ENE3031 Computer Simulation — Practice Exam (Midterm)

Fall 2014

Name:

- You will have 1 hours and 50 minutes.
- This exam is closed book and closed notes. Calculators are not allowed. No scrap paper is allowed. Make sure that there is nothing on your desk except pens and erasers.
- If you need extra space, use the back of the page and indicate that you have done so.
- The test must be **stapled** when you turn in your test.
- **Show your work** except problem #1. A correct answer without work will receive zero.
- **We will not select among several answers**. Make sure it is clear what part of your work you want graded. Otherwise, zero point will be given to the problem.

1. (20 points, 2 points each) Short-Answer Questions. Just write your answer.

   (a) Suppose X has p.d.f. $f(x) = cx^2, 0 < x < 3$. Find c. ANSWER:________

   (b) Suppose X has p.d.f. $f(x) = 2x, 0 < x < 1$. Find $\mathsf{Var}(X)$. ANSWER:________

   (c) If $\mathsf{E}[X] = 7$, what is $\mathsf{E}[3X - 4]$? ANSWER:________

   (d) If $\mathsf{E}[X] = 7$ and $\mathsf{Var}(X) = 3$, what is $\mathsf{Var}(3X - 4)$? ANSWER:________

   (e) Suppose X is an RV with c.d.f. $F(x)$. Whats the distribution of $F(X)$? ANSWER:________

   (f) If U_1 and U_2 are i.i.d. $Unif(0, 1)$, what is the distribution of $-6\ln(U_1) - 6\ln(1 - U_2)$? ANSWER:________

   (g) TRUE or FALSE? The Erlang distribution is a special case of the Gamma. ANSWER:________

   (h) Consider a Poisson process with rate 0.2 arrivals per minute. Find the probability that the second interarrival time is less than one minute. ANSWER:________

   (i) TRUE or FALSE? If X and Y are independent, they have zero covariance. ANSWER:________

   (j) Suppose X is a Bernoulli(p) random variable. Find $\mathsf{Corr}(X, X^2)$. ANSWER:________

2. (20pts) Suppose that two types of customers arrive at a single-server queue. Type-B customers have priority over Type-As (though nobody gets preempted if theyre already being served.) Otherwise, service is FIFO within each type class. Assume the system starts out empty and idle.

Customer	Type	Interarrival time	Service time
1	A	3	10
2	A	5	8
3	B	8	4
4	B	4	6
5	B	11	2
6	A	6	8
7	A	5	7
8	B	3	4
9	A	6	8
10	A	7	2

(a) When does the last customer leave the system?

(b) What is the average number of customers in the system?

(c) What is the average additional amount of time type-Bs would have to spend in the system if they didnt have priority?

3. (20pts) Questions on Uniform Generation.

(a) Consider the multiplicative random number generator $X_n = aX_{n-1} \bmod 7$. If the multiplier $a = 5$ and the seed $X_0 = 1$, what is X_{217}?

(b) For the multiplicative random number generator given in part (a) above, which works better: a = 4 or a = 5? Say why.

(c) Consider the following n = 30 pseudo-random numbers. (Read from left to right, and then down.)

0.29 0.37 0.46 0.69 0.90 0.93 0.99 0.86 0.72 0.47
0.30 0.18 0.29 0.38 0.69 0.76 0.91 0.62 0.41 0.30
0.11 0.45 0.72 0.88 0.65 0.55 0.31 0.27 0.15 0.92

Conduct a runs up-and-down test on these numbers and state what the conclusion of the test is. Use $\alpha = 0.10$. (Use $z_{\alpha/2} = z_{0.05} = 1.645$).

(d) Same question as (c) above, except do a χ^2 goodness-of-fit test when $k = 3$ and $\alpha = 0.10$. (Use $\chi^2_{\alpha,\ k-1} = \chi^2_{0.10,\ 2} = 0.4605$)

4. (20pts) Questions on Random Variate Generation.

   (a) Suppose the random variable X has p.d.f. $f(x) = 3x^2$, for $0 \le x \le 1$. Give an inverse transform method for generating realizations of X.

