Operations Management I

Inventory Management (3)

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Inventory Management

Dynamic Lot Sizing ◄--------

- Basics
- Problem Description
- Mathematical Formulation
- Solution Algorithms

Hopp and Spearman, 2008, **Factory Physics**, McGraw Hill. **(Chapter 2)**

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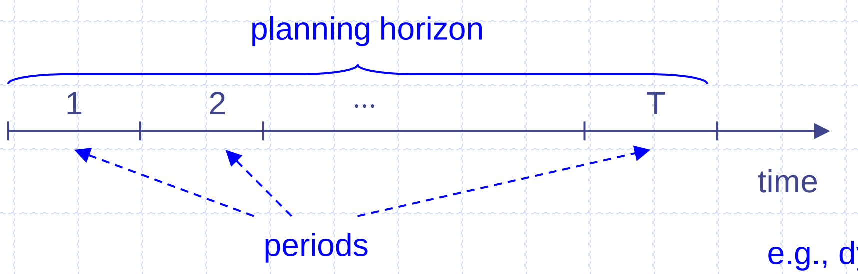
Dynamic Lot Sizing

Overview

- Basic
  - ✓ Dynamic and deterministic demands
    (over the planning horizon with discrete time periods)
    ◄- - - - - Relax the assumption of constant-demand rate of static models



✓ Multi-period and
✓ non constant demand rate

e.g., dynamic demands

1	2	…	T
20	50	…	30

  - ✓ Decsion

    Determining production lot size in each period for the objective of minimizing the sum of production, setup, and inventory holding costs over the planning horizon

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Dynamic Lot Sizing

Problem Description ← Simplest production planning model

e.g., lot sizing in MRP

- Decision variable

  Number of products to be produced in each period (production quantity in each period)

- Objective

  can be removed if time-invariant

  Minimizing the sum of production, setup, and inventory holding costs over the planning horizon.

- Assumptions

  ✓ Single product type
  ✓ Backlogging is not allowed.

Example

Customer demand is known over a 10-week planning horizon. (time varing demand)

	Planning periods (week)									
	1	2	3	4	5	6	7	8	9	10
Demand	20	50	10	50	50	10	20	40	20	30
Production cost	10	10	10	10	10	10	10	10	10	10
Setup cost	100	100	100	100	100	100	100	100	100	100
Holding cost	1	1	1	1	1	1	1	1	1	1

Cost factors (Production cost, Setup cost, Holding cost)

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Mathematical Formulation

Minimize $\sum_{t=1}^{T}(A_t \cdot y_t + c_t \cdot Q_t + h_t \cdot I_t)$

subject to

$I_t = I_{t-1} + Q_t - D_t$ for all t ◄----- Inventory balance constraints

$Q_t \leq M \cdot y_t$ for all t ◄----- $Q_t = 0$ when $y_t = 0$

$y_t \in \{0,1\}$ for all t

$Q_t \geq 0$ for all t

Parameters

- D_t demand in period t (units)
- A_t setup cost to produce a lot in period t ($)
- c_t unit production cost, not counting setup or inventory costs ($/unit)
- h_t holding cost to carry a unit of inventory from period t to t + 1 ($/unit•period)
- M large number

Decision variables

- Q_t lot size in period t
- y_t = 1 if there is a setup in period t, and 0 otherwise

Approaches – Overview

- Heuristics
  - ✓ Lot-for-lot
  - ✓ Fixed order quantity, etc.
- Optimal
  - ✓ Dynamic programming algorithm of Wagner and Whitin (1958)

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Solution Algorithms – Heuristics (1)

- Lot-for-lot (LFL)

  Produces exactly what is required in each period ($Q_t = D_t$)

  Maximum setup cost, and minimum inventory holding cost

Example (A_t = 100, h_t = 1 for all t)

	Time Period (week)										
	1	2	3	4	5	6	7	8	9	10	Total
D_t	20	50	10	50	50	10	20	40	20	30	300
Q_t	**20**	**50**	**10**	**50**	**50**	**10**	**20**	**40**	**20**	**30**	300
I_t	0	0	0	0	0	0	0	0	0	0	0
Setup cost	100	100	100	100	100	100	100	100	100	100	1000
Holding cost	0	0	0	0	0	0	0	0	0	0	0
Total cost	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**100**	**1000**

$$\sum D_t = \sum Q_t = \text{constant}$$

Production cost is the same (and hence needs not be considered)

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◈ Dynamic Lot Sizing

Solution Algorithms – Heuristics (2)

- Fixed order quantity

  Produces a fixed amount at each time when a setup is performed (e.g, EOQ, total demand / n, etc.)

