Facilities planning

Facility location

Notation

m: existing facilities

n:new facilities

$\mathbf{x_j} = (x_j, y_j)$: location of new facility j

$\mathbf{p_i} = (a_i, b_i)$:location of existing facility i

t_{ij}:trip between $\mathbf{x_j}$ and $\mathbf{p_i}$ per month

$d(\mathbf{x_j},\mathbf{p_i})$:distance between $\mathbf{x_j}$ and $\mathbf{p_i}$

total distance per month = $t_{ij}\ d(\mathbf{x_j},\mathbf{p_i})$

v_{ij}:average travel speed from facility i to j

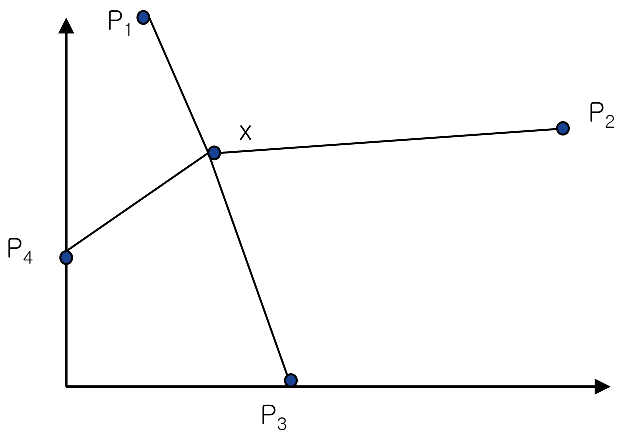
average travel time = $t_{ij}\ d(\mathbf{x_j},\mathbf{p_i})/v_{ij}$

c_{ij}:cost per unit travel time

total travel cost per month = $c_{ij}t_{ij}\ d(\mathbf{x_j},\mathbf{p_i})/v_{ij}$

let $w_{ij} = c_{ij}t_{ij}\ /v_{ij}$

want to minimize $f(x_1,..,x_n) = g(w_{ij}d(\mathbf{x_j},\mathbf{p_i}))$



Facility location problems

Facility location problems

- Single facility, multiple facilities

- single facility problem : $n = 1$

- multiple facility problem : $n > 1$

- Distance : rectilinear, Euclidean

- rectilinear distance $d(x_j, p_i) = | x_j - a_i | + | y_j - b_i |$

- Euclidean distance $= \sqrt{\left((x_j - a_i)^2 + (y_j - b_i)^2\right)}$

- Optimization : minisum, minimax

minisum problem $\rightarrow$ minimize $\sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} d(x_j, p_i)$

minimax problem $\rightarrow$ minimize $\max_{i,j} \left(w_{ij} d(x_j, p_i) \right)$

single facility minisum problem with rectilinear distance

$$\text{minimize } f(x, y) = \text{minimize} \sum_{i=1}^{m} w_i \left(| x - a_i | + | y - b_i | \right)$$

The objective function can be rewritten as

$$\text{minimize } f(x, y) = \text{minimize} \left(f_1(x) + f_2(y) \right)$$

where $f_1(x) = w_i \left(| x - a_i | \right)$ and $f_2(y) = w_i \left(| y - b_i | \right)$

Therefore we can solve for x and y independently.

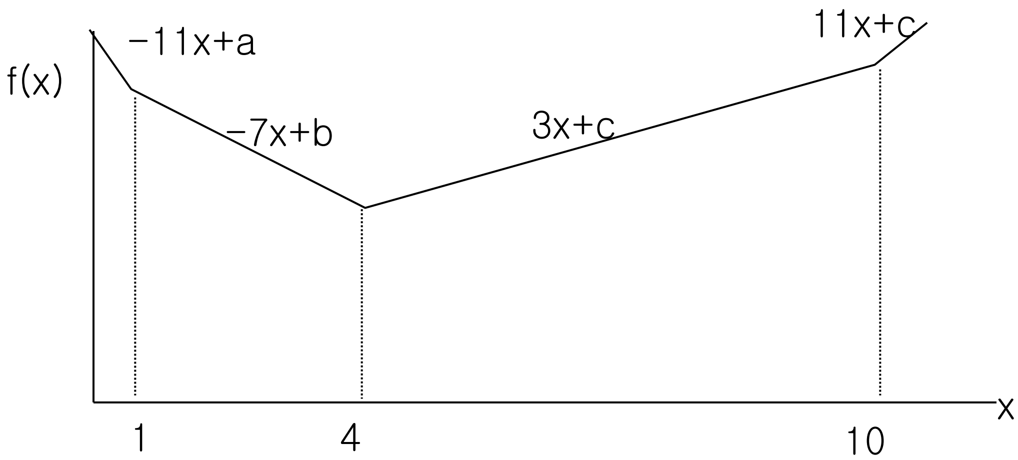
example)

A test station is to be added to an existing plant. The station will receive 4 loads per day from location (10,8), 2 from (1,4), 3 from (4,6) and 2 from (4,9).

Where should the station be located to minimize total distance traveled?

$$\text{minimize } f(x, y) = 4 | x - 10 | + 2 | x - 1 | + 3 | x - 4 | + 2 | x - 4 | +$$
$$4 | y - 8 | + 2 | y - 4 | + 3 | y - 6 | + 2 | y - 9 |$$

single facility minisum problem with rectilinear distance



Optimal location exists among existing locations

Optimal property

Median Location (x, y)

$$\sum_{x_i<x} w_i \le \sum_{i=1}^{m} \frac{w_i}{2} \text{ and } \sum_{x_i>x} w_i \le \sum_{i=1}^{m} \frac{w_i}{2}$$

$$\sum_{y_i<y} w_i \le \sum_{i=1}^{m} \frac{w_i}{2} \text{ and } \sum_{y_i>y} w_i \le \sum_{i=1}^{m} \frac{w_i}{2}$$

