Rating system for Scrabble

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This note suggests a rating system to be used in the game of Scrabble, which is based on the Glicko system [1]. Rather than just using information about win and loss, we base the rating system on the spread (the difference in score between the two players).

Let τ and b be positive constants, whose value is determined before the rating system is used. The value of τ depends on the language which is used, and should be estimated to ensure that the rating system is as exact as possible. (For Swedish it appears that $\tau = 90$ gives good results.) On the other hand, b is an arbitrary constant, denoting the number of rating points that corresponds to one point in the game. If, for instance, $b = 5$, it means that if a player is rated 100 points above another player, the spread in games played between the two players should on average be 20 points in favour of the first player.

We assume that every player has a playing strength θ which is unknown for us. The rating of every player consists of two numbers, μ, which is an estimate of θ, and σ, which is the standard deviation (rating deviation) of the estimate.

1 Calculating the rating

Let μ and σ be the rating and rating deviation of a player prior to a tournament, and assume that he plays against m players, where the ith opponent has rating μ_i and rating deviation σ_i. Let s_i be the spread in the game against the ith opponent from the point of view of the player (*i.e.* positive if the player won and negative if he lost).

Then $\nu_i = \mu_i + bs_i$ is the player's rating achievement in the ith game. We let $\rho_i = b^2\tau^2 + \sigma_j^2$. The square of the player's rating deviation after the tournament is then

$$\sigma'^2 = \left(\frac{1}{\sigma^2} + \sum_{i=1}^{m} \frac{1}{\rho_i}\right)^{-1},$$

while the new rating is

$$\mu' = \sigma'^2 \left(\frac{\mu}{\sigma^2} + \sum_{i=1}^{m} \frac{\nu_i}{\rho_i} \right).$$

1.1 New players

When new players enter the system, they are given a rating μ_0 and a rating devation σ_0.

1.2 Inactive periods

The rating deviation of a player decreases when he or she participates in tournaments, but increases in inactive periods. The square of the rating deviation increases linearly with time, but cannot be larger than σ_0. Thus

$$\sigma' = \sigma_0 \wedge \sqrt{\sigma^2 + cd^2}$$

where d is the number of days since the last tournament the player participated in, σ is the rating deviation after that tournament, and c is a constant.

1.3 The values of the constants

In the current implementation of the system, $b = 5$, $\tau = 90$, $\mu_0 = 1500$, $\sigma_0 = 400$ and $c = 10$.

2 Justification

This section is based on Glickman [1].

Before the rating period begins, we assume that θ has a priori distribution of

$$\theta \mid \mu, \sigma \sim \mathcal{N}(\mu, \sigma^2)$$

where μ and σ^2 are known. We also assume that the ith opponent has a strength θ_i with a priori distribution

$$\theta_i \mid \mu_i, \sigma_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$$

with known μ_i and σ_i^2. If s_i is the spread in the game between the player and the ith opponent, the hypothesis on which this rating system is based says that

$$s_i \mid \theta, \theta_i \sim \mathcal{N}\left(\frac{\theta - \theta_i}{b}, \tau^2 \right).$$

Let $\boldsymbol{s}$ denote the outcomes of all the games the player has played during the rating period. We wish to calculate the density function $f(\theta \mid \boldsymbol{s})$ of the posterior distribution of θ. This is given by

$$\int \cdots \int f(\theta \mid \theta_1, \ldots, \theta_m, \boldsymbol{s}) \varphi(\theta_1 \mid \mu_1, \sigma_1^2) \ldots \varphi(\theta_m \mid \mu_m, \sigma_m^2) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_m,$$

where $\varphi(\cdot)$ is the density function of the normal distribution with given expectation and variance and

$$f(\theta \mid \theta_1, \ldots, \theta_m, \boldsymbol{s}) \propto \varphi(\theta \mid \mu, \sigma^2) L(\theta, \theta_1, \ldots, \theta_m \mid \boldsymbol{s}).$$

Here $L(\theta, \theta_1, \ldots, \theta_m \mid \boldsymbol{s})$ is the likelihood function for all the parameters. We then get

$$\begin{aligned}
f(\theta \mid \boldsymbol{s}) &\propto \int \cdots \int \varphi(\theta \mid \mu, \sigma^2) L(\theta, \theta_1, \ldots, \theta_m \mid \boldsymbol{s}) \varphi(\theta_1 \mid \mu_1, \sigma_1^2) \cdots \varphi(\theta_m \mid \mu_m, \sigma_m^2) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_m \\
&= \varphi(\theta \mid \mu, \sigma^2) \int \cdots \int L(\theta, \theta_1, \ldots, \theta_m \mid \boldsymbol{s}) \varphi(\theta_1 \mid \mu_1, \sigma_1^2) \cdots \varphi(\theta_m \mid \mu_m, \sigma_m^2) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_m \\
&\propto \varphi(\theta \mid \mu, \sigma^2) \prod_{j=1}^{m} \int \varphi(\boldsymbol{s} \mid (\theta - \theta_j)/b, \tau^2) \varphi(\theta_j \mid \mu_j, \sigma_j^2) \mathrm{d}\theta_j.
\end{aligned}$$

