Electrical Engineering

HW 4 – Chapter 5, Solution

<1>

$$i_C(0^+) = \frac{V_1}{R_1}$$

$$i_C(0^+) = \frac{15\ V}{0.5\ k\Omega}$$

This equation yields $i_C(0^+) = 30\ mA$.

<2>

$$V_C\left(0^+\right) = V_C\left(0^-\right) = -7V$$

$$i_C(t) = I_0 = 17mA$$

$$V_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t)dt = \frac{1}{C}\left(\int_{-\infty}^{0} i_C(t)dt + \int_0^t I_0 dt\right)$$

$$= V_C\left(0^+\right) + \frac{I_0}{C}\int_0^t dt = V_C\left(0^+\right) + \frac{I_0}{C} t \,\Big|_0^t$$

$$= -7 + \frac{17\times10^{-3}}{0.55\times10^{-6}} t = -7 + 30.91\times10^{3} t$$

<3>

To determine $i_L(t)$ for $t > 0$, use the full equation for the response of current through an inductor:

$$i_L(t > 0) = i_L(\infty) + [i_L(0) - i_L(\infty)]e^{-t/\tau}$$

To utilize this equation, three values are required: $i_L(\infty)$, $i_L(0)$, and τ. To calculate $i_L(0)$, recognize that L is in steady-state (i.e., short-circuit) when $t < 0$. Therefore, the $i_L(0)$ is the same as the current through R_1 and R_2. Note that the voltage drop across R_1 and R_2 is the difference between V_{S1} and V_{S2}:

$$i_L(0) = \frac{V_{S1} - V_{S2}}{R_1 + R_2}$$

Substitute known values:

$$i_L(0) = \frac{9\ V - 12\ V}{2.2\ \Omega + 4.7\ \Omega}$$

Simplify:

$$i_L(0) = -0.43\ A$$

To determine $i_L(\infty)$, recognize, again, that L is in steady-state (i.e., short-circuit) as $t \to \infty$. Therefore, $i_L(\infty)$ is V_{S2} over the addition of R_2 and R_3:

$$i_L(\infty) = \frac{V_{S2}}{R_2 + R_3}$$

Substitute known values:

$$i_L(\infty) = \frac{12\ V}{4.7\ \Omega + 18\ k\Omega}$$

Simplify:

$$i_L(\infty) = 0.67\ mA$$

For the time constant, calculate the equivalent resistance seen by L. To accomplish this goal, remove L from the circuit and turn V_{S2} into a short-circuit. Then, L sees R_2 in series with R_3:

$$R_{eq} = R_2 + R_3$$

Substitute known values:

$$R_{eq} = 4.7\ \Omega + 18\ k\Omega$$

Simplify:

$$R_{eq} \approx 18\ k\Omega$$

Calculate τ as the time constant for an RL circuit:

$$\tau = \frac{L}{R_{eq}}$$

$$\tau = \frac{120\ mH}{18\ k\Omega}$$

Simplify:

$$\tau = 6.66\ \mu s$$

Plug $i_L(0), i_L(\infty)$, and τ into the equation for $i_L(t)$ for $t > 0$:

$$i_L(t > 0) = 0.67\ mA + [-0.43\ A - 0.67\ mA]e^{-t/6.66\ \mu s}$$

<4>

There are two conditions to examine: $t < 0$ and $t > 0$. When $t < 0$, there is no source attached to the circuit. Therefore, there exists no current through the capacitor when $t < 0$.

The current through a charging capacitor at any time, t, is given by the equation:

$$i_C(t) = \frac{V}{R_{eq}} e^{-t/R_{eq}C}$$

To fully realize this equation, the time constant, τ, and the equivalent resistance, R_{eq}, must be determined.

Find R_{eq} seen by the capacitor. That is, remove C and make V_1 a short. Then, combine the resistors.

Combine R_1 and R_2 in parallel:

$$R_{1,2,eq} = \frac{R_1 R_2}{R_1 + R_2}$$

Substitute known values:

$$R_{1,2,eq} = \frac{300\ m\Omega * 1.2\ \text{k}\Omega}{1.2\ k\Omega + 300\ m\Omega}$$

$$R_{1,2,eq} \approx 0.3\ \Omega$$

Combine $R_{1,2,eq}$ and R_3 in series. Note that this combination results in the total equivalent resistance, R_{eq}.