Fact : Every median location solves the single facility rectilinear location problem

sol) $\sum w_i = 4+2+3+2 = 11, \frac{\sum w_i}{2} = 5.5$

$$\begin{array}{l} x:1\ \ 4\ \ 10 \\ w:2\ \ 5\ \ 4 \end{array} \qquad \sum_{x_i<4} w_i \le 5.5 \text{ and } \sum_{x_i>4} w_i \le 5.5$$

$$\begin{array}{l} y:4\ \ 6\ \ 8\ \ 9 \\ w:2\ \ 3\ \ 4\ \ 2 \end{array} \qquad \sum_{y_i<8} w_i \le 5.5 \text{ and } \sum_{y_i>8} w_i \le 5.5$$

$(x^*, y^*) = (4,8)$

Justification of Median location

Assume that there exists an optimal location, x^* that is not a median location.
We will show that this statement leads to a contradiction with the definition of a median location

Let

$S_1 = \{i : x_i < x^*\}, S_2 = \{i : x_i > x^*\}$ and $S_3 = \{i : x_i = x^*\}$

Since x^* is not a median location either

$$\sum_{i \in S_1} w_i > \frac{\sum_{i=1}^{m} w_i}{2} \quad \text{or} \quad \sum_{i \in S_2} w_i > \frac{\sum_{i=1}^{m} w_i}{2}$$

Assume $\displaystyle\sum_{i \in S_1} w_i > \frac{\sum_{i=1}^{m} w_i}{2}$

Justification of Median location(cont.)

In the neighborhood of x^*, the objective can be written

$$f_1(x) = \sum_{i \in S_1} w_i (x - x_i) + \sum_{i \in S_2} w_i (x_i - x) + \sum_{i \in S_3} w_i |x - x_i|$$

Since x^* is optimal, $f_1(x^* - \varepsilon) > f_1(x^*)$

$$f_1(x^* - \varepsilon) - f_1(x^*) = \sum_{i \in S_1} w_i ((x^* - \varepsilon) - x_i) + \sum_{i \in S_2} w_i (x_i - (x^* - \varepsilon)) + \sum_{i \in S_3} w_i |(x^* - \varepsilon) - x_i|$$

$$- \sum_{i \in S_1} w_i (x^* - x_i) + \sum_{i \in S_2} w_i (x_i - x^*) + \sum_{i \in S_3} w_i |x^* - x_i|$$

$$= \left[-\sum_{i \in S_1} w_i + \sum_{i \in S2} w_i + \sum_{i \in S3} w_i \right] \varepsilon$$

Justification of Median location(cont.)

$$\text{Using the fact that } \sum_{i \in S_1} w_i > \frac{\sum_{i=1}^{m} w_i}{2} \rightarrow 2\sum_{i \in S_1} w_i > \sum_{i=1}^{m} w_i$$

$$\rightarrow 2\sum_{i \in S_1} w_i > \sum_{i \in S_1} w_i + \sum_{i \in S_2} w_i + \sum_{i \in S_3} w_i \rightarrow -\sum_{i \in S_1} w_i + \sum_{i \in S2} w_i + \sum_{i \in S3} w_i < 0$$

$$f_1(x^* - \varepsilon) - f_1(x^*) = \left[-\sum_{i \in S_1} w_i + \sum_{i \in S2} w_i + \sum_{i \in S3} w_i \right] \varepsilon < 0$$

Thus $f_1(x^* - \varepsilon) < f_1(x^*)$ is a contradiction to the assumed optimality of x^*

Contour line

The optimal location found by optimal property may not be feasible.

Other machiens may be there, access to the location may be insufficient, etc.

Contour Line : every point on the contour line has the same value of the objective function

p : number of unique x_i locations

q : number of unique y_i locations

c_j : x coordinate of unique existing location j

r_j : y coordinate of unique existing location j

C_j : sum of the weights associated with coordinate c_j

R_j : sum of the weights weight associated with coordinate r_j

$$\text{minimize} \sum_{j=1}^{p} C_j \left| x - c_j \right| + \sum_{j=1}^{q} R_j \left| y - r_j \right|$$

Contour line

For any point (x, y) in rectangle $[s, t]$

where $c_t \le x \le c_{t+1}$ and $r_s \le y \le r_{s+1}$

$$f(x,y) = \sum_{j=1}^{p} C_j \left| x - c_j \right| + \sum_{j=1}^{q} R_j \left| y - r_j \right|$$

$$= \sum_{j=1}^{t} C_j (x - c_j) + \sum_{j=t+1}^{p} C_j (c_j - x) + \sum_{j=1}^{s} R_j (y - r_j) + \sum_{j=s+1}^{q} R_j (r_j - y)$$

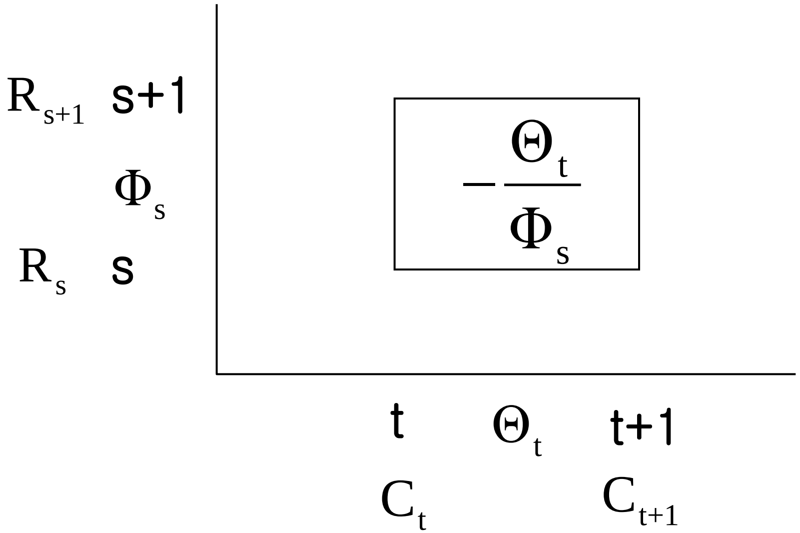
$$= \left(\sum_{j=1}^{t} C_j - \sum_{j=t+1}^{p} C_j \right) x - \left(\sum_{j=1}^{t} C_j c_j - \sum_{j=t+1}^{p} C_j c_j \right)$$

