$(a,b) \to 0,$
$[a,b] \to c10$

Chapter 1

Function of One Variable

Exercise 1.1

(1) Evaluate the difference quotient for given function.

(i) $f(x) = 4-3x;\ \frac{f(3+h)-f(3)}{h}$

$f(3+h) = \frac{4-3(3+h)-4+9}{h}$

$= \frac{4-9-3h-4+9}{h}$

$= -3$ Ans

(ii) $f(x) = \frac{x+3}{x+1};\ \frac{f(x)-f(1)}{x-1}$

$\Rightarrow \frac{\frac{x+3}{x+1} - \frac{1+3}{1+1}}{x-1} = \frac{\frac{x+3}{x+1}-2}{x-1}$

$\Rightarrow \frac{-1}{x-1}$ Ans

(2) Find the domain of the function.

(i) $f(x) = \frac{x+4}{x^2-9}$

$= \frac{x+4}{x^2-3^2} = \frac{x+4}{(x+3)(x-3)}$

Domain $= (-\infty,-3)\cup(-3,3)\cup(3,\infty)$

(ii) $f(x) = \frac{2x^3-5}{x^2+x-6}$

$= \frac{2x^3-5}{(x+3)(x-2)}$

Domain $= (-\infty,-3)\cup(-3,2)\cup(2,\infty)$

(iii) $f(t) = \sqrt[3]{2t-1}$

Domain $= (-\infty,\infty)$

(iv) $g(t) = \sqrt{3-t} - \sqrt{2+t}$

Domain $= [-2,3]$

(v) $f(p) = \sqrt{2-\sqrt{p}}$

Domain $= [0,4]$

(vi) $G(x) = \frac{3x+|x|}{x}$

Domain $= (-\infty,0)\cup(0,\infty)$

(3) Find domain & range of the function.

(i) $h(x) = \sqrt{4-x^2}$

Domain $= [-2,2]$

Range $= [0,2]$

(ii) $f(x) = \sqrt{x-5}$

Domain $= [5,\infty)$

Range $= [0,\infty)$

(iii) $gx = \frac{2x+1}{x-3}$

Domain $= (-\infty,3)\cup(3,\infty)$

Range $(-\infty,2)\cup(2,\infty)$

$(a,b) \to o,$
$[a,b] \to c1c$
(1)

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$= \frac{\cancel{4}-9-3\cancel{h}-\cancel{4}+9}{\cancel{h}}$

$= -3$ Ans

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$\Rightarrow \frac{\frac{x+3}{x+1}-\frac{1+3}{1+1}}{x-1} = \frac{\frac{x+3}{x+1}-2}{x-1}$

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$= \frac{2x^3-5}{(x+3)(x-2)}$

Domain $= (-\infty,-3)\cup(-3,2)\cup($

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Domain $= [-2,2]$

Range $= [0,2]$

ii) $f(x) = \sqrt{x-5}$

Domain $= [5,\infty)$

Range $= [0,\infty)$

iii) $g x = \frac{2x+}{x-}$

Domain $= (-\infty,3)$

Range $(-\infty,2)$

(2)

(4) Identify which one is graph of function.

(i) $y = x^3 + 2$

Given eqn $y = x^3 + 2$

For any vertical line say $x = 1$

$y = 1^3 + 2 = 3$, curve meet single point $y = 3$

Thus $y = x^3 + 2$ is function.

(ii) $y = x + 2$

For any vertical line say $x = 1$

$y = 1 + 2 = 3$, curve meet at $y = 3$

$\therefore y = x + 2$ is function

(iii) $x = y^2$

For any horizental line say $y = 1$

$x = 1^2 = 1$, curve meet single point $x = 1$

Thus $x = y^2$ is function.

(iv) $y = x^2$

For any vertical line say $x = 1$

$y = 1^2 = 1$, curve meet at $y = 1$

Thus $y = x^2$ is function.

(v) $x^2 + y = 5$

$y = 5 - x^2$

For vertical line $x = 1$

$y = 5 - 1 = 4$, curve meet at $y = 4$

Thus $x^2 + y = 5$ is function.

(vi) $x = y^2 - 2$

$y = \sqrt{x+2}$

For vertical line $x = 2$ say,

$y = \sqrt{2+2} = 2$, curve meet at $y = 4$

Thus $x = y^2 - 2$ is a function

(vii) $y = -\sqrt{x+2}$

For vertical line $x = 2$ say,

$y = -\sqrt{2+2} = -2$, curve meet at $y = -2$

Thus $y = -\sqrt{x+2}$ is function.

(5) Determine whether following functions even or odd or nei

i) $f(x) = \dfrac{x^2}{x^4 + 1}$

So, $f(-x) = \dfrac{(-x)^2}{(-x)^4 + 1} = \dfrac{x^2}{x^4 + 1}$ $\therefore f(-x) = f(x)$ which is even fun

$\cancel{h(x) = h(-x)} \rightarrow$ even $= f(-x) = f(x)$
$\cancel{h(x) = -h(x)} \rightarrow$ odd. $= f(-x) = -f(x)$

(3)

ii) $g(x) = x\|x\|$	iii) $h(x) = 1 + x^3 - x^5$
$g(-x) = -x\|-x\| = -x\|x\|$	$h(-x) = 1 - x^3 + x^5$
$= -(x\|x\|)$	which is neither odd no[r]
$\because g(-x) = -g(x)$ is odd function.	even function.
iv) $g(x) = 3$	i) $f(x) = 2\|x\| + 1$
$g(-x) = -3 \quad \because g(-x) = -g(x)$	$f(-x) = 2\|-x\| + 1 = 2\|x\| + 1$
It is odd function.	$f(-x) = f(x)$ so, it is even func[tion]

[6] If f & g are both even function then prove that $f \cdot g$ also even function.

→ For, $f \cdot g \Rightarrow f(-x) = f(x)$ & $g(-x) = g(x)$

Let $f \cdot g = h$

$h(-x) = f(x) \cdot g(x)$ [$\because f(-x) = f(x)$ & $g(-x) = g(x)$]

$f \cdot g(x) = h(x)$ Hence proved

[7] A rectangle has perimeter 20m. Express the area of rectangle as a function of the ~~length~~ breadth of one it's side.

→ Here, $P = 2(l+b)$. [Where, $P = 20$ given]

So, $20 = 2(l+b)$

$l+b = 10$

$l = 10 - b$

Again, Area $= l \times b$

$= (10-b) \times b$

$= 10b - b^2$ Hence, Expr[essed]

[8] A rectangle has area $16m^2$. Express the perimeter o[f] the rectangle as a function of the length of one of it's si[des]

→ Area $(A) = 16m^2$

$l \times b = 16m^2$ To express it into perimeter of rectangle

$l = \frac{16}{b}$ we know, Perimeter (P) $= 2(l+b)$

or, $b = 16/l$

$$= 2\left(l + \frac{16}{l}\right)$$

$$= \frac{2(l^2+16)}{l}$$

$$= 2l + \frac{32}{l} \quad \underline{\text{Ans}}$$

Q No: 9 An open rectangular box with volume $2m^3$ has a square base. Express the surface area of the box a function of the length of a side of the base.

→ Here, Volume (V) $= 2m^3$

$l \times b \times h = 2$

$a \times a \times h = 2$ [Becoz it has square base]

$h = 2/a^2$

Now, side of a base

$$(S) = a^2 + 4ah$$

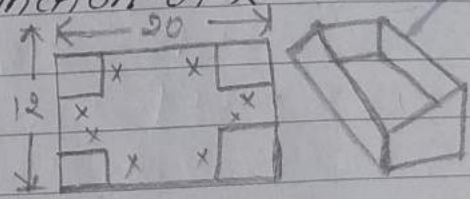
$$= a^2 + 4\cancel{a} \times \frac{2}{a^{\cancel{2}}}$$

$$= a^2 + \frac{8}{a} \quad \underline{\text{Ans}}$$

Q No: 10 Normal window has the shape of a rectangula surmounted by a semicircle. If the perimeter of the window is 30ft, express the area A of the window as function of the width x of window.

→ Here,

Q No: 11 A box with an open top is to be constructed from a rectangular piece of cardboard with dimension 12m, by 20m, by cutting out equal squares of side x at each corner & then folding up the sides as in the figures. Express the volume V of the box as function of x.



→

Q No: 12 In a certain state the maximum speed permitted on freeways is 65 ml/h & the minimum speed is 40 m/h. The fine for violating these limits is \$15 for every mile per hour above the max^m speed or below the min^m speed. Express the amount of the fine F as function of driving speed x.

Q.No:13. An electricity company charges its customers a base rate of $10 a month plus 6 cents per kilowatt hour (kwh) for the first 1200 kwh & 7 cents per kwh for all usage over 1200 kwh. Express the monthly cost E as function of amount x of electricity used.

Exercise 1.2 Linear Mathematical Model

1. Find an equation of the family of linear functions such that $f(2) = 1$.

→

2. Recent studies indicate that the average surface temp of the earth has been rising steadily some scientists have modeled the temperature by the linear function $T = 0.02t +$

8. 50,
(a)
(b)
→
(a)→
3.
a)
b)
c)
Pro

8.50, where T is tempt in $0°C$ & t represent years since 1900.

(a) What do slope & T-intercepts represents?

(b) Use the eqn to predict the average global surface tempt in 2100.

→ Here,

(a) → $T = 0.02t + 8.50$ — (i)

Slope (m) represents 0.02

& T intercept (T) = 8.50

(b) → $2100 = 0.02t + 8.50$

$t = \frac{2100 - 8.50}{0.02}$

---

3. The manager of a furniture factory finds that it costs \$2200 to manufacture 100 chairs in one day & \$4800 to produce 300 chairs in one day.

a) Express to cost as a function of the numbers of chairs produced, assuming that it is linear. Then sketch the graph.

b) What is the slope of the graph & what does it represents?

c) What is Y-intercept of the graph & what does it represent?

⇒ Here, The linear cost $(c) = bx + a$ $[\because y = mx + c]$

Acc. to Question,

$2200 = b \times 100 + a$ — (i)

And $4800 = b \times 300 + a$ — (ii)

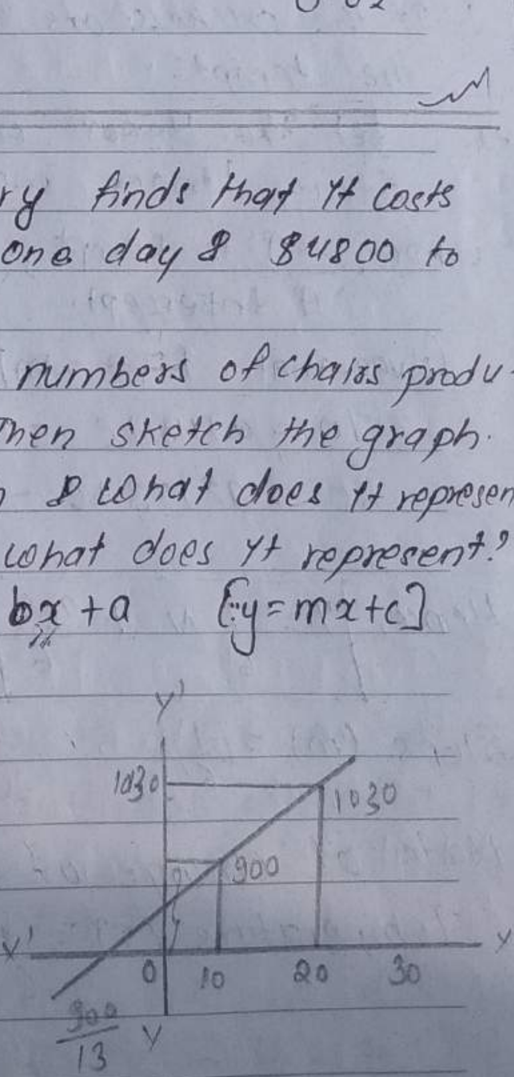
From ~~Adding~~ (i) & (ii) we get,

$4800 = 300b + 2200 - 100b$

$b = \frac{4800 - 2200}{200} = \frac{2600}{200}$

$\therefore b = 13$

& $a = 2200 - 100 \times 13 = 900$



$y = 13x + 900$

4. Biologist has noticed that the chirping rate of crickets of certain species is related to temperature & the relationship appears to be nearly linear. A cricket produces 113 chirps per minute at 70°F & 173 chirps per minutes at 80°F.

a. Find a linear eqn that models the tempt T as a function of the number of chirps per minutes N.

b. What is the slope of graph? What does it represent?

c. If the crickets are chirping at 150 chirps per minute, estimate the tempt?

→ ~~Let The Linear cost(c) =~~

Since, Linear model So, $T = mN + c$ — (i)

where, T is tempt, m is slope N is chirps per minute & c is T intercept.

Given that ~~$T = mN$~~ $+ c$

$70 = a \times 113 + c$ — (ii)

$80 = a \times 173 + c$ — (iii)

from eqn (ii) & (iii)

$80 = 173a + 70 - 113a$

$a = 1/6$

$c = -113 \times \frac{1}{6} + 70$

$= \frac{307}{6}$

• Hence, $\boxed{T = \frac{1}{6} N + \frac{307}{6}}$

• Slope (m) $= \left(\frac{1}{6}\right)$

• Rate of change of tempt 1/6

• Temperature (T) $= \frac{1}{6} \times 150 + \frac{307}{6} = 76.16°C$

8. In certain part of the world, the number of death N per week have been observed to be linearly related to the average concentration x of sulphur dioxide in the air. Suppose there are 97 death when $x = 100 mg/m^3$ & 110 death when $x = 500 mg/m^3$.

→ Since, linear model so, $N = ax + b$ — (i)

where, N is no. of death per week, a = ~~concer~~ slope & b is N intercept.

Given that, $97 = a \times 100 + b$ — (ii)

$110 = a \times 500 + b$ — (iii)

From (ii) & (iii)

$110 = 500a + 97 - 100a$ & $b = 97 - 100 \times \frac{13}{400}$

$a = \frac{13}{400}$ $= \frac{375}{4}$

- So, Total Eqn $N = \frac{13}{400}x + \frac{375}{4}$.

- $N = \frac{13}{400} \times 300 + \frac{375}{4} = 104$ Ans

96. The monthly cost of driving a car depends on the number of miles driven. Lynn found that in May it cost her \$380 to drive 480 ml & in June it cost her \$460 to drive 800 ml.

a) Express the monthly cost (c) as a function of the distance driven d, assuming that a linear relationship gives a suitable model.

b) Use part (a) to predict the cost of driving 1500 miles per month.

c. Draw the graph of the linear function. What does the slope represents?

d. What does the C-intercept represent?

⇒ Here,

Since, linear model so, $C = ax + b$ — (i)

where, C is cost, a is slope & b is cost intercept

$380 = 480b + a$ — (i)

$460 = 800b + a$ — (ii)

$80 = 320b$

$b = \frac{1}{4} = 0.25$

$C = 480 \times \frac{1}{4} + a$

$$\boxed{C = 0.25 + 260}$$

$380 = 120 + a$

$a = 260$

7. Temperature measured in degree Fahrenheit is linear function of tempr measured in degree celcius. Use the fact that $0°$ celsius is equal to $32°$ Fahrenheit & $100°$ celsius is equal to $212°$ Fahrenheit to write an equation for this linear function.

a) Using this function convert $15°$ celsius to Fahrenheit

b) Convert $68°$ Fahrenheit to Celsius.

c) What tempr is the same in both the Celsius & Fahrenheit scale?

(ii)

→ Here, Since Linear model so, $\frac{C}{100} = \frac{F-32}{180}$ — (i)

$$\frac{C}{5} = \frac{F-32}{9}$$

$$F = \frac{9}{5}C + 32$$

b) → $68F^\circ = (?)C^\circ$

$$68 = \frac{9}{5}C + 32$$

$$\frac{(68-32)\times 5}{9} = C$$

$$C = 20^\circ C$$

c) ⇒ $\frac{x}{100} = \frac{x-32}{180}$

$$\frac{x}{5} = \frac{x-32}{9}$$

$$x = -40 \text{ Ans}$$

Q~~No~~5. Since the begining of the year, the price of a bottle of soda at a local discount supermarket has been rising at a constant rate of 2 cents per month. By November first, the price has reached \$1.56 per bottle. Express the price of the soda as function of time & determine the price at the begining of the year

1.3 Combination of Functions.

f, g

$(f+g)(x) = f(x) + g(x)$

$(f-g)(x) = f(x) - g(x)$

$(fg)(x) = f(x) \cdot g(x)$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \cdot g(x) \neq 0$

Domain = Intersection of the domain of f & domain of g

$f(x) = \sqrt{x}$

$g(x) = \sqrt{2-x}$

$(f+g)(x) = f(x) + g(x)$

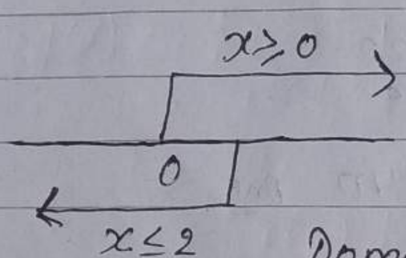
$= \sqrt{x} + \sqrt{2-x}$

domain of $\sqrt{x}$ ie. $x \geq 0$, $A = [0, \infty)$

domain of $\sqrt{2-x}$ ie, $2 - x \geq 0$

$B = [-\infty, 2]$ $\quad 2 \geq x$

$x \leq 2.$



Domain $= A \cap B = [0, 2]$

Composite functions.

If f & g are functions, then composite function $f \circ g$ is defined by $f \circ g(x) = f(g(x))$.

Note that the domain of $f \circ g(x)$ is intersection of domain of $g(x)$ & $f(g(x))$.

Q1 If $f(x) = \sqrt{x}$ & $g(x) = \sqrt{2-x}$

Find the each function & it's domain.

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$.

Solution,

a) $f \circ g(x) = f(g(x)) = f(\sqrt{2-x}) = \sqrt{\sqrt{2-x}} = (2-x)^{1/4}$

To find domain of $f \circ g$.

First we find domain of $g(x) = \sqrt{2-x}$

Domain of $g(x)$ is $\quad 2 - x \geq 0$

$x \leq 2$ ie. $A = (-\infty, 2]$.

& domain of $f(g(x)) = (2-x)^{1/4}$

$2-x \geq 0$

$x \leq 2$ i.e. $B = (-\infty, 2]$.

Here, $A \cap B = (-\infty, 2]$ is domain of fog.

b) $gof(x) = g(f(x)) = g\sqrt{x} = \sqrt{2-\sqrt{x}}$

Domain of $f(x) = \sqrt{x}$ i.e. $x \geq 0$ i.e. $A = [0, \infty)$

The domain of $f\,g(x) = \sqrt{2-\sqrt{x}}$

$2-\sqrt{x} \geq 0$

$\sqrt{x} \leq 2$

$0 \leq x \leq 4$

So, $B = [0, 4]$

$\because$ Domain of gof is $A \cap B = [0, 4]$

(c) $fof(x) = f(f(x) = f\sqrt{x} = \sqrt{\sqrt{x}} = x^{1/4}$

The domain of $f(x)$ is $A = [0, \infty)$

The domain of $f(f(x)$ is $B = [0, \infty)$

Domain of fof is $A \cap B = [0, \infty)$

(d) $gog(x) = gg(x) = g\sqrt{2-x} = \sqrt{2-\sqrt{2-x}}$

Domain of $g(g(x)$ is $2-\sqrt{2-x} \geq 0$

i.e. $\sqrt{2-x} \leq 2$

$0 \leq 2-x \leq 4$

$-2 \leq -x \leq 2$

$2 \geq x \geq -2$

$\therefore$ Domain of $g(g(x)$ i.e. $B = [-2, 2]$

$\therefore$ Domain of gog is $A \cap B$ $B = [-2, 2]$

(14)

Exercise 1.3

1. Find $f+g$, $f-g$, $f\cdot g$ & f/g & state their domain.

i) $f(x) = x^3 + 2x^2$

$g(x) = 3x^2 - 1$

$\rightarrow f+g = f(x)+g(x) = x^3+2x^2+3x^2-1$

$= x^3+5x^2-1$

$\cancel{= x(x^2+5x-1)}$

$f-g = f(x)-g(x) = \cancel{x(}\, x^3+2x^2-3x^2+1$

$= x^3-x^2+1$

$f(x)\cdot g(x) = (3x^2-1)\times(x^3+2x^2)$

$= x^3(3x^2-1)+2x^2(3x^2-1)$

$= 3x^5-x^3+6x^4-2x^2$

$= 3x^5+6x^4-x^3-2x^2$

$\frac{f(x)}{g(x)} = \frac{x^3+2x^2}{3x^2-1}$ $\cancel{= x(x^2+2x)}$

- Domain for $(f+g)$, $(f-g)$ & $f\cdot g = R$

Domain exist for all real no. so, domain is (-

- Domain for f/g is $R-\{\pm\frac{1}{\sqrt{3}}\}$

$3x^2-1 \neq 0$

$x^2 = \frac{1}{3}$

$x = \pm\frac{1}{\sqrt{3}}$

ii) $f(x) = \sqrt{3-x}$, $g(x) = \sqrt{x^2-1}$

$f+g = f(x)+g(x) = \sqrt{3-x}+\sqrt{x^2-1}$

$f-g = f(x)-g(x) = \cancel{\sqrt{3-x}}\ \sqrt{3-x}-\sqrt{x^2-1}$

$f\cdot g = f(x)\cdot g(x) = \sqrt{3-x}\cdot\sqrt{x^2-1}$

$= \sqrt{(3-x)(x^2-1)}$

$f/g = \frac{f(x)}{g(x)} = \frac{\sqrt{3-x}}{\sqrt{x^2-1}}$; $x \neq \pm 1$

Domain for $(f+g), (f-g)$ & (fg) is $(-\infty, 1]$
Domain for f/g is $(-\infty, 1)$

iii) $f(x) = \sqrt{x}$; $g(x) = \sqrt{1-x}$

$f+g = f(x) + g(x) = \sqrt{x} + \sqrt{1-x}$

$f-g = f(x) - g(x) = \sqrt{x} - \sqrt{1-x}$

$f(x) \cdot g(x) = \sqrt{x} \cdot \sqrt{1-x} = \sqrt{x(1-x)}$

$f/g = \dfrac{f(x)}{g(x)} = \sqrt{\dfrac{x}{1-x}}$ $x \neq 1$

Domain for $(f+g)$, $(f-g)$ & (fg) is
Domain for (f/g) is

iv) $f(x) = \sqrt{x+1}$, $g(x) = \sqrt{x-1}$

$f+g = \sqrt{x+1} + \sqrt{x-1}$

$f-g = \sqrt{x+1} - \sqrt{x-1}$

$f \cdot g = \sqrt{x+1} \cdot \sqrt{x-1} = \sqrt{x^2-1}$

$f/g = \dfrac{\sqrt{x+1}}{\sqrt{x-1}}$, $x \neq 1$

Domain for $(f+g)$, $(f-g)$ & $(f \cdot g)$ is $[1, \infty)$
Domain for (f/g) is $(1, \infty)$

3. Find $f \circ g \circ h$

1) $f(x) = 3x-2$, $g(x) = \sin x$, $h(x) = x^2$

Here,

$(f \circ g \circ h)(x)$

$f \circ g\, h(x) = f \circ (g\,h(x)) = f \circ g(x^2) = f\,g(x^2) = \cancel{f g x^2}$

$= \cancel{f(\sin x^2)} = 3\sin x^2 - 2$

$\Rightarrow \cancel{f \circ g \circ h(x)} = \cancel{f \circ g} \Rightarrow f(\sin x)^2$

