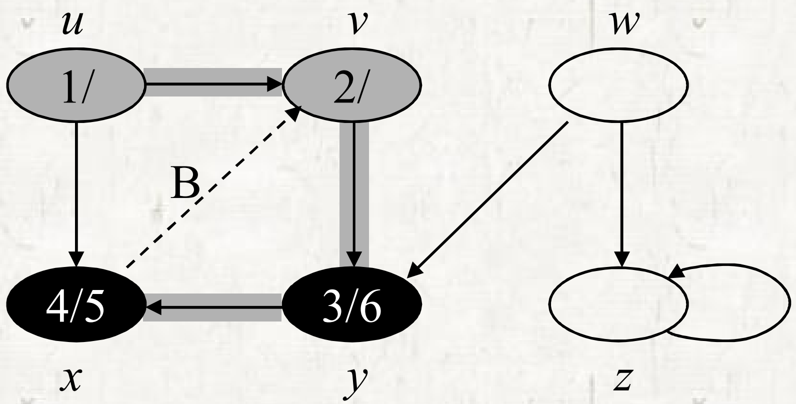
Depth-first search



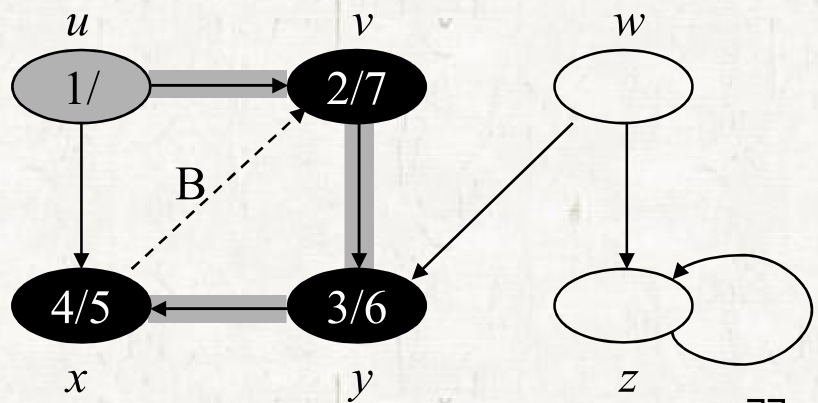
(e)



(f)

