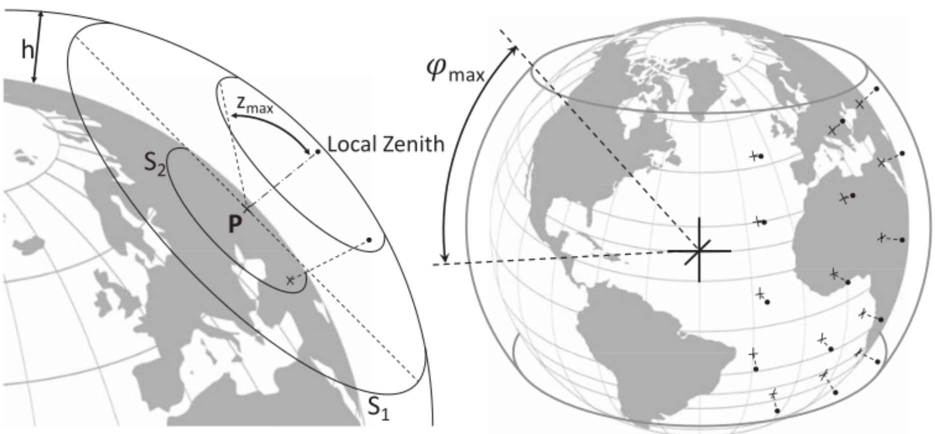
Personal notes

March 9, 2022

1 Derivations of Ragazzoni 2020

1.1 Geometry

Satellite orbital plane has inclination $\phi_{\max}$ wrt equator, i.e. $\phi_{\max}$ is the highest latitude on the surface where the satellite passes at the zenit. The height above the surface is h and the



Earth radius R.
The surface wrapped by the satellite around the earth is a spherical zone area = $2\pi \times$ radius $\times$ height of area

$$S = 2\pi(R+h) \times 2(R+h)\sin\phi_{\max} = 4\pi(R+h)^2 \sin\phi_{\max} \tag{1}$$

Satellites above the horizon. An observer in a point P well inside the area wrapped by the satellite will see a limited of the total surface, i.e. the spherical cusp given by the intersection of the satellite are with the horizon plane in P

$$S_1 = 2\pi(R+h)h \tag{2}$$

For a population of satellites that share the same $\phi_{\max}$ an h the fraction of satellites above the horizon is

$$\eta_1 = \frac{S_1}{S} = \frac{1}{2\sin\phi_{\max}} \frac{h}{R+h} \tag{3}$$

In the limit of low-altitude orbit $h \ll R$ and

$$\eta_1 \approx \frac{1}{2\sin\phi_{\max}} \frac{h}{R} \tag{4}$$

Satellites close to the zenit. In the typical case we are interested on satellite with a zenital angle $z < z_{\max}$. The relative fraction of satellite close to the zenit in low altitude regime is only

$$\eta_2 \approx \frac{1}{2\sin\phi_{\max}} \frac{h^2}{R^2} \tan^2 z_{\max} \tag{5}$$

i.e. drastically reduced.

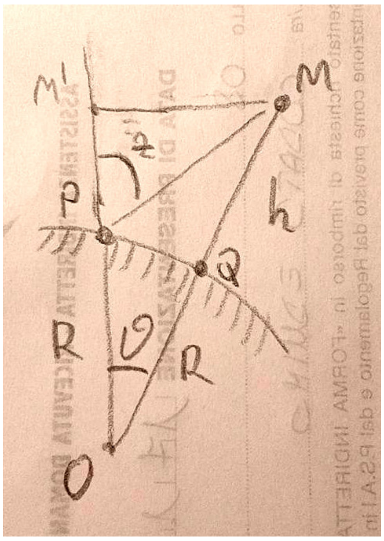
Satellite visibility. To be visible a satellite must be illuminated by the Sun. Relation between h and the satellite-earth-sun phase angle Ψ is

$$\cos\Psi = \frac{R}{R+h} \tag{6}$$

2 Equivalent for meteors

2.1 Geometry/visibility

A meteor has an height h above the point Q on the (spherical) surface of the Earth. The



angular separation between Q and the observing site P is θ. O is the center of the Earth, M the position of the meteor and M' its projection over the vertical of P. We have

$$\begin{aligned} MM' &= (R+h)\sin\theta \\ PM' &= (R+h)\cos\theta - R \end{aligned} \tag{7}$$

between the observed angular zenit distance z and the other parameters is

$$\tan z = \frac{(R+h)\sin\theta}{(R+h)\cos\theta - R} \tag{8}$$

To be meaningful, the denominator at the right hand side must be positive, i.e.

$$\cos\theta \geq \frac{R}{R+h} \tag{9}$$

which set the visibility limit in terms of phase angle between the meteor and the observed, wrt the Earth. Note when $\cos\theta \to R/(R+h)$ then $\tan z \to \pi/2$ i.e. the meteor is observed at the horizon.