$$+ \left(\sum_{j=1}^{s} R_j - \sum_{j=s+1}^{q} R_j \right) y - \left(\sum_{j=1}^{s} R_j r_j - \sum_{j=s+1}^{q} R_j r_j \right)$$

$$= \Theta_t x + \Phi_s y + K_{st}$$

Contour line associated with objective value $= k$

$$k = \Theta_t x + \Phi_s y + K_{st} \rightarrow y = -\frac{\Theta_t}{\Phi_s} x + \frac{k - K_{st}}{\Phi_s}$$



Constructing contour line

Step1: Draw p vertical lines to intersect all x_i and q horizontal lines to intersect all y_i

Step2. Label vertical lines by C_j and horizontal lines by R_j, the sum of the weights of facilities intersected by the lines

$$\text{Step 3: Set } \Theta_0 = -\sum_{j=1}^{p} C_j, \Phi_0 = -\sum_{j=1}^{q} R_j$$

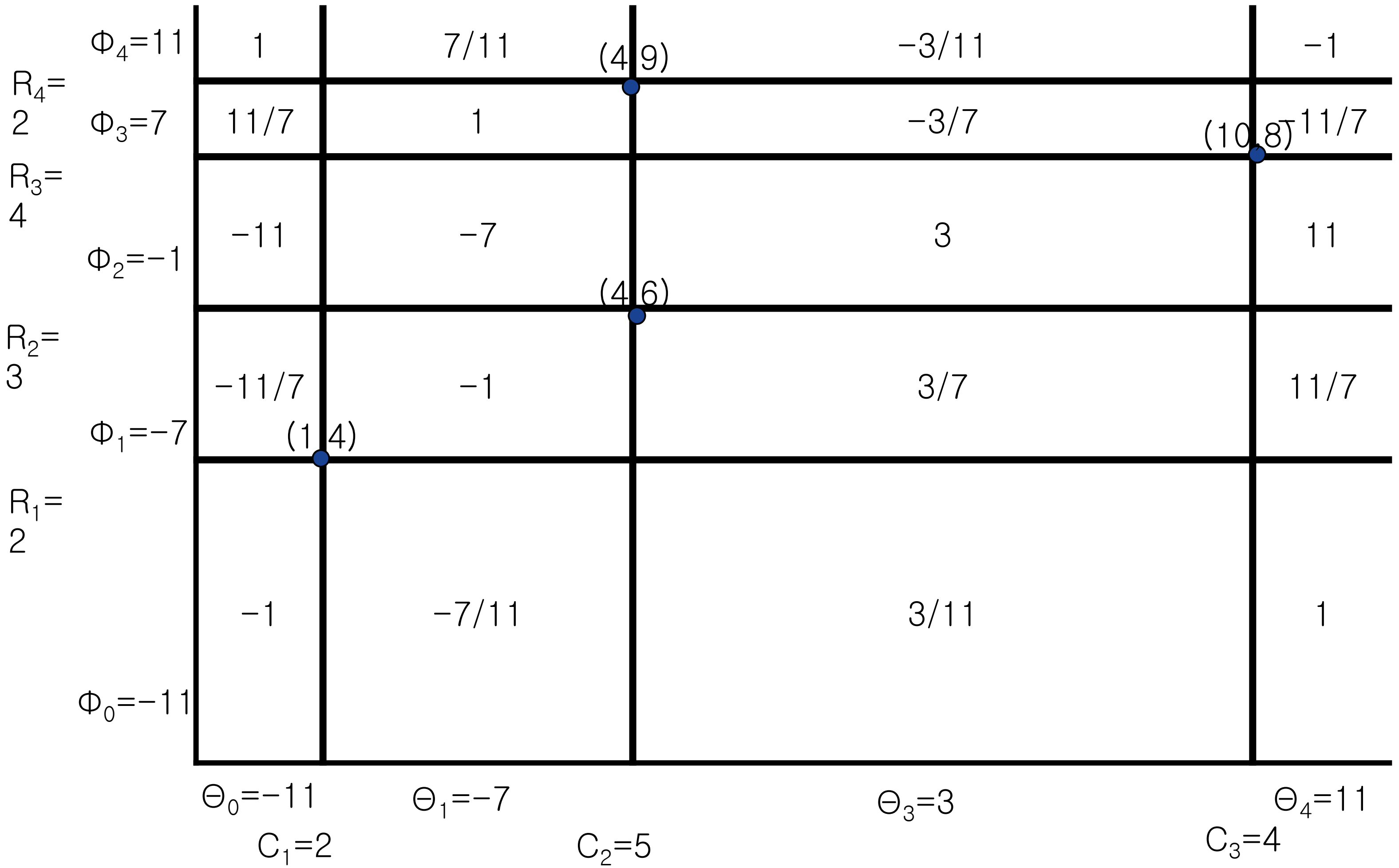
$$\Theta_r = \sum_{j=1}^{r} C_j - \sum_{j=r+1}^{p} C_j \text{ and } \Phi_s = \sum_{j=1}^{s} R_j - \sum_{j=s+1}^{q} R_j$$

