Sorting in Linear Time

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- **Lower bounds for sorting**
- **Counting sort**
- **Radix sort**

Lower bounds for sorting

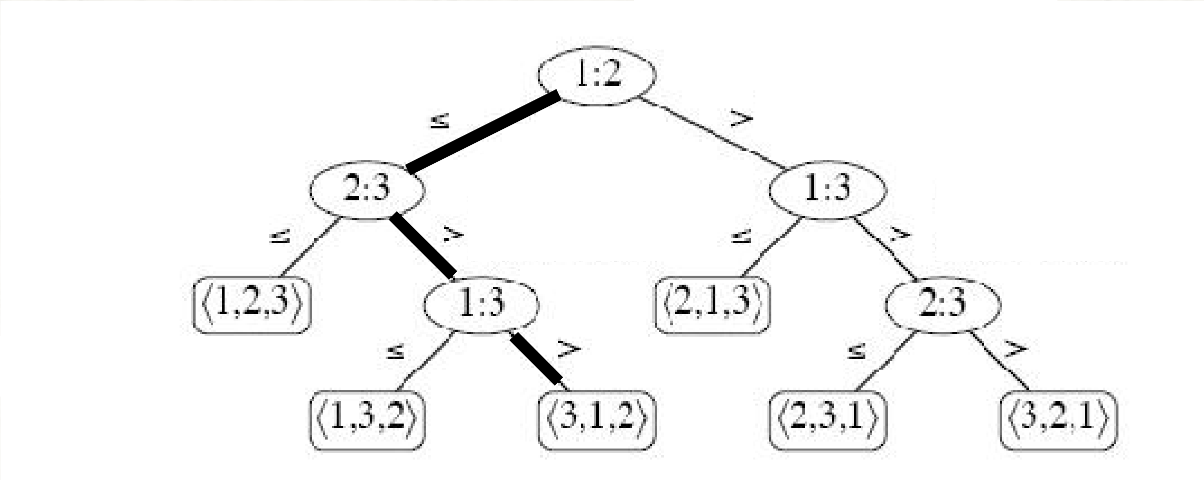
- **Comparison sorts**
  - Sorting algorithms using only comparisons to determine *the sorted order of the input elements*.
  - Use tests such as $a_i < a_j$, $a_i \le a_j$, $a_i = a_j$, $a_i \ge a_j$, or $a_i > a_j$.
  - Heapsort, Mergesort, Insertion sort, Selection sort, Quicksort
- **Lower bounds for (comparison) sorting**
  - Any comparison sort must make $\Omega(n \lg n)$ comparisons in the worst case to sort n elements.

Lower bounds for sorting

- **Comparison sort**
  - we assume without loss of generality that all of the input elements are distinct.
    - The comparisons $a_i \le a_j$, $a_i \ge a_j$, $a_i > a_j$, and $a_i < a_j$ are all equivalent.
    - We assume that all comparisions have the form $a_i \le a_j$

The decision-tree model

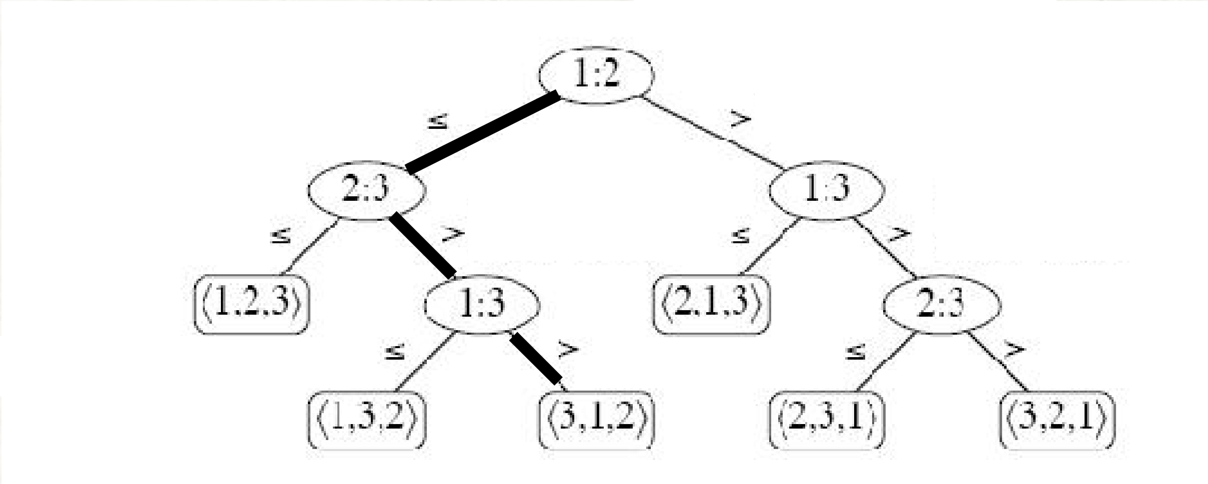
- Comparison sorts can be viewed in terms of ***decision trees***.
  - A full binary tree.
  - Each leaf is a permutation of input elements.
  - Each internal node *i*:*j* indicates a comparison $a_i \le a_j$.



