ENE 3031 -Fall 2014
Homework 1 Solution

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1 Suppose that X is a discrete random variable having probability function $\Pr(X = k) = ck$ for $k = 1, 2, 3$. Find c, $\Pr(X \leq 2)$, $\mathrm{E}[X]$, and $\mathrm{Var}(X)$.

Solution. Since

$$1 \;=\; \sum_{k=1}^{3} \Pr(X = k) \;=\; \sum_{k=1}^{3} ck \;=\; 6c,$$

we have $c = 1/6$. $\square$

Next,

$$\Pr(X \leq 2) \;=\; \sum_{k=1}^{2} \Pr(X = k) \;=\; \sum_{k=1}^{2} k/6 \;=\; 1/2. \quad \square$$

Next,

$$\mathrm{E}[X] \;=\; \sum_{x} x\Pr(X = x) \;=\; \sum_{k=1}^{3} k\Pr(X = k) \;=\; \sum_{k=1}^{3} k^2/6 \;=\; 7/3. \quad \square$$

Similarly,

$$\mathrm{E}[X^2] \;=\; \sum_{x} x^2\Pr(X = x) \;=\; \sum_{k=1}^{3} k^2\Pr(X = k) \;=\; \sum_{k=1}^{3} k^3/6 \;=\; 6.$$

Thus,

$$\mathrm{Var}(X) \;=\; \mathrm{E}[X^2] - (\mathrm{E}[X])^2 \;=\; 5/9. \quad \Box$$

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2 Suppose that X is a continuous random variable having p.d.f. $f(x) = cx$ for $1 \leq x \leq 2$. Find c, $\Pr(X \geq 1)$, $\mathrm{E}[X]$, and $\mathrm{Var}(X)$.

Solution. Since

$$1 \;=\; \int_{-\infty}^{\infty} f(x)\,dx \;=\; \int_{1}^{2} cx\,dx,$$

we have $c = 2/3$. $\Box$

Next, we trivially see that $\Pr(X \geq 1) = 1$. $\Box$

Next,

$$\mathrm{E}[X] \;=\; \int_{-\infty}^{\infty} x f(x)\,dx \;=\; \frac{2}{3}\int_{1}^{2} x^2\,dx \;=\; 14/9. \quad \Box$$

Similarly,

$$\mathrm{E}[X^2] \;=\; \int_{-\infty}^{\infty} x^2 f(x)\,dx \;=\; \frac{2}{3}\int_{1}^{2} x^3\,dx \;=\; 5/2.$$

Thus,

$$\mathrm{Var}(X) \;=\; \mathrm{E}[X^2] - (\mathrm{E}[X])^2 \;=\; 0.08025. \quad \Box$$

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3 #4.7. Suppose that X and Y are jointly continuous random variables with

$$\begin{cases} y - x & \text{for } 0 < x < 1 \text{ and } 1 < y < 2 \\ 0 & \text{otherwise} \end{cases}.$$

a. Compute and plot $f_X(x)$ and $f_Y(y)$.

Solution. The marginal p.d.f. of X is

$$f_X(x) \;=\; \int_1^2 (y-x)\,dy \;=\; \frac{3}{2} - x, \;\; 0 < x < 1. \quad \square$$

The marginal p.d.f. of Y is

$$f_Y(y) \;=\; \int_0^1 (y-x)\,dx \;=\; y - \frac{1}{2}, \;\; 1 < y < 2. \quad \square$$

b. Are X and Y independent?

Solution. Since

$$f(x,y) \;=\; y - x \;\neq\; (\frac{3}{2} - x)(y - \frac{1}{2}) \;=\; f_X(x)f_Y(y),$$

we see that X and Y are *not* independent. $\square$

c. Compute $F_X(x)$ and $F_Y(y)$.

Solution. The marginal c.d.f. of X is

$$F_X(x) \;=\; \int_0^x (\frac{3}{2} - t)\,dt \;=\; \frac{x(3-x)}{2}, \;\; 0 < x < 1. \quad \square$$

The marginal c.d.f. of Y is

$$F_Y(y) \;=\; \int_1^y (t - 1/2)\,dt \;=\; \frac{y(y-1)}{2}, \;\; 1 < y < 2. \quad \square$$

d. Compute $\mathrm{E}[X]$, $\mathrm{Var}(X)$, $\mathrm{E}[Y]$, $\mathrm{Var}(Y)$, $\mathrm{Cov}(X,Y)$, and $\mathrm{Corr}(X,Y)$.