- Others
  - ✓ Fixed order period
  - ✓ Part-period balancing, etc.

Example (A_t = 100, h_t = 1 for all t)

Set the fixed quantity to 100 units (300 units in total devided by 3)

Initial inventory (I_0) = 0

$$I_1 = I_0 + Q_1 - D_1 = 0 + 100 - 20 = 80$$

$$I_t = I_{t-1} + Q_t - D_t$$

	Time Period (week)										
	1	2	3	4	5	6	7	8	9	10	Total
D_t	20	50	10	50	50	10	20	40	20	30	300
Q_t	**100**	**0**	**0**	**100**	**0**	**0**	**100**	**0**	**0**	**0**	300
I_t	80	30	20	70	20	10	90	50	30	0	0
Setup cost	100	0	0	100	0	0	100	0	0	0	300
Holding cost	80	30	20	70	20	10	90	50	30	0	400
Total cost	**180**	**30**	**20**	**170**	**20**	**10**	**190**	**50**	**30**	**0**	**700**

$$I_2 = I_1 + Q_2 - D_2 = 80 + 0 - 50 = 30$$

1000 when lot-for-lot

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Solution Algorithms – Optimal (1)

- Basic property (by Wagner and Whitin (1958)

  Under an optimal lot-sizing policy, either the inventory carried to period *t* from a previous period will be zero or the production quantity in period *t* will be zero.

$$\boxed{I_{t-1} \cdot Q_t = 0 \quad \text{for } t = 1, 2, \ldots, T}$$

$$I_{t-1} > 0 \rightarrow Q_t = 0$$
$$Q_t > 0 \rightarrow I_{t-1} = 0$$

We will produce either nothing or exactly enough to satisfy demand in the current period plus some integer number of future periods.

$$Q_t = \begin{cases} 0 \\ D_t \\ D_t + D_{t+1} \\ \\ D_t + D_{t+1} + \ldots + D_T \end{cases}$$

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Dynamic Lot Sizing

Solution Algorithms – Optimal (2)

- Approaches

✓ Full enumeration

Enumerating all possible combinations of periods in which production occurs

Number of alternatives: 2^{T-1}
(exponential growth)

✓ Wagner and Whitin algorithm ← dynamic programming

Production occurs in a period if the inventory carries to that period is zero.
(decision: how many periods of demand to produce in each period)

e.g. 6-period problem

✓ Period 1: D_1, $\boxed{D_1 + D_2}$, … , $D_1 + D_2, \ldots + D_6$

Period 3: D_3, $D_3 + D_4$, $\boxed{D_3 + D_4 + D_5}$, $D_3 + D_4 + D_5 + D_6$

Period 6: D_6

Period 2: D_2, $D_2 + D_3$, $D_2 + D_3 + D_4$, ...

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Solution Algorithms – Optimal (3)

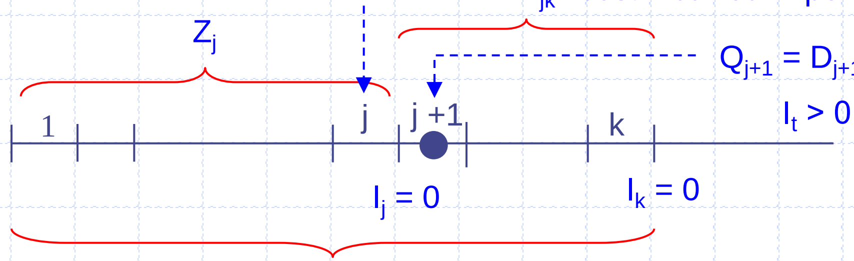
- Wagner and Whitin algorithm
  - ✓ Dynamic programming formulation

last period (prior to period *k*) having an end inventory of zero
(The last production occurs in period j + 1.)