$f \Rightarrow \qquad \Rightarrow 3\sin x^2 - 2.$

(ii) $f(x) = |x-4|$, $g(x) = 2^x$, $h(x) = \sqrt{x}$

$f \circ g \circ h(x)$

$\Rightarrow f \circ g \circ (\sqrt{x}) \Rightarrow f\, g(\sqrt{x}) \Rightarrow f(2^{\sqrt{x}}) \Rightarrow 2^{\sqrt{x}} - 4$ Ans

Q No: 4. Express the function in form of fog if.

i) $f(x) = (2x + x^2)^4$

$= f(2x + x^2)^4$

$f(x) = x^4$

$g(x) = 2x + x^2$

ii) $f(x) = \cos^2 x$

$f(x) = x^2$

$g(x) = \cos x$

iii) $V(t) = \sec(t^2)\tan t^2$

$f(x) = \sec t \cdot \tan t$

$g(x) = t^2$

Q. 2. ~~Express~~ find fog, gof, fof & gog & state their domain.

(i) $f(x) = \sqrt{x}$; $g(x) = x + 1$

- $f \circ g = f \circ (x+1) = \sqrt{x+1}$, · Domain $= x + 1 > 0$
  $x = -1 = [-1, \infty)$
- $g \circ f = g\sqrt{x} = \sqrt{x} + 1$, Domain $= \sqrt{x} + 1 > 0 = [0, \infty]$
- $f \circ f = f\sqrt{x} = \sqrt{\sqrt{x}} = x^{1/4}$, Domain $= [0, \infty)$
- $g \circ g = g(x+1) = x + 1 + 1 = x + 2$, Domain $= (-\infty, \infty)$

(ii) $f(x) = x^2 - 1$; $g(x) = 2x + 1$

(17)

$\cdot fog = f(2x+1) = (2x+1)^2 - 1 = (2x)^2 + 2\cdot 2x\cdot 1 + 1^2 - 1 = 4x^2 + 4x + 1 - 1$

Domain $= 4x^2+4x > 0 \quad x = -1, 0 \quad \therefore D = (-\infty, \infty)$

$\cdot gof = g(x^2-1) = 2(x^2-1)+1 = 2x^2-2+1 = 2x^2-1$

Domain $= 2x^2-1 > 0 = \cdot x = \pm 1 \quad \therefore D = (-\infty, \infty)$

Ans $fof = f(x^2-1) = (x^2-1)^2 - 1 = x^4 - 2x^2 - 1 - 1 = x^4 - 2x^2 - 2$

Domain $= (-\infty, \infty)$

$gog = g(2x+1) = 2(2x+1)+1 = 4x+2+1 = 4x+3$

Domain $= (-\infty, \infty)$

(ii) $f(x) = \sqrt{x}$; $g(x) = \sqrt[3]{1-x}$

$fog = f(\sqrt[3]{1-x}) = f(1-x)^{\frac{1}{3}} = \sqrt{(1-x)^{\frac{1}{3}}} = (1-x)^{\frac{1}{3}\times\frac{1}{2}} = (1-x)^{\frac{1}{6}}$

Domain $= (-\infty, 1]$

$gof = g\sqrt{x} = \sqrt[3]{1-\sqrt{x}} = (1-\sqrt{x})^{\frac{1}{3}}$ & domain is $[0, \infty)$

$f\cdot f(x) = f\sqrt{x} = \sqrt{\sqrt{x}} = x^{\frac{1}{4}}$ & domain is $[0, \infty]$

$gog(x) = \sqrt[3]{1-x} = \sqrt[3]{1-\sqrt[3]{1-x}}$ & domain is $(-\infty, \infty)$

(iii) $f(x) = x + \frac{1}{x}$; $g(x) = \frac{x+1}{x+2}$

$fog = f\left(\frac{x+1}{x+2}\right) = \frac{x+1}{x+2} + \frac{1}{\frac{x+1}{x+2}} = \frac{x+1}{x+2} + \frac{x+2}{x+1} = \frac{(x+1)^2 + (x+2)^2}{(x+1)\cdot(x+2)}$

Domain $= R - \{-1, -2\}$

$gof = g\left(x + \frac{1}{x}\right) = \frac{x + \frac{1}{x} + 1}{x + \frac{1}{x} + 2} = \frac{x^2+1+x}{x^2+1+2x} = \frac{x^2+x+1}{(x+1)^2}$

Domain is $R - \{0, 1\}$

$fof = f\left(x + \frac{1}{x}\right) \quad x + \frac{1}{x} + \frac{1}{x+\frac{1}{x}} = \frac{x^2+1}{x} + \frac{x}{x^2+1} = \frac{x^2 + (x^2+1)^2}{x(x^2+1)}$

$= \frac{x^4+3x^2+1}{x(x^2+1)}$

$\therefore$ Domain is $R - \{0\}$

- $gog(x) = g\left(\frac{x+1}{x+2}\right) = \frac{\cancel{x+1+1}}{\cancel{x+2+2}} \Rightarrow \frac{\cancel{x+2}}{x+3}$

$$= \frac{\frac{x+1}{x+2}+1}{\frac{x+1}{x+2}+2} = \frac{\frac{x+1+x+2}{x+2}}{\frac{x+1+2x+4}{x+2}} = \frac{2x+3}{3x+5}$$

$\therefore$ Domain $= R - \{-2, -5/3\}$

v> $f(x) = \sqrt{x+1}$, $g(x) = \frac{1}{x}$

$\rightarrow fog = f\,\frac{1}{x} = \frac{1}{\sqrt{x+1}}$ $\therefore$ domain $= \{-1, 0]$

$gof = g\sqrt{x+1} = \sqrt{\frac{1}{x}+1}$ $\therefore$ domain $= (-1, \infty)$

$fof = f\sqrt{x+1} = \sqrt{\sqrt{x+1}+1}$ $\therefore$ domain $= (-1, \infty)$

$gog = g\,\frac{1}{x} = \frac{1}{1/x} = x$ $\therefore$ domain $= R - \{0\}$

vi> $f(x) = x^2$, $g(x) = 1-\sqrt{x}$

$fog = f(1-\sqrt{x}) = (1-\sqrt{x})^2 = 1-2\sqrt{x}+x$ $\therefore D = [0, \infty$

$gof = g(x^2) = g\ (1-(\sqrt{x})^2 = 1-x$ $\therefore D = (-\infty,$

$fof = f(x^2) = x^4$ $\therefore D = (-\infty, \infty$

$g \cdot g = g(1-\sqrt{x}) = 1-\sqrt{1-\sqrt{x}}$ $\therefore D = [0, 1]$

5. Express the function in form $fogoh$ if

(i) $R(x) = \sqrt{\sqrt{x}-1}$ $f \circ g \circ h = ?$ $f(x) = \sqrt{x}$ $g(x) = x-1$ $h(x) = \sqrt{x}$	(ii) $H(x) = \sqrt[8]{2+\lvert x\rvert}$ $f(x) = \sqrt[8]{x}$ $g(x) = 2+x$ $h(x) = \lvert x\rvert$

iii) $H(x) = \sec x^4(\sqrt{x})$

$f(x) = x^4$ Given, $H(x) = \sec x^4(\sqrt{x})$ so, formula of F
$g(x) = \sec x$ in x say & add 4 then take sec & that makes
$h(x) = \sqrt{x}$ under root of x so, $f(x)=x^4$, $g(x)=\sec x$, $h(x)=\sqrt{x}$

Hence, $f \circ g \circ h(x) = f \circ g(hx) = f \circ g \sqrt{x}$
$= f \sec \sqrt{x}$
$= \sec x^4 \sqrt{x} = H(x)$.

Q No:6 Suppose the graph of f is given. Write equation for the graphs that are obtained from the graph of f as follows:

i) Shift 3 unit upwards Let eqn of graph be $y = f(x)$ Acc. to. question, Reqd eqn is $y = f(x) + 3$	ii) Shift 2 unit to the right Let eqn of graph be $y = f(x)$ Acc. to eqn reqd eqn is $y = f(x - 2)$
iii) Reflect about y-axis $y = f(x)$ Let, To reflect about y-axis. $y = f(-x)$.	iv) Strech vertically by a factor of 3. $y = f(x)$ suppose, Acc. to. question, reqd eqn is $y = 3f(x)$

v) Compressed horizentally by a factor of 1.

Let, $y = f(x)$ be the eqn of graph.
Acc. to. question, reqd eqn is
$y = 1.f(x) = y = f(x)$. Ans

Q.No:7 Explain how each graph is obtained from the graph of $y = f(x)$

i) $y = f(x) + 8$

$\Rightarrow$ Shift distance 8 ~~distance~~ unit upwards.

ii) $y = f(x+8)$

$\Rightarrow$ Shift 8 unit distance left.

iii) $y = f(8x)$

$\Rightarrow$ Compress horizontally by factor of 8.

iv) $y = -f(x) - 1$

$\Rightarrow$ ~~Strech horizentally~~ Reflection towards ~~y-axis~~ x-axis & shift 1 units downward.

v) $y = 8f(\frac{1}{8}x)$

$\Rightarrow$ strech horizentally by a factor of 8 followed by strech vertically by a factor of 8.

Q.No: Book If y is even & g is odd function & both are defined on R then find which of the following combine function are even or odd or neither what

$\Rightarrow$ Given,

$f(-x) = f(x)$

$g(-x) = -g(x)$ Then, Acc. to question,

$$f.g(-x) = f(-x) \times g(-x)$$
$$= f(x) \times -g(x)$$
$$= -f(x).g(x)$$
$$= -fg(x)$$

which is odd function.

And also find (2)

(i) f/g (ii) f^2 (iii) $g g$ (iv) $f o g$ (v) $g o f$ (vi) $f o f$ (vii) $g o g$.

i) → Here, f/g: $\frac{f}{g}(-x) = \frac{f(-x)}{g(-x)}$.

or, $= \frac{f(x)}{-g(x)} = -\frac{f(x)}{g(x)}$

which is odd function.

ii) $f^2 = f \times f$

$= f(-x) \times f(-x)$.

$= f(x) \times f(x)$

$= f^2(x)$.

which is even function

iii) $g g = g \times g$

$= g(-x) \times g(-x) = -g(x) \times -g(x)$

$= g(x)$

which is even function

iv) $f o g = f o g(-x)$

$= f(-x) \times -g(x)$

$= -f(x) \cdot g(x)$

$= -fg(x)$

which is odd function

v) $g o f = ?$

$g \circ f(-x) = g(-x) \times f(-x)$

$= -g(x) \times f(x)$

$= -g f(x)$

which is odd function

vii) $g o g$

$g \circ g(-x) = g(-x) \times g(-x)$

$= -g(x) \times -g(x)$

$= g^2(x)$

vi) $f o f = ?$

$f \cdot f(-x) = f(-x) \times f(-x)$

$= f(x) \times f(x)$

$= f^2(x)$

Q.No:8. Find the new function by using given transformation on given functions.

i) $f(x) = -\sqrt{x}$ shifted right by 3.
$\rightarrow \quad f(x) = -\sqrt{x-3}$

ii) $y = 2x-7$ shifted up by 7
$\rightarrow \quad f(x) = 2x-7+7 = 2x \quad \therefore y = 2x$

(iii) $y = x^2 - 1$ stretched vertically by a factor of 3
$\rightarrow \quad y = 3(x^2-1) = 3x^2 - 3$

(iv) $y = \sqrt{x+1}$ compressed horizentally by factor 4.
$\rightarrow \quad y = \sqrt{4x+1}$

(v) $y = \frac{1}{2}(x+1)+5$ shifted down by 5 followed by right 1
$\rightarrow y = \frac{1}{2}(x+1)+5-5 = \frac{1}{2}(x+1-1) = \frac{1}{2}x$

(vi) $f(x) = \frac{1}{x^2}$ shifted left by 2 followed by down 1.
$\rightarrow f(x) = \frac{1}{(x+2)^2} - 1$

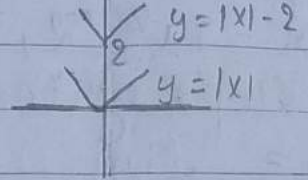
(vii) $f(x) = x^3 - 4x^2 - 10$ compress vertically by 2 followed
$\rightarrow$ by reflection about x-axis.

Soln Given, function $f(x) = x^3 - 4x^2 - 10$
When the function $f(x)$ is compressed vertically by $c=2$
$$f(x) = \frac{f(x)}{c} = \frac{x^3 - 4x^2 - 10}{2}$$

Again, function is reflect about x-axis $f(x) = -f(x)$
$$\therefore f(x) = -\frac{(x^3 - 4x^2 - 10)}{2}$$

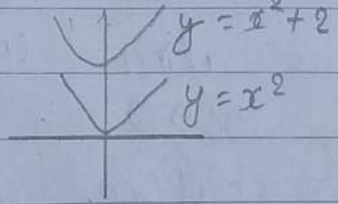
Q.No:9. Find the appropriate transformation used thus obtained ~~vertically by 2~~ new function as below, & graph the function by ~~followed by reflection about x-axis~~ hand not by plotting points

i) $y = |x| - 2$



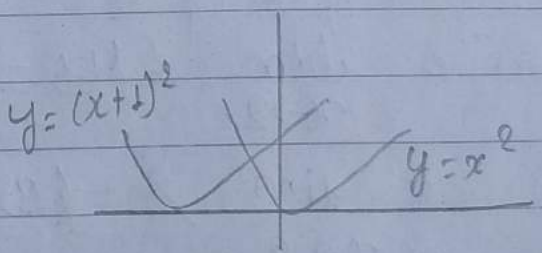
→ original function $f(x) = |x|$,
shift the distance 2unit
down words.

ii) $y = x^2 + 2$



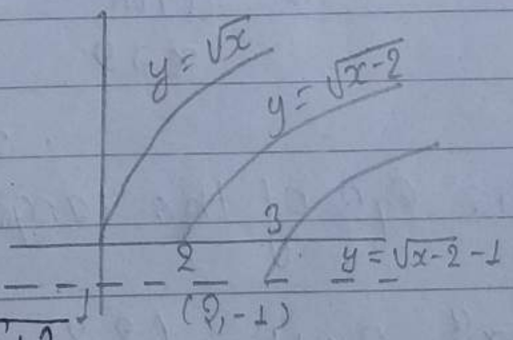
→ Original function $f(x) = x^2$
Shift the distance 2unit
upwards.

iii) $y = (x+1)^2$



→ original function $f(x) = x^2$;
shifting the graph at distance 1
unit left.

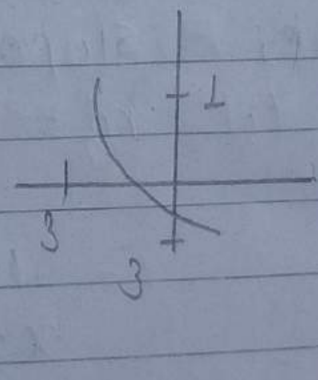
iv) $y = \sqrt{x-2} - 1$



→ Original function $f(x) = \sqrt{x}$
Shifting a distance 2 unit right
followed by shifting 1 unit down

v) $1 - 2\sqrt{x+3}$

→ Original function $f(x) = 1 - 2\sqrt{x+3}$
Shift left by 3 unit stretch vertically by 2, reflect about x-axis & shift
1 unit upwards.



Chapter 3

Derivative

$\left\{ \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right\}$ (2a) formula

Exercise 3.1

1. Find an eqn of the tangent line to the curve at the given point.

a) $y = 4x - 3x^2$ $(2, -4)$

→ Here, Slope of the tangent at $(2, -4)$

is $m = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$

$= \lim_{x \to 2} \frac{4x - 3x^2 - (4 \times 2 - 3 \times 2^2)}{x - 2}$

$= \lim_{x \to 2} \frac{4x - 3x^2 - 8 + 12}{x - 2}$

$= \lim_{x \to 2} \frac{4x - 3x^2 + 4}{x - 2}$

$= \lim_{x \to 2} \frac{-(3x^2 - 4x - 4)}{x - 2}$

$= \lim_{x \to 2} \frac{-(3x + 2)\cancel{(x - 2)}}{\cancel{(x - 2)}}$

$= 3 \times 2 + 2 = -8$

∴ The eqn of the tangent at $(2, -4)$ is $y + 4 = -8(x - 2)$

*b) $y = x^3 - 3x + 1$, $(2, 3)$

→ Here, Slope of the tangent at $(2, 3)$

is $m = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$

$= \lim_{x \to 2} \frac{x^3 - 3x + 1 - \{2^3 - 3 \times 2 + 1\}}{x - 2}$

$= \lim_{x \to 2} \frac{x^3 - 3x + \cancel{1} - [8 + 6 - \cancel{1}]}{x - 2}$

$= \lim_{x\to 2} \frac{x^3 - 3x - 2}{x-2}$ This derivative is.

$= \lim_{x\to} 3x^2 - 3 = 12 - 3 = 9 = m$

$\therefore$ The eq^n of the tangent at $(2,3)$ is

$y + 3 = -9(x-2)$

$y + 3 = -9x + 18$ or, $9x + y - 15 = 0$

c) $y = \sqrt{x}$, $(1,1)$

Here, Slope of the tangent at $(1,1)$ is

$$m = \lim_{x\to 1} \frac{f(x) - f(1)}{x-1}$$

$$= \lim_{x\to 1} \frac{\sqrt{x} - 1}{x-1} \times \frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$= \lim_{x\to 1} \frac{\cancel{\sqrt{x}} - \cancel{\sqrt{x}} + \cancel{(x-1)}}{\cancel{(x-1)}(\sqrt{x}+1)}$$

$$= \lim_{x\to 1} \frac{1}{\sqrt{x}+1}$$

$$\therefore m = \frac{1}{2}$$

$\therefore$ $y - 1 = \frac{1}{2}(x-1)$

$x = 2y - 1$ is the req^d eq^n.

d) $y = \frac{2x+1}{x+2}$, $(1,1)$

Here, slope of the tangent at $(1,1)$ is

$$m = \lim_{x\to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x\to 1} \frac{\frac{2x+1}{x+2} - \frac{2\times 1+1}{1+2}}{x-1}$$

$$m = \lim_{x\to 1} \frac{2x+1-1(x+2)}{(x-1)(x+2)}$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x+2)}$$

$$= \frac{1}{1+2} \quad \therefore m = \frac{1}{3}$$

$\therefore$ The reqd eqn is

$$y-1 = \frac{1}{3}(x-1)$$

$$3y-3 = x-1$$

$$\therefore x-3y+2=0 \text{ Ans}$$

Q No: 2. a) Find the slope of tangent to the curve $y = 3+4x^2-2x^3$ at the point where $x=a$.

Here, The slope of the tangent at $x=a$ is

$$m = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\cancel{3}+4x^2-2x^3-\cancel{3}+4a^2-2a^3}{x-a}$$

Q no b) Find the eqn of the tangents to the curve $y = 3+4x^2+2x^3$ at the point (1, 5) & (2, 3).

Q.No: 3. If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after 't' second is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.

→ Here, velocity $(v) = 40$ ft/s

$y = 40t - 16t^2$

Find velocity when $t = 2$.

$$V(2) = \lim_{t \to 2} \frac{f(t) - f(2)}{t - 2}$$

$$= \lim_{t \to 2} \frac{40t - 16t^2 - 16}{t - 2}$$

$= -24$ ft/s Ans

Q.No: 4. The displacement (in meters) of a particle moving in a st. line is given by the eqn of motion $s = 1/t^2$, where t is measured in second. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$, & $t = 3$.

→ Given, Find velocity at $t = a$, $t = 1$, $t = 2$, $t = 3$.

velocity $(t=a)=?$

$m = t^{-2} = -2t^{-3}$

$\therefore m = -2 \times \frac{1}{t^3}$ at $t=a$,

$V = -\frac{2}{a^3}$ m/s.

At $t=3$ $V = \frac{-2}{27}$ m/s

At $t=1$

$m = t^{-2} = -2t^{-3}$

$\therefore m = -2 \times \frac{1}{t^3}$ at $t=1$

$v = -2$

At $t=2$ $V = \frac{-2}{8}$ m/s Ans

5. The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$.

a) Find the average velocity ~~when $t=4$~~ over each time interval: is $[3,4]$

→ Given that $s(t) = t^2 - 8t + 18$ at time period $[3,4]$

we know the average velocity of a particle at time t is

$V_{avg} = \frac{S_f - S_a}{t_f - t_a}$ — (1) $\because$ $t_a \rightarrow$ initial time

$t_f \rightarrow$ final time

$S_f \rightarrow$ functional value at t_f

$S_a \rightarrow$ " " " t_a

(29)

b) Find the instantaneous velocity when $t=4$.

$S = t^2 - 8t + 18$ Given,

$S = 4^2 - 8 \times 4 + 18$ $t = 4$

$S = 16 - 32 + 18$

$S = 2$

Exercise 3·2 (30)

1. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers of locations as of October 1 are given).

year	2004	2005	2006	2007	2008
N	8569	10241	12440	15011	16680

a. Find the average rate of growth
i) from 2006 to 2008.

→ Average rate $= \frac{\cancel{12440}\ 16680 - 12440}{2008 - 2006} = \frac{4240}{2} = 2120$

Thus, in 2008 to 2006, the avg rate of increase of N is 2120 locations/years.

b. Estimate the instantaneous rate of growth in 2006 by taking the average of two average rates of change. What are its units?

→ Here, the base year is 2006. Here the average rate from 2006 to 2005 is

Avg. rate $= \frac{12440 - 10241}{2006 - 2005} = 2199$

By (a)(i) the avg. rate from 2006 to 2008 is 2120.
∴ The instantaneous rate of growth in 2006 is.

Rate of growth $= \frac{2199 + 2120}{2} = 2159.5$

c. Estimate the instantaneous rate of growth in 2007 & Compare it with the growth rate in 2006. What do you conclude?

→ Here, Avg rate $= \frac{15011 - 12440}{2007 - 2006}$ $= \frac{2571 + 2120}{2} = 1177.55$

$= 2571$

Growth rate is smaller than 200

2. The cost (in Rs) of producing x ounces of gold from a new gold mine is $C = f(x)$ rupees.

a) What is the meaning of the derivative $f'(x)$? What are its units?

→ In the sense of economics, $f(x)$ is the production cost & $f'(x)$ be the rate of change of production cost with respect to the produced meter. Therefore,

$$f'(x) = \lim_{x \to 0} \frac{\Delta C}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}.$$

where, $C = f(x)$, be units for $f'(x)$

b) What does the statement $f'(800) = 17$ mean?

→ The statement $f'(800) = 17$ means that after the production of 800 ounce gold, the rate of the production cost is increasing by 17 rupee per ounce.

Exercise 3.3

1. Find the derivative of the function using the definition of derivative. State the domain of the functions & the domain of its derivative.