Step4: for each rectangular segment [r, s], compute the slope of contour line by

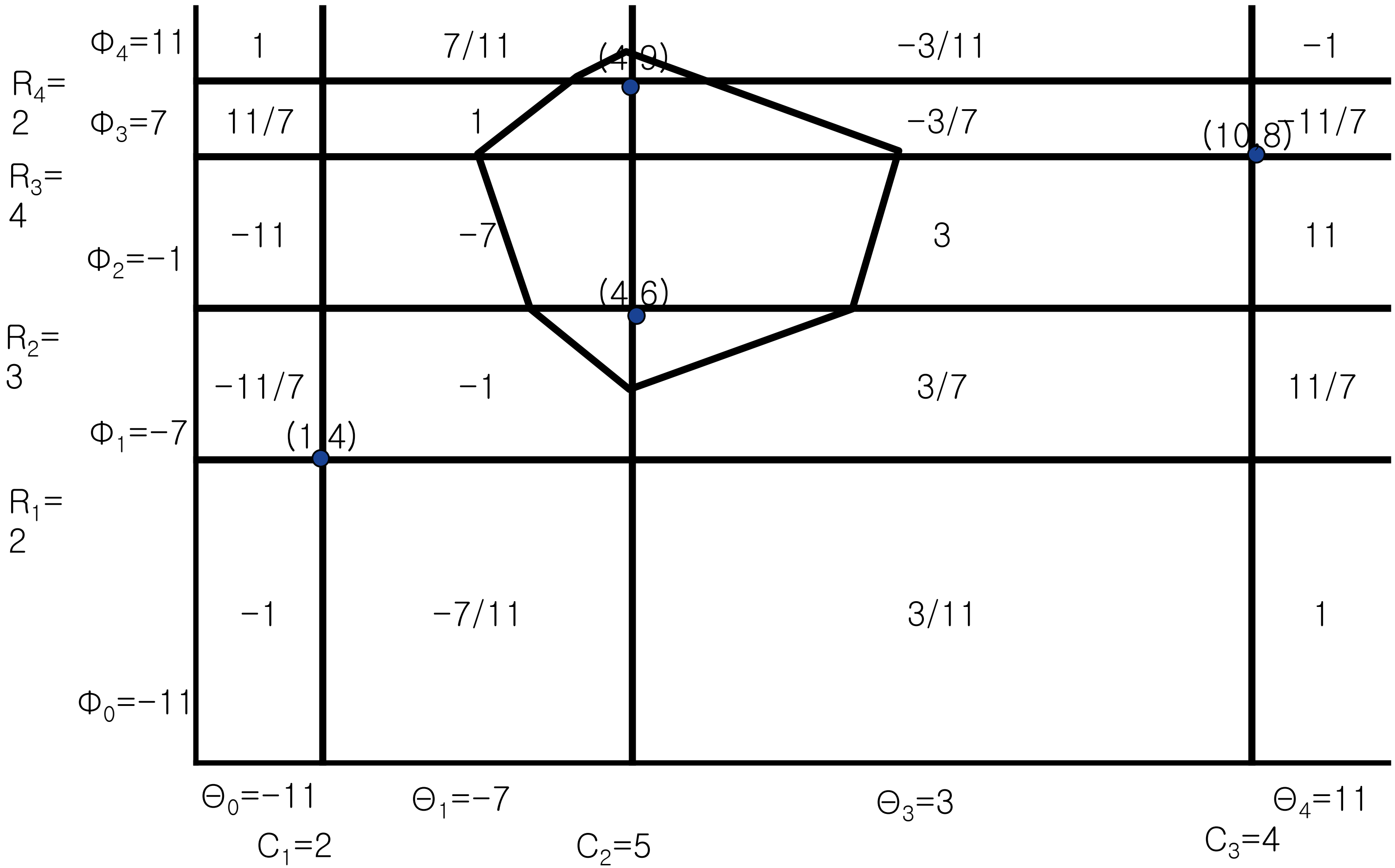
$$s_{rs} = -\frac{\Theta_r}{\Phi_s}$$

Step5: Select any point (x, y) and draw the contour that starts and ends at (x, y) using slope s_{rs} in each segment. Step 5 can be repeated as many times as desired to produce the contour map

