Logistic Regression and Perceptron

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Some slides adapted from Dan Jurfasky, Brendan O'Connor and Marine Carpuat

NB & LR

- Both are linear models

$$z = \sum_{i=0}^{|X|} w_i x_i$$

- Training is different:
  - NB: weights are trained independently
  - LR: weights trained jointly

Linear Models

- Compute Features:

$$f(d_i) = x_i = \begin{pmatrix} \text{count(“nigerian”)} \\ \text{count(“prince”)} \\ \text{count(“nigerian prince”)} \end{pmatrix}$$

- Assume we are given some weights:

$$w = \begin{pmatrix} -1.0 \\ -1.0 \\ 4.0 \end{pmatrix}$$

Linear Models

- Compute Features
- We are given some weights
- Compute the dot product:

$$z = \sum_{i=0}^{|X|} w_i x_i$$

- Intuition: weighted sum of features
- All Linear models have this form

Naïve Bayes as a Log-Linear Model

$$P(\text{spam}|D) \propto P(\text{spam}) \prod_{w \in D} P(w|\text{spam})$$

$$P(\text{spam}|D) \propto P(\text{spam}) \prod_{w \in \text{Vocab}} P(w|\text{spam})^{x_i}$$

$$\log P(\text{spam}|D) \propto \log P(\text{spam}) + \sum_{w \in \text{Vocab}} x_i \cdot \log P(w|\text{spam})$$

features

weights

Logistic Regression

- (Log) Linear Model – similar to Naïve Bayes
- Doesn’t assume features are independent
- Correlated features don’t “double count”

Logistic Regression

- Compute the dot product:

$$z = \sum_{i=0}^{|X|} w_i x_i$$

linear combination

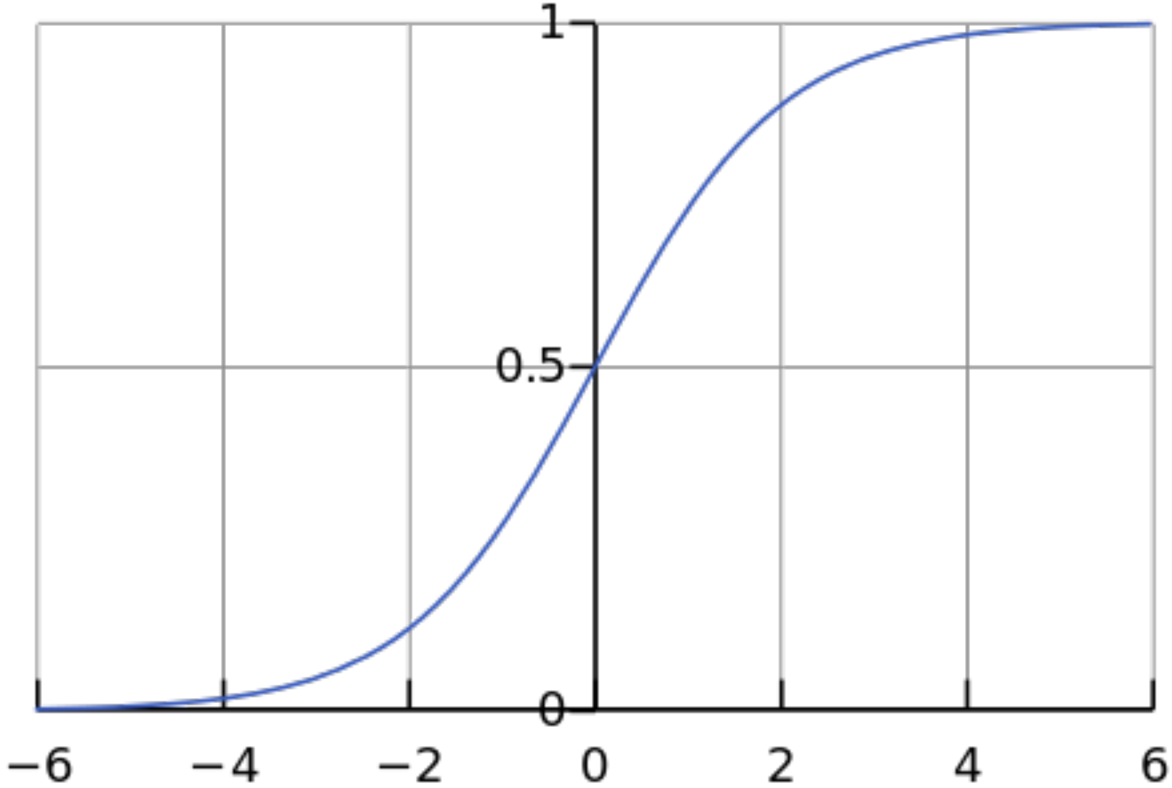
- Compute the logistic function:

convert into probabilities between [0, 1]

$$P(\mathrm{spam}|x) = \frac{e^z}{e^z + 1} = \frac{1}{1 + e^{-z}}$$

exponential/log space

The Logistic function



$$P(\text{spam}|x) = \frac{e^z}{e^z + 1} = \frac{1}{1 + e^{-z}}$$

NB vs. LR

- Both compute the dot product

- NB: sum of log probabilities

- LR: logistic function

NB vs. LR: Parameter Learning

- NB: Learn conditional probabilities **independently** by counting
- LR: Learn feature weights **jointly**

LR: Learning Weights

- Given: a set of feature vectors and labels

- Goal: learn the weights

LR: Learning Weights

$$
\begin{array}{rcc}
 & \begin{array}{ccccc} \phantom{x_{11}} & \phantom{x_{12}} & \text{Feature} & & \phantom{x_{1n}} \end{array} & \text{Labels} \\
\text{Document} & \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{d1} & x_{d2} & x_{d3} & \cdots & x_{dn} \end{bmatrix} & \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}
\end{array}
$$

Q: what parameters should we choose?

- What is the right value for the weights?

- Maximum Likelihood Principle:
  - Pick the parameters that maximize the probability of the y labels in the training data given the observations x.

Maximum Likelihood Estimation

$$w_{\mathrm{MLE}} = \operatorname{argmax}_w \log P(y_1, \ldots, y_d | x_1, \ldots, x_d; w)$$

$$= \operatorname{argmax}_w \sum_i \log P(y_i | x_i; w)$$

$$= \operatorname{argmax}_w \sum_i \log \begin{cases} p_i, & \text{if } y_i = 1 \\ 1 - p_i, & \text{if } y_i = 0 \end{cases}$$

logistic function

$$\boxed{p_i = \sigma(\textstyle\sum_j w_j x_j)}$$

$$= \operatorname{argmax}_w \sum_i \log p_i^{\mathbb{I}(y_i=1)} (1 - p_i)^{\mathbb{I}(y_i=0)}$$

Maximum Likelihood Estimation

$$= \operatorname{argmax}_w \sum_i \log p_i^{\mathbb{I}(y_i=1)} (1-p_i)^{\mathbb{I}(y_i=0)}$$

