Logistic Regression

Instructor: Wei Xu

Some slides adapted from Dan Jurfasky and Brendan O'Connor

Naïve Bayes Recap

- Bag of words (order independent)

- Features are assumed independent given class

$$P(x_1, \ldots, x_n|c) = P(x_1|c) \ldots P(x_n|c)$$

Q: Is this really true?

The problem with assuming conditional independence

- Correlated features -> double counting evidence
  - Parameters are estimated independently

- This can hurt classifier accuracy and calibration

Logistic Regression

- (Log) Linear Model – similar to Naïve Bayes
- Doesn’t assume features are independent
- Correlated features don’t “double count”

What are “Features”?

- A feature function, f
  - Input: Document, D (a string)
  - Output: Feature Vector, X

What are “Features”?

$$f(d) = \begin{pmatrix} \text{count(“boring”)} \\ \text{count(“not boring”)} \\ \text{length of document} \\ \text{author of document} \\ \vdots \end{pmatrix}$$

Doesn’t have to be just “bag of words”

Feature Templates

- Typically “feature templates” are used to generate many features at once

- For each word:
  - ${w}_count
  - ${w}_lowercase
  - ${w}_with_NOT_before_count

Logistic Regression: Example

- Compute Features:

$$f(d_i) = x_i = \begin{pmatrix} \text{count}(\text{“nigerian”}) \\ \text{count}(\text{“prince”}) \\ \text{count}(\text{“nigerian prince”}) \end{pmatrix}$$

- Assume we are given some weights:

$$w = \begin{pmatrix} -1.0 \\ -1.0 \\ 4.0 \end{pmatrix}$$

Logistic Regression: Example

- Compute Features
- We are given some weights
- Compute the dot product:

$$z = \sum_{i=0}^{|X|} w_i x_i$$

Logistic Regression: Example

- Compute the dot product:

linear combination

$$z = \sum_{i=0}^{|X|} w_i x_i$$

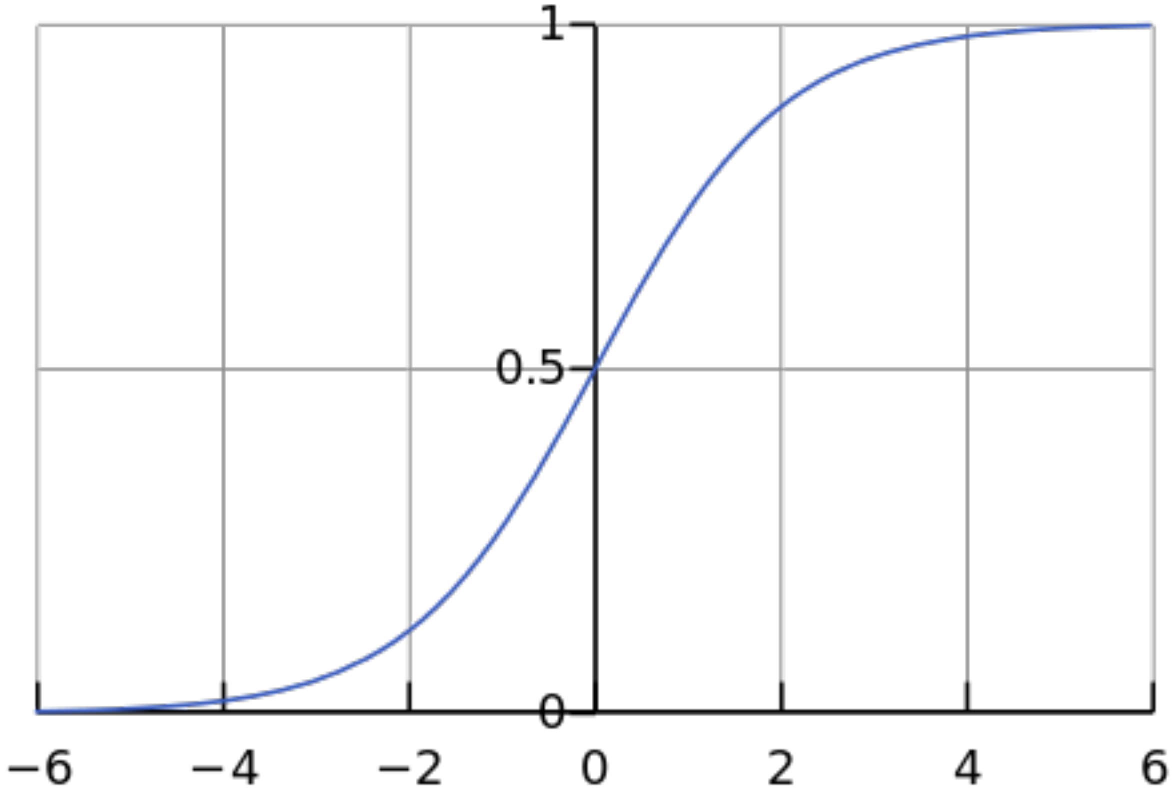
- Compute the logistic function:

convert into probabilities between [0, 1]

$$P(\text{spam}|x) = \frac{e^z}{e^z + 1} = \frac{1}{1 + e^{-z}}$$

exponential/log space

The Logistic function

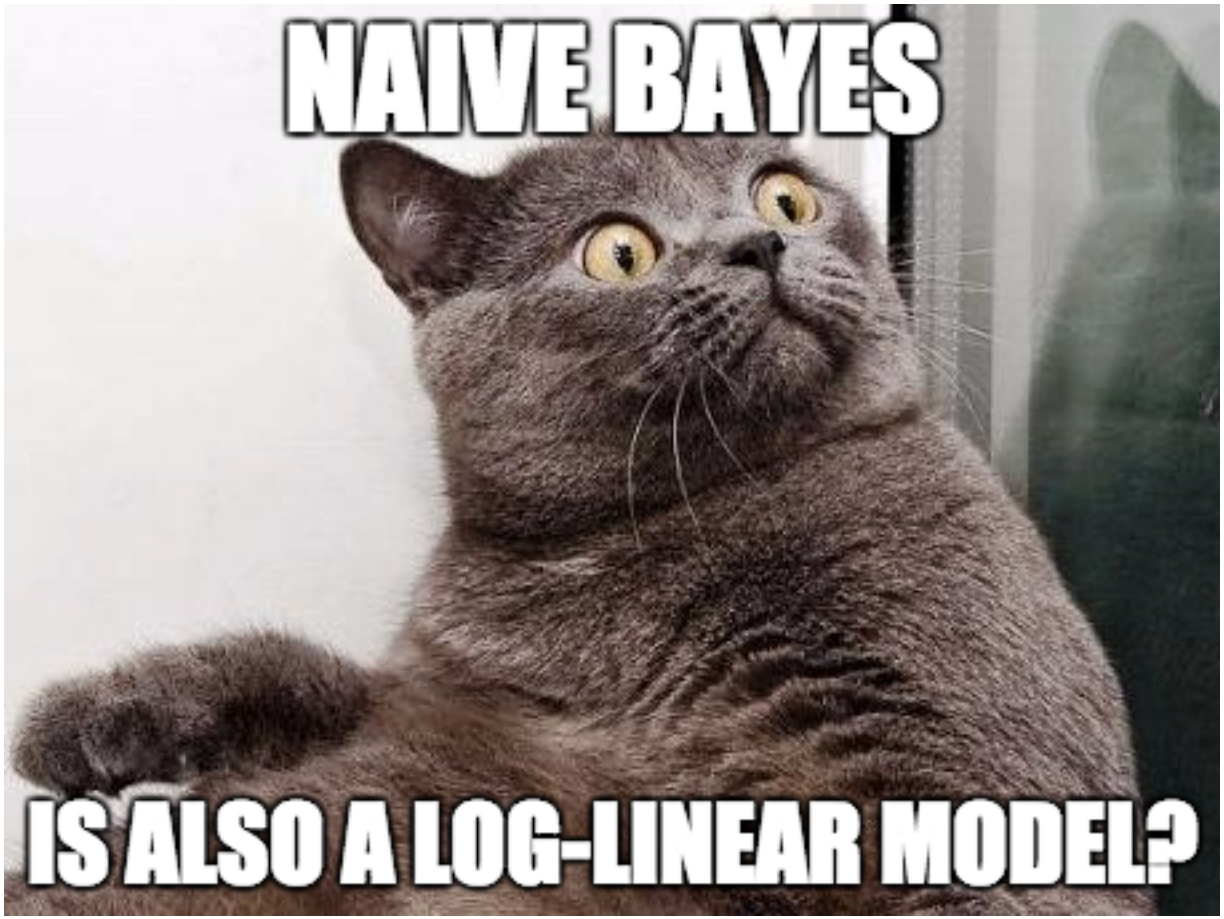


$$P(\text{spam}|x) = \frac{e^z}{e^z + 1} = \frac{1}{1 + e^{-z}}$$

The Dot Product

$$z = \sum_{i=0}^{|X|} w_i x_i$$

- Intuition: weighted sum of features
- All Linear models have this form


NAIVE BAYES
IS ALSO A LOG-LINEAR MODEL?

Naïve Bayes as a log-linear model

- Q: what are the features?

- Q: what are the weights?

Naïve Bayes as a Log-Linear Model

$$P(\text{spam}|D) \propto P(\text{spam}) \prod_{w \in D} P(w|\text{spam})$$

$$P(\text{spam}|D) \propto P(\text{spam}) \prod_{w \in \text{Vocab}} P(w|\text{spam})^{x_i}$$

$$\log P(\text{spam}|D) \propto \log P(\text{spam}) + \sum_{w \in \text{Vocab}} x_i \cdot \log P(w|\text{spam})$$

Naïve Bayes as a Log-Linear Model

$$\log P(\text{spam}|D) \propto \log P(\text{spam}) + \sum_{w \in \text{Vocab}} x_i \cdot \log P(w|\text{spam})$$

In both naïve Bayes and logistic regression we compute the dot product!

NB vs. LR

- Both compute the dot product
- NB: sum of log probabilities
- LR: logistic function

NB vs. LR: Parameter Learning

- Naïve Bayes:
  - Learn conditional probabilities **independently** by counting
- Logistic Regression:
  - Learn weights **jointly**

LR: Learning Weights

- Given: a set of feature vectors and labels
- Goal: learn the weights

LR: Learning Weights

$$
\begin{array}{rcc}