1. $\boxed{f(x) = \frac{1}{2}x - \frac{1}{3}}$

The derivation of a function 'f' at a point a is denoted by $f'(a)$ & is defined as $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Acc. to Question,

$$\text{or, } \lim_{a \to 0} \frac{f(x+a) - f(x)}{a}$$

$\frac{1}{2}x - \frac{1}{3}$

1. $f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{\left(\frac{1}{2}(x+h) - \frac{1}{3}\right) - \frac{1}{2}x - \frac{1}{3}}{h}$

$= \lim_{h \to 0} \frac{\frac{1}{2}(x+h) - \frac{1}{3} - \frac{1}{2}x + \frac{1}{3}}{h}$

$= \frac{\frac{1}{2}\cancel{h}}{\cancel{h}} = \frac{1}{2}$ $\quad \because$ Domain $= (R, R)$

2. $\boxed{f(x) = mx + b}$

$f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(m(x+h) + b) - (mx+b)}{h}$

$= \lim_{h \to 0} \frac{mx + hm + b - mx - b}{h}$

$= \therefore f'(x) = m$

$\because$ Domain $= (R, R)$

3. $\boxed{f(x) = x^2 - 2x^3}$

$f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\lim_{h \to 0} = \frac{(x+h)^2 - 2(x+h)^3 - (x^2 - 2x^3)}{h}$

$\lim_{h \to 0} = \frac{\cancel{h}(2x + h - 2h^2 - 6x^2 - 6hx)}{\cancel{h}}$

$f'(x) = 2x - 6x^2$

$\because$ Domain $= R, R$

4. $\boxed{g(t) = \frac{1}{\sqrt{t}}}$

$g(t) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{\sqrt{t+h}} - \frac{1}{\sqrt{t}}}{h}$

$= \lim_{h \to 0} \frac{\frac{\sqrt{t} - \sqrt{t+h}}{\sqrt{t+h} \cdot \sqrt{t}} \times \frac{\sqrt{t} + \sqrt{t+h}}{\sqrt{t} + \sqrt{t+h}}}{h} = \lim_{h \to 0} \frac{t - (t+h)}{h \times (\sqrt{t} \cdot \sqrt{t+h}) \cdot \sqrt{t} + \sqrt{t+h}}$

$= \lim_{h \to 0} = \frac{-\cancel{h}}{\cancel{h}\, t\sqrt{t+h} + t\sqrt{t+h}} = \frac{-1}{2t\sqrt{t}}$ $\quad \because$ Domain $= (0, \infty)$, $(0, \infty)$

(33)

5. $\boxed{g(x) = \sqrt{9-x}}$

$$g'(x) = \lim_{h\to 0} \frac{\sqrt{9-(x+h)} - \sqrt{9-x}}{h}$$

$$g'(x) = \lim_{h\to 0} \frac{\sqrt{9-x-h} - \sqrt{9-x}}{h} \times \frac{\sqrt{9-x-h} + \sqrt{9-x}}{\sqrt{9-x-h} + \sqrt{9-x}}$$

$$g'(x) = \lim_{h\to 0} \frac{9-x-h-9+x}{h\cdot \sqrt{9-x-h} + \sqrt{9-x}} = \frac{-1}{2\sqrt{9-x}}$$

$\because$ Domain is $(-\infty, 9)$, $(-\infty, 9)$

6. $\boxed{f(x) = \dfrac{x^2-1}{2x-3}}$

$$f(x) = \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h)^2-1}{2(x+h)-3} - \frac{\{x^2-1)}{\{2x-3)}\right]$$

$$f(x) = \lim_{h\to 0} \frac{1}{h} \frac{x^2+2hx-1}{2x+2h-3} - \frac{x^2-1}{2x-3} = f(x) \lim_{h\to 0} \frac{(2x-3)(x^2+2hx-1)-(x^2-1) \times 2x+2h}{(2x+2h-3)\times(2x-3)}$$

$$f(x) = \lim_{h\to 0} \frac{1}{h}\left[\frac{2x(x^2+2hx-1)-3(x^2+2hx-1) - \{x^2(2x+2h-3)-1(2x+2h-3)\}}{(2x+2h-3)\times(2x-3)}\right]$$

$$f(x) = \lim_{h\to 0} \frac{1}{h}\left[\frac{2x^3+4hx^2-2x-3x^2-6hx+3-\{2x^3+2hx^2-3x^2-2x-2h+3\}}{(2x+2h-3)\times(2x-3)}\right]$$

$$f(x) = \lim_{h\to 0} \frac{1}{h}\left[\frac{\cancel{2x^3}+4hx^2-\cancel{2x}-\cancel{3x^2}-6hx+\cancel{3}-\cancel{2x^3}-2hx^2+\cancel{3x^2}+\cancel{2x}+2h-\cancel{3}}{(2x+2h-3)\times(2x-3)}\right]$$

$$f(x) = \lim_{h\to 0} \frac{1}{h}\left[\frac{2hx^2-6hx+2h}{(2x+2h-3)(2x-3)}\right]$$

$$f(x) = \lim_{h\to 0} \frac{1}{\cancel{h}} \times \cancel{h}\frac{(2x^2-6x+2)}{(2x+2h-3)(2x-3)} = \frac{2x^2-6x+2}{(2x-3)^2}$$

7. $f(x) = x^{3/2}$

$f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f(x) = \lim_{h \to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h}$

$= \lim_{h \to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} \times \sqrt[3]{x}$

$= \lim_{h \to 0} \frac{1}{h} [(x+h)^{3/2} - 3(x+h) x^{1/2} + 3(x+h)^{1/2}(x) + x^{3/2}]$

$= \frac{1}{h} \lim_{h \to 0} \quad x^{3/2} - \cancel{3x^{3/2}} + \cancel{3x^{3/2}} + x^{3/2}$

8. $f(x) = x^4$

$f(x) = \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h \to 0} \frac{((x+h)^2)^2 - (x^2)^2}{h}$

$= \lim_{h \to 0} \frac{((x+h)^2 + x^2)((x+h)^2 - x^2)}{h} = \lim_{h \to 0} \frac{(2x^2 + 2hx + h^2)(2x^2 + 2hx + h^2 - x^2)}{h}$

$= \lim_{h \to 0} \frac{4hx^3 + 4h^2x^2 + 2h^3x + 2h^2x^2 + 2h^3x + h^4}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(4x^3 + 4hx^2 + 2h^2x + 2hx^2 + 2h^2x + h^3)}{\cancel{h}}$

$\therefore f(x+h) = 4x^3$

Domain of $f(x) = R$

Domain of $f(x+h) = R$.

Exercise 3.4 (35)

1. Differentiate the function:

a) $f(x) = 2^{40}$

$\frac{d(x)}{d(t)} = d(2^{40})$

$= 0$ Ans

b) $f(x) = e^5$

$\frac{dx}{dt} = \frac{d(e^5)}{dt}$

$= e^5$ Ans

c) $F(x) = 3/4 x^8$

$\frac{dx}{dt} = \frac{3}{4} \times 8x^7 = 6x^7$

Ans

2. Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 1$ where the tangent is horizontal.

Soln: $f(x) = y = 2x^3 + 3x^2 - 12x + 1$

$f'(x) = y' = 6x^2 + 6x - 12$

$= 6x^2 + 6x - 12$

$= 6(x^2 + x - 2)$

$= (x^2 + x - 2)$

$= x^2 + (2-1)x - 2$

$= x^2 + 2x - x - 2$

$= x(x+2) - 1(x+2)$

$= (x+2)(x-1)$

$\therefore x = -2$ & 1

When $x = 1$

$y = 2x^3 + 3x^2 - 12x + 1$

$= 2 \times 1 + 3 \times 1 - 12 \times 1 + 1$

$= 2 + 3 - 12 + 1$

$= -6$

When $x = -2$

$y = 2 \times (-2)^3 + 3 \times (2)^2 - 12 \times -2 + 1$

$= 2 \times -8 + 3 \times 4 + 24 + 1$

$= -16 + 12 + 24 + 1$

$= -4 + 1 + 24 = 21$

So, ~~the~~ points (x, y) are: $(1, -6)$ $(-2, 21)$.

3. Show that the curve $y = 2x^e + 3x + 5x^3$ has no tangent with slope 2.

$\rightarrow$ $f(x) = y = 2x^e + 3x + 5x^3$

$f'(x) = y' = 2x^e + 3 + 15x^2$

Acc. to Question

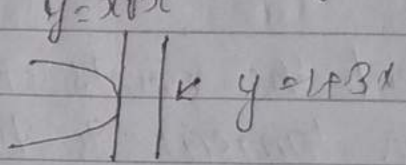
2

Slope $(m) = 2$ so, $2x^6 + 3 + 15x^2 = 2$

$2x^6 + 15x^2 + 1 = 0$

This eqn can't be solved so, no tangent line with slope 2

4. Find an equation of the tangent line to the curve $y = x\sqrt{x}$ that is parallel to the line $y = 1 + 3x$.



$f(x) = y = x\sqrt{x}$

$= x \cdot x^{1/2}$

$f'(x) = y' = x \cdot \frac{1}{2} x^{-1/2} + x^{1/2}$

$= \frac{x}{\sqrt{2x}} + \sqrt{x}$

$(m_1)\ y = \frac{\sqrt{x}}{2} + \sqrt{x}$ —— (i)

$m_1 = \frac{\sqrt{4}}{2} + \sqrt{4} = 3$

$(m_2) = 3$ —— (ii)

$\because$ (i) & (ii) parallel

$\frac{\sqrt{x}}{2} + \sqrt{x} = 3$.

or $\sqrt{x} + 2\sqrt{x} = 6$

$\sqrt{x} = 2$

$\therefore x = 4$

Putting value of x

$y = 4\sqrt{4}$

$= 8$

$\therefore$ Point is $(4, 8)$.

Now, for Eqn of tangent

$y - y_1 = m(x - x_1)$

$y - 8 = 3(x - 4)$

$y - 8 = 3x - 12$

$y = 3x - 4$

which is the required Equation

Q No. 5: Find an equation of the normal line to the parabola $y = x^2 - 5x + 4$ that is parallel to the line $x - 3y =$

2. → Here, $y = x^2 - 5x + 4$ — (i)

$$\frac{dy}{dx} = 2x - 5$$

$$m_1 = 2x - 5 = 2 \times \frac{8}{3} - 5 = \frac{16}{3} - 5 = \frac{1}{3} = -3$$

m_2 from $x - 3y = 5$

$$\frac{1}{3}x - \frac{5}{3} = y \qquad m_2 = \frac{1}{3}$$

$m_1 = m_2$

$2x - 5 = 1/3$

$6x - 15 = 1$

$x = 8/3$

putting value of x in (i)

$$y = \frac{64}{9} - \frac{40}{3} + 4$$

$$y = \frac{-20}{9}$$

$\therefore$ (point $(x, y) = (8/3, -20/9)$

Now,

$$y - y_1 = m(x - x_1)$$

$$y + \frac{20}{9} = -3\left(x - \frac{8}{3}\right)$$

$$\frac{9y + 20}{9} = -3\left(\frac{3x - 8}{3}\right)$$

$$\frac{9y + 20}{3} = -9x + 24$$

$$9y + 20 = -27x + 72$$

$$y = -3x - \frac{52}{9}$$

Q No: 6. a) Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.

$\rightarrow$ $y = x^2 + x$ $\quad f(1) = 2\times1+1 = 3$

$m_1 = \frac{dy}{dx} = 2x+1$

$m_2 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y+3}{x-2}$ So the slope of the tangent line to $f(x)$ $y = x^2 + x$ at $P(2, -3)$ is 3. ∴ the eq^n to the

b) Show that there is no line through the point $(2, 7)$ that is tangent to the parabola. Then draw a diagram to see why?

$\rightarrow$

7. Differentiate:

a) $g(x) = \sqrt{x}\, e^x$

$$\frac{d(gx)}{dx} = \sqrt{x}\cdot\frac{de^x}{dx} + e^x\cdot\frac{dx^{1/2}}{dx}$$

$$= \sqrt{x}\cdot e^x + e^x\cdot\tfrac{1}{2}x^{-1/2}$$

$$= \sqrt{x}e^x + \frac{1}{2\sqrt{x}}e^x$$

$$= e^x\left(\frac{2x+1}{2\sqrt{x}}\right) \text{ Ans}$$

b) $y = \dfrac{e^x}{1-e^x}$

$$\frac{dy}{dx} = \frac{(1-e^x)\cdot\frac{de^x}{dx} - e^x\cdot\frac{d(1-e^x)}{dx}}{(1-e^x)^2}$$

$$= \frac{(1-e^x)\cdot e^x + e^x\, e^x}{(1-e^x)^2}$$

$$= \frac{e^x(1-\cancel{e^x}+\cancel{e^x})}{(1-e^x)^2} \text{ Ans}$$

c) $G(x) = \dfrac{x^2-2}{2x+1}$

$$\frac{d(gx)}{dx} = \frac{(2x+1)\cdot\frac{d(x^2-2)}{dx} - (x^2-2)\cdot\frac{d(2x+1)}{dx}}{(2x+1)^2}$$

$$= \frac{(2x+1)(2x) - (x^2-2)\cdot 2}{(2x+1)^2}$$

$$= \frac{2(x^2+x+2)}{(2x+1)^2} \text{ \#}$$

e) $f(t) = \dfrac{2t}{2+\sqrt{t}}$

$$\frac{df(t)}{dt} = \frac{(2+\sqrt{t})\cdot 2 - 2t\cdot\frac{1}{2}t^{-1/2}}{(2+\sqrt{t})^2}$$

$$= \frac{4+2\sqrt{t}-\sqrt{t}}{(2+\sqrt{t})^2}$$

$$= \frac{4+\sqrt{t}}{(2+\sqrt{t})^2} \text{ Ans}$$

d) $y = \dfrac{x+1}{x^3+x-2}$

$$\frac{dy}{dx} = \frac{(x^3+x-2)\cdot 1 - (x+1)(3x^2+1)}{(x^3+x-2)^2}$$

$$= \frac{(x^3+x-2) - (3x^3+x+3x^2+1)}{(x^3+x-2)^2}$$

$$= \frac{-2x^3-3x^2-3}{(x^3+x-2)^2} \text{ \#}$$

f) $f(x) = \dfrac{x^2}{1+2x}$

$$\frac{df(x)}{dx} = \frac{(1+2x)\cdot 2x - x^2\cdot 2}{(1+2x)^2}$$

$$= \frac{2x(1+2x-x)}{(1+2x)^2}$$

$$= \frac{2x(1+x)}{(1+2x)^2} \text{ Ans}$$

f) $f(x) = \dfrac{1-xe^x}{x+e^x}$

$$\frac{d(f(x))}{dx} = \frac{-(x+e^x)\cdot(xe^x+e^x) - (1-xe^x)\cdot(1+e^x)}{(x+e^x)^2}$$

$$= \frac{-(x^2\cdot e^x + xe^x + xe^{2x} + e^{2x}) - (1+e^x - xe^x - xe^{2x})}{(x+e^x)^2}$$

$$= \frac{-x^2e^x + \cancel{xe^x} \mp xe^{2x} \mp e^{2x} - 1 - e^x + \cancel{xe^x} + \cancel{xe^{2x}}}{(x+e^x)^2}$$

$$= \frac{x^2e^x + 2xe^x + 2xe^{2x} + e^{2x} - 1 - e^x}{(x+e^x)^2}$$

$$= \frac{(x^2+2x-1)\,e^x(e^{2x}-1)}{(x+e^x)^2} \quad \# \quad \to \frac{-(x^2+1)e^x - e^{2x} - 1}{(x+e^x)^2}$$

Q NO:8 Differentiate:

(a) $f(x) = 3x^2 - 2\cos x$

$= 6x + 2\sin x$

(b) $g(\theta) = e^\theta(\tan\theta - \theta)$

$\dfrac{dg(\theta)}{d\theta} = e^\theta\cdot(\sec^2\theta - 1) + (\tan\theta - \theta)\cdot e^\theta$

$= e^\theta(\tan^2\theta + \tan\theta - \theta)$

(c) $y = \dfrac{x}{2-\tan x}$

$\dfrac{dy}{dx} = \dfrac{2-\tan x\cdot 1 - x\cdot(-\sec^2 x)}{2-\tan x}$

$= \dfrac{2-\tan x + x\sec^2 x}{2-\tan x}$ #

(d) $y = \dfrac{\cos x}{1-\sin x}$

$\dfrac{dy}{dx} = \dfrac{(1-\sin x) - \sin x + \cos x\cdot\cos x}{(1-\sin x)^2}$

$= \dfrac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2}$

$= \dfrac{1-\sin x}{(1-\sin x)^2} = \dfrac{1}{1-\sin x}$ #

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ed. $y = \frac{\cancel{\cos x}}{\cancel{1-\sin x}} \quad \frac{1-\sec x}{\tan x}$

$$\frac{dy}{dx} = \frac{1-\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} = \frac{\cos x - 1}{\cancel{\cos x}} \times \frac{\cancel{\cos x}}{\sin x} = \frac{\cos x - 1}{\sin x}$$

$$= \frac{\sin x \cdot -\sin x - (\cos x - 1)\cdot \cos x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x + \cos x}{\sin^2 x}$$

$$= \frac{-(1-\cos x)}{\sin^2 x} = \frac{-(1-\cos x)}{(1-\cos x)(1+\cos x)}$$

$$= \frac{-1}{1+\cos x} \#$$

Q No: 9 Find an equation of the tangent line to the curve $y = 3x + 6\cos x$ at the point $(\pi/3, \pi+3)$

$\rightarrow \frac{dy}{dx} = 3 - 6\sin x$

$= 3 - 6\sin 60$

$= 3 - \cancel{6}^{3} \times \frac{\sqrt{3}}{\cancel{2}}$

$= 3 - 3\sqrt{3}$

From,

$y - y_1 = m(x - x_1)$

$y - \pi - 3 = 3 - \sqrt{3}(x - \pi/3)$

$y - (\pi + 3) = 3 - 3\sqrt{3}\,(x - \pi/3)$ #

or, $y = (\pi + 3) + (3 - 3\sqrt{3})(x - \pi/3)$ #

(10) An object with weight 'W' is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ

With plane, then the magnitude of the force is
$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$ where, $\mu \to$ constant called coefficient of friction.

(a) Find the rate of change of F with respect to θ.

(b) When is this rate of change equal to 0?

$\Rightarrow$ Here,

(a) ~~dr~~ $F = \frac{\mu w}{\mu \sin\theta + \cos\theta}$

$\mu \to$ constant called coefficient of friction.

$$\frac{dF}{d\theta} = \mu w \frac{[\mu \sin\theta + \cos\theta \cdot \frac{d1}{d\theta} - 1 \times d(\mu \sin\theta + \cos\theta)]}{(\mu \sin\theta + \cos\theta)^2}$$

$$= \frac{\mu w(\mu \cos\theta - \sin\theta)}{(\mu \cos\theta + \sin\theta)^2}$$

$$= \cancel{\mu w(\mu \cos\theta - s}$$

(b) $\frac{dF}{d\theta} = 0$

$$\frac{-\mu w(\mu \cos\theta - \sin\theta)}{(\mu \cos\theta + \sin\theta)^2} = 0$$

or $-\mu w(\mu \cos\theta - \sin\theta) = 0$

$\mu \cos\theta = \sin\theta$

$\mu = \tan\theta$

$\therefore \theta = \tan^{-1}(\mu)$.

11. An object at the end of a vertical spring is stretched 4cm beyond its rest position & released at time $t = 0$

It's position at time t is $s = f(t) = 4\cos t$. And the velocity & accln at time 't' & use them to analyze the motion of the object.

→ Here, $s = 4\cos t$

$$\frac{ds}{dt} = v = 4 \cdot -\sin t$$

$$\therefore v = -4\sin t$$

$$a = \frac{dv}{dt} = -4\cos t$$

Rolle's Theorem

Mean Value Theorem.

Statement: If a function $f(x)$ is
(a) Continious in $[a, b]$
(b) differentiable in (a, b)
(c) $f(a) = f(b)$. Then there exist at least one point c which belongs to (a, b) such that, $f'(c) = 0$.

$y = f(x)$

Since $f(x)$ is continous in $[a, b]$
$\therefore$ It is bounded in closed interval $[a, b]$
Suppose that m & M be least value & greatest value of $f(x)$ in $[a, b]$

* When $f(x)$ is constant function then

$$f(a) = f(b) = m = M$$

$$\therefore f'(x) = 0 \quad \forall x \in (a, b)$$

* When, $m \neq M$ then there exist a point c in an open interval (a, b) where, $f(c) = M$

$$f(c+h) < f(c)$$

i.e. $f(c+h) - f(c) < 0$

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \leq 0$$

$$Rf'(c) \leq 0 \quad — (i)$$

Similarly when, $f(c-h) < f(c)$

i.e. $f(c-h) - f(c) < 0$

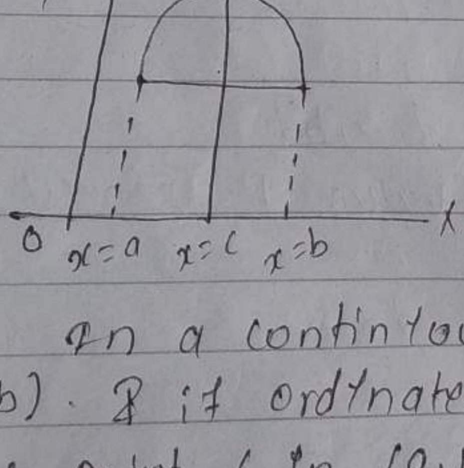
i.e. $\dfrac{f(c-h) - f(c)}{-h} > 0$

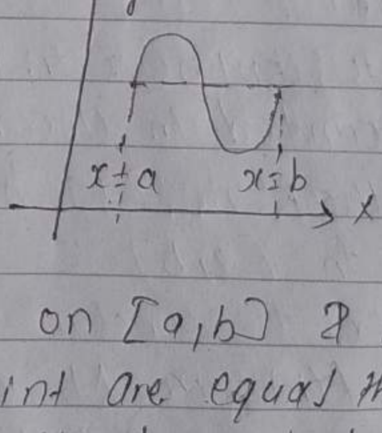
$$\lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} \geq 0$$

$$Lf'(c) \geq 0$$

$f(x)$ exist in (a,b) $\because f'(c) = Rf'(c) = Lf'(c)$

* Similarly when $f(c) = m$
we can prove that $f'(c) = 0$
Hence, in all cases $f'(c) = 0$.





In a continuous curve on $[a,b]$ & differentiable (a,b). & if ordinates end point are equal there is atleast one point c in (a,b) where the tangent is parallel to x-axis.

Implicit Differentiation : 45

Formula :→

- $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}\operatorname{cosec}^{-1}(x) = \frac{-1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{1-x^2}$

$\boxed{\text{Exercise : 3.5}}$

1. Write the composite function in the form of $f(g(x))$. [Identify the inner function $u = g(x)$ & the other function $y = f(u)$]. Then find the derivative dy/dx.