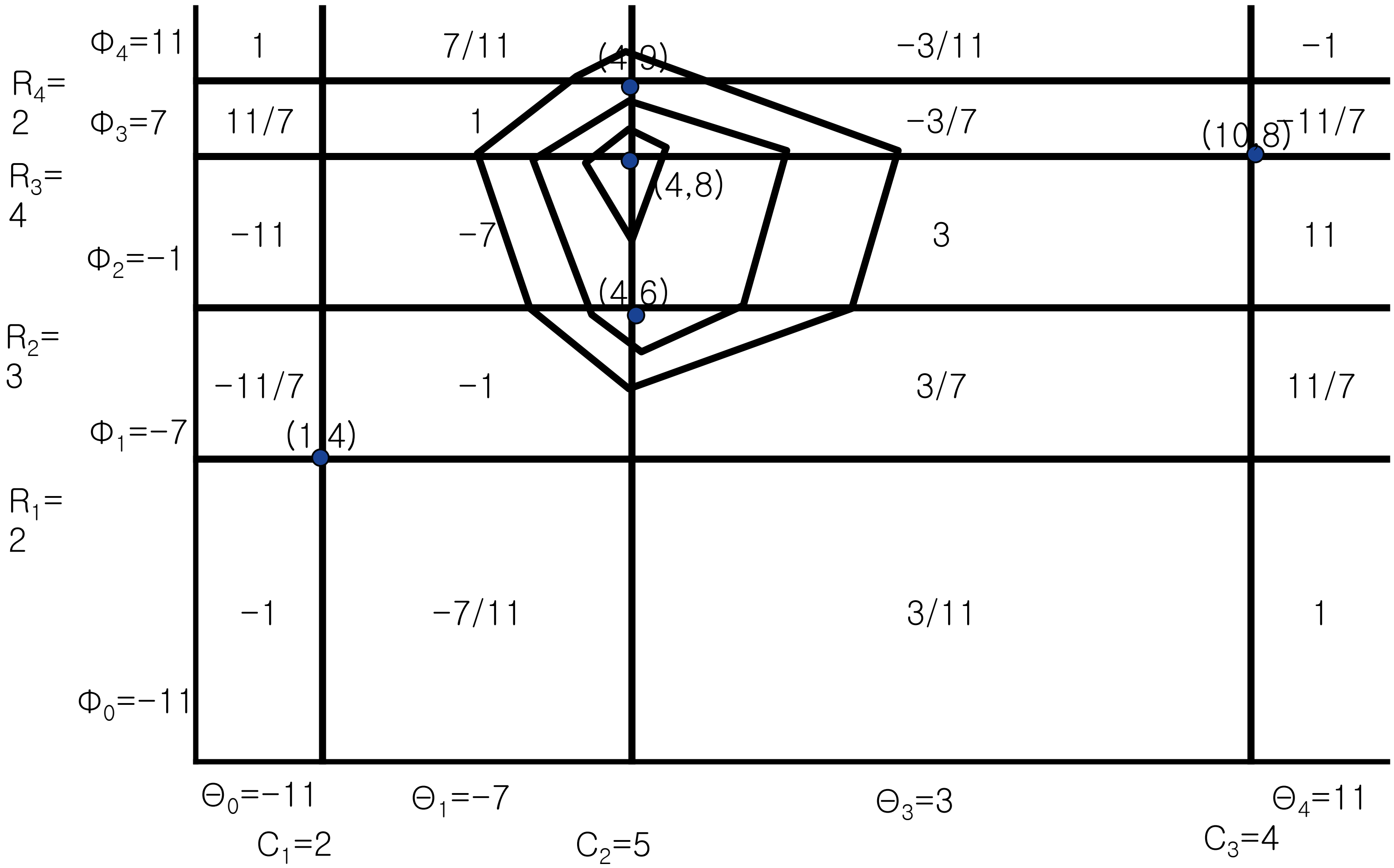
Example


Φ4=11
Φ3=7
Φ2=−1
Φ1=−7
Φ0=−11
R4= 2
R3= 4
R2= 3
R1= 2
1
7/11
−3/11
−1
11/7
1
−3/7
−11/7
−11
−7
3
11
−11/7
−1
3/7
11/7
−1
−7/11
3/11
1
(4,9)
(10,8)
(4,6)
(1,4)
Θ0=−11
Θ1=−7
Θ3=3
Θ4=11
C1=2
C2=5
C3=4

Example(cont.): Contour line starting from (4,5)



Example(cont.): Finding an optimal solution (4,8)



Minimax location problem with Rectilinear distance

$$\min\left\{\max\left(w_i(|x-a_i|+|y-b_i|)+h_i, \forall i\right)\right\}$$

h_i is the time to prepare to go to location i from location (x, y)