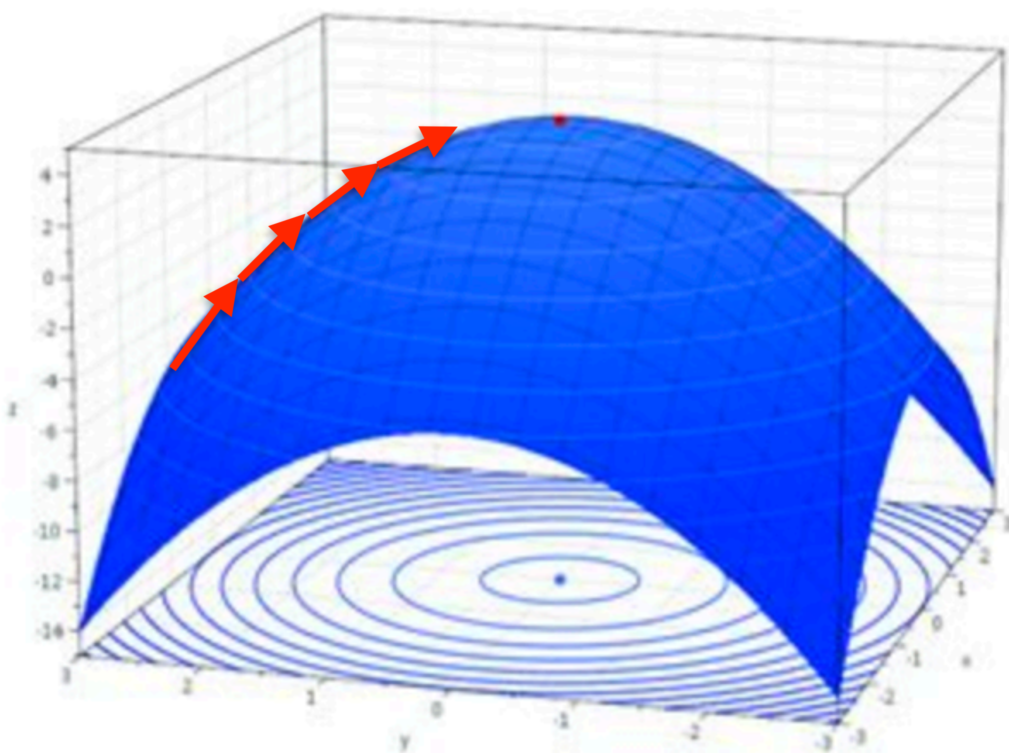
$$= \operatorname{argmax}_w \sum_i y_i \log p_i + (1-y_i) \log(1-p_i)$$

- Unfortunately there is no closed form solution
  - (like there was with naïve Bayes)

Maximum Likelihood Estimation

- Solution:
  - Iteratively climb the log-likelihood surface through the derivatives for each weight
- Luckily, the derivatives turn out to be nice

Gradient Ascent



Gradient Ascent

Loop While not converged:

For all features **j**, compute and add derivatives

$$w_j^{\text{new}} = w_j^{\text{old}} + \eta \frac{\partial}{\partial w_j} \mathcal{L}(w)$$

$\mathcal{L}(w)$: Training set log-likelihood

$\left(\frac{\partial \mathcal{L}}{\partial w_1}, \frac{\partial \mathcal{L}}{\partial w_2}, \ldots, \frac{\partial \mathcal{L}}{\partial w_n} \right)$: Gradient vector

LR Gradient

$$w_{\mathrm{MLE}} = \operatorname{argmax}_w \underbrace{\sum_i y_i \log p_i + (1 - y_i) \log(1 - p_i)}_{\mathcal{L}}$$

$$\frac{\partial \mathcal{L}}{\partial w_j} = \sum_i (y_i - p_i) x_j$$

Exercise

Derivative of Sigmoid

$$
\begin{aligned}
\frac{d}{dx}\sigma(x) &= \frac{d}{dx}\left[\frac{1}{1+e^{-x}}\right] \\
&= \frac{d}{dx}\left(1+\mathrm{e}^{-x}\right)^{-1} \\
&= -(1+e^{-x})^{-2}(-e^{-x}) \\
&= \frac{e^{-x}}{(1+e^{-x})^2} \\
&= \frac{1}{1+e^{-x}} \cdot \frac{e^{-x}}{1+e^{-x}} \\
&= \frac{1}{1+e^{-x}} \cdot \frac{(1+e^{-x})-1}{1+e^{-x}} \\
&= \frac{1}{1+e^{-x}} \cdot \left(1 - \frac{1}{1+e^{-x}}\right) \\
&= \sigma(x) \cdot (1-\sigma(x))
\end{aligned}
$$

Logistic Regression: Pros and Cons

- Doesn’t assume conditional independence of features
  - Better calibrated probabilities
  - Can handle highly correlated overlapping features
- NB is faster to train, less likely to overfit

MultiClass Classification

- Q: what if we have more than 2 categories?
  - Sentiment: Positive, Negative, Neutral
  - Document topics: Sports, Politics, Business, Entertainment, …

Q: How to easily do Multi-label classification?