Solution. Without going through some of the tedious details, we find that

$$\begin{aligned}
\mathrm{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\,dx \;=\; \int_0^1 x(3/2-x)\,dx \;=\; 5/12. \quad \square \\
\mathrm{E}[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx \;=\; \int_0^1 x^2(3/2-x)\,dx \;=\; 1/4. \\
\mathrm{Var}(X) &= \mathrm{E}[X^2]-(\mathrm{E}[X])^2 \;=\; 11/144 \quad \square \\
\mathrm{E}[Y] &= \int_{-\infty}^{\infty} y f_Y(y)\,dy \;=\; \int_1^2 y(y-1/2)\,dy \;=\; 19/12. \quad \square \\
\mathrm{E}[Y^2] &= \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy \;=\; \int_1^2 y^2(y-1/2)\,dy \;=\; 31/12. \\
\mathrm{Var}(Y) &= \mathrm{E}[Y^2]-(\mathrm{E}[Y])^2 \;=\; 11/144 \quad \square
\end{aligned}$$

With a little more work, we find that

$$\begin{aligned}
\mathrm{E}[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dy\,dx \;=\; \int_0^1\int_1^2 xy(y-x)\,dy\,dx \;=\; 2/3 \\
\mathrm{Cov}(X,Y) &= \mathrm{E}[XY]-\mathrm{E}[X]\mathrm{E}[Y] \;=\; 1/144 \quad \square \\
\mathrm{Corr}(X,Y) &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \;=\; 1/11 \quad \square
\end{aligned}$$

4

#4.18. (Bonus — This takes some work.) If $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with mean μ and variance σ^2, then compute $\mathrm{Cov}(\bar{X}, S^2)$, where $\bar{X} = \sum_{i=1}^n X_i/n$ is the sample mean and $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$ is the sample variance. When will this covariance be equal to 0?

Solution. We begin with some preliminary results. First,

$$\begin{aligned}
\mathrm{Cov}(\bar{X}, S^2) &= \mathrm{Cov}\left(\bar{X}, \frac{1}{n-1}\sum_{i=1}^n (X_i-\bar{X})^2\right) \\
&= \frac{1}{n-1}\mathrm{Cov}\left(\bar{X}, \sum_{i=1}^n X_i^2 - n\bar{X}^2\right) \\
&\ \text{(by standard algebra)} \\
&= \frac{1}{n-1}\left[\mathrm{Cov}\left(\bar{X}, \sum_{i=1}^n X_i^2\right) - n\mathrm{Cov}(\bar{X}, \bar{X}^2)\right]. \qquad (1)
\end{aligned}$$

Second,

$$\begin{aligned}
\mathrm{Cov}\left(\bar{X}, \sum_{j=1}^n X_j^2\right) &= \frac{1}{n}\,\mathrm{Cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^n X_j^2\right) \\
&= \frac{1}{n}\sum_{i=1}^n \sum_{j=1}^n \mathrm{Cov}(X_i, X_j^2) \\
&= \frac{1}{n}\sum_{i=1}^n \mathrm{Cov}(X_i, X_i^2) \quad (\text{since Cov} = 0 \text{ if } i \neq j) \\
&= \mathrm{Cov}(X_1, X_1^2) \quad (\text{since the } X_i\text{'s are i.i.d.}). \qquad (2)
\end{aligned}$$

Third,

$$\begin{aligned}
n^3\,\mathrm{Cov}(\bar{X}, \bar{X}^2) &= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \mathrm{Cov}(X_i, X_j X_k) \\
&= \sum\sum\nolimits_{i\neq k}\mathrm{Cov}(X_i, X_i X_k) + \sum\sum\nolimits_{i\neq j}\mathrm{Cov}(X_i, X_j X_i) + \sum_{i=1}^n \mathrm{Cov}(X_i, X_i^2) \\
&= 2\sum\sum\nolimits_{i\neq k}\mathrm{Cov}(X_i, X_i X_k) + \sum_{i=1}^n \mathrm{Cov}(X_i, X_i^2) \quad (\text{by symmetry}) \\
&= 2n(n-1)\mathrm{Cov}(X_i, X_i X_k) + n\mathrm{Cov}(X_i, X_i^2) \quad (\text{since } X_i\text{'s are i.i.d.}) \\
&= 2n(n-1)\big(\mathrm{E}[X_i^2 X_j] - \mathrm{E}[X_i]\mathrm{E}[X_i X_k]\big) + n\mathrm{Cov}(X_i, X_i^2) \\
&= 2n(n-1)\big(\mathrm{E}[X_i^2]\mathrm{E}[X_j] - \mathrm{E}[X_i]\mathrm{E}[X_i]\mathrm{E}[X_i]\big) + n\mathrm{Cov}(X_i, X_i^2) \quad (X_i\text{'s are i.i.d.}) \\
&= 2n(n-1)\mathrm{Var}(X_i)\mathrm{E}[X_i] + n\mathrm{Cov}(X_i, X_i^2). \qquad (3)
\end{aligned}$$

