Final Project

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1 Problem B

$$\begin{aligned}
\rho c\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} &= f \quad \text{ on } \Omega \times (0,T) \\
u &= g \quad \text{ on } \Gamma_g \times (0,T) \\
\kappa\frac{\partial u}{\partial x}n_x &= h \quad \text{ on } \Gamma_h \times (0,T) \\
u|_{t=0} &= u_0 \quad \text{ in } \Omega.
\end{aligned}$$

In the 1D case, we can consider the following options.

$$f = \sin(l\pi x), \quad u_0 = e^x, \quad u(0,t) = u(1,t) = 0, \quad \kappa = 1.0.$$

So, we can get the fomulation

$$\rho c\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2} = sin(l\pi x)$$

Use explicit difference scheme.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{u_i^{n+1} - u_i^n}{\Delta t} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}
\end{aligned}$$

A explicit numerical solution can be obtained.

$$u_i^{n+1} = \frac{\kappa\Delta t}{\rho c\Delta x^2}u_{i+1}^n + (1 - \frac{2\kappa\Delta t}{\rho c\Delta x^2})u_i^n + \frac{\kappa\Delta t}{\rho c\Delta x^2}u_{i-1}^n + \frac{\Delta t sin(l\pi x)}{\rho c}$$

Use implicit difference scheme.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{u_i^{n+1} - u_i^n}{\Delta t} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}
\end{aligned}$$

A implicit numerical solution can be obtained.

$$CFL = \frac{\kappa\Delta t}{\rho c\Delta x^2}$$

$$-CFLu_{i-1}^{n+1} + (1 + 2CFL)u_i^{n+1} - CFLu_{i+1}^{n+1} = u_i^n + \frac{\kappa\Delta t}{\rho c}sin(l\pi x)$$

To solve the implicit equation, we need to solve the diagonal matrix first.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -CFL & 1+2CFL & -CFL & 0 & \cdots & 0 \\ 0 & -CFL & 1+2CFL & -CFL & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \\ 0 & 0 & \cdots & -CFL & 1+2CFL & -CFL \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_0^{n+1} \\ u_1^{n+1} \\ u_2^{n+1} \\ \cdots \\ u_{n-1}^{n+1} \\ u_n^{n+1} \end{bmatrix} = \begin{bmatrix} u_0^n + \frac{\kappa \Delta t}{\rho c} sin(l\pi x) \\ u_1^n + \frac{\kappa \Delta t}{\rho c} sin(l\pi x) \\ u_2^n + \frac{\kappa \Delta t}{\rho c} sin(l\pi x) \\ \cdots \\ u_{n-1}^n + \frac{\kappa \Delta t}{\rho c} sin(l\pi x) \\ u_n^n + \frac{\kappa \Delta t}{\rho c} sin(l\pi x) \end{bmatrix}$$

then we calculate the analytical solution.the solution of the partial differential equation will converge to a steady state solution as time $t \to \infty$. In particular, the steady state is characterized by $\partial u / \partial t = 0$.

$$\begin{aligned} \kappa \frac{\partial^2 u}{\partial x^2} + sin(l\pi x) &= 0 \\ \frac{\partial u}{\partial x} - \frac{cos(l\pi x)}{l\pi} + c_1 &= 0 \\ u - \frac{sin(l\pi x)}{l^2\pi^2} + c_1 x + c_2 &= 0 \end{aligned}$$

form the initial condition we can get $c_1 = \frac{sin(l\pi)}{l^2\pi^2}, c_2 = 0$.

$$u = \frac{sin(l\pi x)}{l^2\pi^2} - \frac{sin(l\pi)}{l^2\pi^2} x$$