Two Types of MultiClass Classification

- Multi-label Classification
  - each instance can be assigned more than one labels
- Multinominal Classification
  - each instance appears in exactly one class (classes are exclusive)

Multinominal Classification

- Pretty straightforward with Naive Bayes.

$$P(\text{spam}|D) \propto P(\text{spam}) \prod_{w \in D} P(w|\text{spam})$$

Log-Linear Models

$$P(y|x) \propto e^{w \cdot f(x,y)}$$

$$P(y|x) = \frac{1}{Z(w)} e^{w \cdot f(x,y)}$$

Multinominal Logistic Regression

$$P(y|x) \propto e^{w \cdot f(x,y)}$$

$$P(y|x) = \frac{1}{Z(w)} e^{w \cdot f(x,y)}$$

$$\boxed{P(y|x) = \frac{e^{w \cdot f(x,y)}}{\sum_{y' \in Y} e^{w \cdot f(x,y')}}}$$

Multinominal Logistic Regression

- Binary (two classes):
  - We have one feature vector that matches the size of the vocabulary
- Multi-class in practice:
  - one weight vector for each category

w_{pos} $\quad$ w_{neg} $\quad$ w_{neut}

Can represent this in practice with one giant weight vector and repeated features for each category.

Maximum Likelihood Estimation

$$w_{\mathrm{MLE}} = \operatorname{argmax}_w \log P(y_1, \ldots, y_n | x_1, \ldots, x_n; w)$$

$$= \operatorname{argmax}_w \sum_i \log P(y_i | x_i; w)$$

$$= \operatorname{argmax}_w \sum_i \log \frac{e^{w \cdot f(x_i, y_i)}}{\sum_{y' \in Y} e^{w \cdot f(x_i, y')}}$$

(a.k.a) Softmax Regression

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Softmax function

From Wikipedia, the free encyclopedia

In mathematics, the **softmax function**, or **normalized exponential function**,[1]:198 is a generalization of the logistic function that "squashes" a K-dimensional vector $\mathbf{z}$ of arbitrary real values to a K-dimensional vector $\sigma(\mathbf{z})$ of real values in the range (0, 1) that add up to 1. The function is given by

$$\sigma(\mathbf{z})_j = \frac{e^{z_j}}{\sum_{k=1}^{K} e^{z_k}} \quad \text{for } j = 1, \ldots, K.$$

(a.k.a) Maximum Entropy Classifier

- or MaxEnt

- Math proof of “LR=MaxEnt”:
  - [Klein and Manning 2003]
  - [Mount 2011]

http://www.win-vector.com/dfiles/LogisticRegressionMaxEnt.pdf

Multiclass LR Gradient

$$\frac{\partial \mathcal{L}}{\partial w_j} = \sum_{i=1}^{D} f_j(y_i, d_i) - \sum_{i=1}^{D} \sum_{y \in Y} f_j(y, d_i) P(y|d_i)$$