Let $z = \max\left((w_i |x-a_i|+|y-b_i|)+h_i, \forall i\right)$

$$\min \ z$$

$$\text{st} \quad w_i(|x-a_i|+|y-b_i|)+h_i \le z, \forall i$$

Equivalent formulation

$$\min \ z$$

$$\text{st} \quad |x-a_i|+|y-b_i| \le \frac{z-h_i}{w_i}, \forall i$$

Unweighted Problem(UP) : $w_i = 1, h_i = 0, \forall i$

Unweighted Problem with Attends(UPA) : $w_i = 1, h_i > 0, \exists i$

Weighted Problem with Attends(WPA) : $w_i \neq w_i, h_i > 0, \exists i$

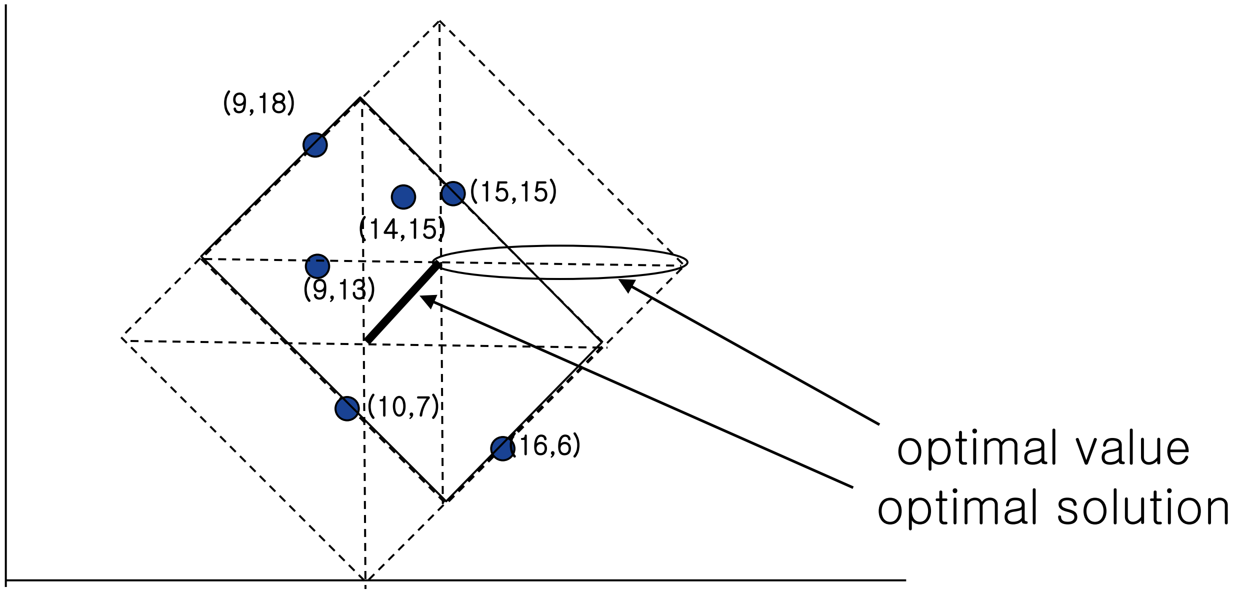
Geometric interpretation of UP

$$\min \; z$$

$$\text{st} \quad |x - a_i| + |y - b_i| \le z, \forall i$$

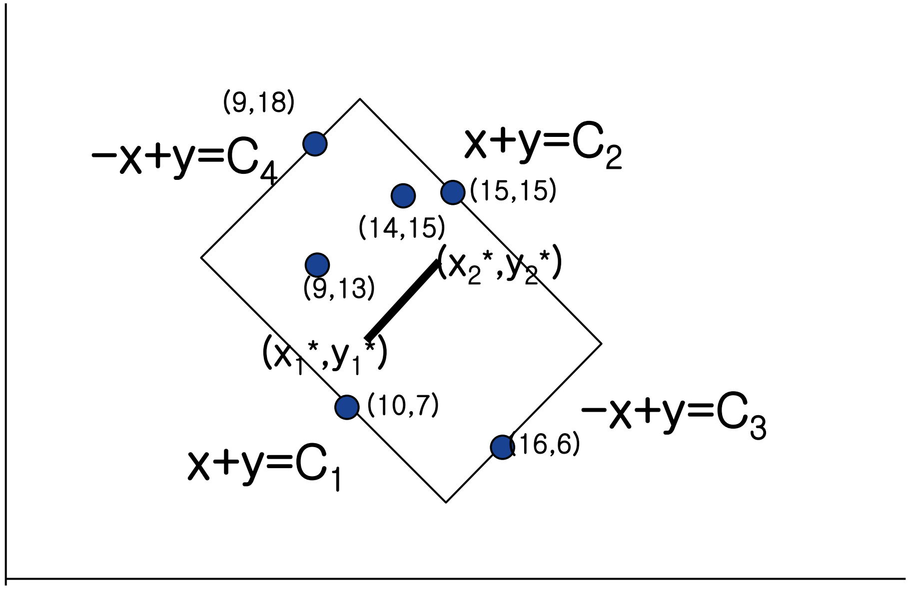
The problem is to find a diamond of minimum radius that will contain all the existing facility locations $\Rightarrow$ center point is (x, y)

i	1	2	3	4	5	6
a	9	9	10	14	15	16
b	13	18	7	15	15	6

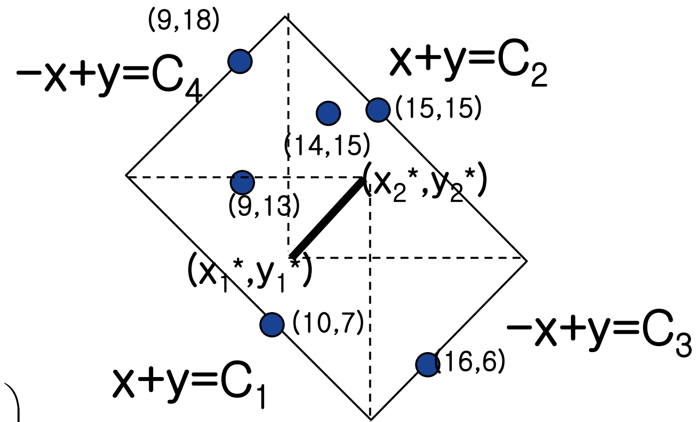


Solution procedure for UP

$$C_1 = \min(a_i + b_i, \forall i) = \min(22,27,17,29,30,22) = 17$$
$$C_2 = \max(a_i + b_i, \forall i) = \max(22,27,17,29,30,22) = 30$$
$$C_3 = \min(-a_i + b_i, \forall i) = \min(4,9,-3,1,0,-10) = -10$$
$$C_4 = \max(-a_i + b_i, \forall i) = \max(4,9,-3,1,0,-10) = 9$$
$$C_5 = \max(C_2 - C_1, C_4 - C_3) = \max(13,19) = 19$$



Solution procedure for UP(cont.)



$$(x_1^*, y_1^*) = \left(\frac{C_2 - C_4}{2}, \frac{C_2 + C_3}{2} \right)$$

$\Leftarrow$ (intersection of $-x + y = C_4$ and $x + y = C_2$, intersection of $x + y = C_2$ and $-x + y = C_3$)

$$(x_2^*, y_2^*) = \left(\frac{C_1 - C_3}{2}, \frac{C_1 + C_4}{2} \right)$$

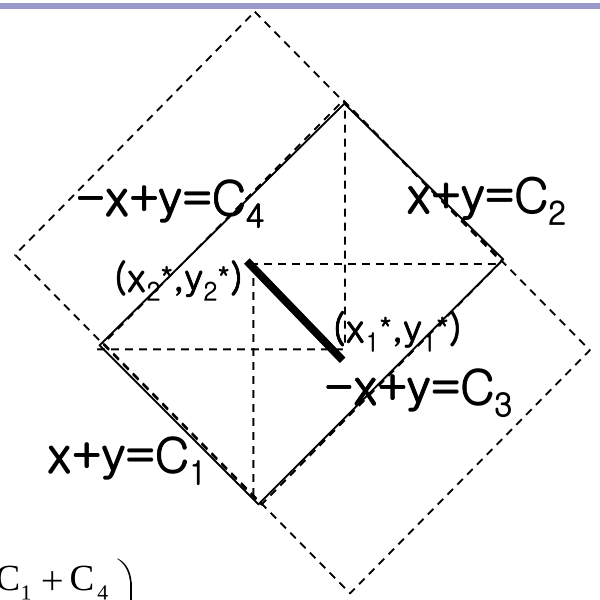
$\Leftarrow$ (intersection of $x + y = C_1$ and $-x + y = C_3$, intersection of $x + y = C_1$ and $-x + y = C_4$)

$$(x_1^*, y_1^*) = \left(\frac{C_2 - C_4}{2}, \frac{C_2 + C_4 - (C_4 - C_3)}{2} = \frac{C_2 + C_4 - C_5}{2} \Leftarrow \text{since } C_4 - C_3 \geq C_2 - C_1 \right)$$

$$(x_2^*, y_2^*) = \left(\frac{C_1 - C_3}{2}, \frac{C_1 + C_3 + C_4 - C_3}{2} = \frac{C_1 + C_3 + C_5}{2} \Leftarrow \text{since } C_4 - C_3 \geq C_2 - C_1 \right)$$

$$(x_1^*, y_1^*) = (10.5, 10), (x_2^*, y_2^*) = (13.5, 13), z = x_1^* + C_4 - 10 = 9.5$$