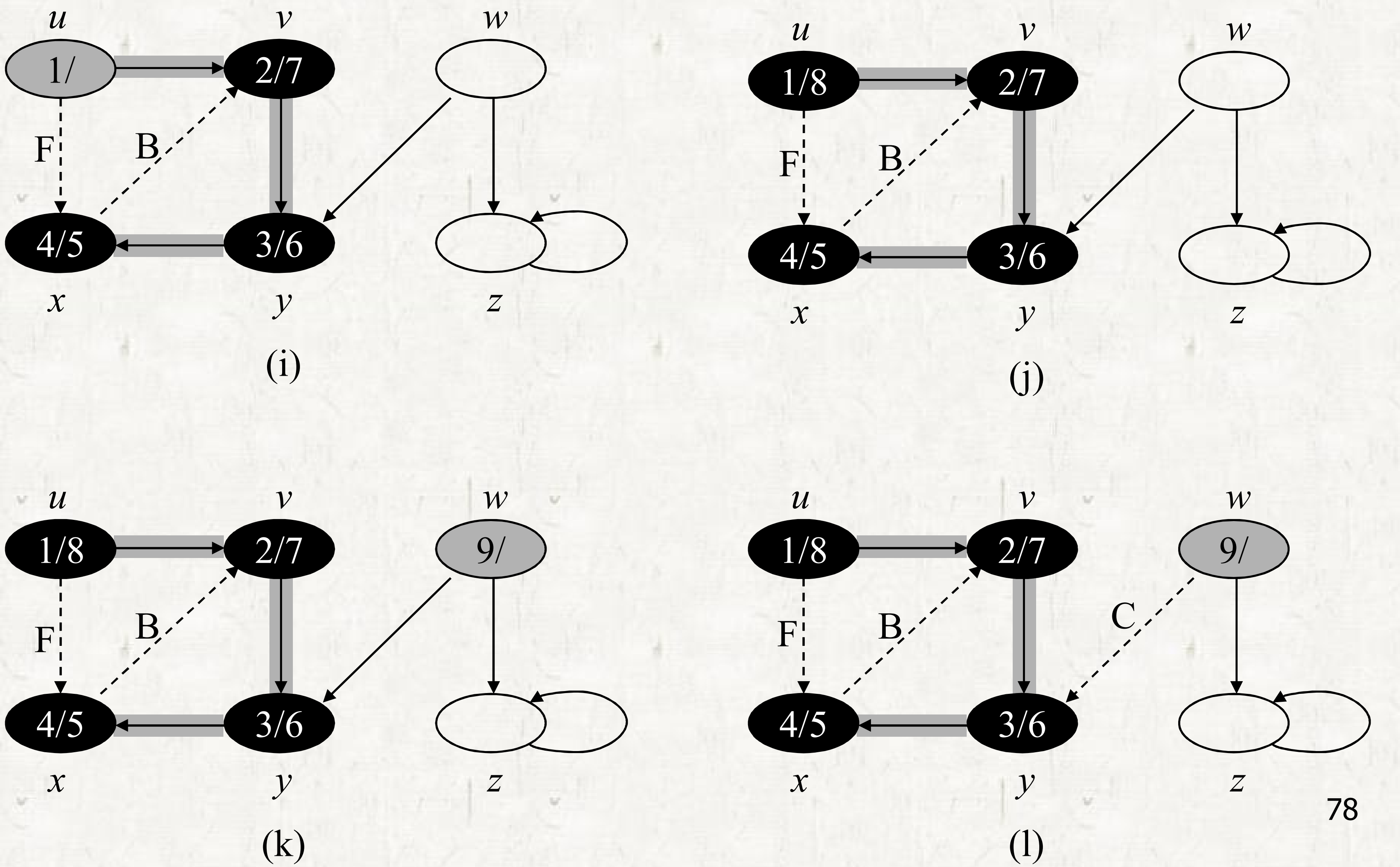


(g)