Low altitude approximation. The formula above becomes

$$\tan z \approx \frac{\sin\theta}{\cos\theta - 1} = \tan\left(\frac{\theta}{2}\right) \rightarrow\; z \approx \theta/2 \tag{10}$$

in fact if $(R+h) \approx R$ we can act as if earth surface and the spherical surface at the height of the meteor were the same one. Then the z is a circumference angle while θ is a center angle insisting on the same arc $\widehat{PQ}$ so they will be one half of the other. The visibility limit becomes

$$\cos\theta \geq \left(1+\frac{h}{R}\right)^{-1} \approx 1-\frac{h}{R} \tag{11}$$

Ablation typically starts at $80-90\,\mathrm{km}$ [Ceplecha+98] so assuming $(h+R) \approx R$ is legit at first approximation.

2.2 Intrinsic motion

Shape of the trajectory. Before entering the atmosphere the motion of a meteoroid consist on its intrinsic solar orbital motion + acceleration due to Earth attraction. Typical velocity ranges from 11.2 km/s (pure Earth attraction) to 72.8 km/s (solar + Earth attraction).

During the ablation the trajectory is a straight line, because we neglect

- the effect of gravitational attraction of the Earth, which is much smaller than the air drag exerted by the atmosphere;
- the effect of the curvature of the planet. This is because during the burn meteors covers hundreds of km $\ll R$.

In general the motion along this straight line is not uniform as the meteoroid is being slew down by air drag.

Equation of motion. Motion on the trajectory is described by the set of differential equation [Cepleca+98]

$$\begin{aligned} a &= \frac{\mathrm{d}v}{\mathrm{d}t} = -\Gamma A \rho_m^{-2/3} \rho m^{-1/3} v^2 \\ \dot{m} &= \frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{\Lambda A}{2\xi} \rho_m^{-2/3} m^{2/3} v^3 \end{aligned} \tag{12}$$

where

- Γ is the drag coefficient (fraction of momentum transferred to the air from the body, ranges from 0 for no exchange of momentum, to 2 for a perfectly elastic impact),
- Λ the heat transfer coefficient (fraction of kinetic energy converted into heat),
- ξ the energy for the ablation of a unit mass,
- $A = Sm^{-2/3}\rho_m^{2/3} = S/V^{2/3}$ the shape factor (S the cross-section, for cube $A = 1$, for a sphere $A \approx 1.2$, getting more elongated toward the direction of motion makes A grow),
- ρ_m the bulk density,

- m the mass and
- v the velocity.

Usually independent parameters are gathered in the two terms [Cepleca+98]

$$\begin{aligned} \sigma &= \frac{\Lambda}{2\xi\Gamma} \\ K &= \Lambda A \rho_m^{-2/3} \end{aligned} \tag{13}$$

respectively the ablation and the shape-density coefficients. Motion equation become [adapted from Cepleca+98]

$$\begin{aligned} a &= -K\rho m^{-1/3} v^2 \\ \dot{m} &= -K\sigma m^{2/3} v^3 \end{aligned} \tag{14}$$

Dependence on the height. Note that $\rho = \rho(h) = \rho[h(t)]$, i.e. we need a further relation to express the variation of atmospheric density as a function of the height. A good approximation of the atmospheric density profile is

$$\rho = \rho_0 e^{-h/H} \tag{15}$$

where the scale height H is

$$H = \frac{P_0}{g\rho_0} \sim 8\,\mathrm{km} \tag{16}$$

with P the pressure. The pedix represents the value at the sea level, for $h = 0$.

We introduce the third equation of motion

$$v_h = \frac{\mathrm{d}h}{\mathrm{d}t} = \cos(z_0)\, v \tag{17}$$

where z_0 is the zenit angle of the meteor radiant (i.e. inclination of the trajector wrt the local vertical).