 & \begin{array}{ccccc} \phantom{x_{11}} & \phantom{x_{12}} & \text{Feature} & & \phantom{x_{1n}} \\ & & \downarrow & & \end{array} & \begin{array}{c} \text{Labels} \\ \downarrow \end{array} \\
\text{Document} \rightarrow & \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{d1} & x_{d2} & x_{d3} & \cdots & x_{dn} \end{bmatrix} & \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}
\end{array}
$$

Q: what parameters should we choose?

- What is the right value for the weights?

- Maximum Likelihood Principle:
  - Pick the parameters that maximize the probability of the y labels in the training data given the observations x.

Maximum Likelihood Estimation

$$w_{\mathrm{MLE}} = \mathrm{argmax}_w \log P(y_1, \ldots, y_d | x_1, \ldots, x_d; w)$$

$$= \mathrm{argmax}_w \sum_i \log P(y_i | x_i; w)$$

$$= \mathrm{argmax}_w \sum_i \log \begin{cases} p_i, & \text{if } y_i = 1 \\ 1 - p_i, & \text{if } y_i = 0 \end{cases}$$

$$= \mathrm{argmax}_w \sum_i \log p_i^{\mathbb{I}(y_i=1)} (1 - p_i)^{\mathbb{I}(y_i=0)}$$

Maximum Likelihood Estimation

$$= \operatorname{argmax}_w \sum_i \log p_i^{\mathbb{I}(y_i=1)} (1 - p_i)^{\mathbb{I}(y_i=0)}$$

$$= \operatorname{argmax}_w \sum_i y_i \log p_i + (1 - y_i) \log(1 - p_i)$$

- Unfortunately there is no closed form solution
  - (like there was with naïve Bayes)

Closed Form Solution

- a Closed Form Solution is a simple solution that works instantly without any loops, functions etc
- e.g. the sum of integer from 1 to n

```
s= 0
for i in 1 to n
    s = s + i
end for
print s
```