Solution procedure for UP(cont.)



$$(x_1^*, y_1^*) = \left(\frac{C_2 - C_4}{2}, \frac{C_1 + C_4}{2} \right)$$

$\Leftarrow$ (intersection of $-x + y = C_4$ and $x + y = C_2$, intersection of $x + y = C_1$ and $-x + y = C_4$)

$$(x_2^*, y_2^*) = \left(\frac{C_1 - C_3}{2}, \frac{C_2 + C_3}{2} \right)$$

$\Leftarrow$ (intersection of $x + y = C_1$ and $-x + y = C_3$, intersection of $x + y = C_2$ and $-x + y = C_3$)

$$(x_1^*, y_1^*) = \left(\frac{C_2 - C_4}{2}, \frac{C_2 + C_4 - (C_2 - C_1)}{2} = \frac{C_2 + C_4 - C_5}{2} \Leftarrow \text{since } C_2 - C_1 \geq C_4 - C_3 \right)$$

$$(x_2^*, y_2^*) = \left(\frac{C_1 - C_3}{2}, \frac{C_1 + C_3 + C_2 - C_1}{2} = \frac{C_1 + C_3 + C_5}{2} \Leftarrow \text{since } C_2 - C_1 \geq C_4 - C_3 \right)$$

LP formulation

$$\min\ z$$
$$\text{st}\quad |x-a_i|+|y-b_i|\le z, \forall i$$

$$|x-a_i|\le z\text{-}|y-b_i|, \forall i \Rightarrow -z+|y-b_i|\le x-a_i\le z\text{-}|y-b_i|, \forall i$$

$$\Rightarrow \begin{cases} |y-b_i|\le z+x-a_i, \forall i \\ |y-b_i|\le z-x+a_i, \forall i \end{cases} \Rightarrow \begin{cases} -z-x+a_i\le y-b_i\le z+x-a_i, \forall i \\ -z+x-a_i\le y-b_i\le z-x+a_i, \forall i \end{cases} \Rightarrow \begin{cases} x+y-z\le a_i+b_i, \forall i \\ x+y+z\ge a_i+b_i, \forall i \\ \text{-}x+y-z\le -a_i+b_i, \forall i \\ \text{-}x+y+z\ge -a_i+b_i, \forall i \end{cases}$$

Let $C_1=\min(a_i+b_i, \forall i), C_2=\max(a_i+b_i, \forall i), C_3=\min(-a_i+b_i, \forall i), C_4=\max(-a_i+b_i, \forall i)$

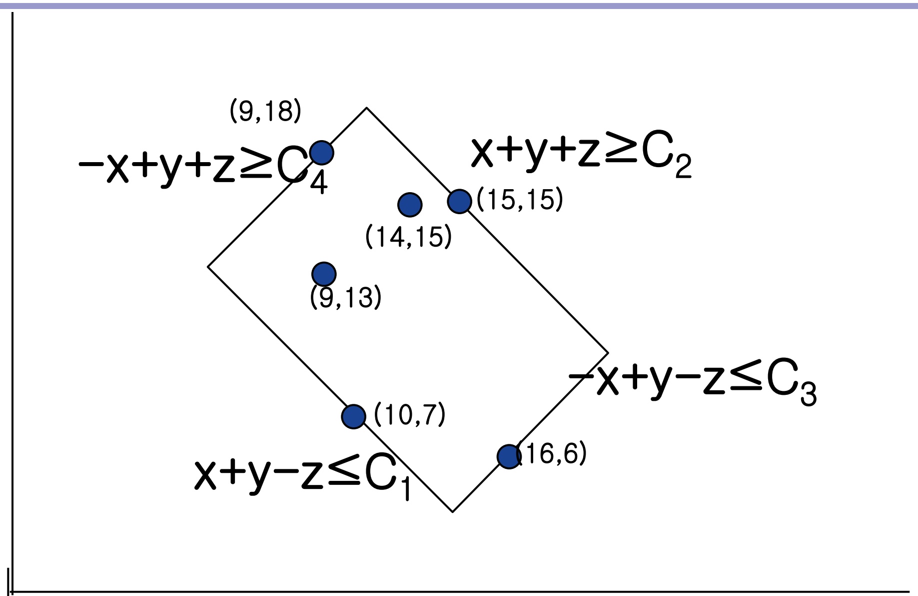
$$\Rightarrow \begin{cases} x+y-z\le C_1 \sim (1) \\ x+y+z\ge C_2 \sim (2) \\ \text{-}x+y-z\le C_3 \sim (3) \\ \text{-}x+y+z\ge C_4 \sim (4) \end{cases}$$

$$-(1)+(2)\Rightarrow z\ge \frac{C_2-C_1}{2} \text{ and } -(3)+(4)\Rightarrow z\ge \frac{C_4-C_3}{2}$$