(a) $y = \sqrt{4+3x}$

Soln: $y = \sqrt{4+3x}$ as $f(g(x))^3$

$g(x) = 4+3x$

$f(u) = \sqrt{u}$

$f\{g(x)\} = \sqrt{4+3x}$

Now,

$y = \sqrt{4+3x}$

$$\frac{dy}{dx} = \frac{d}{dx}(4+3x)^{1/2} = 1/2\,(4+3x)^{-1/2} \times 3$$

$$= \frac{3}{2\sqrt{4+3x}}$$

(b) $y = \tan(\sin x)$

$y = \tan(\sin x)$

$g(x) = \sin x$

$f(u) = \tan u$

Hence, $f\{g(x)\} = \tan(\sin x)$

Now,

$$y = \tan(\sin x)$$

$$\frac{dy}{dx} = \frac{d}{dx}\tan(\sin x) = \sec^2 \cdot (\sin x) \cdot \cos x$$

$$= \cos x \cdot \sec^2(\sin x)$$

(c) Q. $f(x) = (4x - x^2)^{100}$

Given that,

$f(x) = (4x - x^2)^{100}$

So, $g(x) = 4x - x^2$

$f(u) = u^{100}$

Hence, $f\{g(x)\} = (4x - x^2)^{100}$

Now,

$$\frac{dy}{dx} = \frac{d}{dx}(4x - x^2)^{100} = 100(4x - x^2)^{99} \cdot (4 - 2x)$$

(d) $f(z) = \dfrac{1}{z^2+1}$

Give that,

$g(x) = 3$

$f(u) = \dfrac{1}{u^2+1}$

$$\frac{df}{dz} = \frac{d}{dz}\left(\frac{1}{z^2+1}\right) = \frac{-1}{(z^2+1)^2} \cdot 2z = \frac{-2z}{(z^2+1)^2}$$

(e)

Given that,

$y = \cos(a^3 + x^3)$

So, $g(x) = a^3 + x^3$; $f(u) = \cos u$

$$\frac{dy}{dx} = \frac{d\cos(a^3+x^3)}{dx} = -\sin(a^3+x^3)\cdot 3x^2$$
$$= -3x^2 (\sin(a^3+x^3))$$

(f) $y = a^3 + \cos^3 x$

Given that,

$y = a^3 + \cos^3 x$

$g(x) = x$

$f(u) = a^3 + \cos^3(u)$

$$\frac{dy}{dx} = \frac{d}{dx}(a^3 + \cos^3 x)$$
$$= 0 + 3\cos^2 x \cdot (-\sin x)$$
$$= -3\cos^2 x \cdot \sin x$$

(g) $h(t) = (t+1)^{2/3} \cdot (2t^2-1)^3$

So; ~~g(x)~~ Here,

$h(t) = (t+1)^{2/3} \cdot (2t^2-1)^3$

So, $g(x) = t$ & $f(u) = (u+1)^{2/3} \cdot (2u^2-1)^3$

$$\frac{dh}{dt} = \frac{d(t+1)^{2/3}}{dt} \cdot (2t^2-1)^3$$

$$\frac{dh}{dt} = (t+1)^{2/3} \cdot \frac{d}{dt}(2t^2-1)^3 + (2t^2-1)^3 \frac{d}{dt}(t+1)^{2/3}$$
$$= (t+1)^{2/3} \cdot 3(2t^2-1)^2 \cdot 4t + (2t^2-1)^3 \cdot \frac{2}{3}(t+1)^{-1/3}$$

(h) $y = \left(\frac{x^2+1}{x^2-1}\right)^3$

So, $g(x) = \left(\frac{x^2+1}{x^2-1}\right)$ & $f(u) = u^3$

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$$\frac{dy}{dx} = 3\left(\frac{x^2+1}{x^2-1}\right)^2 \left[\frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}\right]$$

$$\frac{dy}{dx} = 3\left(\frac{x^2+1}{x^2-1}\right)^2 \frac{(x^2-1)2x - (x^2+1)\cdot 2x}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{3(x^2+1)^2\cdot(-4x)}{(x^2-1)^4}$$

$$= \frac{-12x(x^2+1)^2}{(x^2-1)^4}$$

(9) $\frac{e^u - e^{-u}}{e^u + e^{-u}}$

$$\frac{dy}{du} = \frac{d}{du}\left(\frac{e^u - e^{-u}}{e^u + e^{-u}}\right)$$

$$= \frac{(e^u + e^{-u})\frac{d}{du}(e^u - e^{-u}) - (e^u - e^{-u})\frac{d}{du}(e^u + e^{-u})}{(e^u + e^{-u})^2}$$

$$= \frac{(e^u + e^{-u})(e^u + e^{-u}) - (e^u - e^{-u})(e^u - e^{-u})}{(e^u + e^{-u})^2}$$

$$= \frac{4e^u - e^{-u}}{(e^u + e^{-u})^2} \qquad = \frac{4}{(e^u + e^{-u})^2}$$

(2) Find an equation of the tangent line to the curve $y = 2/(1+e^{-x})$ at the point $(0,1)$

Soln Given eqn of curve;

$$y = \frac{2}{(1+e^{-x})} \quad \text{--- (i)}$$

Req^d eq^n of tangent to the curve (i) at point (0,1) be.

$$y-1 = \left(\frac{dy}{dx}\right)\cdot(x-0) \quad \text{— (ii)}$$

Diff. Eq^n (i) w.r.t x we get;

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2}{1+e^{-x}}\right) = 2\frac{d}{dx}(1+e^{-x})^{-1}$$

$$= -2(1+e^{-x})^{-2}\cdot -e^{-x}$$

$$= \frac{2e^{x}}{(1+e^{-x})^{2}}$$

Now, $\left(\frac{dy}{dx}\right)_{(0,1)} = \frac{-2}{(1+1)^2} = 1/2$

Then, eq^n (ii) becomes

$y-1 = 1/2\,(x)$

$y = x/2 + 1$ which is the required eq^n of the tange at point (0,1) of the curve (i).

(3) Find the x-co-ordinates of all points on the cur $y = \sin 2x - 2\sin x$ at which the tangent line is horizontal.

→ Soln: Given equation of the curve:

$$y = \sin 2x - 2\sin x \quad \text{— (i)}$$

Differentiating eq^n (i) w.r.t. 'x'

$$\frac{dy}{dx} = 2\cos 2x - 2\cos x$$

Now, the tangent is horizontal to the curve

i.e. $\frac{dy}{dx} = 0$

$2\cos 2x - 2\cos x = 0$

or, $\cos 2x - \cos x = 0$

or, $2\cos^2 x - 1 - \cos x = 0$

or, $(\cos x - 1)(2\cos x + 1) = 0$

Either, $\cos x = 1$ or, $2\cos x = -1/2$

$x = 2n\pi$ $\cos x = \cos(\pi + \pi/3)$

$x = \frac{2\pi}{3} + 2n\pi$ & $\pi/3 + 2n\pi$

Reqd x-coordinate are, $2\pi n$ or, $\frac{2\pi}{3} + 2\pi n$ or, $\frac{\pi}{3} + 2n\pi$

(4) If $f(x) = f(gx)$, where $f(-2) = 8$, $F'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, & $g'(5) = 6$, find $f'(5)$.

Soln Given that:

$f(x) = f(g(x))$

& $f(-2) = 8$, $F'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$ & $g'(5) = 6$

$F'(5) = ?$ we have,

$f(x) = f(g(x)$ Diff. w.r. to x we get,

$f'(x) = f' g(x)$, $g'(x)$ of $x = 5$

$f'(5) = f'(g(5)\ g'(5)$

$= f'(-2) \cdot 6$

$= 4 \times 6$

$f'(5) = 24$

(5) (a) Find y' by implict differentiation.

(b) Solve the eqn explicity for y & differentiable to get y' in terms of x.

(c) Check that your solutions to parts (a) & (b) are consistent by substituting the expression for y into your solution for part (a).

(a) → (i) $xy + 2x + 3x^2 = 4$

soln: Given eqn

$$xy + 2x + 3x^2 = 4$$

Differentiating w.r.t. (x) we get,

$$x \cdot \frac{dy}{dx} + y + 2 + 6x$$

$$\frac{dy}{dx} = \frac{-y-2-6x}{x} \quad \text{— (i)}$$

→ $xy + 2x + 3x^2 = 4$

$$y = \frac{4-3x^2-2x}{x} \quad \text{— (ii)}$$

Diff. w.r.to. (x)

$$\frac{dy}{dx} = \frac{x\frac{d}{dx}(4-3x^2-2x) - (4-3x^2-2x)\cdot\frac{d}{dx}}{x^2}$$

$$= \frac{x(-6x-2) - 4 + 3x^2 + 2x}{x^2}$$

$$= \frac{-6x^2 - 2x - 4 + 3x^2 + 2x}{x^2}$$

$$= \frac{-4-3x^2}{x^2}$$

→ Substituting eqn (ii) in (i);

$$\frac{dy}{dx} = \frac{\frac{-4+3x^2+2x}{x} - 2 - 6x}{x}$$

$$= \frac{-4+3x^2+2x-2x-6x^2}{x^2} = \frac{-4-3x^2}{x^2} \text{ (Consistent)}$$

5(i) ⇒ Soln $\frac{1}{x} + \frac{1}{y} = 1$

Diff w.r.t (x) we get, $\frac{-1}{x^2} - \frac{1}{y^2}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-y^2}{x^2} \quad \text{— (i)}$$

(b) Soln: $\frac{1}{x} + \frac{1}{y} = 1$

$1/y = \frac{x-1}{x}$

$y = \frac{x}{x-1}$ — (ii)

$$\frac{dy}{dx} = \frac{(x-1)\frac{dx}{dx} - x\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{(x-1)-x}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

(c) Soln;

Substituting eqn (ii) in (i)

$$\frac{dy}{dx} = -\frac{\left(\frac{x}{x-1}\right)^2}{x^2} = \frac{-x^2}{(x-1)^2 - x^2} = \frac{-1}{(x-1)^2} \text{ Consistent.}$$

5(iii) Soln:

Given eqn: $\cos x + \sqrt{y} = 5$

Diff. w.r.t x, we get;

$$-\sin x + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\sqrt{2}y\,(\sin x) \quad \text{— (i)}$$

b) 5

Soln: $\cos x + \sqrt{y} = 5$

$\sqrt{y} = 5 - \cos x$

Sq. both sides; $y = 25 - 10\cos x + \cos^2 x$ — (i)

Diff. w.r.to 'x'

$$\frac{dy}{dx} = 10\sin x - 2\cos x \cdot \sin x$$
$$= 2\sin(5 - \cos x).$$

c)

Soln, Substituting (ii) in (i)

$$\frac{dy}{dx} = 2\sqrt{(5-\cos x)^2}\ \sin x$$
$$= 2(5-\cos x)(\sin x)$$
$$= 2\sin x(5-\cos x).$$

(6) Find $\frac{dy}{dx}$ by implicit differentiation.

(a) $2x^3 + x^2y - xy^3 = 2$

Diff. w.r.t. x

$$6x^2 + 2xy + x^2\frac{dy}{dx} - 2xy\frac{dy}{dx} - y^3 = 0$$

$$\frac{dy}{dx} = \frac{y^3 - 6x^2 + 2xy}{x^2 - 3xy^2}$$

(b)

Here, $xe^y = x - y$;

diff w.r.to. 'x'.

$$e^y\frac{dx}{dx} + x\frac{de^y}{dx} = \frac{dx}{dx} - \frac{dy}{dx}$$

$$e^y + xe^y \cdot \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$(xe^y + 1)\frac{dy}{dx} = 1 - e^y$$

$$\frac{dy}{dx} = \frac{1 - e^y}{xe^y + 1}$$

(c) $x^2y^2 + x\sin y = 4$

Soln: $x^2y^2 + x\sin y = 4$

Diff w.r.t. x

$$2xy^2 + 2x^2y \cdot \frac{dy}{dx} + \sin y + x\cos y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2xy^2 - \sin x}{2x^2y + x\cos y}$$

(d) $4\cos x \cdot \sin y = 1$

Soln: $4\cos x \sin y = 1$

diff w.r.t. (x)

$$4\cos x \cdot \cos y \frac{dy}{dx} - 4\sin y \cdot \sin x = 0$$

$$\frac{dy}{dx} = \frac{\sin x \cdot \sin y}{\cos x \cdot \cos y} = \tan x \cdot \tan y.$$

(e) $x\sin y + y\sin x = 1$

$$x\cos y \cdot \frac{dy}{dx} + \sin y \frac{dx}{dx} + \sin x \frac{dy}{dx} + y\cos x = 0$$

$$(x\cos y + \sin x)\frac{dy}{dx} = -y\cos x - \sin y$$

$$\frac{dy}{dx} = \frac{-y\cos x - \sin y}{x\cos y + \sin x}$$

(i) $\tan(x-y) = \dfrac{y}{1+x^2}$

Diff w.r. to. x

$$\sec^2(x-y) - \sec^2(x-y)\frac{dy}{dx} = \frac{(1+x^2)\frac{dy}{dx} - 2xy}{(1+x^2)^2}$$

$$(1+x^2)^2\sec^2(x-y) - (1+x^2)^2\sec^2(x-y)\frac{dy}{dx} = (1+x^2)\frac{dy}{dx} - 2xy$$

$$(1+x^2)^2\sec^2(x-y) + 2xy = [(1+x^2) + (1+x^2)^2\sec^2(x-y)]\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{(1+x^2)^2\sec^2(x-y) + 2xy}{(1+x^2) + (1+x^2)^2\sec^2(x-y)}$$

(9) Find the derivative of function simplify where possible.

(a) $y = \tan^{-1}(x^2)$

Diff. w.r.t 'x'

$$\frac{dy}{dx} = \frac{1}{1+x^4}\cdot 2x \quad = \frac{2x}{1+x^4}$$

(b) $g(x) = \sqrt{x^2-1}\,\sec^{-1}x$

$$g'(x) = \sqrt{x^2-1}\cdot\frac{1}{x\sqrt{x^2-1}} + \sec^{-1}x\left(\frac{1}{2\sqrt{x^2-1}}\cdot 2x\right)$$

$$= \frac{1}{x} + \frac{x\sec^{-1}x}{\sqrt{x^2-1}}$$

(i) $y = \sin^{-1}(2x+1)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(2x+1)^2}} \times 2$$

$$= \frac{1}{\sqrt{-x^2-x}}$$

(ii) $y = \tan^{-1}(x - \sqrt{1+x^2})$

put $x = \tan\theta$ —— (i) then,

$$y = \tan^{-1}(\tan\theta - \sqrt{1+\tan^2\theta})$$

$$= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sin\theta - 1}{\cos\theta}\right)$$

$$= \tan^{-1}\left[\frac{2\sin\theta/2 \cdot \cos\theta/2 - (\sin^2\theta/2 + \cos^2\theta/2)}{\cos^2\theta/2 - \sin^2\theta/2}\right]$$

$$= \tan^{-1}\left(\frac{-(\sin^2\theta/2 - 2\sin\theta/2 \cdot \cos\theta/2 + \cos^2\theta/2)}{\cos^2\theta/2 - \sin^2\theta/2}\right)$$

$$= -\tan^{-1}\left(\frac{-(\sin\theta/2 - \cos\theta/2)^2}{(\cos\theta/2 + \sin\theta/2)(\cos\theta/2 - \sin\theta/2)}\right)$$

$$= -\tan^{-1}\left(\frac{1 - \cot\theta/2}{1 + \cot\theta/2}\right)$$

$$= -\cot^{-1}\left(\frac{1 + \cot\theta/2}{1 - \cot\theta/2}\right)$$

$$= -\cot^{-1}\left(\frac{1 + \cot\pi/4 \cdot \cot\theta/2}{1 - \cot\pi/4 \cdot \cot\theta/2}\right)$$

$$= \cot^{-1}\cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$y = \frac{\pi}{4} + \frac{\theta}{2} \quad \text{—— (ii)}$$

Diff ① w.r.t. θ

$$\frac{dy}{d\theta} = \frac{1}{2}$$

Then, $\frac{dy}{dx} = \frac{1/2}{(1+x^2)} = \frac{1}{2(1+x^2)}$

⑥ $F(\theta) = \sin^{-1}\sqrt{\sin\theta}$

$$f'(\theta) = \frac{1}{\sqrt{1-\sin\theta}} \cdot \frac{1}{2\sqrt{\sin\theta}} \cdot \cos\theta$$

$$= \frac{\cos\theta}{2\sqrt{\sin\theta \cdot \sin 2\theta}}$$

⑦ $y = x\sin^{-1}x + \sqrt{1-x^2}$

$$\frac{dy}{dx} = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \cdot -2x$$

$$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$= \sin^{-1}x$$

⑧ $y = \tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right.$

Here, $\frac{dy}{dx} = \frac{1}{1+\frac{1-x}{1+x}} \frac{\left[\sqrt{1+x} \cdot \frac{d}{dx}\sqrt{1-x} - \sqrt{1-x}\frac{d}{dx}\sqrt{1+x}\right]}{1+x}$

$$= \frac{1}{1+\frac{1-x}{1+x}} \frac{\left[\sqrt{1+x}\ \frac{-1}{2\sqrt{1-x}} - \sqrt{1-x} \cdot \frac{1}{2\sqrt{1+x}}\right]}{1+x}$$

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$$= \frac{1}{2}\left[\frac{-(\sqrt{1+x})^2 - (\sqrt{1-x})^2}{2\sqrt{1-x^2}}\right]$$

$$= \frac{1}{2} \frac{-1}{\sqrt{1-x^2}} \qquad \therefore \frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}} \quad -1 < x < 1$$

(10) Soln: $x^2 + xy + y^2 + 1 = 0$

Diff w.r.t. 'x'

$$2x + y + x\frac{dy}{dx} + 2y \cdot \frac{dy}{dx} + 0 = 0$$

$$2x + y + (x + 2y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x - y}{(x+2y)}$$

(11) Find equations of both the tangent lines to the ellipse $x^2 + 4y^2 = 36$ that pass through point $(12, 3)$.

Soln: Eqn of ellipse is $x^2 + 4y^2 = 36$ — (i)

Let (a, b) be the point on the tangent line & curve.

So, Eqn (i) becomes

$$a^2 + 4b^2 = 36 \quad \text{— (ii)}$$

Diff. Eqn (i) w.r.t. x

$$2x + 8y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{4y}$$

Slope of the tangent line is now, $-a/4b$

Thus, Eqn of tangent is $y - 3 = \frac{-a}{4b}(x - 12)$ — (iii)

Further, since (x, y) is also in the line of tangent the eqn becomes now replace (x, y) with (a, b)

$$b-3 = \frac{-a}{4b}(a-12)$$

$$4b^2 = -a^2 + 12a + 12b$$

or, $4b^2 + a^2 = 12a + 12b$

or, $36 = 12(a+b)$

or, $a+b = 3$

or, $b = 3-a$

Again eqn (i) becomes,

$$a^2 + 4(3-a)^2 = 36$$

or, $a^2 + 4(9 - 6a + a^2) = 36$

or, $a^2 + 36 - 24a + 4a^2 = 36$

or, $5a^2 - 24a = 0$

$a(5a-24) = 0$

$\therefore a = 0$ or, $a = 24/5$

When, $a = 0 \Rightarrow b = 3$

When, $a = \frac{24}{5} \Rightarrow b = -9/5$

Then, eqn (ii) becomes with $(0, 3)$

$$y-3 = \frac{0}{4 \cdot 3}(x-12)$$

$$y-3 = 0 \Rightarrow y = 3 \quad \text{— (iv)}$$

And eqn (ii) becomes with $\left(\frac{24}{5}, \frac{-9}{5}\right)$

$$y-3 = \frac{-24/5}{4 \times -9/5}(x-12)$$

or, $y-3 = \frac{24}{5} \times \frac{5}{36}(x-12)$

or, $3y - 9 = 2x - 24$

or, $3y - 2x + 15 = 0$ —— (v)

Thus, eqⁿ (i) & (ii) are equation of tangent to the ellipse (i) from point $(12,3)$.

Q No 12 Differentiate the function

(a) $f(x) = x\ln(x) - x$

$f'(x) = x \times \frac{1}{x} + \ln x \times 1 - 1$

$f'(x) = 1 + \ln(x-1) = \ln x$

(b) $f(x) = \sin(\ln(x))$

$f'(x) = \cos(\ln x) \times 1/x \times 1$

$f'(x) = \cos(\ln x) \times \frac{1}{x} \times 1$

$f'(x) = \cos\left(\frac{\ln x}{x}\right)$

(c) $f(x) = \ln(\sin^2 x)$

$f'(x) = \frac{1}{\sin^2 x} \cdot 2\sin x \cdot \cos x$

$= \frac{2\cos x}{\sin x} = 2\cot x.$

(d) $f(x) = \sin x \ln(5x)$

$f'(x) = \cos x \cdot \ln(5x) + \sin x \times \frac{1}{5x} \times 5$

$= \cos x \cdot \ln(5x) + \frac{\sin x}{2}$

Q.No: 13 Find an equation of the tangent line to the curve at the given point.

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(a) $y = \ln(x^2 - 3x + 1)$, $(3, 0)$

Soln: Given, eqn is

$$y = \ln(x^2 - 3x + 1) \quad \text{—(i)}$$

Diff (i) w.r.t. x

$$\frac{dy}{dx} = \frac{1}{x^2 - 3x + 1} \cdot (2x - 3) = \frac{2x - 3}{x^2 - 2x + 1}$$

Gradient at point (3,0) of the tangent to be determined will be.

$$\frac{dy}{dx}_{(3,0)} = \frac{6 - 3}{9 - 9 + 1} = 3$$

Now,

Eqn of the slope - point formula which is eqn of the tangent to eqn (i) is.

$$y - y_1 = 3(x - x_1)$$

or, $y - 0 = 3(x - 3)$

or, $y = 3x - 9$

or, $3x - y - 9 = 0$ is the reqd eqn of tangent.

(b) ~~Soln~~ $y = x^2 \ln x$ —(i)

Soln: $\frac{dy}{dx} = 2x \ln x + \frac{x^2}{x} = 2x \ln x + x$

Gradient at point (1, 0) of the tangent to be determined will be,

$$\frac{dy}{dx}_{(1,0)} = 2 \ln 1 + 1 = 1$$

Now, Eqn of slope - point formula which is eqn of the tangent to eqn (i) is; $y - 0 = 1(x - 1)$

$x - y - 1 = 0$ is reqd eqn.

(1) Find an equation of the tangent line to the hyperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_0, y_0)

Soln → Given, eqn of hyperbola is;

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{—— (i)}$$

Diff (i) w.r.t x we get;

$$\frac{2x}{a^2} - \frac{2y}{b^2} - \frac{dy}{dx} = 0$$

Now, we can find $\left(\frac{dy}{dx}\right)$ at our point (x_0, y_0). that gives the slope of the tangent line & now we can find the eqn of the tangent line;

Now, eqn of two point formula which is eqn of tangent to (i) is;

$$y - y_0 = \frac{x_0 b^2}{y_0 a^2}(x - x_0)$$

or, $(y - y_0) y_0 a^2 = x_0 b^2 (x - x_0)$

or, $x x_0 b^2 - y y_0 a^2 = x_0^2 b^2 - y_0^2 a^2$

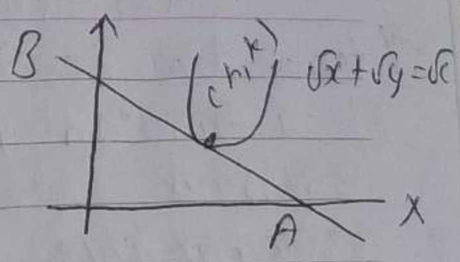
Divide throug $a^2 b^2$

$$\frac{x x_0}{a^2} - \frac{y y_0}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}$$

$$\frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1 \quad \text{R.H.S is same as eqn (i)}$$

Proved

(8)



Soln:

Let (h,k) be the point of contact of tangent line & the curve,

Given curve, $\sqrt{x}+\sqrt{y}=\sqrt{c}$ — (i)

Diff. (i) w.r.t. 'x'

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-\sqrt{y}}{\sqrt{x}}$$

Slope of tangent at (h,k) is

$$\frac{dy}{dx}=\frac{-\sqrt{k}}{\sqrt{h}}$$

Now, Eqn. of tangent at (h,k) is

$$y-k=\frac{\sqrt{k}}{\sqrt{h}}(x-h)$$

or, $\sqrt{h}(y-k)=\sqrt{x}(x-h)$

or, $\sqrt{12x}+\sqrt{h}y=\sqrt{kh}(\sqrt{h}+\sqrt{k})$

So, X-intercept $=\sqrt{h}(\sqrt{h}+\sqrt{k})$ when $y=0$

Y-intercept $=\sqrt{k}(\sqrt{h}+\sqrt{k})$ when $x=0$

And, Sum of intercepts $=(\sqrt{h}+\sqrt{k})(\sqrt{h}+\sqrt{k})$

$=\sqrt{c}\cdot\sqrt{c}$

$=c$ #

Exercise 3·6

Mean Value Theorem

If a function $f(x)$ is

1. Continous in $[a,b]$
2. Differentiable in (a,b)
3. $f(a)=f(b)$, there exist at least one value c belongs to (a,b) such that $f'(c)=0$.

① Verify that the function satisfies the three hypothesis of Rolle's theorum on the given interval. Then find all the numbers c that satisfy the conclusion of Rolle's Theorem.