Plugging (**??**) and (**??**) into (**??**), we get

$$\begin{aligned}
\mathrm{Cov}(\bar{X}, S^2) &= \frac{\mathrm{Cov}(X_1, X_1^2)}{n-1} - \frac{2(n-1)\mathrm{Var}(X_1)\mathrm{E}[X_1] + n\mathrm{Cov}(X_1, X_1^2)}{n(n-1)} \\
&= \frac{1}{n}\big(\mathrm{Cov}(X_1, X_1^2) - 2\mathrm{Var}(X_1)\mathrm{E}[X_1]\big) \\
&= \frac{1}{n}\big(\mathrm{E}[X_1^3] - \mathrm{E}[X_1]\mathrm{E}[X_1^2] - 2\mathrm{E}[X_1^2]\mathrm{E}[X_1] + 2(\mathrm{E}[X_1])^3\big) \\
&= \frac{\mathrm{E}[X_1^3] - 3\mu\mathrm{E}[X_1^2] + 2\mu^3}{n}. \quad \Box
\end{aligned}$$

Wow! That took a long time, eh? In any case, note that the numerator of the above quantity is the *skewness* of X_i. Further, it's possible to re-write the above quantity as

$$\mathrm{Cov}(\bar{X}, S^2) \;=\; \frac{\mathrm{E}[(X_1-\mu)^3]}{n}.$$

It is easy to see that if X_1 is symmetric about μ, then the skewness is 0. $\square$

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5 #4.23. Suppose that the following 10 observations come from some distribution (not highly skewed) with unknown mean μ.

$$7.3 \quad 6.1 \quad 3.8 \quad 8.4 \quad 6.9 \quad 7.1 \quad 5.3 \quad 8.2 \quad 4.9 \quad 5.8$$

Compute $\bar{X}$, S^2, and an approximate 95% confidence interval for μ.
Solution. After the standard calculations, we find that $\bar{X} = 6.38$, $S^2 = 2.16$, and the approximate 95% confidence interval is given by

$$\begin{aligned}
\mu &\in \bar{X} \pm t_{\alpha/2,n-1}\sqrt{S^2/n} \\
&= 6.38 \pm t_{0.025,9}\sqrt{2.16/10} \\
&= 6.38 \pm 2.262\sqrt{2.16/10} \\
&= 6.38 \pm 1.05. \quad \square
\end{aligned}$$

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6 #4.26. A random variable X has the *memoryless property* if, for all $s, t > 0$,

$$\Pr(X > t+s | X > t) \;=\; \Pr(X > s).$$

Show that the exponential distribution has the memoryless property.
Solution. As we did in class, we have

$$\begin{aligned}
\Pr(X > t+s | X > t) &= \frac{\Pr(X > t+s,\, X > t)}{\Pr(X > t)} \\
&= \frac{\Pr(X > t+s)}{\Pr(X > t)} \\
&= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} \\
&= e^{-\lambda t} \\
&= \Pr(X > s). \quad \square
\end{aligned}$$

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7 #4.27. A geometric distribution with parameter p $(0 < p < 1)$ has the p.m.f.

$$f(x) \;=\; (1-p)^x p, \qquad x = 0, 1, 2, \ldots.$$

Show that this distribution has the memoryless property.