Therfore $z\ge \max(\frac{C_2-C_1}{2}, \frac{C_4-C_3}{2})$

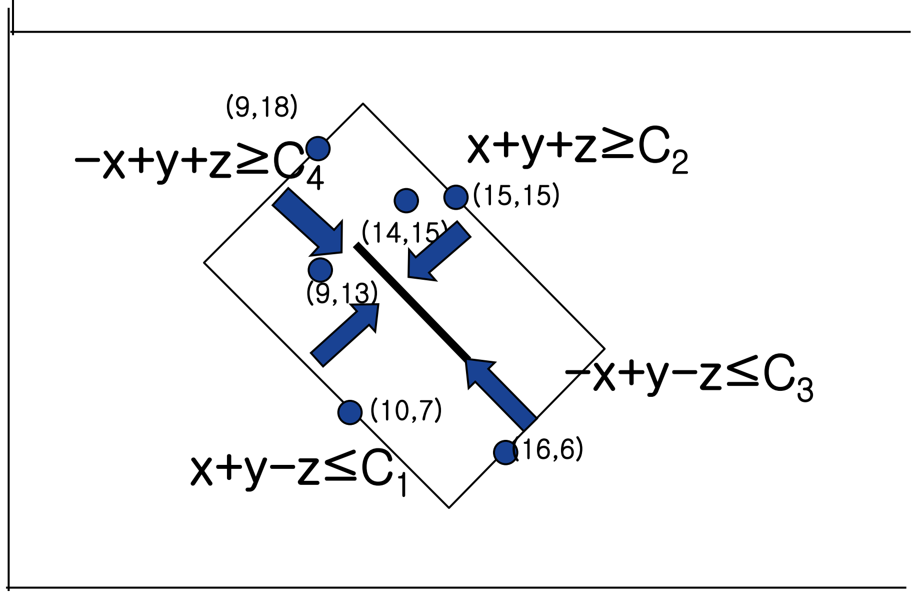
Let $C_5=\max(C_2-C_1, C_4-C_3)$ then $z\ge \frac{C_5}{2}$

Geometric interpretation of LP



infeasible

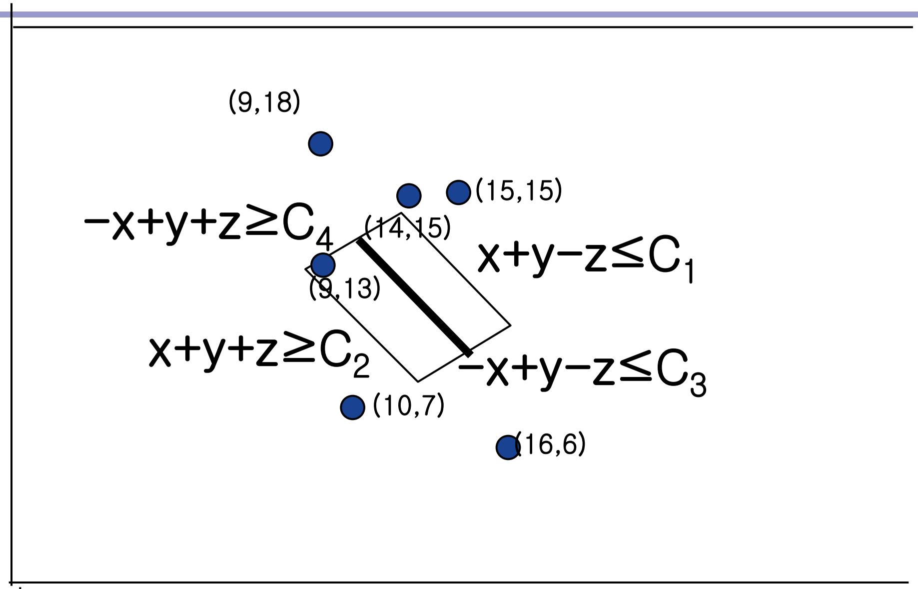
increase z



infeasible

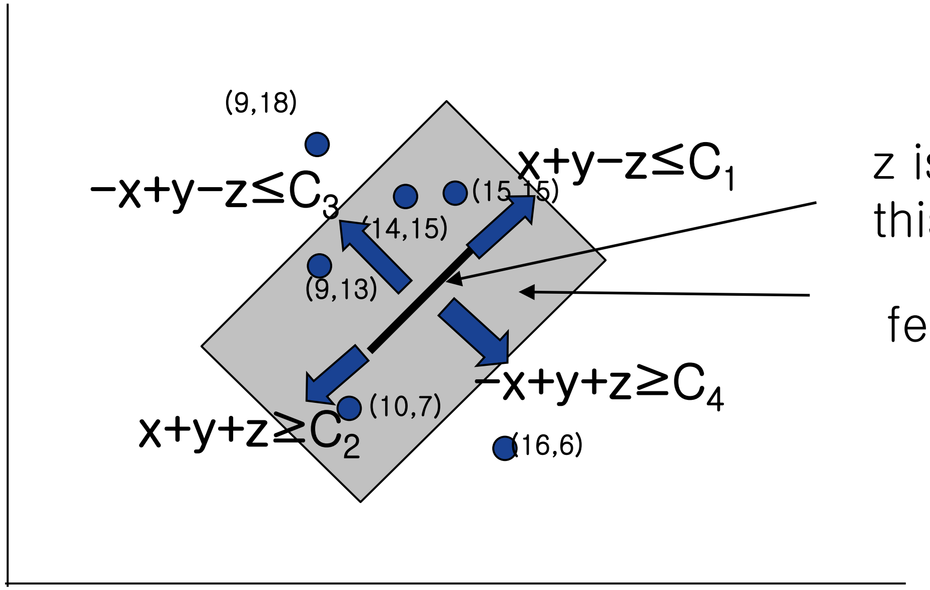
Geometric interpretation of LP(cont.)

increase z



infeasible

increase z



z is minimized at this line

feasible region

Example 10.3

Table 10.3 *Data for Example 10.3*

i	a_i	b_i	$a_i + b_i$	$-a_i + b_i$
1	0	0	0	0
2	4	6	10	2
3	8	2	10	−6
4	10	4	14	−6
5	4	8	12	4
6	2	4	6	2
7	6	4	10	−2
8	8	8	16	0
$c_1 = 0$	$c_2 = 16$	$c_3 = -6$	$c_4 = 4$	$c_5 = 16$

HW#4

- 10.1(4th edition, 10.1)(Assume minisum optimization problem)
- 10.4(4th edition, 10.8) (Do not consider total value of the houses.)