(a) $f(x)=5-12x+3x^2$, in $[1,3]$

Since, $\lim_{x\to1^+} f(x)=f(1)$ & $\lim_{x\to3^-} f(x)=f(3)$

. Thus $f(x)$ is continous at end points & all interior points of $[1,3)$

i) $f(x)$ is continious at the point $[1,3]$

ii) $f'(x)=-12+6x=6x-12$ exist in $(1,3)$,
So, $f(x)$ is differentiable in $(1,3)$

iii) $f(1)=f(3)$

Thus, all three condition are holds on $f(x)$. Hence, $\exists$ point $c\in(1,3)$ such that, $f'(c)=0$

or, $6c-12=0$

$\therefore c=3$

(b) $f(x)=x^3-x^2-6x+2$, in $[0,3]$

Since, $\lim_{x\to0^+} f(x)=f(0)$ & $\lim_{x\to3^-} f(x)=f(3)$

Thus, $f(x)$ is continious at end point & all interior points of $[0,3]$

i) $f(x)$ is continuous in $[0,3]$

ii) $f'(x) = 3x^2 - 2x - 6$, exist in $(0,3)$, so $f(x)$ is differentiable.

iii) $f(0) = f(3) = 0$.

Thus, all three condition are holds on $f(x)$.

Hence, $\exists$ a point $c \in (-1,1)$ such that

$f'(c) = 0$ $\quad 3c^2 - 2c - 6 = 0$

$$c = \frac{6}{3c-2}$$

⑥ $f(x) = \cos 2x$, in $[\pi/8, 7\pi/8]$

Since,

$\lim_{x \to \pi/8^+} f(x) = f(\pi/8)$ & $\lim_{x \to 7\pi/8^-} f(x) = f\left(\frac{7\pi}{8}\right)$

i) $f(x)$ is not continuous at all interior point of $\left[\frac{\pi}{8}, \frac{7\pi}{8}\right]$

ii) $f'(x) = -\frac{\sin 2x}{2}$

It is not differentiable so, it is not continuous,

$f'(c) = 0$

$-\frac{\sin 2c}{2} = 0$ or, $\sin 2c = \sin 0$ or, $2c = \pi$ or, $c = \pi/2$

Roll's theorem verified #

2) Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1,1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem.

Since

$\lim_{x \to -1^+} f(x) = -f(x)$ & $\lim_{x \to 1^-} f(x) = f(1)$

$f(x)$ is not continious, when we put the value of x in $f(x)$. Both value is not continious at that point. And $f'(x)$ is also not exist at the point $(-1,1)$. So, it should not verified the Rolle's theorem.

4. Verify that the function satisfies the three hypothesis the mean value theorem on the given interval. Then find all the numbers c that satisfy the conclusion of the data.

ⓐ $f(x) = 2x^2 - 3x + 1$, in $[0, 2]$

Since, $f(x) = 2x^2 - 3x + 1$ is continious on $[0, 2]$ & $f'(x) = 4x - 3$ so, differentiable on $(0, 2)$, $f(x) = 2x^2 - 3x$ satisfy the value of both condition for mean value theorem so, there exist $c \in (0, 2)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$4c - 3 = \frac{f(0) - f(2)}{2 - 0}$$

$$4c - 3 = \frac{1 - 3}{2}$$

$c = 1/2$ Hence, mean value is $c = 1/2$

ⓑ $f(x) = x^3 + x - 1$ $[0, 2]$

Since, $f(x) = x^3 + x - 1$ is continious on $[0, 2]$ & $f'(x) = 3x^2 + 1$ on differentiable on $[0, 2]$ & $f(x) = x^3 + x - 1$ satisfy the both condition for mean value theorem.

So there $c \in (0, 2)$ such that,

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$3c^2 + 1 = \frac{f(2) - f(0)}{2-0} \quad \& \quad 3c^2 + 1 = -5$$

$$3c^2 = 6$$

$$3c^2 + 1 = \frac{9+1}{2} \qquad c = \pm\sqrt{2}$$

$$c = \pm 2/\sqrt{3} \ \#$$

(4) $f(x) = \dfrac{x}{x+2}$ in $[1, 4]$

Since, $f(x) = \dfrac{x}{x+2}$ is continuous on $[1, 4]$ & $f'(x)$ $f'(x) = \dfrac{x+2-x}{(x+2)^2} = \dfrac{2}{(x+2)^2}$, so, differentiable on, $[1, 4]$, Thus $f(x) = \dfrac{x}{x+2}$. Satisfy the both condition for mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b-a} \qquad \frac{2}{(c+2)^2} = \frac{\frac{2}{3} - \frac{1}{3}}{4-1}$$

$$\frac{2}{(c+2)^2} = \frac{1/3}{3/1}$$

$$\frac{c}{c+2} = \frac{1}{9}$$

$$c^2 + 2c + 4 = 18$$

$$c^2 + 4c - 14 = 0$$

(5) Let $f(x) = (x-3)^{-2}$. Show that there is no value of c in $(1, 4)$, such that $f(4) - f(1) = f'(c)$ $(4, -1)$. Why does this not contradict the mean value theorem?

available

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$\rightarrow \frac{f(4)-f(1)}{3} = f'(c)$

$f(a) = (1)^{-2} = 1$

$f(1) = (1-3)^{-2}$

$= \frac{1}{2^2} = \frac{1}{4}$

$f(1) = 1/4$, $\lim x \to 1$

$f(x) = (4-3)^{-2} = 1^{-2} = 1$

$\lim_{x\to 4} f(x) = (4-3)^2 = 1$

$\because f(x) = f(4)$

$\because f(x)$ is continuous on $[1,4]$

Now, $f'(x) = \frac{-2}{(x-3)^2} = \frac{-2}{(x-3)^3}$

$\because f'(c) = \frac{f(4)-f(1)}{4-1}$ or, $\frac{1-1/4}{3}$ or, $\frac{-2}{(c-3)^3} = \frac{\cancel{3}}{\cancel{3}\cdot 4}$

or, $\frac{-2}{(c-3)^2} = \frac{1}{4}$

$\because (c-3)^{-3} = 1/8$

$(c-3)^{-3} = (-2)^{-3}$

$c-3 = 2$

$c = 1$

$\boxed{\text{Q No 6}}$ Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2,2]$

$\rightarrow f'(x) = 3x^2 - 15$

At $f(x) = 0$ $\quad x^3 - 15x + c = 0$

At point $[-2,2]$

$f(-2) = -8 + 30 + c = 0$

$f(-2) = c + 22$

$f(2) = -22 + c$

$c \in (-2,2)$

$f'(c) = \frac{f(2)-f(-2)}{2+2}$

$= \frac{-22+c-22+c}{4}$

$= \frac{-44}{4} = -11$

$3c^2 - 15 = 0$

$3c^2 = -11 + 15$

$c = \pm\frac{2}{\sqrt{3}}$

Q NO: 7

If $f(1) = 10$ & $f'(x) = 2$ for $1 \le x \le 4$, how small can $f(4)$ possibly be?

Ans Soln;

$f(1) = 10$ Since, the function is continious &
$f'(x) = 2$ differentiable, so it satisfy the mean value theorem.

$\therefore f'(c) = \frac{f(4) - f(1)}{4-1}$ or, $2 = \frac{f(4) - 10}{3}$ or, $6 = f(4) - 10$

$f(4) = 16$

$\therefore f(4)$ will be 16.

Q NO: 8

Use the mean value theorem to prove the inequality $(\sin a - \sin b) \le |a-b|$. for all a & b.

Let us suppose $f(x) = \sin x$

$f'(x) = \cos x$

By using mean value theory $f'(c) = \frac{\sin b - \sin a}{b-a}$

Also,

$\cos c \le 1$

$-1 \le \cos c \le 1$ $\therefore \frac{\sin b - \sin a}{b-a} \le 1$

$|\cos c| \le 1$

$\sin a - \sin b \le a-b$ proved

9) Use the method of Example 3 to prove the identity $2\sin^{-1}x = \cos^{-1}(1-2x^2)$ for $x \geq 0$

Indeterminant forms & L'Hospital

If a $f(x)$ $f(x) \longrightarrow = \frac{\phi(x)}{\psi(x)}$ & $\lim_{x \to a} \phi(x) = \lim_{x \to a} \psi(x) = 0$

This $f(x)$ is said to be indeterminant form at $x = a$ other indeterminant form an $0 \times \infty$, $\frac{\infty}{\infty}$, $\infty - \infty$, 0^0, 0^∞, ∞^∞, 1^∞ etc.

L-Hospital rule,

$$\lim_{x \to 4} \frac{f(x)}{g(x)} \left(\text{for } \frac{0}{0}\right) = \lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ if exist}$$

Exercise 3.7

1. Find the limit. Use L' Hospital Rule where appropriate. If there is a more elementary method consider using it. If L'Hospital Rule does not apply, explain why.

(a) $\lim_{x \to 1} \frac{x^2 - 1}{x^2 - x}$ $\left(\frac{0}{0} \text{ form}\right)$

By using L hospital rule,

$$\lim_{x \to 1} \frac{2x}{2x-1} = \frac{2}{2 \times 1 - 1} = \frac{2}{1} = 2$$

(b) $\lim_{x \to (\pi/2)^+} \frac{\cos x}{1 - \sin x}$

By using L.H. Rule.

$$\lim_{x \to (\pi/2)^+} \frac{\sin x}{\cos x}$$

(c) $\lim_{x \to (\pi/2)^+} = \tan x$

$= \tan 90'$

$= \infty$

(d) $\lim_{\theta \to \pi/2} \frac{1 - \sin\theta}{1 + \cos 2\theta}$

$= \frac{1}{4 \sin\theta}$

$= 1/4$

(e) $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$

$$\lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{\sqrt{x}}} = \frac{1}{x} \times \frac{1}{\sqrt{x}} = \frac{1}{\infty}$$

$= 0$

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f) $\boxed{\lim_{x\to 0} \frac{\ln x}{x}}$

$\lim_{x\to 0^+} \frac{1/x}{x}$

$\lim_{x\to 0^+} = \frac{1}{x^2} = \frac{1}{0} = \infty$

g) $\boxed{\lim_{x\to\infty} \frac{\ln\sqrt{x}}{x^2}}$

$\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}}{\frac{x^2}{1}}$

$\lim_{x\to\infty} \frac{1}{\sqrt{x}} \times 2x$

$\lim_{x\to\infty} \frac{2\sqrt{x}\times\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}}$

$\Rightarrow 2\times\infty = \infty$

h) $\boxed{\lim_{x\to 0} \frac{\tanh x}{\tan x}}$

$\lim_{x\to 0} \frac{\frac{\sinh x}{\cosh x}}{\frac{\sin x}{\cos x}}$

$\lim_{x\to 0} \frac{\sinh x \cdot \cos x}{\cosh x \cdot \sin x}$: $\left(\frac{0}{0}\right)$ form.

$\frac{h\cos\cancel{hx} \cdot \sin\cancel{x}}{h\sin\cancel{x} \cdot \cos\cancel{x}}$

$= 1$

i) $\boxed{\lim_{x\to 0} \frac{\sin^{-1} x}{x}}$

$\lim_{x\to 0} \frac{\frac{1}{1-x^2}}{x}$

$\lim_{x\to 0} \frac{x}{1-x^2}$

$\lim_{x\to 0} \frac{1-x^2 \cdot 1 - x \cdot -2x}{(1-x^2)^2}$

$\lim_{x\to 0} \frac{-2x^2 - 1 + x^2}{(1-x^2)^2}$

$\lim_{x\to 0} \frac{-x^2-1}{(1-x^2)^2} = \frac{-1}{1} =$

j) $\boxed{\lim_{x\to\infty} x\sin(\pi/x)}$

$\lim_{x\to\infty} \cos\frac{\pi}{x}\cdot\pi + \sin\pi/x$

$0 + \cos\pi/\infty \times \pi$

$0 + \cos 0 \times \pi$

$0 + 1\times\pi$

π

(k) $\lim_{x \to \infty} \sqrt{x}\, e^{-x/2}$

$\lim_{x \to \infty} \frac{x^{1/2}}{e^{x/2}}$

$\lim_{x \to \infty} \frac{1}{e^{x/2} \cdot x^{1/2}}$

$= 1/\infty = 0$

(m) $\lim_{x \to 0^+} \sin x \ln x$

$\lim_{x \to 0^+} \frac{\sin x}{\frac{1}{\ln x}}$

$\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{\sin x}}$

$= \frac{1/x}{1/\cos x}$

$= \frac{1}{x} \times \cos x$

$\lim_{x \to 0^+} \frac{\cos x}{x}$

$\lim_{x \to 0} = \frac{\sin x}{1}$

$= \sin 0$

$= 0$

(l) $\lim_{x \to 0} \cot 2x \sin 6x$

$\lim_{x \to 0} \frac{\sin 6x}{\tan 2x}$ $\left(\frac{0}{0}\right)$ form

$\lim_{x \to 0} \frac{\cos 6x \cdot \cancel{6}\,3}{\sec^2 2x \cdot \cancel{2}}$

$= \frac{3 \times 1}{\frac{1}{\cos^2 2 \times 0}} = \frac{3 \times 1}{1/1} = 3 \times 1/3$

(n) $\lim_{x \to \infty} x^3 e^{-x^2}$

$\lim_{x \to \infty} \frac{x^3}{e^{x^2}}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$\lim_{x \to \infty} \frac{3x^2}{e^{x^2} \cdot 2x}$

$\lim_{x \to \infty} \frac{3}{e^{x^2} \cdot 2x}$

$= \frac{3}{8}$

$= 0$

(o) $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

$\lim_{x \to 1} \frac{x \ln x - x + 1}{(x-1) \ln x}$

$\lim_{x \to 1} \frac{x \cdot 1/x + \ln x - 1}{\frac{(x-1)}{x} + \ln x}$

$\lim_{x \to 1} \frac{\ln x}{\frac{x-1}{x} + \ln x}$

$$\lim_{x\to 1} \frac{\frac{1}{x}\cdot x}{1+x\cdot\frac{1}{x}\cdot \ln x}$$

$$\lim_{x\to 1} \frac{1}{2+\ln x}$$

$$=\frac{1}{2}+\ln 1$$

$$=\frac{1}{2+0}=\frac{1}{2}$$

2) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

$$\lim_{x\to 0^+}\frac{e^x-1-x}{x\cdot(e^x-1)}\left(\frac{0}{0}\text{ form}\right)$$

$$\lim_{x\to 0^+}\frac{e^x-1}{xe^x+e^x-1}$$

$$\lim_{x\to 0^+}\frac{e^x}{xe^x+e^x+e^x}$$

$$\frac{e^0}{e^0+e^0}$$

$$=\frac{1}{1+1}=\frac{1}{2} \text{ Ans}$$

5) $\lim_{x\to\infty}(x-\ln x)$

P) $\lim_{x\to 0}(\operatorname{cosec}x-\cot x)$

$$\lim_{x\to 0}\frac{1}{\sin x}-\frac{\cos x}{\sin x}$$

$$\lim_{x\to 0}\frac{1-\cos x}{\sin x}\left(\frac{0}{0}\right)\text{ form}$$

$$\lim_{x\to 0}\frac{\sin x}{\cos x}=\tan x$$

$$=\tan 0=0.$$

r) $\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)$

$$\lim_{x\to 0}\left(\frac{1}{\tan x}-\frac{1}{x}\right)$$

$$\lim_{x\to 0}\frac{x-\tan x}{x(\tan x)}\left(\frac{0}{0}\text{ form}\right.$$

$$\lim_{x\to 0}\frac{1-\sec^2 x}{\tan x+x\sec^2 x}$$

$$\lim_{x\to 0}\frac{-\sec^2 x}{\sec^2 x+x2\sec x+\sec^2}$$

$$\lim_{x\to 0}\frac{-2\sec x}{2\sec^2 x+2\sec x+2x\sec}$$

$$\therefore=\frac{-1}{\sec x+1+x\tan x}$$

$$=\frac{1}{\sec x+1+x\tan x}$$

$$=\frac{1}{1+1}$$

$$=\frac{1}{2}$$

i) $\lim_{x \to 0^+} x^{\sqrt{x}}$

$\lim_{x\to 0^+} \ln y = \sqrt{x} \log x$

Let $y = \lim_{x \to 0^+} [x^{\sqrt{x}}]$

Taking log on both side

$\log y = \lim_{x\to 0^+} \log x^{\sqrt{x}}$

$= \lim_{x \to 0} \sqrt{x} \log x$

$= \lim_{x\to 0} \sqrt{x} \cdot \frac{1}{\sqrt{x}} + \log x \cdot \frac{1}{2\sqrt{x}}$

$\log y = 0$

$y = e^0$ $\quad \lim_{x\to 0^+} [x^{\sqrt{x}}] = 1$

$y = 1$

v) $\lim_{x\to 0} (1-2x)^{1/x}$

$\lim_{x\to 0} = \lim (1-2x)^{1/x}$

$= \frac{1}{x} \ln (1-2x)$ (0/0 form)

$= 1 \times \frac{1}{(1-2x)} \times 2$

$= \frac{2}{1-2x}$

$\lim_{x\to 0} = 2$

$\lim_{x\to 0} y = e^{-2}$

$= e^{-2}$

u) $\lim_{x\to 0^+} (\tan 2x)^x$

$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \log (\tan 2x)^x$

$\lim_{x\to 0^+} \ln y = x \log \tan 2x$

$= \frac{x \log \sin 2x}{\log \cos \cdot 2x}$

$\lim_{x\to 0} = 1 \times \frac{\frac{1}{\sin 2x} \times 2}{\frac{1}{\cos 2x} \times 2}$

$\frac{\frac{2}{\sin 2x}}{\frac{2}{\cos 2x}}$

$\lim_{x\to 0^+} \cot 2x$

$\lim \cot 2 \times 0 = \cot 0^\circ = \frac{1}{\tan 0} =$

w) $\lim_{x\to\infty} x^{1/x}$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \ln x^{1/x}$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{1}{x} \ln(x)$ $\left(\frac{\infty}{\infty}\right)$

$\ln y = \lim_{x\to\infty} \frac{1/x}{1}$

$y = e$

$= 1$

Exercise 4.2.

1. Find two numbers whose difference is 100m & whose product is a minimum.

→ Soln Let x & y be the two number.

Difference of two number $= x-y$

$\cancel{x-y=100}$ or, $x = 100+y$

or, $x-100 = y$ — (i)

$f'(x) = x(x-100)$

$f'(x) = x^2-100x$

$= 2x-100$

$f''(x) = 2$

For minimum & maximum

$f'(x) = 0$

$2x-100 = 0$

$2x = 100$

$x = 50$

$f'(x) > 0$

$x = 50$

$y = x-100$

$y = -50$

2. Find the demensions of rectangle with perimeter 100m whose area is as large as possible.

→ Soln perimeter of rectangle $= 100$

perimeter of rectangle $= 2(l+b)$

or, $2(l+b) = 100$

$l+b = 50$

$b = 50-l$

Area of rectangle $= l\times b$

$= (50-l)\times l$

$= 50l - l^2$

Let us suppose x & y be length & breadth.

$y = x-50$

or, $x\times y = x(x-50)$

$= x^2 - 50x$

$f'(x) = 2x-50$

$f''(x) = 2$

For maxima & minima

$f'(x) = 0$

$2x - 50 = 0$

$2x = 50$

$x = 25$

$y = x - 50$

$y = 25 - 50$

$= -25$

$\therefore$ Max^{ma} & min^{ma} $(25, -25)$ #

3. Show that of all the rectangles with a give area, the one with smallest perimeter is a sq

→ Let p be the fixed perimeter of a rectangle with length x & height y. So that $p = 2x + 2y$. The a is $A = x \times y$ we can write this as a funct of x by solving $P = 2x + 2y$ for y.

$P = 2x + 2y$

$2y = P - 2x$, $y = (P/2 - x)$

$A = f(x) = x(P/2 - x) = (\frac{P}{2}x - x^2)$

$f'(x) = P/2 - 2x$

The critical point occur when the derivative d not exist or when derivative is zero.

$\frac{P}{2} - 2x = 0$

$P/2 = 2x$

$P = 4x$ which proves the given rectangle is square.

4. The highway department is planning to built a pi[...] park for motorists along a major highway. The pa[...] is to be rectangular with an area 5000 square yar[...] & is to be fenced off on the three sides not adj[...] to the highway. What is least amount of fencing required for this job? How long & wide shou[...] the park be for the fencing to be maximized?

Soln →

Length $= l$, width $= w$, Area $= l \times b$

Acc. to. Question,

$l \times w = 5000$ — (i)

Perimeter of the fench $(P) = 2l + w$

Minimum length when $\frac{dP}{dw} = 0$

$\frac{dp}{dw} = \frac{d(2l + w)}{dw}$

$0 = 2 - \frac{5000}{l^2}$

$2l^2 = 50000$

$l^2 = 2500$

$l = 50$

So, width is 50m

Hence, putting the value of width in eq^n 1st we ge[t]

$w \times 50 = 5000$

$w = 100m$ #

5. A farmer with 750 ft of fencing wants to en[close] a rectangular area & then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

Soln →

If the d. Perimeter $= 2\times l + 5\times w = 750$

$$2l + 5w = 750$$

$$w = \frac{750-2l}{5}$$

$$w = 150 - \frac{2}{5}\times l$$

Area of each pen $= \frac{l}{4} w$

$$= \frac{2}{4}\left(150 - \frac{2}{5}\times l\right)$$

$$= \frac{l}{4}\times 150 - \frac{2}{5}\times\frac{l}{4}\times l$$

$$= 37.5\times l - l^2/10$$

Area is max^m when $\frac{dA}{dl} = 0$

$$37.5 - \frac{1}{\cancel{10}_5}\times \cancel{2}l = 0$$

$$37.5 = \frac{1}{5} l$$

$$l = 187.5$$

Area of one pen $= \frac{l}{4}\times w = \frac{187.5}{4}\times 75$

$$= 3516\ ft^2$$

Area of four pen $= 14062.5\ ft^2$ #

6. A Normal window has the shape of a rectangle surrounded by a semicircle. (Thus the diame-

ter of the semicircle is equal to width of the rectangle. If the perimeter of the window is 30ft. find the dimensions of the window so that the greatest possible amount of light is admitted.

Soln → x = width of rectangle & diameter of the semicircle.

- $r \to \frac{x}{2} \Rightarrow$ radius of circle.
  - Area of semicircle $= \frac{\pi r^2}{2}$

$$= \pi\left(\frac{x}{2}\right)^2 \cdot \frac{1}{2}$$

$$= \frac{\pi x^2}{8}$$

- Circumference of semicircle

$$= \frac{2\pi r}{2} = \pi r = \frac{\pi x}{2}$$

- Perimeter of window = perimeter of semicircle + perimeter of 3 sides of rectangle.

$$30 = \frac{\pi x}{2} + x + 2 \times l$$

$$2l = 30 - \frac{\pi x}{2} - x$$

$$l = 15 - \frac{\pi x}{4} - \frac{x}{2}$$

- Area of rectangle $= l \times x$

$$= \left(15 - \frac{\pi x}{4} - \frac{x}{2}\right) \times x$$

$$= \left(15x - \frac{\pi x^2}{4} - \frac{x^2}{2}\right)$$

Total area = Area of semicircle + Area of rectangle

$$= \frac{\pi x^2}{8} + 15x - \frac{\pi x^2}{4} - \frac{x^2}{2}$$

$$= 15x + \pi x^2 - \frac{2\pi x^2}{8} - \frac{4x^2}{8}$$

Now,

$$\frac{dA}{dx} = d\left(15x - x^2\,\frac{(\pi + 4)}{8}\right)$$

$$\frac{dA}{dx} = 15 - \frac{2x(\pi+4)}{8\ 4}$$

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Area is max^m when $\frac{dA}{dx} = 0$

$$0 = 15 - \frac{2x}{8}(\pi + 4)$$

Area is min^m when $\frac{dA}{dx} = 0$

$$0 = 15 - x\frac{(\pi+4)}{4}$$

$$15 = x\frac{(\pi+4)}{4}$$

$$x(\pi+4) = 60$$

$$x = \frac{60}{(\pi+4)}$$

$\therefore$ Radius $= \frac{x}{2}$

$$= \frac{60}{2(\pi+4)} = \frac{30}{\pi+4}$$

which is radius.