M_{jk} cost incurred in period $j + 1$ through k

$Q_{j+1} = D_{j+1} + D_{j+2} + \dots + D_k$

$I_t > 0$ for $t = j + 1, j + 2, \dots, k - 1$



Z_k minimum cost program for periods $1, 2, \dots k$,
when $I_k = 0$ is required

$$Z_k = \min_{0 \le j < k} \left\lfloor Z_j + M_{jk} \right\rfloor$$

where $Z_0 = 0$

Selecting the last period having zero ending inventory for k-period subproblem

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Dynamic Lot Sizing

Solution Algorithms – Optimal (4)

- Wagner and Whitin algorithm

✓ Calculation of M_{jk} ← cost incurred in period j + 1 through k

$$Z_k = \min_{0 \le j < k} \lfloor Z_j + M_{jk} \rfloor$$

where $Z_0 = 0$

Inventory holding cost = $\sum_{t=j+1}^{k-1} h_t \cdot I_t$ ← $I_t = Q_{j+1} - \sum_{r=j+1}^{t} D_r = \sum_{r=t+1}^{k} D_r$

for $t = j + 1, \ldots, k - 1$

last period (prior to period k) having an end inventory of zero

… j | j + 1 | j + 2 | … | k – 1 | k …

$I_j = 0$ $I_k = 0$

Setup and production costs = $A_{j+1} + c_{j+1} \cdot Q_{j+1}$ ← $Q_{j+1} = D_{j+1} + D_{j+2} + \ldots + D_k$

$$M_{jk} = A_{j+1} + c_{j+1} \cdot Q_{j+1} + \sum_{t=j+1}^{k-1} h_t \cdot I_t$$

$$= A_{j+1} + c_{j+1} \cdot Q_{j+1} + \sum_{t=j+1}^{k-1} h_t \cdot \left(\sum_{r=t+1}^{k} D_r \right)$$

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◈ **Dynamic Lot Sizing**

Solution Algorithms – Optimal (5)

- Wagner and Whitin algorithm

✓ Procedure (forward algorithm)

Step 1. Determine in sequence the values $Z_1, Z_2, \ldots, Z_T$, where

$$Z_k = \min_{0 \le j < k} \left[Z_j + M_{jk} \right] \quad \text{(Initially, } Z_0 = 0\text{)}$$

$$M_{jk} = A_{j+1} + c_{j+1} \cdot Q_{j+1} + \sum_{t=j+1}^{k-1} h_t \cdot \left(\sum_{r=t+1}^{k} D_r \right)$$

$$Q_{j+1} = D_{j+1} + D_{j+2} + \ldots + D_k$$

Step 2. Use j_T' (from Z_T) to work backward to extract the optimal lot sizes

j_t' the last period in which the inventory level is 0 for the t-period subproblem
(The last production occurs in period $j_t^* = j_t' + 1$)

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◈ Dynamic Lot Sizing

Solution Algorithms – Optimal (6)

- Wagner and Whitin algorithm
  - ✓ Example

	Planning periods (week)									
	1	2	3	4	5	6	7	8	9	10
D_t	20	50	10	50	50	10	20	40	20	30
A_t	100	100	100	100	100	100	100	100	100	100
h_t	1	1	1	1	1	1	1	1	1	1

Production cost is not considered since it is constant over the planning horizon.

➢ 1-period subproblem

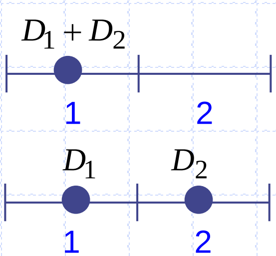
$$Z_1 = Z_0 + M_{01} = Z_0 + A_1 = 0 + 100 = 100 \quad \longleftarrow \quad Z_0 = 0$$

$$j_1^* = j_1' + 1 = 1 \ (j_1' = 0)$$

The last production occurs in period 1.

➢ 2-period subproblem

$$Z_2 = \min \begin{cases} Z_0 + M_{02} = Z_0 + A_1 + h_1 D_2 = 0 + 100 + 1(50) = 150 \\ Z_1 + M_{12} = Z_1 + A_2 = 100 + 100 = 200 \end{cases}$$



$$j_2^* = j_2' + 1 = 1 \ (j_2' = 0)$$

The last production occurs in period 1.