Final equations. The final set of ODE that solve the motion of a meteor is

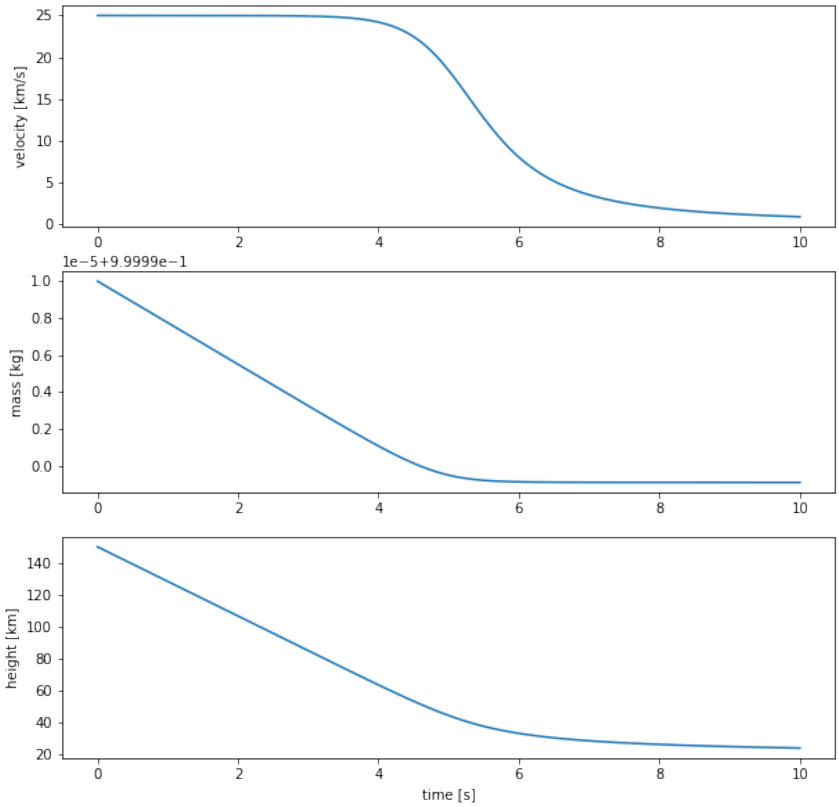
$$\begin{aligned} a &= -K\rho_0 e^{-h/H} m^{-1/3} v^2 \\ \dot{m} &= -K\sigma m^{2/3} v^3 \\ v_h &= \cos z_0\, v \end{aligned} \tag{18}$$

with K and σ that comes from the properties and composition of the body, z_0 from the pre-atmospheric motion while ρ_0 and H are two constants dependent on the atmospheric model.

Note we assumed, more or less implicitly:

- straight-line trajectory (see above);
- constant inclination z_0 (direction of the vertical does not changes significantly below the short meteor path);
- σ and K constant. It is assumed that the shape and properties does not change significantly during ablation.

Solution. An example for $\rho_m = 3000\,\mathrm{kg/cm^3}$, $\Lambda = 1$, $A = 1$, $\sigma = 0.03\,\mathrm{s^2/km^2}$ and $z_0 = 30°$. Initial condition velocity 25 km/s, mass 1.0 kg and 150 km of altitude. Integration over 10 s, 300 time-steps.



2.3 Apparent motion

Due to the short path and low altitude approximation, the motion can be naturally expressed using a rectangular coordinate system in the neighborhood of the meteor, i.e. assuming locally the Earth to be flat right below the meteor.

An object at height h can actually be observed up to a distance $\approx \sqrt{2Rh}$ (in low altitude approximation) from its projection on the ground. In the case of meteors with height up to a hundred of km, they can be observed up to 10^3 km, which is no longer negligible wrt the curvature.

If θ is the angular separation between the vertical of the meteor and the observer the apparent inclination of the meteor is $z = z_0 - \theta$

3 Meteor background – simplified

Assumptions:

- Constant isotropic trajectories for any observer on the Earth.

- Cumulative density flux ϕ (meteors/area/s) as a function of the absolute magnitude (100 km) M

$$\log\phi = -a + bM \tag{19}$$

with a and b positive constant. From [Cook+78].