7. A box with a square base & open top must have a volume of $32000\ cm^3$. Find the dimension of the box that minimize the amount of matrial is used.

→ Sol^n,

Let the volume of square be V & width x, base x, & height h cm is :-

The amount of material is derectly proportional to the surface area, so, we will minimize the amount of material minimizing the surface area.

The surface area of the box described is:

$A = x^2 + 4xh$

we need A as a function of x alone, we will use the fact that,

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$v = x^2 h = 32000\, cm^3$

which gives us, $h = \frac{32000}{x^2}$, so the area becomes,

$A = x^2 + 4x\left(\frac{32000}{x^2}\right)$

$= x^2 + \frac{128000}{x}$

$\frac{dA}{dx} = 2x - \frac{128000}{x^2}$

$\frac{2x^3 - 128000}{x^2} = 0$

$x^3 = 64000$

$x = 40\, cm$

To find height,

$h = \frac{32000}{1600}\, cm$

$= 20\, cm$

i.e. $h = \frac{320}{x^2}$

8. If $1200\, cm^2$ of a material is available to make a box with a square base & an open top. Find the largest possible volume of box:

Soln → Let a be the base of box, & h be height of the box.

Volume $= a^2 h$

$1200 = a^2 h$

$1200 = a^2 + 4ah$

$h = \frac{300}{a} - \frac{a}{4}$

$v = a^2 h$

$= a^2\left[\frac{300}{a} - \frac{a}{4}\right]$

$= 300a - \frac{a^3}{4}$

For the maxm value,

$v' = 300 - \frac{1}{4} \times 3a^2$

$0 = 300 - \frac{3a^2}{4}$

$100 = \frac{3a^2}{4}$

$a^2 = 400$

$a = 20$

To check the

$v'' = \left(-\frac{3}{4}\right) 2a$

$= -\frac{3}{2} a$

$$v''(2a) = \frac{-3}{2}(2a) = \frac{-3}{2} a$$

$$v''(20) = \frac{-3}{2} \times 20$$

$-30 < 0$ it shows v is max^m

$$v = 300 \times 20 - \frac{1}{4} \times 20^3$$

$$v = 6000 - 2000 = 4000 \text{ cm}^3$$

9. A rectangular storage container with an open top to have a volume of $10 m^3$. The length of its base is twice the width. Material for the base costs $10 per square meter material for the sides cost $6 per square meter. Find the cost of materials for the cheapest such container.

Sol^n →

Let the width of the base be x,

The length is $2x$.

The height is $\frac{10}{2x^2}$

Cost of base is $2x^2 \times 5 = 10x^2$

Cost of sides is $3 \times \left(2 \times x \times \frac{10}{2x^2}\right) + 2 \times 2x + \frac{10}{2x^2}$

$= \frac{30}{x} + \frac{20}{x}$

$= \frac{50}{}$

10. A cylindrical can without a top is made to contain $V cm^3$ of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

Soln→

The volume V of a cylinder is given by,

$V = \pi r^2 h$ — (i)

The surface area, A of an open-top cylindrical container is given by:

$A = 2\pi r h + \pi r^2$ — (ii)

Since volume V is constant, rearranging the eqn for the volume in terms of the height h,

$h = \frac{V}{\pi r^2}$ Substituting, $h = \frac{V}{\pi r^2}$ in (i)

$$A = 2\pi r \left(\frac{V}{\pi r^2}\right) + \pi r^2$$

$$= \frac{2V}{r} + \pi r^2$$

To minimize the cost, you must use the minimum amount of material. To find the minimum surface area, set derivative of so to zero

$$\frac{dA}{dr} = \frac{-2V}{r^2} + 2\pi r = 0$$

$$2\pi r = \frac{2V}{r^2}$$

$$r = \left(\frac{V}{\pi}\right)^{1/3}$$ — (iii)

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Putting the value of iii in eq^n (i) we get,

$$h = \frac{V}{\pi r^2} = \frac{V}{\pi \left(\frac{V}{\pi}\right)^{\frac{1}{3}\times 2}} = \frac{V}{\pi \left(\frac{V}{\pi}\right)^{2/3}} = \frac{V}{\pi \times \frac{V^{2/3}}{\pi^{2/3}}} = \frac{V^{1/3}}{\pi^{1/3}}$$

$$h = \left(\frac{V}{\pi}\right)^{1/3} = r$$

(II) Find the point on the line $y = 2x+3$ that is closed to the origin.

→ Sol^n Line segment is shortest when it is perpendicul to the line $y = 2x+3$

We know slope of the line $y = 2x+3$ is 2.

So, slope of the line perpendicular to it will be $-\frac{1}{2}$

$$y = 0 = \frac{-1}{2}(x-0)$$

$$y = -\frac{1}{2}x$$

The intersection of this perpendicular line with our original will be the point which is the closest to the origi

so,
$$2x + 3 = -\frac{1}{2}x$$
$$4x + 6 = -x$$
$$5x = -6$$
$$x = -6/5$$

For value of y put the value of x in

$$y = 2x+3$$
$$= 2\times\left(\frac{-6}{5}\right)+3 = 3/5$$

∴ The reaction point is $\left(\frac{-6}{5}, \frac{3}{5}\right)$

(12) Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$

Soln → Suppose $P(3,0) = P(x_1, y_1)$

$x_2\ y_2 = x, \sqrt{x}$ from $y = \sqrt{x}$

Using distance formula.

$$\sqrt{(\sqrt{x}-0)^2 + (x-3)^2}$$
$$= \sqrt{x + x^2 - 6x + 9}$$
$$= \sqrt{x^2 - 5x + 9}$$

To minimize the distance find the derivative of it & equal to 0.

So, $S = \sqrt{x^2 + 9 - 5x} = (x^2 - 5x + 9)^{1/2}$

$$\frac{ds}{dx} = (2x-5) \times \frac{1}{2(x^2-5x+9)^{1/2}}$$

$$\frac{ds}{dx} = \frac{2x-5}{2\sqrt{x^2-5x+9}}$$

$0 = 2x - 5$

$x = 5/2$

Putting value of x in $y = \sqrt{x}$ we get,

$$\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$$

(13) *Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$

→ $4x^2 + y^2 = 4$

$y^2 = 4 - 4x^2$

$\therefore y = \pm\sqrt{4 - 4x^2}$

Using distance formula

~~d =~~

$$d = \sqrt{(1-x)^2 + (0 - \pm\sqrt{4-4x^2})^2}$$

$$d = \sqrt{1 - 2x - x^2 + 4 - 4x^2}$$

$d =$

$d = \sqrt{1-2x+x^2+4-4x^2}$

$d^2 = -3x^2-2x+5.$

Distance function is parabola that open downwards so its max value occur at vertex (x).

x-co-ordinate vertex is $x = -b/2a$

$x = \frac{2}{-6} = \frac{-1}{3}.$

We have x-coordinate of point that is farthest to $(1,0)$

$y = \pm\sqrt{4-4(-\frac{1}{3})^2}$

$y = \pm\sqrt{4-4/9}$

$y = \pm\frac{4}{3}\sqrt{2}$ Points are $\left(\frac{-1}{3}, \frac{4}{3}\sqrt{2}\right)$ & $\left(\frac{-1}{3}, \frac{-4}{3}\sqrt{2}\right)$

(14) At which points on the curve $y = 1+40x^3-3x^5$ does the tangent line have the largest slope.

$y = 1+40x^3-3x^5$

slope $= \frac{dy}{dx} = y'$

$y' = 120x^2 - 15x^4$ — (i)

Denoting the eqn (i) as, $f(x) = 120x^2-15x^4$ — (ii)

We want to find the value of x for which $f(x)$ is ~~min~~ max^m.

Differentiating eqn (ii).

$f'(x) = 240x - 60x^3$

$f'(x) = 4x - x^3$

$x^3 - 4x = 0$

$x(x^2-4) = 0$

$x = (-2, 0, 2)$

Again differentiating $s'(x)$ to get
$s''(x) = 240 - 180x^2$

Evaluate $s''(x)$ for three critical numbers.

$s''(-2) = 240 - 180 \times 4 = -40$

$s''(0) = 240 - 0 = 240$

$s''(2) = 240 - 180 \times 4 = 480$

$s''(x)$ is negative at $x = \pm 2$, hence there is local maxima at $x = \pm 2$.

Local maxima is absolute maxima because at $x = \pm 2$.

Local maxima is absolute maxima becoz

$\lim_{x \to \pm\infty} s(x) = \pm\infty$

Hence, the slope is maximum when $x = \pm 2$,
Putting the value of x in $y = 1 + 40x^3 - 3x^5$.
we get $y = 255$ or -233.
The points are $(2, 255)$ & $(2, -233)$

(15) Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius.

Soln $\Rightarrow$ From the above questions.

$x^2 + y^2 = r^2$

$y^2 = r^2 - x^2$

$y = \sqrt{r^2 - x^2}$

The area will be $A = 2x \times 2y = 2x \times 2\sqrt{r^2 - x^2}$
$= 4x\sqrt{r^2 - x^2}$

Differentiating w.r.t. x we get,

$$A' = 4\sqrt{r^2-x^2} + \frac{4x(-2x)}{2\sqrt{r^2-x^2}} = 4\sqrt{r^2-x^2} - \frac{8x^2}{2\sqrt{r^2-x^2}}$$

$$= 4\sqrt{r^2-x^2} - \frac{4x^2}{\sqrt{r^2-x^2}} = 0$$

or, $x = \frac{1}{\sqrt{2}} r$

The value of y will given by.

$$y = \sqrt{r^2 - \left(\frac{1}{\sqrt{2}} r\right)^2} = \sqrt{\frac{1}{2} r^2} = \frac{1}{\sqrt{2}} r.$$

Thus the shape will be square of dimension $\frac{1}{\sqrt{2}} r$ by $\frac{1}{\sqrt{2}} r$. $\therefore$ Area $= \frac{1}{2} r^2$ #

(16) Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Soln $\Rightarrow$ Let (x, y) be the co-ordinate in the first quardant of the corner of the rectangle of the ellipse.

By symmetry, $A = 4xy$,

$A = 4x\left[b \times \left(1 - \frac{x^2}{a^2}\right)^{1/2}\right]$ by the eq^n of ellipse.

$$= \left(\frac{4b}{a}\right) \times x\left(a^2 - x^2\right)^{1/2}$$

Now, we take A' & set it equal to 0

$$A' = \left(\frac{4b}{a}\right) * \left[\left(1 - a^2 - x^2\right)^{1/2} + x\left(\frac{1}{2}\right) x - 2x\right)\left(a^2 - x^2\right)^{-1/2}\right]$$

$$= \left(\frac{4b}{a}\right) * \left[a^2 - x^2\right)^{1/2} - x^2\left(a^2 - x^2\right)^{-1/2}\right]$$

$$= \frac{4b}{a} \times \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$$

Setting A' to 0 we get,

$$a^2 - 2x^2 = 0$$
$$x = \frac{a}{\sqrt{2}}$$

So, the max area is; $A(x) = \left(\frac{4b}{a}\right) \times x\,(a^2 - x^2)^{1/2}$

$$A\left(\frac{a}{\sqrt{2}}\right) = \frac{4b}{a} \times \left(\frac{a}{\sqrt{2}}\right) \cdot a^2 (a\sqrt{2})^2)^{1/2}$$
$$= \frac{4b}{a} \times \frac{a}{\sqrt{2}} \times \frac{a}{\sqrt{2}} = 2ab$$

(17) Find the area of the largest rectangle that can be inscribed in a right triangle with leg of length 3cm & 4cm in two sides of the rectangle lie along to the leg

Sol^n →

The hypotenuse of the triangle has eqn.

$$y = \left(\frac{-3}{4}\right) x + 3$$

Thus a points on the line is $x, (-3/4), (x+3)$

Thus, rectangle will have a vertex $(0, 0)$ & a vertex opposite $\left(x, \left(\frac{-3}{4}\right) x + 3\right)$.

The area of the rectangle is base × height

= (x coordinate × y co-ordinate) in this case.

$$\text{Area} = x \times \left(-3/4\right) x + 3$$

$$\text{Area} = \left(\frac{-3}{4}\right) x^2 + 3x$$ max area occurs when derivative is zero

$\left(-\frac{3}{2}\right) x + 3 = 0$ is at $x = 2$, we have max area.

$2 \times \left[\left(-\frac{3}{4} \right) \times 2 + 3 \right] = 2 \times \frac{3}{2} = 3.$

Ex: 4.3

1) Suppose the tangent line to curve $y = f(x)$ at the point $(2,5)$ has the equation $y = 9-2x$. If Newton's method is used to locate root of the equation $f(x) = 0$ & the initial approximation is $x_1 = 2$. find the second approximation x_2.

→ Given Eq^n :

$y = 9-2x$ at point $(2,5)$

$y' = -2$

$x_1 = 2$

$x_2 = x$

we have, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 2 - \frac{9-2\times 2}{-2}$

$= 9/2$

2. Use Newton's method with the specified initial approximation x_1 to x_3 the third approximation to the root of the given equation. (Give your answer to four decimal places.

(i) $\frac{1}{3}x^3 + \frac{1}{2}x^2 + 3 = 0$, $x_1 = -3$

→ $f'(x) = \frac{1}{3} \times 3x^2 + \frac{1}{2} \times 2x$

$= x^2 + x$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$x_2 = -3 - \frac{(-9 + 9/2 + 3)}{9-3}$

$x_2 = -3 - \left(\frac{-18+9+6}{6}\right)$

$x_2 = \frac{-6+1}{2}$

$= -5/2 = -2.75$

$x_3 = x_2 + 0$

$= -2.75$

3. Starting with $x_1 = 2$, find the third approximation x_3 to the root of the equation $x^3 - 2x - 5 = 0$.

Given → $x_1 = 2$. $y' = 3x^2 - 2$

$f'(x) = 3x^2 - 2$ then,

$x_2 = x_1 - \frac{f(x)}{f'(x)}$

$= 2 + 1/10$

$= \frac{21}{10} = 2.1$

$x_3 = x_2 - \frac{f(x)}{f'(x)}$

$= 2.1 - \frac{(2.1)^3 - 2.1 - 5}{3 \times (2.1)^2 - 2}$

$= 1.139$

$= 2.0946.$

$\therefore x_3 = 2.0946.$

4. Use Newton's method with initial approximation $x_1 = 1$ to find x_2, the second approximation to the root of the equation $x^4 - x - 1 = 0$. Explain how the method works by first graphing the function & its tangent line at $(1, -1)$

→ Given Eqⁿ:

$x_1 = 1$

$f(x) = x^4 - x - 1$

$f'(x) = 4x^3 - 1$

$x_2 = x_1 - \frac{f(x)}{f'(x)}$

$= 1 - \frac{-1}{3}$

$= 1 + 1/3$

$= 4/3$

$= 1.33$ #

5. Use Newton's method to approximate the give number correct to eight decimal places :-

as $\sqrt[5]{20}$

$x = \sqrt[5]{20}$

$x^5 - 20 = 0$

$f(x) = x^5 - 20$

$f'(x) = 5x^4$ $\qquad f(1) = -19,\ f(2) = 12,$ Here, $f(1)\cdot f(2) < 0$

So it lies in between 1 & 2.

$x_2 = x_1 - \frac{f(x_1)}{f'(x)} = 1 - \frac{[1^5 - 20]}{5\times 1^4} = 1 + \frac{19}{5} = 4.8$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 4.8 - \frac{[(4.8)^5 - 20]}{5\times(4.8)^4} = 3.847502041$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 3.847535204 - \frac{(3.847535204)^5 - 20}{5\times(3.8475352041)^4}$

$= 3.096280967.$

$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 3.096280967 - \frac{(3.096280967)^5 - 20}{5\times(3.096280967)^4} = 2.520545741$

$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 2.520545141 - \frac{(2.520647411)^5 - 20}{5\times(2.520545774)^4} = 2.115538419$

$x_7 = x_6 - \frac{f(x_6)}{f'(x_6)} = 2.11553841 - \frac{(2.115538419)^5 - 20}{5\times(2.520545774)^4} = 2.115538419$

$x_8 = x_7 - \frac{f(x_7)}{f'(x_7)} = 1.8921299 - \frac{(1.8921299)^5 - 20}{5\times(2.115538419)^4} = 1.8921299$

$x_9 = x_8 - \frac{f(x_8)}{f'(x_8)} = 1.81483706 - \frac{(1.81483706)^5 - 20}{5\times(1.81483706)^4} = 1.820564213.$

$x_{10} = x_9 - \frac{f(x_9)}{f'(x_9)} = 1.820564213 - \frac{(1.820564213)^5 - 20}{5\times(1.820564213)^4} = 1.820564212$

$\therefore \sqrt[5]{20} \approx 1.8205642$ #

$-\cos t$

[Chapter 5] $--\sin t + \sin t$

Exercise 5.1

Antiderivative:

(1) A particle is moving with the given data. Find the position of the particle.

a) $v(t) = \sin t - \cos t$, $s(0) = 0$.

$s(t) = \int \sin t - \cos t$

$s(t) = -\cos t - \sin t + C_s$

$s(0) = -\cos(0) - \sin(0) + C_s$

$0 = -\cos 0 - \sin 0 + C_s$

$v \to acc$ $\quad 0 = -1 - 0 + C_s$

$C_s = 1$

$-\cos t - \sin t + 1$ position of the

$s(0)^{=0}$ $\cos t$

$= -\sin t$

$10\sin t + 3\cos t$, $s(0) = 0$, $s(2\pi) = 12$

$v(t) = \int 10$

$s \to \frac{ds}{dt} \to v \to a \to \frac{dv}{dt}$

$v(t) = \int 10\sin t + 3\cos t$

$= -10\cos t + 3\sin t + C_v$

$0 = -10\cos 0 + 3\sin(0) + C_v$

$0 = -10 + 0\ C_v$

$C_v = 10$

$\cos$

$-10\cos t + 3\sin t + 10$

$12 = -10\sin t - 3\cos t + 10t + C_s$

$12 = -10\sin 2\pi - 3\cos 2\pi + 10 \times 2\pi + C_s$

$(90 \times 4 + \theta)$

95

$\sin\theta - (90 \times 4 + \theta)$

$12 = -10\sin 360 - 3\cos 360 + 10 \times 2\pi + C_s$

$12 = - \quad 0 - 3 + 10 \times 2\pi + C_s$

$12 = -3 + 2\pi \times 10 + C_s$

$12 - 3 = + 2\pi \times 10 + C_s$

$9 = 2\pi \times 10 + C_s$

$-C_s + 9 = 2\pi \times 10 - 9$

(c) $a(t) = 6t + 4$, $v(0) = -6$ cm/s, $s(0) = 9$ cm

Soln:-

$v t = \int 6t + 4$

$= \frac{6t^2}{2} + 4t + C_v$

$= 3t^2 + 4t + C_v$

$v(0) = 3t^2 + 4t + C_v$

$-6 = 3 \times 0 + 4 \times 0 + C_v$

$C_v = -6$

$v(t) = \cancel{6t} \; 3t^2 + 4t - 6$

$s(t) = \int 3t^2 + 4t - 6$

$= \frac{3t^3}{3} + \frac{4t^2}{2} - 6t$

$= t^3 + 2t^2 - 6t + C_s$

$9 = 0 + 0 - 0 + C_s$

$C_s = 9$

$t^3 + 2t^2 - 6t + 9$

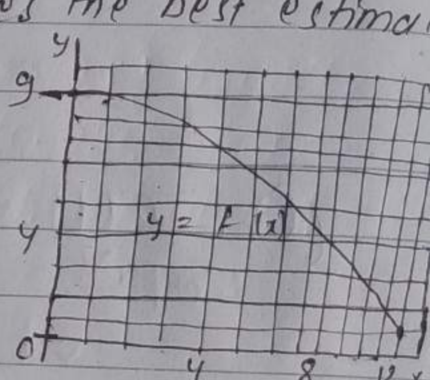
1. a) Use six rectangles to find estimates of each type for the under the given graph of f from $x=0$ to $x=12$.

i) L_6 ii) R_6 iii) M_6.

b) Is L_6 an underestimate or overestimate of true of area

c) Is R_6 an underestimate or overestimate of the true ar

d) Which of the numbers, L_6, R_6 or M_6 gives the best estima Explain.



→ Here, $x_1 = 0$, $x_2 = 12$ $n = 6$

$$\Delta x = \frac{x_2 - x_1}{n} = \frac{12-0}{6} = 2$$

(i) $L_6 = \Delta x\,(f(0) + f(2) + f(4) + f(6) + f(8) + f(10))$
$= 2\,(9 + 8.9 + 8.2 + 7.2 + 6 + 4)$
$= 84.9$ (overestimate)

(ii) $R_6 = \Delta x\,(f(2) + f(4) + f(6) + f(8) + f(10) + f(12))$
$= 2\,(8.9 + 8.2 + 7.2 + 6 + 4 + 1$
$= 70.8$ (underestimate)

(iii) $M_6 = \Delta x\,(f(1) + f(3) + f(5) + f(7) + f(11))$
$= 2\,(9 + 8.6 + 7.9 + 6.8 + 3.1)$
$= 72.9$ (Best estimate).

2. a) Estimate the area under the graph of $f(x) = \cos x$ from $x=0$ to $x=\pi/2$ using four approximating rectangles & right end points. Sketch the graph & the rectangles. Is your estimate an underestimate or overestimate?

b) Repeat part (a) using left end points.

→ $f(x) = \cos x$

$$\Delta x = \frac{\pi/2 - 0}{4} = \frac{\pi}{8}$$

$x_1 = \frac{\pi}{8}$, $x_2 = \frac{2\pi}{8}$, $x_3 = \frac{3\pi}{8}$, $x_4 = \frac{4\pi}{8}$, $x_n = \frac{n\pi}{8}$

Acc. to defn, the area is.

$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \Delta x [f(x_1) + f(x_2) + f(x_3) + f(x_4)]$

$= \lim_{n\to\infty} \Delta x \left(\cos\frac{\pi}{8} + \cos\frac{\pi}{4} + \cos\frac{3\pi}{8} + \cos\frac{\pi}{2}\right)$

$= \lim_{n\to\infty} \frac{\pi}{8} \left(\cos 0.3925 + \cos 0.787 + \cos 1.1775 + \cos 90\right)$

$=$ ~~0.99 + 0.99 + 0.99 + 0~~

$= \frac{\pi}{8}(0.9238 + 0.7071 + 0.3826)$

$= 0.790$ # which is underestimate.

(b) $L_4 = \Delta x [f(x_0) + f(x_1) + f(x_2) + f(x_3)]$

$= \frac{\pi}{8} [1 + 0.9238 + 0.7071 + 0.3826]$

$= 1.18$ # which is overestimate.

3. a) Estimate the area under the graph of $f(x) = \sqrt{x}$ from $x=0$ to $x=4$ using four approximating rectangles & right endpoints. sketch the graph & the rectangle. Is your estimate an ~~is~~ underestimate or an overestimate?