Solution. I actually use a slightly different definition of the geometric than Law and Kelton's. Namely, let's interpret X as the number of Bern(p) *trials until we get a success*. Thus, if you see FFFS, then $X = 4$. This means that

$$f(x) \;=\; (1-p)^{x-1} p, \qquad x = 1, 2, \ldots,$$

which is a little different than the problem statement — sorry about that! In any case, the c.d.f. is

$$\begin{aligned}
F(x) &= \Pr(X \le x), \quad x = 1, 2, \ldots \\
&= \sum_{k=1}^{x} q^{k-1} p \\
&= p \sum_{k=0}^{x-1} q^k p \\
&= p\left(\sum_{k=0}^{\infty} q^k - \sum_{k=x}^{\infty} q^k \right) \\
&= p\left(\sum_{k=0}^{\infty} q^k - q^x \sum_{k=0}^{\infty} q^k \right) \\
&= p(1-q^x) \sum_{k=0}^{\infty} q^k \\
&= \frac{p(1-q^x)}{1-q} \\
&= 1 - q^x.
\end{aligned}$$

This implies that

$$\begin{aligned}
\Pr(X > n+m \mid X > m) &= \frac{\Pr(X > n+m,\ X > m)}{\Pr(X > m)} \\
&= \frac{\Pr(X > n+m)}{\Pr(X > m)} \\
&= \frac{q^{n+m}}{q^m} \\
&= q^n \\
&= \Pr(X > n),
\end{aligned}$$

thus proving that the distribution is memoryless. □

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8 Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. $\text{Exp}(\lambda)$.

(a) Find the m.g.f. of X_i.

Solution. As we did in class,

$$\begin{aligned}
M_X(t) &= \mathrm{E}[e^{tX}] \\
&= \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx \\
&= \lambda \int_0^\infty e^{(t-\lambda)x}\, dx \\
&= \frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda. \quad \square
\end{aligned}$$

(b) Use m.g.f.'s to find the distribution of $Y = \sum_{i=1}^n X_i$.

Solution. Since the X_i's are i.i.d., our class notes tell us that

$$M_Y(t) = (M_X(t))^n = \left(\frac{\lambda}{\lambda - t}\right)^n \quad \text{for } t < \lambda.$$

By uniqueness of m.g.f.'s (at least for our class), we see that Y is $\text{Erlang}_n(\lambda)$ (or $\text{Gamma}(n, \lambda)$). □

(c) Suppose $\lambda = 1$. Use the Central Limit Theorem to find the approximate value of $\Pr\left(100 \le \sum_{i=1}^{100} X_i \le 110\right)$.

Solution. As we did in class, using the CLT and the fact that the X_i's are i.i.d. Exp(1), we have

$$\begin{aligned}\sum_{i=1}^{100} X_i &\approx \mathrm{Nor}\left(\mathrm{E}\left[\sum_{i=1}^{100} X_i\right], \mathrm{Var}\left(\sum_{i=1}^{100} X_i\right)\right)\\ &\sim \mathrm{Nor}(100\mathrm{E}[X_1], 100\mathrm{Var}(X_1))\\ &\sim \mathrm{Nor}(100, 100).\end{aligned}$$

Thus, we have

$$\begin{aligned}\Pr\left(100 \le \sum_{i=1}^{100} X_i \le 110\right) &= \Pr\left(\frac{100-100}{10} \le \frac{\sum_{i=1}^{100} X_i - 100}{10} \le \frac{110-100}{10}\right)\\ &\doteq \Pr\left(0 \le \mathrm{Nor}(0,1) \le 1\right)\\ &= \Phi(1) - \Phi(0)\\ &= 0.3415. \quad \Box\end{aligned}$$

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9 Generate 1000 pairs of i.i.d. U(0,1)'s, $(U_{1,1}, U_{1,2}), \ldots, (U_{1000,1}, U_{1000,2})$, and set

$$X_i = \sqrt{-2\ell\mathrm{n}(U_{i1})}\cos(2\pi U_{i2})$$

and

$$Y_i = \sqrt{-2\ell\mathrm{n}(U_{i1})}\sin(2\pi U_{i2})$$

for $i = 1, 2, \ldots, 1000$.

(a) Make a histogram of the X_i's. Comments?

Solution. You get a standard normal p.d.f. $\Box$

(b) Graph X_i vs. Y_i. Comments?

Solution. You get a bivariate normal p.d.f. $\Box$

(c) Make a histogram of X_i/Y_i. Comments?

Solution. You get a Cauchy p.d.f., also known as a t distribution with 1 degree of freedom. This distribution has fatter tails than the standard normal. $\square$

(d) Make a histogram of $X_i^2 + Y_i^2$. Comments?

Solution. You get an exponential p.d.f. $\square$