A decision tree for insertion sort

The decision-tree model

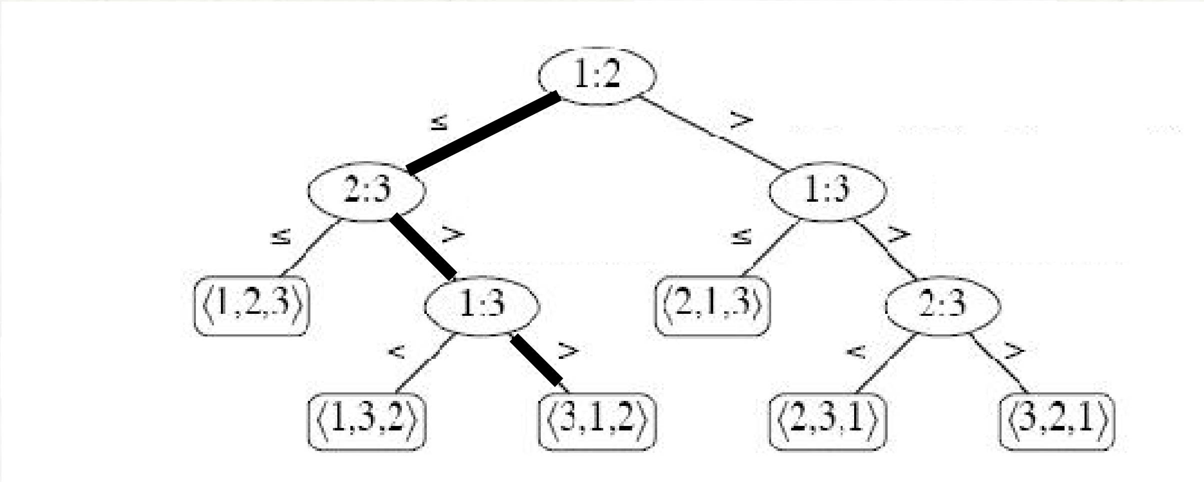
- The left subtree of the node *i:j* includes all permutations for $a_i \le a_j$.
- The right subtree includes all permutations for $a_i > a_j$.



A decision tree for insertion sort

The decision-tree model

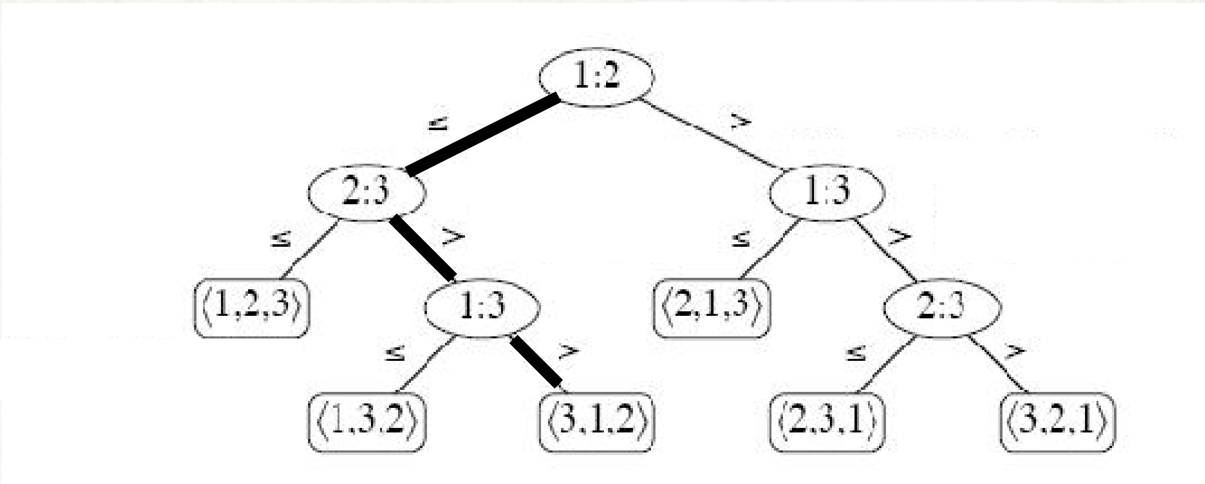
- The execution of the sorting algorithm corresponds to tracing a path from the root of the decision tree to a leaf.