→ Here, $f(x) = \sqrt{x}$

$\Delta x = \frac{4-0}{4} = 1$ $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$, $x_5 = 5$, $x_n = n$

~~Acc. to defn the~~

$R_4 = \Delta x [f(x_1) + f(x_2) + f(x_3) + f(x_4)]$

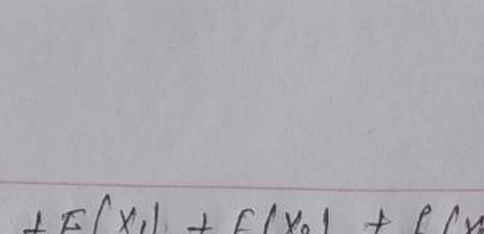
$= 1 [f(1) + f(2) + f(3) + f(4)]$

$= 1 + \sqrt{2} + \sqrt{3} + 2$

$\therefore R_4 = 3 + \sqrt{2} + \sqrt{3}$

$\therefore L_4 = \Delta x \,(f(x_0) + f(x_1) + f(x_2) + f(x_3))$
$= 1(0 + 1 + \sqrt{2} + \sqrt{3})$
$= 1 + \sqrt{2} + \sqrt{3}$

For graph,



It is overestimate.

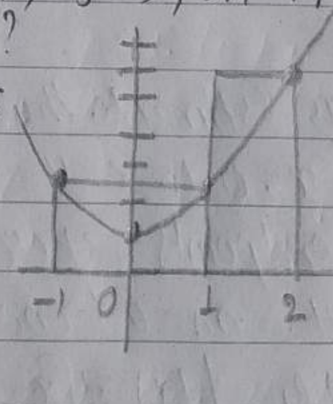
4a) Estimate the area under the graph of $f(x) = 1 + x^2$ from $x = -1$ to $x = 2$ using three rectangles & right endpoints. Then improve your estaimate by using six rectangles. Sketch the curve & the approximating rectangles.
b) Repeat part (a) using left endpoints.
c) Repeart part (a) using midpoints.
d) From your sketches in parts (a)-(c), which appears to be the best estimate?

→ Here, $f(x) = 1 + x^2$

$\Delta x = \frac{2-(-1)}{3} = 1$ $\quad x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4,\ x_5 = 5,\ x_n = n$

putting the value -1 in $f(x) = ?$

$L_3 = \cancel{f}$ $\quad f(-1) = 2,\ f(0) = 1,\ f(1) = 2,\ f(2) = 5$



$R_3 = \Delta x\,(f(x_1) + f(x_2) + f(x_3))$
$= 1\,(f(0) + f(1) + f(2))$
$= 1(1 + 2 + 5) = 8$

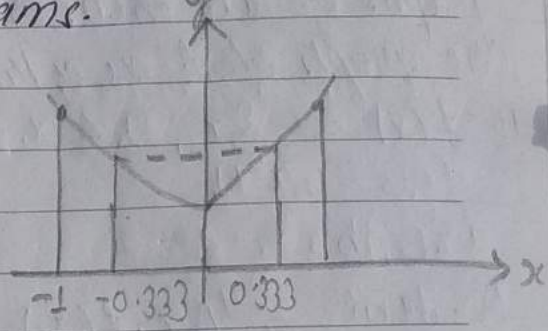
$L_3 = \Delta x\,(f(-1) + f(0) + f(1))$
$= 1\,(2 + 1 + 2)$
$= 5$

$M_3 = \Delta x\,(f(-0.5) + f(0.5) + f(1.5))$
$= \Delta x\,(1.25 + 1.25 + 3.25$
$= 5.75$ #

5. Evaluate the upper & lower sums for $f(x) = 1+x^2$, $-1 \le x \le 1$, with $n = 3$ & 4. Illustrate with diagrams.

→ $f(x) = 1+x^2$,

$\Delta x = \frac{1-(-1)}{3} = 0.666$.



Sub intervals are:

$(-1, -0.333)$

$(-0.333, 0.333)$ Lower sum $= \Delta x\, (f(-0.333) + f(0) + f(0.333))$

$(0.333, 1)$ $= 0.666\, (1.110889 + 1 + 1.110889)$

$= 2.14$

Upper sum $= \Delta x\, (f(-1) + f(0.333) + f(1))$

$= 3.407$ #

Q.7. Speedometer readings for a motorcycle at 12 second intervals are given in the table.

(a) Estimate the distance traveled by the motorcycle during this ~~pe~~ time period using the velocities at the beginning of time intervals.

(b) Give another estimate using the velocities at the end of the time period.

(c) Are your estimates in parts (a) & (b) upper & lower? Explain

t (s)	0	12	24	36	48	60
v (ft/s)	30	28	25	22	24	27

→ Here,

For lower estimate $d = 30 \times 12 + 28 \times 12 + 25 \times 12 + 22 \times 12 + 24 \times 12$

$= 1548$ ft.

For upper estimate. $d = 28 \times 12 + 25 \times 12 + 22 \times 12 + 24 \times 12 = 27 \times 12$

$= 1512$ ft.

Q. Let A be the area of the region that lies under the following graphs of $f(x)$ then, find an expression for the area under the graph of f as a limit. Do not evaluate

a) $f(x) = \frac{2x}{x^2+1}, \ 1 \le x \le 3$

$a = 1, \ b = 3$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot f(x_i)$ where $(x_i = a + i\Delta x)$

$\Delta x = \frac{b-a}{n} = \frac{2}{n}$

$x_i = 1 + \frac{2}{n} \times i$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \times f\left(1+\frac{2i}{n}\right)$

$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(\frac{2 \times 1 + \frac{2i}{n}}{\left(1+\frac{2i}{n}\right)^2+1}\right)$ #

(b) $f(x) = x^2 + \sqrt{1+2x}, \ 4 \le x \le 7$

$a = 4, \ b = 7$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot f(a + i\Delta x)$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} f\left(4 + \frac{3}{n} i\right)$

$= \lim_{n\to\infty} \frac{3}{n} \times \sum_{i=1}^{n} \left[\left(4+\frac{3i}{n}\right)^2 + \sqrt{1+2 \times 4 + \frac{3i}{n}}\right]$

(c) $f(x) = \sqrt{\sin x} \quad 0 \le x \le \pi$

$a = 0 \quad b = \pi$

$\Delta x = \frac{\pi}{n}$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} f\left(0 + \frac{\pi}{n} i\right)$

$= \lim_{n\to\infty} \frac{\pi}{n} \sum_{i=1}^{n} \sqrt{\sin \frac{\pi}{n} i}$ #

(10) Determine a region whose area is equal to the given limit. Do not evaluate the limit.

a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5 + \frac{2i}{n}\right)^{10}$

→ comparing this with

$\lim \sum^{n} \Delta x \cdot f(a + i\Delta x)$

b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$

→ comparing this with

$\lim_{n\to\infty} \sum^{n} \Delta x \cdot f(a + i\Delta x)$

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$a = \cancel{\pi}\ 5,\ \Delta x = \frac{2}{n},\ b = 7$ $= a = 0,\ \Delta x = \frac{\pi}{4n},$

$\Delta x = \frac{b-a}{n}$

$f(x) = x^{10}$

for $\pi \le x \le 7$ $f(x) = \tan x$ $\frac{\pi}{4\cancel{n}} \times \cancel{n} = b$ b

$(0 \le x \le \pi/4)$

Exercise : 5.2

1. If $f(x) = x^2 - 2x,\ 0 \le x \le 3$, evaluate the Riemann with $n = 6$, taking the sample points to be right en what does the Riemann sum represent?

→ Here, $a = 0,\ b = 3,\ n = 6,\ \Delta x = \frac{3-0}{6} = 0.5$

Sub-intervals are, $(0, 0.5), (0.5, 1), (1, 1.5), (1.5, 2), (2, 2.5)$

$R_6 = \Delta x\,(f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f($

$= 0.5\,(0.75 + (-1) + (-0.75) + 0 + (1.25) + 3)$

$= 0.875$ #

② $f(x) = e^x - 2,\ 0 \le x \le 2,\ n = 4.$

→ Here $\Delta x = 2/4 = 0.5$

Sub-intervals are, $(0, 0.5)\ (0.5, 1)\ (1, 1.5)\ (1.5, 2)$

mid points are $(0.25, 0.75, 1.25, 1.75)$

$M_4 = \Delta x\,(f(0.25) + f(0.75) + f(1.25) + f(1.75))$

$= 0.5\{-0.7159 + 0.1170 + 1.49034 + 3.75460\}$

$= 2.322985.$

* Express ~~&~~ use the midpoint Rule with the given value of t approximate the integral.

a) $\int_0^8 \sin\sqrt{x}\, dx, n=4$

$\rightarrow f(x) = \sin\sqrt{x}$

$\Delta x = \frac{b-a}{n} = \frac{8}{4} = 2$

$\therefore$ The four sub-intervals are $(0,2)\ (2,4)\ (4,6)\ (6,8)$

Mid points are $(1,3,5,7)$.

$\therefore M_4 = \Delta x\, (f(1) + f(3) + f(5) + f(7)$

$= 2\,(0.01745 + 0.03022 + 0.039016 + 0.04616$

$= 0.265$ #

b) $\int_0^{\pi/2} \cos^4 x\, dx \quad n=4$

$= f(x) = \cos^4 x \qquad \Delta x = \frac{\frac{\pi}{2} - 0}{4} = 0.39$

$(0, 0.39)\ (0.39, 0.78)\ (0.78, 1.17)\ (1.17, 1.50)$

$\therefore M_4 = \Delta x\, (f(0.195) + f(0.585) + f(0.975) + f(1.365)$

$= 0.39\, (0.9999 + 0.9997 + 0.9994 + 0.988)$

$= 1.559$ #

c) $\int_0^2 \frac{x}{x+1}\, dx,\ n=5$

Here, $a=0,\ b=2,\ n=5$

$\Delta n = 2/5 = 0.4$

Intervals are : $(0, 0.4)\ (0.4, 0.8)\ (0.8, 1.2)\ (1.2, 1.6)\ (1.$

$M_5 = \cancel{(0.2)}\ \Delta x\, (f(0.2) + f(0.6) + f(1.0) + f(1.4) + f$

$= 0.4 \times \{$

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d) $\int_1^5 x^2 e^{-x} dx,\ n=4$

Here, $b=5,\ a=1,\ n=4$.

$\Delta x = 1$ $\because$ Intervals are: $(1,2)\ (2,3),\ (3,4),\ (4,5)$

$M_4 = \Delta x\,(f(1) + f(2.5) + f(3.5) + f(4.5))$

$= \Delta x$

Q No: 6 Express the limit as a definite integral on the given interval.

a) $\lim_{n\to\infty} \sum_{i=1}^{n} x_i \ln(1+x_i^2)\,\Delta x,\ [2,6]$

$a=2,\ b=6$

$f(x) = x \ln(1+x^2)$

Now, In integral form,

$f(x)\,dx = \int_2^6 x \ln(1+x^2)\,dx$

b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\cos x_i}{x_i}\,\Delta x,\ (\pi, 2\pi)$

$a=\pi,\ b=2\pi$

$f(x) = \frac{\cos x}{x}$

Now, In integral form,

$\int_a^b f(x)\,dx = \int_\pi^{2\pi} \frac{\cos x}{x}\,dx$

c) $\lim_{n\to\infty} \sum_{i=1}^{n} [5(x_i^*)^3 - 4x_i^*]\,\Delta x,\ [2,7]$

$a=2,\ b=7$

$f(x) = 5x^3 - 4x$

Now In integral form,

$\int_a^b f(x)\,dx = \int_2^7 5x^3 - 4x\,dx$

7- Use the form of the definition of the integral. Evaluate the integral.

a) $\int_2^5 (4-2x)\,dx$

$a=2,\ b=5,\ \Delta x = \frac{3}{n}$; $f(x) = 4-2x$

$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\Delta x$

$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(2+i\frac{3}{n}\right)\Delta x$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(4-2\left(2+i\frac{3}{n}\right)\right) \times \frac{3}{n}$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left\{4-2\left(\frac{2n+3i}{n}\right)\right\}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \times \frac{-6i}{n}$

$= \lim_{n\to\infty} \frac{-18}{n^2} \sum_{i=1}^{n} i$

$= \lim_{n\to\infty} \frac{-18}{n^2} \frac{n(n+1)}{2}$

$= \lim_{n\to\infty} \frac{-18}{n^2} \times \frac{n^2\left(1+\frac{1}{n}\right)}{2}$

$= -9$ #

(b) $\int_1^4 (x^2-4x+2)\,dx$

soln, $a=4,\ b=1\ \ \Delta x = \frac{3}{n}\ \ \ f(x) = x^2-4x+2$

$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\Delta x$

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$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \left\{ \left(1+\frac{3i}{n}\right)^2 - 4\left(1+\frac{3i}{n}\right) + 1 \right\}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \left(\frac{9i^2}{n^2} - \frac{6i}{n} + 2 \right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \times \left\{ \frac{9}{n^2} \sum_{i=1}^{n} i^2 - \frac{6}{n} \sum_{i=1}^{n} i + 2 \right\}$$

$$= \lim_{n\to\infty} \frac{3}{n} \times \left\{ \frac{9}{n^2} \times \frac{n(n+1)(2n+1)}{6} - \frac{6}{n} \times \frac{n(n+1)}{2} + 2 \right\}$$

$$= \lim_{n\to\infty}$$

(c) $\int_{-2}^{0} (x^2+x)\,dx$

Soln: $a=-2 \quad b=0 \quad f(x)=x^2+x \quad \Delta x = 2/n$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left\{ \left(-2+\frac{2i}{n}\right)^2 + \left(-2+\frac{2i}{n}\right) \right\}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \times \left\{ 4 - \frac{8i}{n} + \frac{4i^2}{n^2} - 2 + \frac{2i}{n} \right\}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \times \left(4 - \frac{6i}{n} + \frac{4i^2}{n^2} \right)$$

$$\lim_{n\to\infty} \frac{2}{n} \left(\sum_{i=1}^{n} 2 - \sum_{i=1}^{n} \frac{6i}{n} + \sum^{n} \frac{(1^2+2^2+3^2+\dots+n^2)}{n^2} \right)$$

$$\lim_{n\to\infty} 2 \left(2 - \frac{6}{n^2} \frac{n(n+1)}{2} + \frac{4}{n^3} \frac{n(n+1)(2n+1)}{6} \right)$$

$$\lim_{n\to\infty} 2\left(2-\frac{6}{n^2}\frac{n^2(1+\frac{1}{n})}{2}+\frac{4}{n^3}\frac{n^3(1+\frac{1}{n})(2+\frac{1}{n})}{6}\right.$$

$$= 2\left(2-\frac{6}{2}+\frac{4\times 1\times 2}{6}\right)$$

$$= 2(2-3+4/3)$$

$$= 2(4/3-1) = 2/3$$

(d) $\int_0^2 (2x-x^3)\,dx$

$a=0,\ b=2 \quad f(x)=2x-x^3 \quad \Delta x = 2/n.$

Soln

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\left(2\times\frac{2i}{n}-\left(\frac{2i}{n}\right)^3\right)$$

$$= \frac{2}{n}\lim_{n\to\infty}\sum_{i=1}^{n}\left\{\frac{4i}{n}-\frac{8i^3}{n^3}\right\}$$

$$= \frac{2}{n}\lim_{n\to\infty}\left(\frac{4}{n}(1+2+3+\dots+n)-\frac{8}{n^3}(1^3+2^3+3\right.$$

$$= \frac{2}{n}\lim_{n\to\infty}\left\{\frac{4}{n}\times n\frac{(n+1)}{2}-\frac{8}{n^3}\times\right.$$

5.3

If the function is continuous on $[a,b]$ then the function g is given by:

$$g(x) = \int_a^x f(t)\,dt$$ continuous on $[a,b]$. and also differentiable $[a,b]$.

then $g'(x) = f(x)$.

Exercise 5.3

2.

(a) $g(x) = \int_1^x \frac{1}{t^3+1}\,dt$.

$$g'(x) = \frac{d}{dx}\int_1^x \frac{1}{t^3+1}\,dt$$

$$= \frac{1}{x^3+1}$$

(b) $g(x) = \int_3^x e^{t^2-t}\,dt$

$$g'(x) = \frac{d}{dx}\int_3^x e^{t^2-t}\,dt$$

$$= e^{x^2-x}.$$

(c) $g(s) = \int_5^s (t-t^2)^8\,dt$

$$g'(s) = \frac{d}{ds}\int_5^s (t-t^2)^8\,dt$$

By using fundamental of theorem of calculus.

$$g'(s) = (s-s^2)^8$$

(d)

(e)

(f)

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(d) $g(r) = \int_0^r \sqrt{x^2+x}\, dx$

$g'(r) = \frac{d}{dr} \int_0^r \sqrt{x^2+x}\, dx$

By using fundamental theorem of calculus,

$g'(r) = \sqrt{r^2+4}$

(e) $F(x) = \int_x^\pi \sqrt{1+\sec t}\, dt$

$F(x) = \int_x^{\pi} \sqrt{1+\sec t}\, dt + \int_0^\pi \sqrt{1+\sec}$

$F(x) = -\int_\pi^x \sqrt{1+\sec t}\, dt$

$F'(x) = -\frac{d}{dx} \int_\pi^x \sqrt{1+\sec t}\, dt$

Using fund. theorem of calc.

$F'(x) = -\sqrt{1+\sec x}$

(g) $h(x) = \int_1^{e^x} \ln t\, dt$

let $u = e^x \therefore du = e^x dx$

$h'(x) = \left[\frac{d}{du} \int_1^u \ln t\, dt\right] \times \frac{du}{dx}$

$\Rightarrow h'(x) = \ln u \times e^x$

$\Rightarrow h'(x) = \ln e^x \times e^x$

$\Rightarrow h'(x) = x \ln e \times e^x$

$\therefore h'(x) = x e^x$

(h) $h(x) = \int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\, dz$

Let $u = \sqrt{x}$ $\quad du = \frac{1}{2\sqrt{x}}\, dx$

Now,

$$h'(x) = \left[\frac{d \int_1^{u} \frac{z^2}{z^4+1}\, dz}{du}\right] \times \frac{du}{dx}$$

$$= \frac{u^2}{u^4+1} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{(\sqrt{x})^2}{(\sqrt{x})^4+1} \times \frac{1}{2\sqrt{x}}$$

$$\therefore h'(x) = \frac{\sqrt{x}}{2(x^2+1)} \text{ ans.}$$

(i) $y = \int_1^{x^4} \cos^2\theta \, d\theta$

Let $u = x^4$ $\quad du = 4x^3\, dx$

Now,

$$y' = \left[\frac{d \int_1^{u} \cos^2\theta\, d\theta}{du}\right] \times \frac{du}{dx}$$

$$= \cos^2 u \times 4x^3$$

$$= \cos^2 x^4 \; 4x^3$$

$$y' = 4x^3 (\cos x^4)^2$$

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(j) $y = \int_{\sin x}^{1} \sqrt{1+t^2}\, dt$

$$y = -\int_{1}^{\sin x} \sqrt{1+t^2}\, dt$$

$$y' = -\left[\frac{d \int_1^{\sin x} \sqrt{1+t^2}\, dt}{d \sin x}\right] \frac{d \sin x}{dx}$$

$$y' = -\sqrt{1+\sin^2 x} \cdot \cos x$$

$$y' = -\cos x \sqrt{1+\sin^2 x}$$

(k) $g(x) = \int_{1-2x}^{1+2x} t \sin t\, dt$

$$g(x) = -\int_0^{1-2x} t \sin t\, dt + \int_0^{1+2x} t \sin t\, dt$$

Now, applying fundamental theorem of calculus,

$$g'(x) = -\frac{d \int_0^{1-2x} t \sin t\, dt}{dx} + \frac{d \int_0^{1+2x} t \sin t\, dt}{dx}$$

$$g'(x) = -(1-2x)\sin(1-2x) + (1+2x)\sin(1+2x)$$

(l) $y = \int_{\cos x}^{\sin x} \ln(1+2v)\, dv$

$$y = -\int_0^{\cos x} \ln(1+2v)\, dv + \int_0^{\sin x} \ln(1+2v)\, dv$$

Now applying fundamental theorem of calculus,

$$y' = -\ln(1+2\cos x) + \ln(1+2\sin x)$$

$$y' = \left[-\frac{d \int_0^{\cos x} \ln(1+2v)\, dv}{d \cos x}\right] \frac{d \cos x}{dx} + \left[\frac{d \int_0^{\sin x} \ln(1+2v)\, dv}{d \sin x}\right] \frac{d \sin x}{dx}$$

$$y' = \sin x \ln(1+2\cos x) + \cos x \ln(1+\sin x)$$

3. Evaluate the integral

(a) $\int_1^4 (5-2t+3t^2)\,dt$

$= \left[5t-t^2+t^3\right]_1^4$

$= \left[5(4-1)-(4^2-1)+(4^3-1)\right]$

$= 15-15+63$

$= 63$

(b) $\int_0^1 \left(1+\frac{1}{2}u^4-\frac{2}{5}u^9\right)du$

$= \left[u+\frac{u^5}{10}-\frac{u^{10}}{25}\right]_0^1$

$= \left[1+\frac{1}{10}-\frac{1}{25}\right]$

$= \frac{50+5-2}{50} = \frac{53}{50}$

(c) $\int_{-5}^{5} e\,dx$

$= e\,[x]_{-5}^{5}$

$= e\,(5-(-5))$

$= 10e$

(d) $\int_0^1 (u+2)(u-3)\,du$

$= \int_0^1 u^2-u-6\,du$

$= \left[\frac{u^3}{3}-\frac{u^2}{2}-6u\right]_0^1$

$= \frac{1}{3}-\frac{1}{2}-6$

$= \frac{2-3-36}{6} = \frac{-37}{6}$

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3. Evaluate the integral

(a) $\int_1^4 (5-2t+3t^2)\,dt$

$= \left[5t - t^2 + t^3\right]_1^4$

$= \left[5(4-1) - (4^2-1) + (4^3-1)\right]$

$= 15 - 15 + 63$

$= 63$

(b) $\int_0^1 \left(1 + \frac{1}{2}u^4 - \frac{2}{5}u^9\right) du$

$= \left[u + \frac{u^5}{10} - \frac{u^{10}}{25}\right]_0^1$

$= \left[1 + \frac{1}{10} - \frac{1}{25}\right]$

$= \frac{50+5-2}{50} = \frac{53}{50}$

(c) $\int_{-5}^{5} e\,dx$

$= e\,[x]_{-5}^{5}$

$= e\,(5-(-5))$

$= 10e$

(d) $\int_0^1 (u+2)(u-3)\,du$

$= \int_0^1 u^2 - u - 6\,du$

$= \left[\frac{u^3}{3} - \frac{u^2}{2} - 6u\right]_0^1$

$= \frac{1}{3} - \frac{1}{2} - 6$

$= \frac{2-3-36}{6} = \frac{-37}{6}$

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(7) $\int_1^9 \frac{x-1}{\sqrt{x}}\,dx$

$= \int_1^9 (\sqrt{x} - x^{-1/2})\,dx$

$= \left[\frac{2}{3}x^{3/2} - 2x^{1/2}\right]_1^9$

$= \left[\frac{2}{3}(9^{3/2} - 1) - 2(9^{1/2} - 1)\right]$

$= \left[\frac{2}{3}(26) - 2(3-1)\right]$

$= \frac{52}{3} - 4$

$= \frac{52-12}{3}$

$= 40/3$

(8) $\int_0^{\pi/4} \sec\theta \tan\theta \, d\theta$

Put $\sec\theta = u$.