Iterative Algorithm

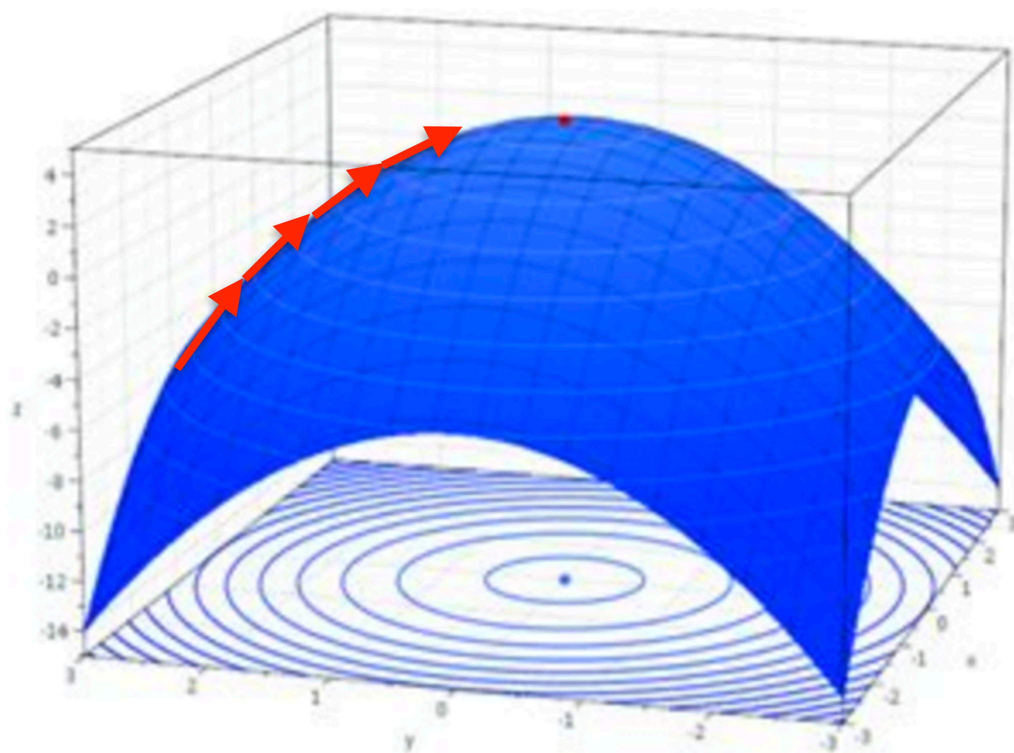
$$s = n(n+1)/2$$

Closed Form Solution

Maximum Likelihood Estimation

- Solution:
  - Iteratively climb the log-likelihood surface through the derivatives for each weight
- Luckily, the derivatives turn out to be nice

Gradient Ascent



Gradient Ascent

Loop While not converged:

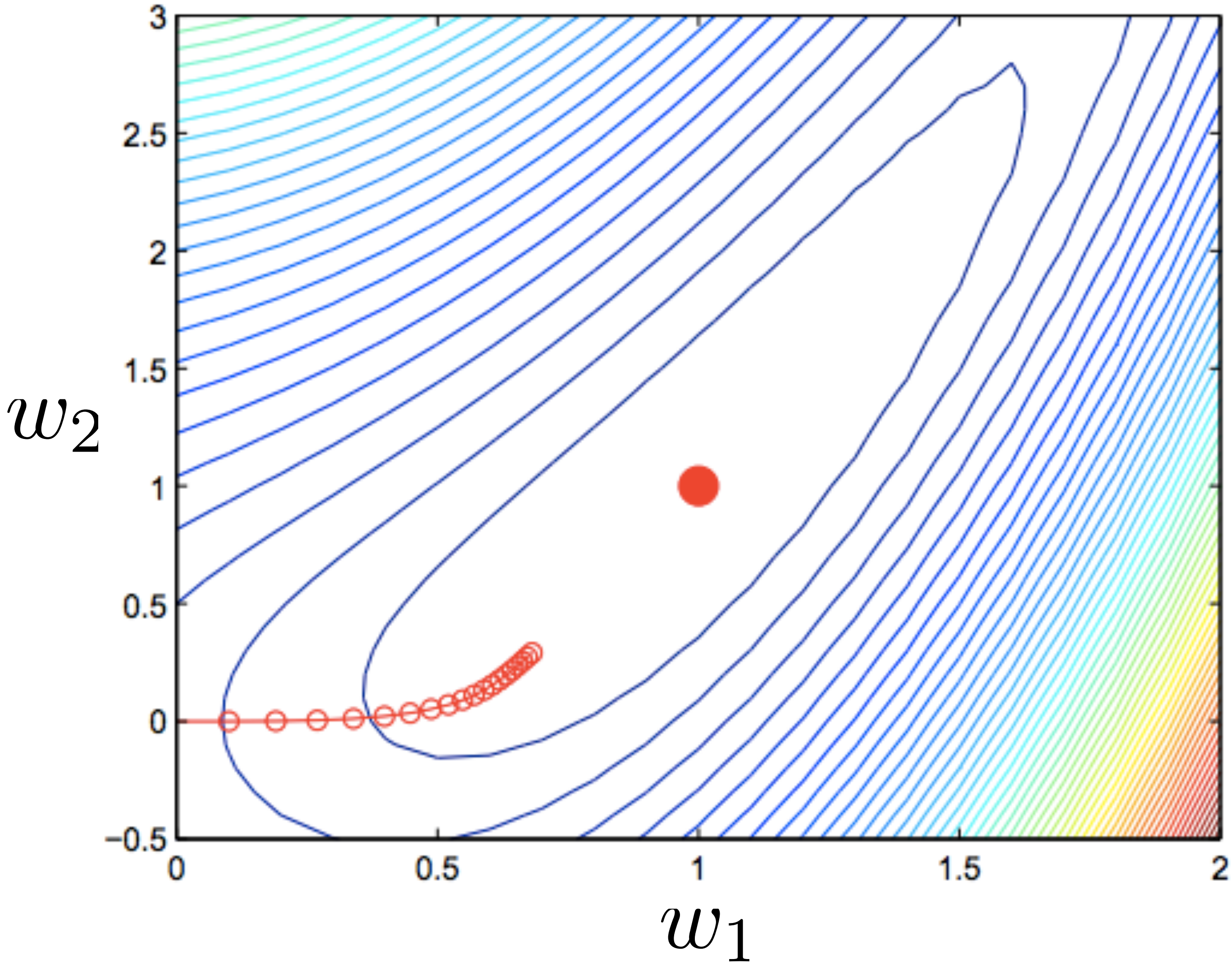
For all features **j**, compute and add derivatives

$$w_j^{\mathrm{new}} = w_j^{\mathrm{old}} + \eta \frac{\partial}{\partial w_j} \mathcal{L}(w)$$