Perceptron Algorithm

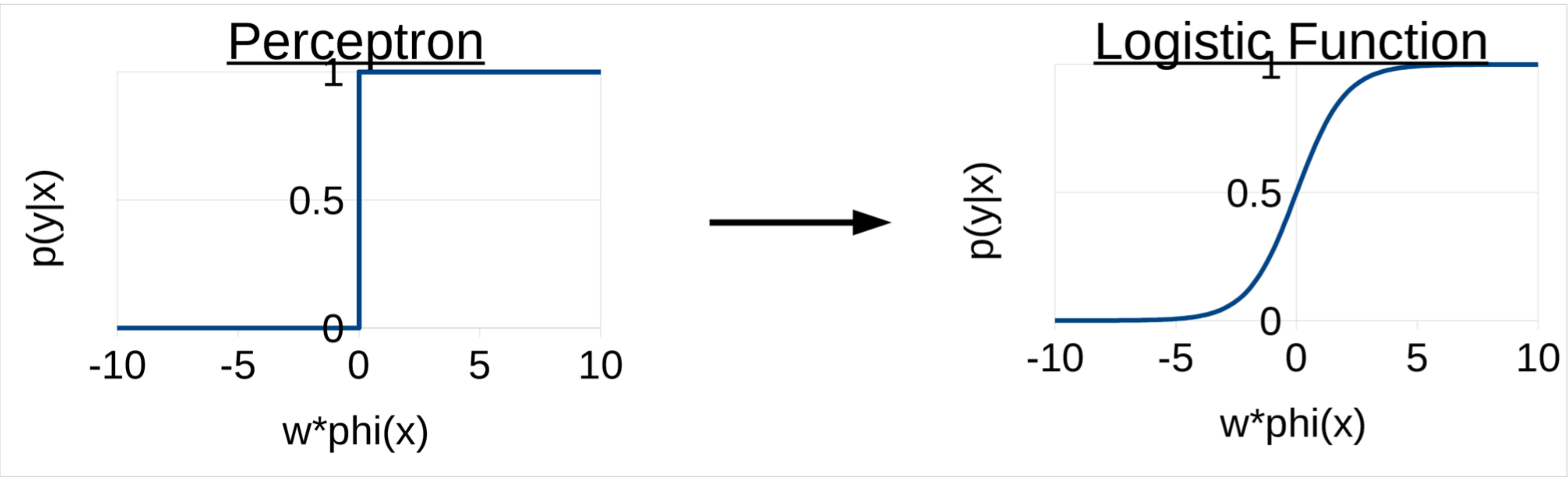
- Very similar to logistic regression
- Not exactly computing gradient

[Rosenblatt 1957]

http://www.peterasaro.org/writing/neural_networks.html

Perceptron Algorithm

- Very similar to logistic regression
- Not exactly computing gradient



$$P(y=1|x)=1 \text{ if } \mathbf{w}\cdot\boldsymbol{\varphi}(x)\geq 0$$
$$P(y=1|x)=0 \text{ if } \mathbf{w}\cdot\boldsymbol{\varphi}(x)<0$$

$$P(y=1|x)=\frac{e^{\mathbf{w}\cdot\boldsymbol{\varphi}(x)}}{1+e^{\mathbf{w}\cdot\boldsymbol{\varphi}(x)}}$$

Online Learning

- Update parameters for each training example (when predication is wrong)

```
for I iterations
    for each labeled pair x, y in the data
        phi = CREATE_FEATURES(x)
        y' = PREDICT_ONE(w, phi)
        if y' != y
            UPDATE_WEIGHTS(w, phi, y)
```

Online Learning

- The Perceptron is an online learning algorithm.
- Logistic Regression is not:

$$w_{\mathrm{MLE}} = \mathrm{argmax}_w \log P(y_1, \ldots, y_d | x_1, \ldots, x_d; w)$$

Perceptron Algorithm

- Very similar to logistic regression
- Not exactly computing gradient

```
Initalize weight vector w = 0
Loop for K iterations
    Loop For all training examples x_i
        if sign(w * x_i) != y_i
            w += (y_i - sign(w * x_i)) * x_i
```

Perceptron Notes

- Guaranteed to converge if the data is linearly separable

- Only hyperparameter is maximum number of iterations

- Parameter averaging will greatly improve performance

Differences between LR and Perceptron

- Online learning vs. Batch

- Perceptron doesn’t always make updates