   (b) Use as many of the following $U(0,1)$s (left to right) as necessary to generate a $Pois(4)$ r.v.
   0.79 0.92 0.83 0.04 0.09 0.13 0.49 0.75 0.83 0.02 0.11 0.23

   (c) Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. $Exp(\lambda)$. How can I generate a realization of $Y = \max\{X_1, X_2, \ldots, X_n\}$ using exactly one $Unif(0,1)$ random number?

5. (20pts) The production line consists of two stations (station 1 and 2) and one rework station (station 3). An engineer recorded some time study data between 8am and 6pm over one week (5 days). The number of arrival customers follows a Poisson process, and the total number of arrivals to station 1 is 1000. At Station 1, average service time of each server at Station 1 is 1/15 hours and 2 servers are available. Average service time of Station 2 is 1/40 hours, average service time of Station 3 is 1/60 hours, and a single server is available at both Station 2 and 3. At Station 1, 40% jobs are found to have defects, defected jobs are sent to the rework station (station 3), and the rest of jobs (60% jobs) are directly sent to station 2. At station 3, only 90% are salvaged at the rework station and sent to station 2 and 10% need, rework again. All service times follow exponential distributions. Find the average waiting time and the average number of customers in front of each station.

Some Formula

•
$$F(x) \;=\; \begin{cases} \int_{-\infty}^{x} f(t)\,dt, & \text{if } X \text{ is continuous} \\ \sum_{\{y|y\le x\}} f(y) \;=\; \sum_{\{y|y\le x\}} \Pr(X=y), & \text{if } X \text{ is discrete} \end{cases}.$$

•
$$\mu \;\equiv\; \mathsf{E}[X] \;\equiv\; \begin{cases} \sum_x xp(x) & \text{if } X \text{ is discrete} \\ \int_{\Re} xf(x)\,dx & \text{if } X \text{ is cts} \end{cases}$$

$$\mathsf{E}[g(X)] \;\equiv\; \begin{cases} \sum_x g(x)p(x) & \text{if } X \text{ is discrete} \\ \int_{\Re} g(x)f(x)\,dx & \text{if } X \text{ is cts} \end{cases}$$

$$\mathsf{E}[g(X,Y)] \;\equiv\; \begin{cases} \sum_y \sum_x g(x,y)p(x,y) & \text{if } X \text{ is discrete} \\ \int_{\Re}\int_{\Re} g(x,y)f(x,y)\,dxdy & \text{if } X \text{ is cts} \end{cases}$$

• $\mathsf{Var}[X] \equiv \mathsf{E}[(X-\mu)^2] = \mathsf{E}[(X-\mathsf{E}[X])^2] = \mathsf{E}[X^2] - (\mathsf{E}[X])^2$

• $p_X(x) = \sum_y p(x,y); p_Y(y) = \sum_x p(x,y); f_X(x) = \int_y f(x,y)dy; f_Y(y) = \int_x f(x,y)dx.$

•
$$F(x,y) \;=\; P(X\le x, Y\le y) \;=\; \begin{cases} \sum\sum_{s\le x, t\le y} f(s,t) & \text{discrete} \\ \\ \int_{-\infty}^{y}\int_{-\infty}^{x} f(s,t)\,ds\,dt & \text{continuous} \end{cases}$$

•
$$\mathsf{E}[Y|X=x] \;\equiv\; \begin{cases} \sum_y yf(y|x) & \text{discrete} \\ \int_{\Re} yf(y|x)\,dy & \text{continuous} \end{cases}$$

•
$$\mathsf{Cov}(X,Y) \;\equiv\; \sigma_{XY} \;\equiv\; \mathsf{E}[(X-\mathsf{E}[X])(Y-\mathsf{E}[Y])] = \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y].$$

•
$$\rho \;=\; \mathrm{Corr}(X,Y) \;\equiv\; \frac{\mathsf{Cov}(X,Y)}{\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)}} \;=\; \frac{\sigma_{XY}}{\sigma_X\sigma_Y}.$$

• Mgf: $M_X(t) = \mathsf{E}[e^{tX}]$ and $\mathsf{E}[X^k] = \frac{d^k}{dt^k}\mathsf{E}[e^{tX}]|_{t=0}$.

