VAE Notes

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Definitions

Bayes

$$p(z \mid x) = \frac{p(x,z)}{\sum_z p(x,z)} = \frac{p(x,z)}{p(x)}$$

Chain

$$p(x,z) = p(x \mid z)p(z)$$

Marginalization

$$\sum_z p(x,z) = p(x)$$

"Evidence"

X data (also called "observations")

"Prior"

$p(z)$ "before you see the data"

"Posterior"

$p(z \mid x)$ "after you see the data"

Log Rules

$$\log(AB) = \log(A) + \log(B)$$

$$\log(\frac{A}{B}) = \log A - \log B$$

"Expectation"

Or "Average"...

$$\mathop{\mathbb{E}}_{p(x)}[f(x)] = \sum_x \overbrace{p(x)}^{\text{probability of } x \text{ under } p} \underbrace{f(x)}_{\text{eval of function at } x}$$

The Two Goals of VAE

1. **Density Estimation**
   Estimate $p(x)$ for $x \in$ dataset.

2. **Inference**
   Given $p(z \mid x)$, find the best z for any x.

Deriving the ELBO

To optimize, we need an objective. The objective in VAEs is ELBO. We want to maximize $\log p(x)$.

Introduce latent variable

$$\begin{aligned} \log p(x) &= \log \sum_z p(x, z) && \text{by marginalization} \\ &= \log \sum_z \frac{p(x, z)}{q(z \mid x)} q(z \mid x) && \text{by times by 1} \end{aligned}$$

We call $q_\phi(z \mid x)$ a "approximate posterior". It has parameters ϕ. Often q is Gaussian, so $\phi = \{\mu, \sigma\}$. We want $q(z \mid x) = p(z \mid x)$.

$$\begin{aligned} &= \log \mathop{\mathbb{E}}_{q(z|x)} \left[\frac{p(x, z)}{q(z \mid x)} \right] && \text{def of } \mathbb{E} \\ &\geq \mathop{\mathbb{E}}_{q(z|x)} \left[\log \frac{p(x, z)}{q(z \mid x)} \right] && \begin{array}{c}\text{Jensen's Inequality} \\ {\scriptstyle \log(A+B) \geq (\log A + \log B)}\end{array} \\ &= \mathop{\mathbb{E}}_{q(z|x)} \left[\log \frac{p(x \mid z) p(z)}{q(z \mid x)} \right] && \text{chain} \\ &= \mathop{\mathbb{E}}_{q(z|x)} \left[\log p(x \mid z) + \log \frac{p(z)}{q(z \mid x)} \right] && \text{log rule} \\ &= \mathop{\mathbb{E}}_{q(z|x)} \Big[\log p(x \mid z) \Big] + \underbrace{\mathop{\mathbb{E}}_{q(z|x)} \left[\log \frac{p(z)}{q(z \mid x)} \right]}_{-\,\mathrm{KL}\,(q(z|x)||p(z))} && \text{linearity} \end{aligned}$$

Gradients of ELBO

$$\text{ELBO} \triangleq \mathop{\mathbb{E}}_{q_\phi(z|x)} \Big[\log p_\theta(x, z) - \log q_\phi(z \mid x) \Big]$$

$$\nabla_\theta \text{ELBO} = \nabla_\theta \Big(\mathop{\mathbb{E}}_{q_\phi(z|x)} \Big[\log p_\theta(x, z) - \log q_\phi(z \mid x) \Big] \Big)$$

$$= \nabla_\theta \Big(\sum_z q_\theta(z \mid x) \Big[\log p_\phi - \log q_\theta(z \mid x) \Big] \Big)$$

$$= \nabla_\theta \Big(\sum_z q_\theta(z \mid x) \log p_\theta(x, z) \Big) - \cancel{\nabla_\theta \Big(\sum_z q_\phi(z \mid x) \log q_\theta(z \mid x) \Big)}$$

$$= \sum_z \nabla_\theta (q_\theta(z \mid x) \log p_\theta(x, z))$$

$$\nabla_\phi \text{ELBO} = \nabla_\phi \mathop{\mathbb{E}}_{q_\phi(z|x)} \Big[\log p_\phi(x, z) - \log q_{\phi(z|x)} \Big]$$

We could do this...

$$= \nabla_\phi \sum_z q_\phi(z \mid x) \log p_\phi(x, z) - \nabla_\phi \sum_z q_\phi(z \mid x) \log q_\phi(z \mid x)$$

This results in a dead end. Here we introduce the reparameterization trick...

$$\nabla_\phi \mathop{\mathbb{E}}_{p(z)} \Big[\log p_\theta(x, f(z)) - \log q_\phi(f(z) \mid x) \Big]$$

Where f is...

$$f = \mu + \sigma \cdot \epsilon, \ \epsilon \sim p(z)$$

$$\mathop{\mathbb{E}}_{p(\epsilon)} \Big[\nabla_\phi \log p_\phi(x, f(\epsilon)) - \nabla_\phi \log q_\phi(f(\epsilon) \mid x) \Big]$$

Now in practice, we have something like this:

$$\mu, \ \sigma = \overbrace{\text{encoder}(x)}^{q_\phi(z|x)}$$

$$\epsilon \sim N(0, 1)$$

$$z = \sigma \cdot \epsilon + \mu, \ z \sim q_\phi(z \mid x)$$

$$x = \text{decoder(z)}, \ p_\theta(x \mid z)$$

$$(x, z, \mu, \sigma) \Rightarrow \text{compute ELBO}$$