$$R_{eq} = R_{1,2,eq} + R_3$$

Substitute known values:

$$R_{eq} = 0.3\ \Omega + 1.2\ k\Omega$$

Simplify:

$$R_{eq} \approx 1.2\ k\Omega$$

Recognize the RC circuit and use R_{eq} to calculate the time constant:

$$\tau = R_{eq} C$$

Substitute known values:

$$\tau = 1.2\ k\Omega * 200\ \mu F$$

Simplify:

$$\tau = 0.24\ s$$

Finally, plug V_1, R_{eq}, and τ into the equation for current through a charging capacitor:

$$i_C(t > 0) = \frac{10\ V}{1.2\ k\Omega} e^{-t/0.24\ s}$$

$$i_C(t > 0) = 0.008e^{-t/0.24\,s}$$

Combine results:

$$i_C(t) = \begin{cases} 0\,A & t \leq 0 \\ 0.008e^{-t/0.24\,s}\,A & t > 0 \end{cases}$$

<5>

The conditions that exist at $t < 0$ have no effect on the long-term DC steady state conditions at $t \to \infty$. In the long-term DC steady state the inductor may be modeled as a short circuit and the capacitor as an open circuit. In this case, the inductor short circuits the R_1 branch and the R_2 C branch. Thus, the voltage across these branches and the current through them are zero. In other words all of the current produced by the 9V source travels through the inductor in this case.

$$i_L(\infty) = \frac{V_{S2}}{R_{S2}} = \frac{9}{290} = 31.03\,mA$$

Of course, this current is also the current traveling through the 290 Ω resistor.

$$i_{RS2}(\infty) = i_L(\infty) = 31.03\,mA$$

And since the voltage across the inductor in the long-term DC steady state is zero (short circuit)

$$0 + V_C(\infty) + i_{R2}(\infty)R_2 = 0$$

$$V_C(\infty) = 0$$

<6>

For t > 0, KCL at the upper left node yields:

$$\frac{v_C}{1\,\Omega} + i_C + i_L = 0$$

or

$$\frac{v_C}{1\,\Omega} + C\frac{dv_C}{dt} + i_L = 0$$

KVL around the right mesh yields:

$$v_C = v_L + i_L(4\,\Omega)$$

or

$$v_C = L\frac{di_L}{dt} + i_L(4\,\Omega)$$

Plug (2) into (1) to find a 2nd-order ODE in the inductor current.

$$LC\frac{d^2 i_L}{dt^2} + \frac{di_L}{dt}\left((4\,\Omega)\,C + \frac{L}{1\,\Omega}\right) + i_L\left(\frac{4\,\Omega}{1\,\Omega} + 1\right) = 0$$

Plug in for L and C and divide both sides by the coefficient of the inductor current to find:

$$0.2\frac{d^2 i_L}{dt^2} + 0.8\frac{di_L}{dt} + i_L = 0$$

Compare to the generalized 2nd-order ODE to find:

$$\frac{1}{\omega_n^2} = 0.2 \qquad \text{and} \qquad \frac{2\zeta}{\omega_n} = 0.8$$

The result is

$$\omega_n = \sqrt{5} \approx 2.24\,\text{rad/sec} \qquad \text{and} \qquad \zeta = 0.4\sqrt{5} \approx 0.89 < 1$$

The damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 1$$

The transient response is underdamped such that the solution for the inductor current is

$$i_L(t) = e^{-\zeta\omega_n t}\left[C_1\cos(\omega_d t) + C_2\sin(\omega_d t)\right]$$

or

$$i_L(t) = e^{-2t}\left[C_1\cos(t) + C_2\sin(t)\right]$$

Since $i(0) = 2.5$ A

$$C_1 = 2.5\,\text{A}$$

Also, since $v_C(0) = 10$ V, equation (2) yields:

$$\left.\frac{di_L}{dt}\right|_{t=0} = \frac{v_C(0) - 4i_L(0)}{L} = \frac{10\,\text{V} - 4(2.5\,\text{A})}{L} = 0$$

where

$$\left.\frac{di_L}{dt}\right|_{t=0} = -2C_1 + C_2$$

Thus:

$$C_2 = 2C_1 = 5.0\,\text{A}$$

<7>

$V(0^-) = V(0^+) = 0$ Applying KVL: $\dfrac{d^2V}{dt^2} + 4\dfrac{dV}{dt} + 4V = 48$

Solving the differential equation: $V = k_1 e^{-2t} + k_2 t e^{-2t} + 12$

From the initial condition:

$V(0) = 0 \Rightarrow k_1 = -12$

$i_L(0) = C\dfrac{dV(0)}{dt} \Rightarrow 6 + \dfrac{k_2}{4} = 3 \Rightarrow k_2 = -12$