A decision tree for insertion sort

The decision-tree model

- the worst-case number of comparisons
  = the height of its decision tree.



A decision tree for insertion sort

The decision-tree model

- ***Theorem* 8.1**: Any comparison sort algorithm requires $\Omega(n \lg n)$ comparisons in the worst case.

- ***Proof***:
  - Height: h, Number of element: n
  - The number of leaves: $n!$
    - Each permutations for n input elements should appear as leaves.
  - $n! \le 2^h$
  - $\lg(n!) \le h$
  - $\Omega(n \lg n)$ (by equation (3.18) : $\lg(n!) = \Theta(n\lg n)$).

Self-study

- **Exercise 8.1-1**
  - The smallest depth of a leaf in a decision tree
- **Exercise 8.1-3**
  - Decision tree existence
- **Exercise 8.1-4**
  - Lower bound of a decision tree

Counting sort

- **Counting sort**
  - A sorting algorithm using *counting*.

A	0	1	1	0	1	1	0	1
B	0	0	0	1	1	1	1	1

  - Each input element x should be located in the ith place after sorting if the number of elements less than x is i-1.

Counting sort

	1	2	3	4	5	6	7	8
A	2	5	3	0	2	3	0	3

	0	1	2	3	4	5
C	2	0	2	3	0	1

B	0	0	2	2	3	3	3	5

Counting sort

- **Counting sort**
  - Stable
    - Same values in the input array appear in the same order in the output array.

A	2	5	3	0	2	3	0	3

B	0	0	2	2	3	3	3	5

Counting sort

	1	2	3	4	5	6	7	8
A	2	5	3	0	2	3	0	3

	0	1	2	3	4	5
C	2	0	2	3	0	1

C'	2	2	4	7	7	8

Counting sort

	1	2	3	4	5	6	7	8
A	2	5	3	0	2	3	0	3

	0	1	2	3	4	5
C'	2	2	4	7	7	8

	1	2	3	4	5	6	7	8
B							3	

	0	1	2	3	4	5
C'	2	2	4	6	7	8

	1	2	3	4	5	6	7	8
B		0					3	

	0	1	2	3	4	5
C'	1	2	4	6	7	8

	1	2	3	4	5	6	7	8
B		0				3	3	

	0	1	2	3	4	5
C'	1	2	4	5	7	8

Counting sort

```
       COUNTING-SORT(A, B, k)
Θ(k)  { for i = 0 to k
      {     C[i] = 0
Θ(n)  { for j = 1 to A.length
      {     C[A[j]] = C[A[j]] + 1
        ▹ C[i] contains the number of elements equal to i.
Θ(k)  { for i = 1 to k
      {     C[i] = C[i] + C[i - 1]
        ▹ C[i] contains the number of elements less than or equal to i.
      { for j = A.length downto 1
Θ(n)  {     B[C[A[j]]] = A[j]
      {     C[A[j]] = C[A[j]] - 1
```

Counting sort

- The overall time is $\Theta(k+n)$ where k is the range of input integers.