• **Central Limit Theorem:** Suppose $X_1,\ldots,X_n$ are i.i.d. with $\mathsf{E}[X_i]=\mu$ and $\mathsf{Var}(X_i) = \sigma^2$. Then as $n\to\infty$,

$$\frac{\sum_{i=1}^n X_i - n\mu}{\sigma\sqrt{n}} \;=\; \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \;\xrightarrow{\mathcal{D}}\; \mathsf{Nor}(0,1),$$

where "$\xrightarrow{\mathcal{D}}$" means that the c.d.f. $\to$ the $\mathsf{Nor}(0,1)$ c.d.f.

• **Normal Approximation to Binomial** If $X\sim$Bin(n,p) and n is large, then

$$\frac{Y-\mathsf{E}[Y]}{\sqrt{\mathsf{Var}(Y)}} \;\approx\; \mathsf{Nor}(0,1).$$

• **Bernoulli(p) Distribution: success or failure**

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } q \end{cases}$$

$\mathsf{E}[X] = p$, $\mathsf{Var}(X) = pq$, and $M_X(t) = pe^t + q$.
• **Binomial(n, p) Distribution: number of successes out of n trials**
$\mathrm{Bin}(n, p)$ is the sum of n i.i.d. $\mathrm{Bern}(p)$.
$\Pr(Y = k) = \binom{n}{k} p^k q^{n-k}, n = 0, 1, \ldots,\ \mathsf{E}[Y] = np,\ \mathsf{Var}[Y] = npq.$
• **Geometric(p): number of trials until the 1st success**
$\Pr(Z = k) = q^{k-1}p,$ for $k = 1, 2, \ldots,\ M_Z(t) = \frac{pe^t}{1-qe^t},$ for $t < \ell\mathrm{n}(1/q)$
$\mathsf{E}[Z] = \frac{1}{p},$ and $\mathsf{Var}(Z) = \frac{q}{p^2}$.
• **Poisson with rate λ: number of events in a fixed interval**
$\Pr(X = k) = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, 2, \ldots$ and $\mathsf{E}[X] = \mathsf{Var}[X] = \lambda$.
• **Uniform(a, b) Distribution**

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\mathsf{E}[X] = \frac{a+b}{2} \qquad \mathsf{Var}(X) = \frac{(a-b)^2}{12}.$$

• **Exponential with rate λ: lifetime with constant failure rate**

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = 1 - e^{-\lambda x} \quad \mathsf{E}[X] = 1/\lambda, \quad \mathsf{Var}(X) = 1/\lambda^2.$$

• **Standard Normal** Z The p.d.f. of the $\mathsf{Nor}(0, 1)$ is

$$\phi(z) \equiv \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad z \in \Re.$$

$$\Pr\{Z \le z\} \equiv \Phi(z) \equiv \int_{-\infty}^{z} \phi(t)\, dt, \quad z \in \Re.$$

$$X \sim \mathsf{Nor}(\mu, \sigma^2) \Rightarrow Z \equiv \frac{X - \mu}{\sigma} \sim \mathsf{Nor}(0, 1).$$

M/M/1(∞/ ∞)

- Sever Utilization ($stability$) $\rho = \frac{\lambda}{\mu}$
- $L = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$
- $W = \frac{L}{\lambda} = \frac{1}{\mu-\lambda}$
- $L_Q = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$
- $W_Q = \frac{L_Q}{\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$
- $P_n = P(L(t) = n) = (1-\rho)\,\rho^n$

M/M/c(∞/ ∞)

- $\rho = \frac{\lambda}{c\mu}$
- $L_Q = \frac{(\frac{\lambda}{\mu})^c \rho}{c!(1-\rho)^2} \left(\sum_{n=0}^{c-1} \frac{(\frac{\lambda}{\mu})^n}{n!} + \frac{(\frac{\lambda}{\mu})^c}{c!} \cdot \frac{1}{1-\frac{\lambda}{c\mu}} \right)^{-1}$
- $W_Q = \frac{L_Q}{\lambda}$
- $W = W_Q + \frac{1}{\mu}$
- $L = \lambda W = \lambda \left(W_q + \frac{1}{\mu} \right) = L_q + \frac{\lambda}{\mu}$