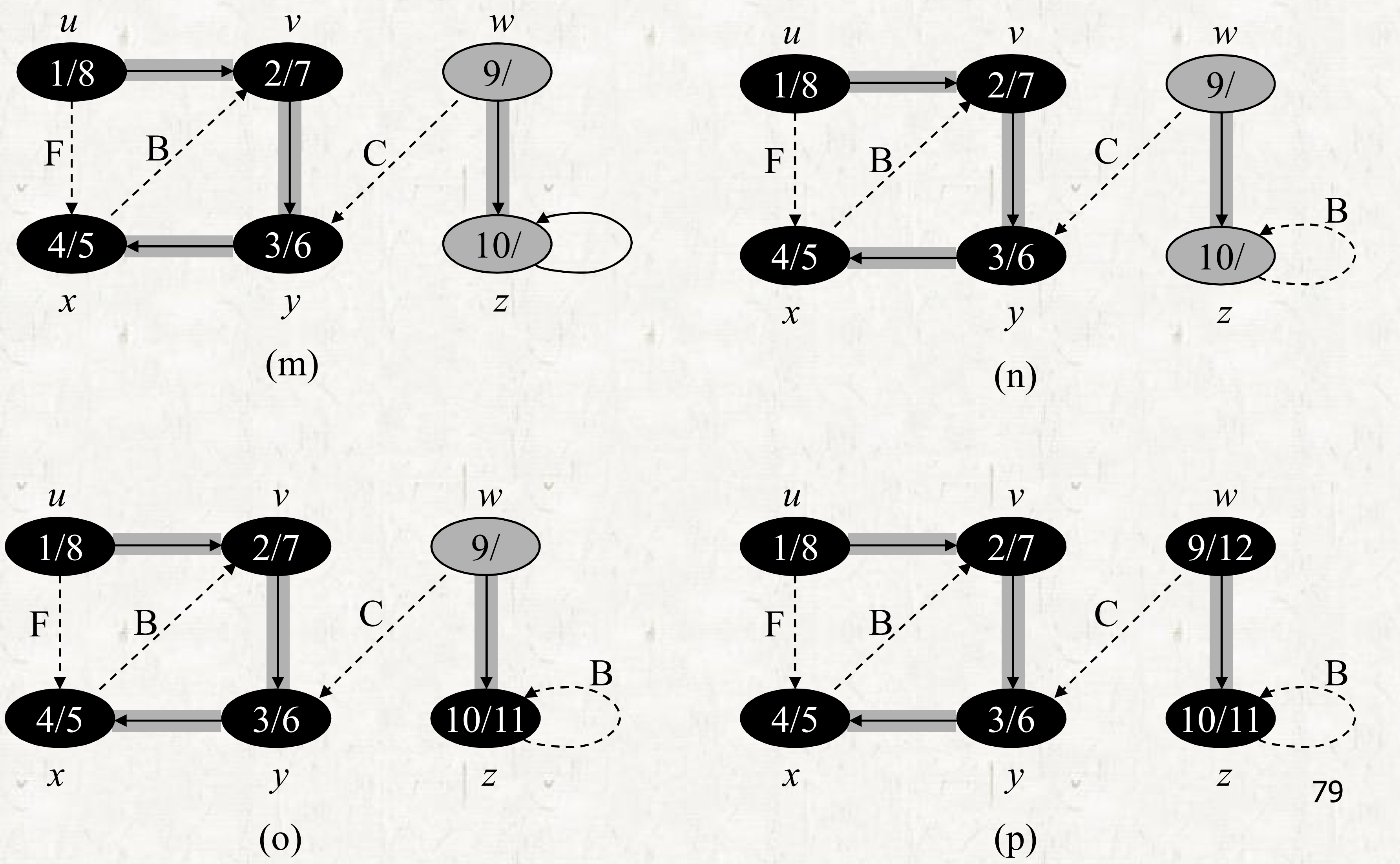


(h)

Depth-first search



Depth-first search



Depth-first search

- In a depth-first search of an ***undirected graph***, every edge of G is either a ***tree edge*** or a ***back edge***.
  - Forward edge?
  - Cross edge?
- Running Time
  - $\Theta(V+E)$

Self-study

- **Exercise 22.3-5 (22.3-4 in the 2nd ed.)**
  - Edge classification
- **Problem 22-2 *a-d***
  - Articulation points