$V(t) = -12e^{-2t} - 12te^{-2t} + 12\,V \quad for\ t > 0$

The maximum value of V is: $V_{max} = V(\infty) = 12V$

<8>

Step 1—DC steady-state response:

Determine the steady-state value of v. The circuit is in steady-state before the switch is thrown. The inductor acts like a short-circuit. Therefore, the $2\ \Omega$ resistor has the full 12 V dropped across it:

$$v(0^-) = 12\ V$$

Step 2—Initial conditions:

(Step 2) Find $v(0^+)$:

Find the initial conditions for the voltage across the resistor: $v(0^+)$ and $dv(0^+)/dt$. Use KVL and note that the voltage across a capacitor cannot change instantaneously:

$$v(0^+) = 12 - v_C(0^+)$$

$$= 12 - 4$$

$$= 8\ V$$

(Step 2) Find $dv(0^+)/dt$:

Take the derivative of the equation obtained using KVL:

$$\frac{dv(0^+)}{dt} = -\frac{dv_C(0^+)}{dt}$$

Use KCL to derive an equation for $dv_C(0^+)/dt$.

$$\frac{v(0^+)}{R} = i_L + C\frac{dv_C(0^+)}{dt}$$

Plug in known values and note that the current through an inductor cannot change simultaneously:

$$\frac{dv_C(0^+)}{dt} = 4 * \left[\frac{8}{2} - 6\right]$$

$$= -8\ V$$

Therefore:

$$\frac{dv(0^+)}{dt} = 8\ V$$

Step 3—Find a second-order differential equation in one state variable:

Use KVL on the resistor and inductor:

$$12 - v - L\frac{di_L}{dt} = 0$$

Take the derivative of the KCL equation plug in the result from KVL on the resistor and capacitor:

$$\frac{di_L}{dt} = \frac{1}{R}\frac{dv}{dt} + C\frac{d^2v}{dt^2}$$

Plug this result into the equation from KVL with the resistor and inductor:

$$12 - v - L\left[\frac{1}{R}\frac{dv}{dt} + C\frac{d^2v}{dt}\right] = 0$$

Simplify:

$$LC\frac{d^2v}{dt} + \frac{L}{R}\frac{dv}{dt} + v = 12$$

$$\Rightarrow \frac{1}{5}\frac{d^2v}{dt} + \frac{2}{5}\frac{dv}{dt} + v = 12$$

Step 4—Determine ζ and ω_n:

Compare the equation derived in Step 3 to the standard from of equation 5.25:

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx}{dt} + x = k_S f(t)$$

and match coefficients.

(Step 4) Find $\boldsymbol{\omega_n}$:

$$\omega_n = \sqrt{5}$$

$$= 2.24\, rad/s$$

(Step 4) Find $\boldsymbol{\zeta}$:

$$\zeta = \frac{\frac{2}{5} * \omega_n}{2}$$

$$= \frac{\frac{2}{5} * 2.24}{2}$$

$$= 0.45$$

Step 5—Determine the transient response:

The response is underdamped, because $\zeta < 1$.

Step 6—Solve for the constants in the differential equation:

Because the response is underdamped, the solution is of the form:

$$v(t) = e^{-\zeta\omega_n t}[\alpha_1 \sin(\omega_d t) + \alpha_2 \cos(\omega_d t)]$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2} = 1.66$ rad/s, according to equation 5.39.

Evaluate the solution to the differential equation using the initial condition for $v(0^+)$:

$$\alpha_2 = 8$$

Take the derivative and evaluate the equation using the initial condition for $dv(0^+)/dt$.

$$8 = -\zeta\omega_n e^{-\zeta\omega_n t}[\alpha_1 \sin(\omega_d t) + \alpha_2 \cos(\omega_d t)] + e^{-\zeta\omega_n t}[\alpha_1\omega_d \cos(\omega_d t) - \alpha_2\omega_d \sin(\omega_d t)]$$

Evaluate at $t = 0$:

$$8 = -\zeta\omega_n\alpha_2 + \alpha_1\omega_d$$

Solve for α_1:

$$\alpha_1 = \frac{8 + \zeta\omega_n\alpha_2}{\omega_d}$$

$$= \frac{8 + 0.45 * 2.24 * 8}{1.66}$$

$$= 9.68$$

Therefore, the equation for v is:

$$v(t) = e^{-1.01t}[9.68 \sin(1.66t) + 8 \cos(1.66t)]$$

$$\boxed{\textbf{Final answer: } v(t) = e^{-1.01t}[9.68 \sin(1.66t) + 8 \cos(1.66t)]}$$