⟵ 70 units (D_1 + D_2 = 20 + 50) in period 1

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Dynamic Lot Sizing

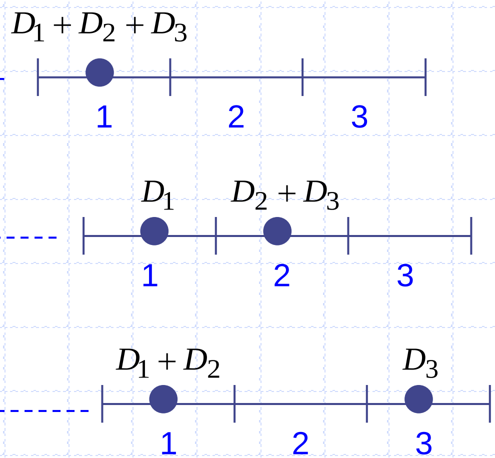
Solution Algorithms – Optimal (7)

- Wagner and Whitin algorithm
  - ✓ Example
    - ➢ 3-period subproblem

	Planning periods (week)									
	1	2	3	4	5	6	7	8	9	10
D_t	20	50	10	50	50	10	20	40	20	30
A_t	100	100	100	100	100	100	100	100	100	100
h_t	1	1	1	1	1	1	1	1	1	1

Production cost is not considered since it is constant over the planning horizon.

$$Z_3 = \min \begin{cases} Z_0 + M_{03} = Z_0 + A_1 + h_1 D_2 + (h_1 + h_2) D_3 \\ \qquad = 0 + 100 + 1(50) + (1+1)(10) = 170 \\ Z_1 + M_{13} = Z_1 + A_2 + h_2 D_3 = 100 + 100 + 1(10) = 210 \\ Z_2 + M_{23} = Z_2 + A_3 = 150 + 100 = 250 \end{cases}$$



$$j_3^* = j_3' + 1 = 1 \ (j_3' = 0)$$

The last production occurs in period 1.

80 units (D_1 + D_2 + D_3 = 20 + 50 + 10) in period 1

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Dynamic Lot Sizing

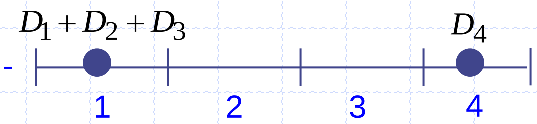
Solution Algorithms – Optimal (8)

- Wagner and Whitin algorithm
  - ✓ Example
    - ➢ 4-period subproblem

	Planning periods (week)									
	1	2	3	4	5	6	7	8	9	10
D_t	20	50	10	50	50	10	20	40	20	30
A_t	100	100	100	100	100	100	100	100	100	100
h_t	1	1	1	1	1	1	1	1	1	1

Production cost is not considered since it is constant over the planning horizon.

$$Z_4 = \min \begin{cases} Z_0 + M_{04} = Z_0 + A_1 + h_1 D_2 + (h_1 + h_2) D_3 + (h_1 + h_2 + h_3) D_4 \\ \qquad = 0 + 100 + 1(50) + (1+1)(10) + (1+1+1)(50) = 320 \\ Z_1 + M_{14} = Z_1 + A_2 + h_2 D_3 + (h_2 + h_3) D_4 \\ \qquad = 100 + 100 + 1(10) + (1+1)(50) = 310 \\ Z_2 + M_{24} = Z_2 + A_3 + h_3 D_4 = 150 + 100 + 1(50) = 300 \\ \boxed{Z_3 + M_{34} = Z_3 + A_4 = 170 + 100 = 270} \end{cases}$$



$$\boxed{j_4^* = j_4' + 1 = 4 \ \ (j_4' = 3)}$$

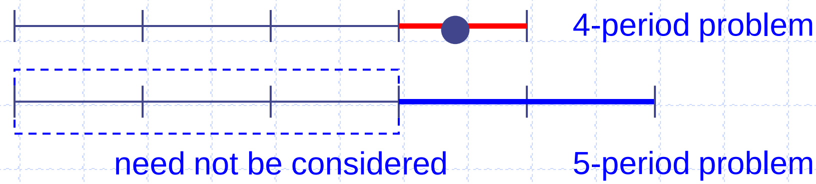
The last production occurs in period 4.