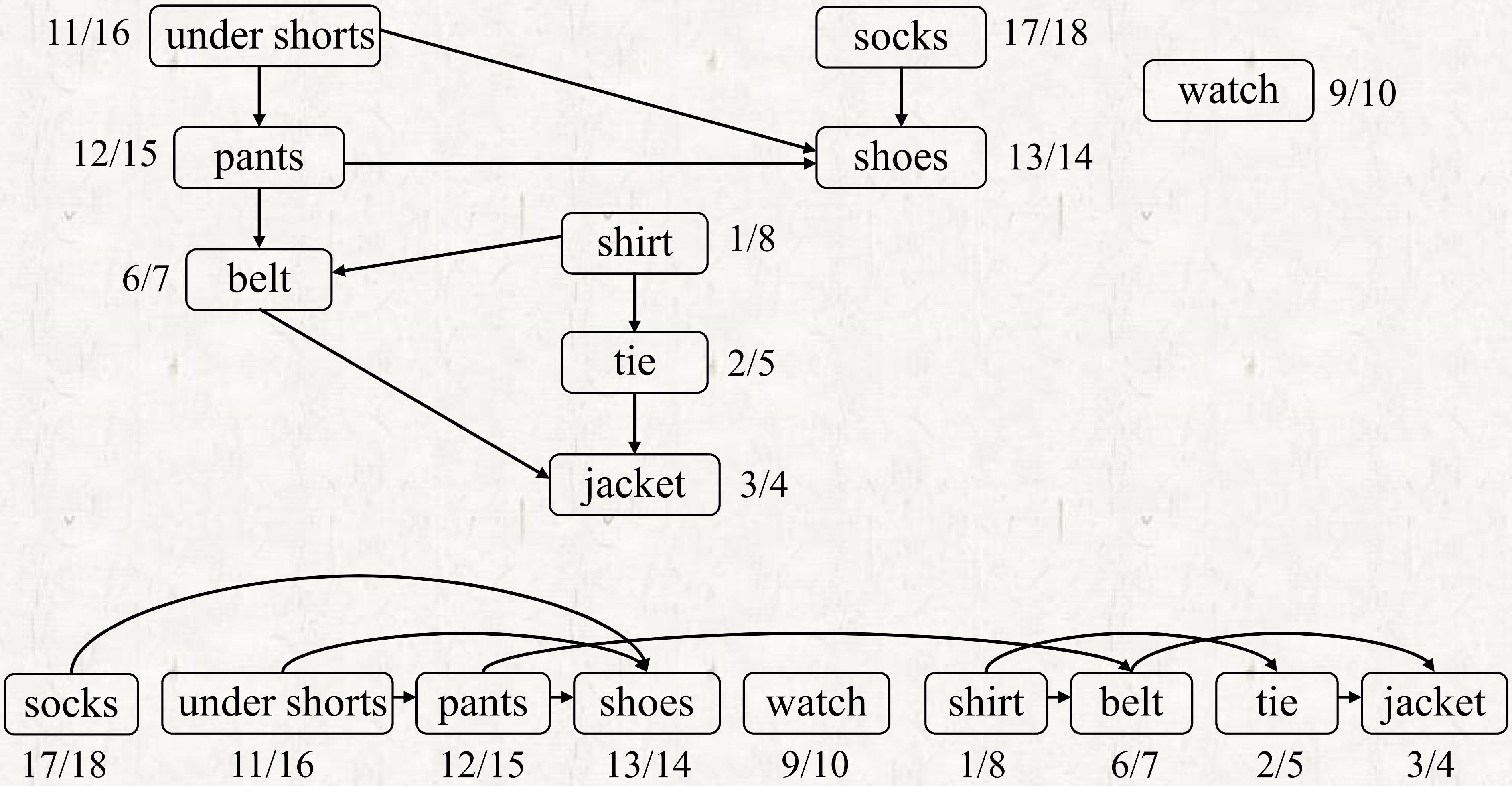
Contents

- ***Graphs***
  - *Graphs basics*
  - *Graph representation*
- ***Searching a graph***
  - *Breadth-first search*
  - *Depth-first search*
- **Applications of depth-first search**
  - Topological sort

Topological sort

- ***Definition***
  - Given a DAG (directed acyclic graph), generate a linear ordering of all its vertices such that all edges go from left to right.

Topological sort



Topological sort

- ***Main ideas***
  - Successively place a node from the ***left*** with **0** ***in-degree***.
  - Successively place a node from the ***right*** with **0** ***out-degree***.
  - Run DFS on G and place the nodes from the ***right*** in the ***increasing order of the finishing time***.
  - $\Theta(V+E)$ time

Topological sort

- ***Correctness***
  - If there is an edge from u to v, then $v.f < u.f$.
  - A directed graph G is ***acyclic*** if and only if a depth-first search of G yields ***no back edges***.

Self-study

- **Exercise 22.4-2**
  - Computing the number of simple paths from *s* to *t* in linear time.
- **Exercise 22.4-3**
  - Cycle detection in an undirected graph.
- **Exercise 22.4-5**
  - Another topological sort algorithm.

Programming Assignment

- ***Depth-first search and its applications***
  - **Exercise 22.3-10 (22.3-9, 2nd ed.) (#1)**
    - Depth-first search with edge classification
  - **Exercise 22.3-12 (22.3-11, 2nd ed.) (#2)**
    - Connected component identification
  - **Topological sort (#3)**
    - The program should detect whether the input is a DAG or not.