- Cumulative density flux ϕ_m as a function of the initial mass m

$$\log\phi_m = -a_m - b_m \log m \tag{20}$$

with a_m and b_m positive constant. From [Cook+78].

Procedure. A tentative approach

1. Choose the limit altitude $h_{\max}$ where meteors can no longer occur (too thin atmosphere).
2. Compute the range of visibility of an object at $h_{\max}$. In LAA it is simply

$$d_{\max} \approx \sqrt{2Rh_{\max}} \tag{21}$$

3. Populate the circular area of radius $d_{\max}$ with a number of meteoroids as a function of the mass provided by ϕ_m. Inverting the log-log relation

$$\phi_m = Am^{b_m} \tag{22}$$

with $A = e^{-a_m}$ a new constant.

4. Assign to each object with a given mass an initial velocity and a random orientation (due to isotropy). Initial velocity can be sampled from a uniform distribution in the range 11.2-72.8 km/s for example.
5. Integrate the motion of each particle starting from $h_{\max}$.
6. Project the motion into the observer spherical reference frame.

Alternative (statistical). Very wide area and (6×10^6 cells) each with thousands of object require statistical treatment.

1. For [Cook+78] the meteor flux through the atmosphere respect

$$\log\phi = -17.89 + 0.534M \tag{23}$$

where Φ is the number flux (meteor/cm^2/s) with an absolute magnitude lower (i.e. brighter) than M.

2. Assume a reference height h^* at which we assume all the ablation occurs, i.e. as if all the meteor background comes from a spherical shell around the Earth at height h^* from the surface.
3. Consider a surface unit on the shall at zenit distance z from seen from the observing point. The distance from the observer is

$$\rho = -R\cos z + \sqrt{(R+h^*)^2 - R^2\sin^2 z} \approx \frac{h^*}{\cos z} \tag{24}$$

4. The relation between intrinsic 100 km magnitude and the apparent one is

$$m = M + 5\log\left(\frac{\rho}{100\,\text{km}}\right) = M - 10 + 5\log\left(h^*\right) - 5\log(\cos z) \tag{25}$$

   with h^* in km.

5. Flux of meteor with magnitude M (within an infinitesimal bin of width dM) are

$$\varphi(M) = \frac{\mathrm{d}\Phi}{\mathrm{d}M} = b \ln 10 \Phi(M) \tag{26}$$

6. We can pass from magnitudes to intensities with

$$I = 10^{-0.4M} \tag{27}$$

   Total intensity per area unit per time unit will be the sum over all meteor of all all magnitudes:

$$I_{\text{tot}} = \int_{-\infty}^{+\infty} \varphi(M) I(M) \mathrm{d}M = 10^{-a} \frac{b}{b-0.4} \left[10^{(b-0.4)M}\right]_{M_{\min}}^{M_{\max}} \tag{28}$$

   we can set $M_{\min} \to -\infty$ and have no problems with the boundaries since $b > 0.4$ and the exponential will tend to 0, when evaluated at $-\infty$.

7. We pass again to total magnitudes computing the total absolute magnitude from from a surface and time unit.

$$M_{\text{tot}} = -2.5\log I_{\text{tot}} = 2.5a - 2.5\log\left(\frac{b}{b-0.4}\right) - 2.5(b-0.4)M_{\max} \tag{29}$$

   We can take the approximation

$$M_{\text{tot}} \approx 2.5a - 2.5\log\left(\frac{b}{b-0.4}\right) \sim 40\,\text{mag} \tag{30}$$

   which is valid because $(b-0.4) \lesssim 1$ and the last term is negligible. Note also $M_{\max}$ is limited: lower masses produces traces of lower magnitudes, in we go below a given mass threshold the particle is too small to mechanically interact with the atmosphere.

8. At $\rho = 100\,\text{km}$ by definition $M = m$. At this distance, if we neglect projection effects, a unit surface is seen under the angular area

$$\Omega = (206265/100)^2 = 4.25 \times 10^6\,\text{arcsec}^2 \sim 0.5 \times 0.5° \tag{31}$$

9. Since surface brightness does not depend on the distance (both flux and the solid angle scales with $1/\rho^2$) at any position

$$S = M_{\text{tot}} + 2.5\log\Omega = 40 + 6 = 46\,\text{mag} \tag{32}$$