We can calculate the integral as

$$\begin{aligned}
&\int \varphi(\boldsymbol{s} \mid (\theta - \theta_j)/b, \tau^2) \varphi(\theta_j \mid \mu_j, \sigma_j^2) \mathrm{d}\theta_j \\
&= \int \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{\left(s_j - \frac{\theta - \theta_j}{b} \right)^2}{2\tau^2} \right) \frac{1}{\sigma_j \sqrt{2\pi}} \exp\left(-\frac{(\theta_j - \mu_j)^2}{2\sigma_j^2} \right) \mathrm{d}\theta_j \\
&\propto \int \exp\left(-\frac{1}{2b^2\tau^2\sigma_j^2} (\rho_j \theta_j^2 + \beta_j \theta_j + \gamma_j) \right) \mathrm{d}\theta_j
\end{aligned}$$

where

$$\begin{aligned}
\rho_j &= b^2\tau^2 + \sigma_j^2, \\
\beta_j &= -2\sigma_j^2\theta + 2(bs_j\sigma_j^2 - \mu_j b^2\tau^2), \\
\gamma_j &= (\theta - bs_j)^2\sigma_j^2 + \mu_j\tau^2\sigma_j^2.
\end{aligned}$$

The integral can be written as

$$\exp\left(-\frac{\rho_j}{2b^2\tau^2\sigma_j^2} \left(\frac{\gamma_j}{\rho_j} - \left(\frac{\beta}{2\rho_j} \right)^2 \right) \right) \int \exp\left(-\frac{\rho_j}{2b^2\tau^2\sigma_j^2} \left(\theta_j + \frac{\beta_j}{2\rho_j} \right)^2 \right) \mathrm{d}\theta_j.$$

The value of the last integral is independent of θ, so the expression is proportional to

$$
\begin{aligned}
&\exp\left(-\frac{\rho_j}{2b^2\tau^2\sigma_j^2}\left(\frac{\gamma_j}{\rho_j}-\left(\frac{\beta}{2\rho_j}\right)^2\right)\right)\\
\propto\ &\exp\left(-\frac{1}{2b^2\tau^2\sigma_j^2\rho_j}\left((\rho_j\sigma_j^2-\sigma_j^4)\theta^2+(-2\rho_j\sigma_j^2 bs_j+2\sigma_j^2(bs_j\sigma_j^2-\mu_j b^2\tau^2))\theta\right)\right)\\
\propto\ &\exp\left(-\frac{\theta^2-2(bs_j+\mu_j)\theta}{2\rho_j}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
f(\theta \mid \boldsymbol{s}) &\propto \varphi(\theta \mid \mu,\sigma^2)\prod_{j=1}^{m}\int \varphi(\boldsymbol{s} \mid (\theta-\theta_j)/b,\tau^2)\varphi(\theta_j \mid \mu_j,\sigma_j^2)\mathrm{d}\theta_j\\
&\propto \exp\left(-\frac{(\theta-\mu)^2}{2\sigma^2}\right)\prod_{j=1}^{m}\exp\left(-\frac{\theta^2-2(bs_j+\mu_j)\theta}{2\rho_j}\right)\\
&= \exp\left(-\frac{(\theta-\mu)^2}{2\sigma^2}-\sum_{j=1}^{m}\frac{\theta^2-2(bs_j+\mu_j)\theta}{2\rho_j}\right)\\
&\propto \exp\left(-\left(\frac{1}{2\sigma^2}+\sum_{j=1}^{m}\frac{1}{2\rho_j}\right)\theta^2+\left(\frac{\mu}{\sigma^2}+\sum_{j=1}^{m}\frac{bs_j+\mu_j}{\rho_j}\right)\theta\right)\\
&= \exp\left(-\frac{1}{2\sigma'^2}\left(\theta^2-2\mu'\theta\right)\right)\\
&\propto \exp\left(-\frac{1}{2\sigma'^2}\left(\theta-\mu'\right)^2\right) = \varphi(\theta \mid \mu',\sigma'^2),
\end{aligned}
$$

where μ' and σ' are as previously defined.

References

[1] Mark Glickman. Parameter estimation in large dynamic paired comparison experiments. *Journal of Applied Statistics*, 28:673–689, 2001.