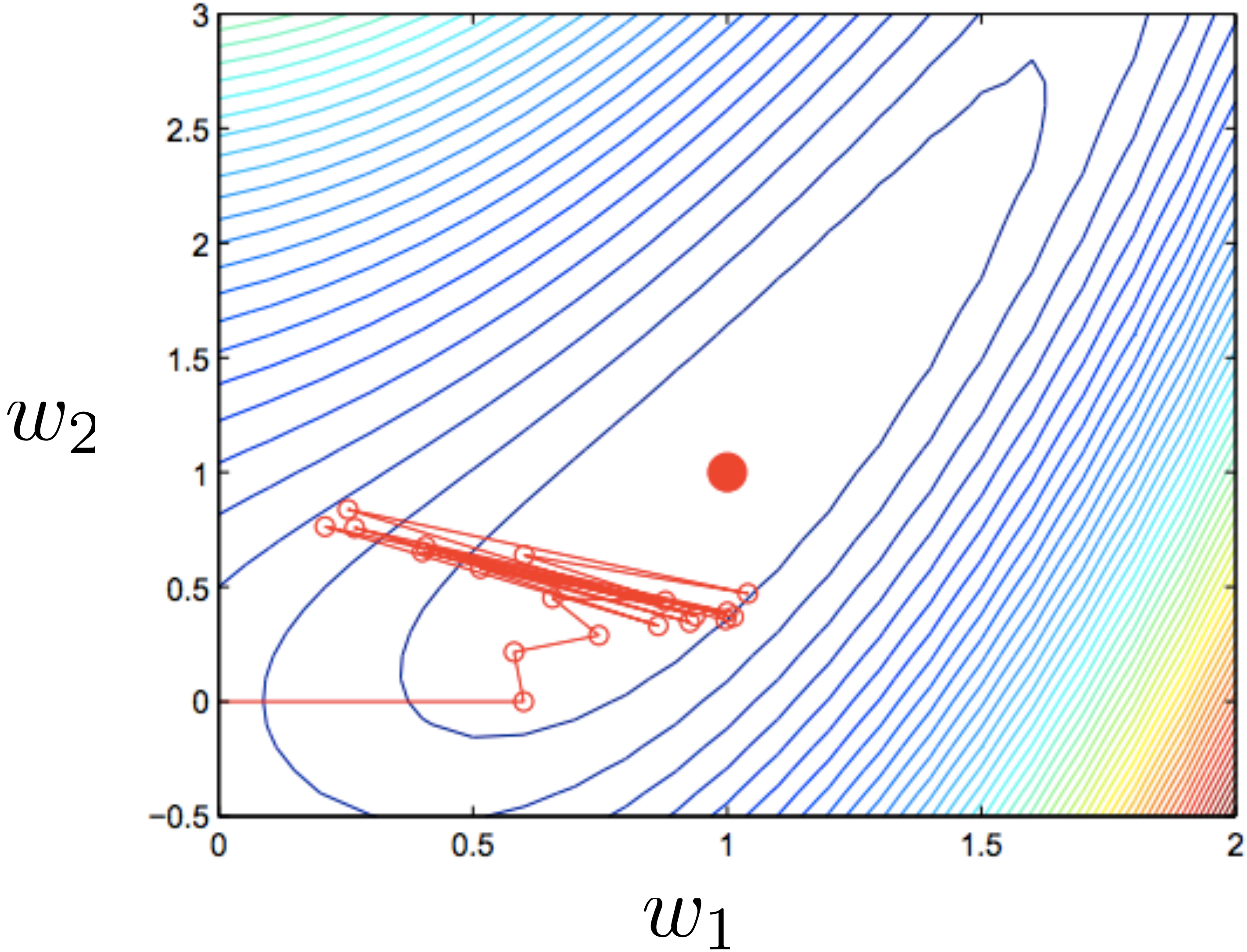
$\mathcal{L}(w)$: Training set log-likelihood

$\left(\frac{\partial \mathcal{L}}{\partial w_1}, \frac{\partial \mathcal{L}}{\partial w_2}, \ldots, \frac{\partial \mathcal{L}}{\partial w_n} \right)$: Gradient vector

Gradient ascent



Gradient ascent



Derivative of Sigmoid

$$
\begin{aligned}
\frac{d}{dx}\sigma(x) &= \frac{d}{dx}\left[\frac{1}{1+e^{-x}}\right] \\
&= \frac{d}{dx}\left(1+\mathrm{e}^{-x}\right)^{-1} \\
&= -(1+e^{-x})^{-2}(-e^{-x}) \\
&= \frac{e^{-x}}{(1+e^{-x})^2} \\
&= \frac{1}{1+e^{-x}} \cdot \frac{e^{-x}}{1+e^{-x}} \\
&= \frac{1}{1+e^{-x}} \cdot \frac{(1+e^{-x})-1}{1+e^{-x}} \\
&= \frac{1}{1+e^{-x}} \cdot \left(1-\frac{1}{1+e^{-x}}\right) \\
&= \sigma(x)\cdot(1-\sigma(x))
\end{aligned}
$$

LR Gradient

$$w_{\mathrm{MLE}} = \mathrm{argmax}_w \sum_i y_i \log p_i + (1 - y_i) \log(1 - p_i)$$

$$\frac{\partial \mathcal{L}}{\partial w_j} = \sum_i (y_i - p_i) x_j$$

Logistic Regression: Pros and Cons

- Doesn’t assume conditional independence of features
  - Better calibrated probabilities
  - Can handle highly correlated overlapping features

- NB is faster to train, less likely to overfit

NB & LR

- Both are linear models

$$z = \sum_{i=0}^{|X|} w_i x_i$$

- Training is different:
  - NB: weights are trained independently
  - LR: weights trained jointly