- 80 units (D_1 + D_2 + D_3 = 20 + 50 + 10) in period 1
- 50 units in period 4

Planning horizon theorem

If $j_t^* = t'$, then the last period in which production occurs in an optimal $t + 1$ period policy must be in the set t', $t' + 1$, …, $t + 1$,

e.g., $j_4^* = 4$



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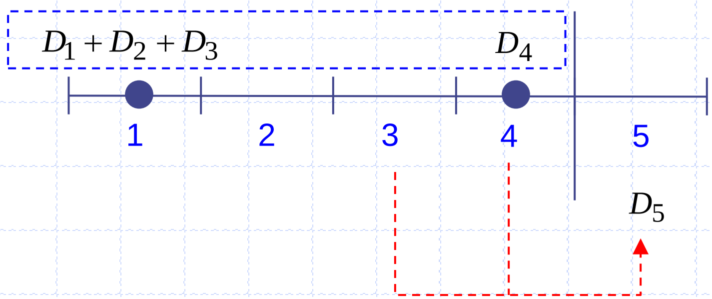
Solution Algorithms – Optimal (9)

- Wagner and Whitin algorithm
  - ✓ Example
    - 5-period subproblem
      - Cheaper to produce for demand of period 5 in period 3 than in period 4?

solution of the 4-period subproblem

80 units in period 1

50 units in period 4



No because $j_4^* = 4$

Therefore, it is unnecessary to consider producing in periods 1, 2, and 3 for the demand in period 5. We need to consider only periods 4 and 5.

Planning horizon theorem

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Solution Algorithms – Optimal (10)

- Wagner and Whitin algorithm
  - ✓ Example
    - ➢ 5-period subproblem

	Planning periods (week)									
	1	2	3	4	5	6	7	8	9	10
D_t	20	50	10	50	50	10	20	40	20	30
A_t	100	100	100	100	100	100	100	100	100	100
h_t	1	1	1	1	1	1	1	1	1	1

Production cost is not considered since it is constant over the planning horizon.

$$Z_5 = \min \begin{cases} \boxed{\begin{array}{l} Z_0 + M_{05} \\ Z_1 + M_{15} \\ Z_2 + M_{25} \end{array}} \\ Z_3 + M_{35} = Z_3 + A_4 + h_4 D_5 = 170 + 100 + 1(50) = 320 \\ Z_4 + M_{45} = Z_4 + A_5 = 270 + 100 = 370 \end{cases}$$

Need not be considered due to the planning horizon theorem

$$j_5^* = j_5' + 1 = 4 \ \ (j_5' = 3)$$

The last production occurs in period 4.

- 80 units (D_1 + D_2 + D_3 = 20 + 50 + 10) in period 1
- 100 units (D_4 + D_5 = 50 + 50) in period 4

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◈ Dynamic Lot Sizing

Solution Algorithms – Optimal (11)

- Wagner and Whitin algorithm
  - ✓ Example

	Planning periods (week)									
	1	2	3	4	5	6	7	8	9	10
D_t	20	50	10	50	50	10	20	40	20	30
A_t	100	100	100	100	100	100	100	100	100	100
h_t	1	1	1	1	1	1	1	1	1	1

Production cost is not considered since it is constant over the planning horizon.

2-period subproblem

Last period with production	Planning horizon *t*									
	1	2	3	4	5	6	7	8	9	10
1	100	**150**	**170**	320						
2		200	210	310						
3			250	300						
4				**270**	**320**	**340**	**400**	560		
5					370	380	420	540		
6						420	440	520		
7							440	**480**	**520**	610
8								500	**520**	**580**
9									580	610
10										620
Z_t^*	100	150	170	270	320	340	400	480	520	580
j_t^*	1	1	1	4	4	4	4	7	7 or 8	8

1-period subproblem

Optimal objective value = 580
- ✓ Lot-for-lot = 1000
- ✓ Fixed order quantity = 700

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Solution Algorithms – Optimal (12)

- Wagner and Whitin algorithm
  - ✓ Example
    - Interpreting the solution

Minimum total cost

$Z_{10}^* = 580$

	Planning horizon t									
	1	2	3	4	5	6	7	8	9	10
Z_t^*	100	150	170	270	320	340	400	480	520	**580**
j_t^*	1	1	1	4	4	4	4	7	7 or 8	8

Optimal lot sizes

$j_{10}^* = 8$ ------> $Q_8 = D_8 + D_9 + D_{10} = 40 + 20 + 30 = 90$

$j_7^* = 4$ ------> $Q_4 = D_4 + D_5 + D_6 + D_7 = 50 + 50 + 10 + 20 = 130$

$j_3^* = 1$ ------> $Q_1 = D_1 + D_2 + D_3 = 20 + 50 + 10 = 80$