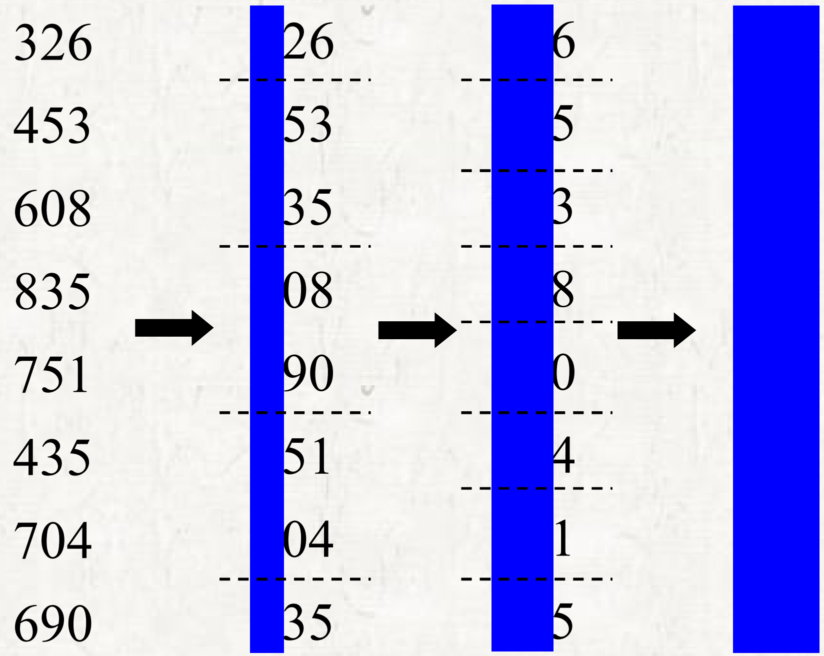
- If $k = O(n)$, the running time is $\Theta(n)$.

Self-study

- **Exercise 8.2-1**
  - A counting-sort example
- **Exercise 8.2-3**
  - Counting-sort stability
- **Exercise 8.2-4**
  - A counting-sort application

Radix sort

- **Radix sort** (MSB → LSB)



Radix sort

- **Radix sort** (MSB ← LSB)

326	69	7
453	75	6
608	45	3
835	70	8
751	83	4
435	43	7
704	32	4
690	60	6

Radix sort

RADIX-SORT(A, d)
1 for i = 1 to d
2 use a ***stable sort*** to sort array A on digit i

- RADIXSORT sorts in $\boldsymbol{\Theta(d(n+k))}$ time when n d-digit numbers are given and each digit can take on up to k possible values.

- When d is constant and $k = O(n)$, radix sort runs in linear time.

Radix sort

- **Changing *d* and *k***

132
453
601
813

d = ?
k = ?

13
45
60
81

d = ?
k = ?

Radix sort

- ***Lemma* 8.4 (Self-study)**

  Given n b-bit numbers and any positive integer $r \leq b$, RADIX-SORT correctly sorts these numbers in $\Theta((b/r)(n + 2^r))$ time.

$$
\left.\overbrace{\begin{array}{cccccccccc}
1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & 1 \\
0 & 1 & & & & & 1 & 0 & 0 & 1 \\
\vdots & \vdots & & & & & & \vdots & \vdots & \vdots \\
0 & 1 & & & & & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & \cdots & \cdots & \cdots & 1 & 0 & 0 & 1
\end{array}}^{b}\right\} n
$$

$$
\underbrace{\quad\quad\quad\quad}_{r} \quad \cdots \quad \underbrace{\quad\quad\quad\quad}_{r} \quad \underbrace{\quad\quad\quad\quad}_{r}
$$

Radix sort

- Computing optimal r minimizing $(b/r)(n + 2^r)$.
  1. $b < \lfloor \lg n \rfloor$

     for any value of r, $(n + 2^r) = \Theta(n)$ because $r \leq b$.

     So choosing $r = b$ yields a running time : $(b/b)(n + 2^b) = \Theta(n)$,

     which is asymptotically optimal.

Radix sort

- Computing optimal r minimizing $(b/r)(n + 2^r)$.

  2. $b \geq \lfloor \lg n \rfloor$

     choosing $r = \lfloor \lg n \rfloor$ gives the best time to within a constant factor, $(b/\lg n)(n+2^{\lg n}) = (b/\lg n)(2n) = \Theta(bn/\lg n)$.

  - As we increase r above $\lfloor \lg n \rfloor$, the 2^r in the numerator increases faster than the r in the dominator.
  - As we decrease r below $\lfloor \lg n \rfloor$, then the b/r term increases and the $n + 2^r$ term remains at $\Theta(n)$.

Radix sort

- Compare radix sort with other sorting algorithms.
  - If $b = O(\lg n)$, we choose $r \approx \lg n$.

  Radix sort: $\Theta(n)$

  Quicksort: $\Theta(n \lg n)$

Radix sort

- The constant factors hidden in the Θ-notation differ.

  1. Radix sort may make fewer passes than quicksort over the *n* keys, each pass of radix sort may take significantly longer.
  2. Radix sort does not sort in place.

Self-study

- **Exercise 8.3-1**
  - Radix sort example
- **Exercise 8.3-2**
  - Stability
- **Exercise 8.3-4**
  - Radix sort application