$\therefore \sec\theta \tan\theta \, d\theta = du$

Now, $\int_0^{\pi/4} \sec\theta \tan\theta \, d\theta$

$= \int_0^{\pi/4} du$

$= [u]_0^{\pi/4}$

$= [\sec\theta]_0^{\pi/4}$

$= \sec\pi/4 - \sec 0$

$= \sqrt{2} - 1$

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(g) $\int_0^3 (2\sin x - e^x)\,dx$

$= [-2\cos x - e^x]_0^3$

$= -2\cos 3 + 2 - e^3 + e^0$

$= -2\cos 3 - e^3 + 3$

(h) $\int_0^1 (x^e + e^x)\,dx$

$= \left[\frac{x^{e+1}}{e+1} + e^x\right]_0^1$

$= \left[\left(\frac{1^{e+1}}{e+1} + e^1\right) - (1)\right]$

$= \frac{e^2+1}{e+1} - 1$

$= \frac{e^2 + \cancel{1} - e - \cancel{1}}{e+1}$

$= \frac{e^2 - e}{e+1}$

(i) $\int_0^1 \cosh t\,dt$

$= [\sinh t]_0^1$

$= [\sinh 1 - \sinh 0]$

$= \sinh 1$

(j) $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}\,dx$

(1c) $\int_{-2}^{2} f(x)\,dx$ where $f(x) = \begin{cases} 2 & \text{if } -2 \le x \le 0 \\ 4-x^2 & \text{if } 0 < x \le 2. \end{cases}$

$= \int_{-2}^{0} f(x) + \int_{0}^{2} f(x)$

$= \int_{-2}^{0} 2 + \int_{0}^{2} 4-x^2$

$= [2x]_{-2}^{0} + [4x - \frac{x^3}{3}]_{0}^{2}$

$= 4 + [8 - 8/3)$

$= \frac{12+24-8}{3}$

$= \frac{12+16}{3} = \frac{28}{3}$

4. Evaluate as a Riemann sum:

$\lim_{n \to \infty} \frac{1}{n}\left(\sqrt{1/n} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \dots + \sqrt{\frac{n}{n}}\right)$

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Exercise 5.4

1. Evaluate the integral.

(a) $\int x^2+x^{-2}\,dx$

$= \frac{x^3}{2} - \frac{1}{x} + c$

(b) $\int \left(x^2+1+\frac{1}{x^2+1}\right)dx$

$= \frac{x^3}{3} + x + \tan^{-1}x + c.$

(c) $\int (1+\tan^2\alpha)\,d\alpha$

$= \int \sec^2\alpha\,d\alpha$

$= \tan\alpha + c.$

(g) $\int_0^1 (x^{10}+10^x)\,dx$

$= \int_0^1 x^{10}\,dx + \int_0^1 10^x\,dx$

put $y = 10^x$ $\quad dy =$

$\log y = x\log 10$

$\frac{1}{y}\frac{dy}{dx} = \log 10$

Now,

$= \int_0^1 x^{10}\,dx + \int_0^1 y \times \frac{1}{y\log 10}\,dy$

$= \left[\frac{x^{11}}{11}\right]_0^1 + \frac{1}{\log_{10}}[y]_0^1$

$= \frac{1}{11} + \frac{10-10^0}{\log 10}$

$= \frac{1}{11} + \frac{9}{\log 10}$ Ans.

(h) $\int_0$

(i) $\int_0^{\pi}$

(j) $\int_{-10}^{1}$

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(h) $\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\, d\theta$

$= \int_0^{\pi/4} \sec^2\theta\, d\theta + \int_0^{\pi/4} 1\, d\theta$

$= [\tan\theta]_0^{\pi/4} + [\theta]_0^{\pi/4}$

$= 1 + \pi/4$

(i) $\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\, d\theta$,

$= \int_0^{\pi/3} \frac{\sin\theta(1+\tan^2\theta)}{\sec^2\theta}\, d\theta$

$= \int_0^{\pi/3} \sin\theta\, d\theta$

$= -[\cos\theta]_0^{\pi/3}$

$= -\left(\frac{\sqrt{3}}{2} - 1\right)$

$= 1 - \frac{\sqrt{3}}{2}$

(j) $\int_{-10}^{10} \frac{2e^x}{\sinh x + \cosh x}\, dx$

$= \int_{-10}^{10} \frac{2e^x}{\frac{e^x - e^{-x} + e^x + e^{-x}}{2}}\, dx$

$= \int_{-10}^{10} \frac{2\cancel{e^x}}{\frac{\cancel{2}\cancel{e^x}}{\cancel{2}}}\, dx$

$= [2x]_{-10}^{10}$

$= 2[10-(-10)] = 40$ Ans.

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(k) $\int_1^2 \frac{(x-1)^3}{x^2}\,dx$

(l) $\int_0^2 |2x-1|\,dx$

$= [\cancel{(}x^2-x)]_0^2$

$= \int_0^2 (2x-1)\,dx + \int_0^2 -(2x-1)\,dx$

$= [x^2-x]_0^2 - [x^2-x]_0^2$

$= 4-2-$

(2) Use a graph to estimate the x-intercepts of the curve $y = 1-2x-5x^4$. Then use this information to estimate the area of region that lies under the curve and above the x-axis.

5.5

1 Evaluate:

(a) $\int \ln\sqrt[3]{x}\, dx$

$= \int 1 . \ln\sqrt[3]{x}\, dx$

$= \ln\sqrt[3]{x} \int 1 dx - \int \left[\frac{d \ln\sqrt[3]{x}}{dx} \int 1 dx\right] dx$

$= \ln\sqrt[3]{x} \cdot x - \int \frac{1}{3x} \cdot x \, dx$

$= x\ln\sqrt[3]{x} - 3x + c$

(b) $\int \sin^{-1}x \, dx$

$= \int 1 . \sin^{-1}x\, dx$

$= \sin^{-1}x \int 1 dx - \int \left\{\frac{d \sin^{-1}x}{dx} \int 1 . dx\right\} dx$

$= \sin^{-1}x \cdot x - \int \frac{x}{\sqrt{1-x^2}} dx$

$= x\sin^{-1}x - \int \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta \, d\theta$

$= x\sin^{-1}x - \int \frac{\sin\theta}{\cos\theta} \cdot \cos\theta \, d\theta$

$= x\sin^{-1}x + \cos\theta + c$

$= x\sin^{-1}x + \sqrt{1-x^2} + c$

$\because \{ x = \sin\theta,$

$\therefore b = \sqrt{1-x^2}$

$\therefore \cos\theta = (\sqrt{1-x^2})/1.$

(c) $\int 5 2^5 dz$

$= 5\int 2^5$

$= 5 \frac{2^5}{\ln 2}$

$= \frac{5 2^5}{\ln 2}$

(d) $\int 2^3 e$

$= 2^3 \int e$

$= 2^3 e$

$= 2^3 e$

$= 2^3 e$

$= 2^3$

(e) $\int x$

$= x\int$

$= x$

$= x($

$=$

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(c) $\int s 2^s ds$

$= s\int 2^s - \int\left[\frac{ds}{ds}\int 2^s\right]ds$

$= s\frac{2^s}{\ln 2} - \int 1\cdot\frac{2^s}{\ln 2}ds$

$= \frac{s2^s}{\ln 2} - \frac{2^s}{(\ln 2)^2} + c$

(d) $\int z^3 e^z dz$

$= z^3\int e^z dz - \int\left[\frac{dz^3}{dz}\int e^z dz\right]dz$

$= z^3e^z - \int 3z^2\cdot e^z dz$

$= z^3e^z - 3[z^2e^z - \int e^z\cdot 2z\, dz$

$= z^3e^z - 3[z^2e^z - 2\{z\int e^z - \int \frac{dz}{dz}\cdot e^z dz\}]$

$= z^3e^z - 3z^2e^z + 6ze^z - 6e^z + c$ an.

(e) $\int x\tan^2 x\, dx$

$= x\int\tan^2 x - \int\{\frac{dx}{dx}\cdot\int\tan^2 x\, dx\}dx$

$= x\int(\sec^2 x - 1) - \int\{1\cdot\int(\sec^2 x - 1\, dx)\}dx$

$= x(\tan x - x) - \int(\tan x - x)dx$

$= x\tan x - x^2 - \ln\cos x + \frac{x^2}{2} + c$

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$\int x \tan^2 x$

$= \tan^2 x \int x\,dx - \int \frac{d \tan^2 x}{dx} \int x\,dx\; dx$

$= \tan^2 x \cdot \frac{x^2}{2} - \int \tan x \cdot \sec^2 x \cdot \int \ldots$

(f) $\int (\sin^{-1}x)^2\,dx$

$= \int 1 \cdot (\sin^{-1}x)^2\,dx$

$= (\sin^{-1}x)^2 \int 1\,dx - \int \left[\frac{d(\sin^{-1}x)^2}{d(\sin^{-1}x)} \times \frac{d \sin^{-1}x}{dx} \int 1\,dx\right] dx$

$= (\sin^{-1}x)^2 \cdot x - \int \sin^{-1}x \times \frac{1}{\sqrt{1-x^2}} \cdot x\,dx$

$\cancel{= (\sin^{-1}x)^2 \cdot x - \left[\frac{x}{\sqrt{1-x^2}} \int \sin^{-1}x - \int \frac{d \frac{x}{\sqrt{1-x^2}}}{dx} \int \sin^{-1}x\right.}$

$= (\sin^{-1}x)^2 \cdot x - \left[\sin^{-1}x \int \frac{x}{\sqrt{1-x^2}}\,dx - \int \left[\frac{d \sin^{-1}x}{dx} \int \frac{x}{\sqrt{1-x^2}}\,dx\right] dx\right.$

$= x(\sin^{-1}x)^2 - \left[\sin^{-1}x(-\cos\theta) - \int \frac{-\cos\theta}{\sqrt{1-x^2}}\,dx\right]$

$= x(\sin^{-1}x)^2 - [\sin^{-1}x(-\cos\theta)] - \sin\theta + c$

$= x(\sin^{-1}x)^2 + \sin^{-1}x\sqrt{1-x^2} - x + c$

(g) $\int_0$

(h)

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(g) $\int_0^{2\pi} t^2 \sin 2t \, dt$

$= \left[t^2 \int \sin 2t \, dt - \int \left[\frac{dt^2}{dt} \int \sin 2t \, dt\right] dt\right]_0^{2\pi}$

$= \left[t^2 \left(\frac{-\cos 2t}{2}\right) - \int 2t \cdot \left(\frac{-\cos 2t}{2}\right)\right]_0^{2\pi}$

$= \left[\frac{t^2}{2}(-\cos 2t) + \left[t \int \cos 2t - \int \left[\frac{d}{dt} \int -\cos 2t \, dt\right] dt\right]\right]_0^{2\pi}$

$= \left[\frac{t^2}{2}(-\cos 2t) - \left[t \cdot \frac{\sin 2t}{2} - \int \frac{\sin 2t}{2}\right]\right]_0^{2\pi}$

$= \left[\frac{t^2}{2}(-\cos 2t) - \frac{t}{2}\sin 2t + \frac{\cos 2t}{4}\right]_0^{2\pi}$

$= \frac{(2\pi)^2}{2}(-\cos 4\pi) - \frac{(0)^2}{2} - \frac{2\pi}{2}\sin 4\pi + \frac{\cos 4\pi}{4} - \frac{1}{4}$

$= -2\pi^2 - 0 - 0 + \frac{1}{4} - \frac{1}{4}$

$= -2\pi^2$

(h) $\int_1^2 x^4 (\ln x)^2 dx$

$= \left[(\ln x)^2 \int x^4 dx - \int \left[\frac{d(\ln x)^2}{dx} \int x^4 dx\right] dx\right]_1^2$

$= \left[(\ln x)^2 \frac{x^5}{5} - 2\int \ln x \cdot \frac{1}{x} \cdot \frac{x^5}{5} dx\right]_1^2$

$= \left[\frac{(\ln x)^2}{5} x^5 - \frac{2}{5}\int \ln x \cdot x^4 dx\right]_1^2$

$= \left[\frac{x^5}{5}(\ln x)^2 - \frac{2}{5}\left[\ln x \int x^4 - \int \left[\frac{d \ln x}{dx} \int x^4 dx\right] dx\right]\right]_1^2$

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$$= \left[\frac{x^5}{5}(\ln x)^2 - \frac{2}{5}\left[\ln x \frac{x^5}{5} - \int \frac{1}{x}\cdot\frac{x^5}{5}dx\right]\right]_1^2$$

$$= \left[\frac{x^5}{5}(\ln x)^2 - \frac{2}{5}\left[\ln x\cdot\frac{x^5}{5} - \frac{x^4}{25}\right]\right]_1^2$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{2}{5}\left[\ln 2\,\frac{32}{5} - \left(\frac{32}{25} - \frac{1}{25}\right)\right]$$

$$= \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125} + c$$

2.

a $\int \sin^3 x \cos^3 x\, dx$

put, $u = \sin^2 x$

$du = \sin x \cos x\, dx$

Now,

$\int u \cos^2 x\, du$

$= \int u(1-u^2)\,du$

$= \int (u - u^3)\,du$

$= \frac{u^2}{2} - \frac{u^4}{4} + c$

$= \frac{(\sin^2 x)^2}{2} - \frac{(\sin^3 x)^3}{4} + c$

$= \sin^4$

$u = \sin x$

$du = \cos x\, dx$

$\int u^2 \cos^2 du$

$= \int u^2(1-u^2)\,du$

$= \int u^2 - u^4\, du$

$= \frac{u^3}{3} - \frac{u^5}{5} + c$

$= \frac{\sin^3}{3} - \frac{\sin^5}{5} + c$

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(8) $\int_0^{\pi} \sin^2 t \cos^4 t \, dt$

Put $\sin t = u$ $\qquad$ $\cos t = u$

$\Rightarrow \cos t \, dt = du$. $\qquad$ $-\sin t \, dt = du$

Now,

$\int_0^{\pi} u^2 \cos^3 t \, du$ $\qquad$ $-\int_0^{\pi} u^4 \sin t \, du$

$= \int_0^{\pi} u^2 (1-u^2)^{3/2} du$ $\qquad$ $-\int_0^{\pi} u^4 (\sqrt{1-u^2}) \, du$

$= \int_0^{\pi} u^2 - u^5 \, du$

$= \left[\frac{u^3}{3} - \frac{u^6}{6}\right]_0^{\pi} + c$

$= \left[\frac{\sin^3 t}{3} - \frac{\sin^6 t}{6}\right]_0^{\pi} + c$

$= \int_0^{\pi} (\sqrt{1-\cos^2 t}) \cos^4 t \, dt$

$= \int_0^{\pi} (\cos^4 t - \cos^6 t) dt$

$= \int_0^{\pi} \cos^4 t \, dt - \int_0^{\pi} \cos^6 t \, dt$

$= \left[\frac{\cos^3 t \sin t}{4} + \frac{3}{4}\left(\frac{\cos t \sin t}{2} + \frac{1}{2} t\right)\right]_0^{\pi} -$

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Improper integral

An integral $\int_a^b f(x)dx$ is improper if

(i) a or b or both limits of integration is infinite

(ii) f(x) becomes infinite at some point in the interval (a,b)

Integrals with infinite limits.

(i) $\int_a^\infty f(x)dx$ is defined as $\lim_{b\to\infty}\int_a^b f(x)dx$ if the limit exists

(ii) $\int_{-\infty}^b f(x)dx$ is defined as $\lim_{a\to-\infty}\int_a^b f(x)dx$ if the limit exists.

(iii) $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$ if the limit exists.

f(x) infinite at some point,

(i) If $f(x)\to\infty$ as $x\to a$ then $\int_a^b f(x)dx = \lim_{h\to 0}\int_{a+h}^b f(x)dx$ $h>0$

(ii) If $f(x)\to\infty$ as $x\to b$ then $\int_a^b f(x)dx = \lim_{h\to 0}\int_a^{b-h} f(x)dx$.

(iii) If $f(x)\to\infty$ as $x\to c$ $(a<c<b)$.

$\int_a^b f(x)dx = \lim_{h\to 0}\int_a^{c-h} f(x)dx + \lim_{h'\to 0}\int_{c+h'}^b f(x)dx$.

* $\int_1^\infty \frac{dx}{x^3}$

$= \lim_{b\to\infty}\int_1^b \frac{dx}{x^3}$

$= \lim_{b\to\infty}\left[-\frac{1}{2x^2}\right]_1^b$

$= \lim_{b\to\infty}\left[-\frac{1}{2}\left(\frac{1}{b^2}-1\right)\right]_1^b$

$= -\frac{1}{2}$ Ans.

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* $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

$= \int_{-\infty}^{a} \frac{1}{1+x^2} dx + \int_{a}^{\infty} \frac{1}{1+x^2} dx$

$= \lim_{b \to -\infty} \int_{b}^{a} \frac{1}{1+x^2} dx + \lim_{c \to \infty} \int_{a}^{c} \frac{1}{1+x^2} dx$

$= \lim_{b \to -\infty} [\tan^{-1} x]_{b}^{a} + \lim_{c \to \infty} [\tan^{-1} x]_{a}^{b}$

$= \lim_{b \to -\infty} [\tan^{-1} a - \tan^{-1} b] + \lim_{c \to \infty} [\tan^{-1} b - \tan^{-1} a]$

$= \tan^{-1} a - \tan^{-1}(-\infty) + \tan^{-1}\infty - \tan^{-1} a$

$= \tan^{-1}\infty + \tan^{-1}\infty$

$= \pi/2 + \pi/2$

$= \pi/4$

* $\int_{0}^{1} \frac{1}{1-x} dx$

when $x = 1$, $\frac{1}{1-x} \to \infty$ so it is improper.

$\therefore \int_{0}^{1} \frac{1}{1-x} dx = \lim_{h \to 0} \int_{0}^{1-h} \frac{1}{1-x} dx$

$= \lim_{h \to 0} -[\log(1-x)]_{0}^{1-h}$

$= \lim_{h \to 0} -[\log(1-(1-h)) - \log(1-0)]$

$= \lim_{h \to 0} -[\log(-h) - \log 1]$

$= -\log 0 - \log 1$

$= -\infty$

CURSIVE

* $\int_0^3 \frac{dx}{(x-1)^{2/3}}$

when $x \to 1$, $\frac{1}{(x-1)^{2/3}} \to \infty$, $0 < 1 < 3$

Now,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \lim_{h \to 0} \int_0^{1-h} \frac{dx}{(x-1)^{2/3}} + \lim_{h' \to 0} \int_{1+h'}^{3} \frac{dx}{(x-1)^{2/3}}$$

Exercise 5.6

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1) (a) $\int_{3}^{\infty} \frac{1}{(x-2)^{3/2}} dx$

$= \lim_{b \to \infty} \int_{3}^{b} \frac{1}{(x-2)^{3/2}} dx$

$= \lim_{b \to \infty} \left[\frac{(x-2)^{-1/2}}{-1/2} \right]_{3}^{b}$

$= -2 \lim_{b \to \infty} \left[\frac{1}{\sqrt{x-2}} \right]_{3}^{b}$

$= -2 \lim_{b \to \infty} \left[\frac{1}{\sqrt{b-2}} - \frac{1}{\sqrt{3-2}} \right]$

$= -2 \left[\frac{1}{\sqrt{\infty-2}} - \frac{1}{1} \right]$

$= -2 \times -1 = 2 \Rightarrow$ converges,

(b) $\int_{-\infty}^{0} \frac{1}{1-4x} dx$

$= \lim_{a \to -\infty} \int_{a}^{0} \frac{1}{1-4x} dx$

$= \lim_{a \to -\infty} -\frac{1}{4} \left[\log(1-4x) \right]_{a}^{0}$

$= \lim_{a \to -\infty} -\frac{1}{4} \left[\log(1-4a) - \log(1-0) \right]$

$= -\frac{1}{4} \left[\log(-\infty) - \log(1) \right]$

$= -\infty$, divergent

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(c) $\int_2^\infty e^{-5p}\,dp$

$= \lim_{a\to\infty} \int_2^a e^{-5p}\,dp$

$= \lim_{a\to\infty} \left[\frac{e^{-5p}}{-5}\right]_2^a$

$= \lim_{a\to\infty} -\frac{1}{5}\left[e^{-5a} - e^{-5\times 2}\right]$

$= \lim_{a\to\infty} -\frac{1}{5}\left[e^{-5a} - e^{-10}\right]$

$= -\frac{1}{5}\left(\frac{1}{e^{\infty}} - \frac{1}{e^{10}}\right)$

$= -\frac{1}{5}\left(\frac{1}{\infty} - \frac{1}{e^{10}}\right)$

$= \frac{e^{-10}}{5}$, convergent.

(d) $\int_0^\infty \frac{x^2}{\sqrt{1+x^3}}\,dx$

let $y = 1+x^3$ $\therefore dy = 3x^2$

Now,

$= \lim_{a\to\infty} \int_0^a \frac{1}{3} \frac{dy}{\sqrt{y}}$

$= \lim_{a\to\infty} \frac{1}{3}\left[y^{1/2}\right]_0^a$

$= \lim_{a\to\infty} \frac{1}{3}\left[(1+x^3)^{1/2}\right]_0^a$

$= \lim_{a\to\infty} \frac{1}{3}\left[(1+a^3)^{1/2} - (1+0)^{1/2}\right]$

$= \frac{1}{3}(1+\infty)^{1/2} - 1$

$= \infty$, divergent

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(e) $\int_{-\infty}^{\infty} xe^{-x^2}dx$

$= \int_{-\infty}^{a} xe^{-x^2}dx + \int_{a}^{\infty} xe^{-x^2}dx$

$= \lim_{b\to-\infty} \int_b^a xe^{-x^2}dx + \lim_{c\to\infty} \int_a^c xe^{-x^2}dx.$

put $x^2 = y$. $\therefore 2xdx = dy,$

$= \lim_{b\to-\infty} \int_b^a \frac{e^{-y}}{2}dy + \lim_{c\to\infty} \int_a^c \frac{e^{-y}}{2}dy,$

$= \lim_{b\to-\infty} -\frac{1}{2}\left[e^{-y}\right]_b^a + \lim_{c\to\infty} -\frac{1}{2}\left[e^{-y}\right]_a^c$

$= \lim_{b\to-\infty} -\frac{1}{2}\left[e^{-x^2}\right]_b^a + \lim_{c\to\infty} -\frac{1}{2}\left[e^{-x^2}\right]_a^c$

$= \lim_{b\to-\infty} -\frac{1}{2}\left[e^{-a^2} - e^{-b^2}\right] + \lim_{c\to\infty} -\frac{1}{2}\left[e^{-c^2} - e^{-a^2}\right]$

$= -\cancel{\frac{e^{-a^2}}{2}} + \cancel{\frac{e^{-\infty}}{2}} - \frac{1}{2}e^{-\infty} + \cancel{\frac{e^{-a^2}}{2}}$

$= 0.$ Convergent,

(f) $\int_0^\infty \sin^2\alpha \, d\alpha$

$= \lim_{b\to\infty} \int_0^b \frac{1}{2}(1-\cos 2\alpha)\,d\alpha$

$= \lim_{b\to\infty} \frac{1}{2}\left[\alpha - \sin 2\alpha\right]_0^b$

$= \lim_{b\to\infty} \frac{1}{2}\left[b - 0 - [\sin 2b - \sin 0]\right]$

$= \lim_{b\to\infty} \frac{1}{2}\left[b - \sin 2b - 0\right]$

$= \infty - \infty$

$= \infty$

(9) $\int_2^\infty \frac{dv}{v^2+2v-3}$

$= \lim_{a\to\infty} \int_2^a \frac{dv}{v^2+3v-v-3}$

$= \lim_{a\to\infty} \int_2^a \frac{dv}{v(v+3)-1(v+3)}$

$= \lim_{a\to\infty} \int_2^a \frac{dv}{(v+3)(v-1)}$

$= \lim_{a\to\infty} \int_2$ Now, $\frac{1}{(v+3)(v-1)} = \frac{A}{v+3} + \frac{B}{v-1}$

$1 = A(v-1) + B(v+3)$

$1 = Av - A + Bv + 3B$

Now, $(A+B)v = 0$ Also, $3B - A = 1.$

$A + B = 0$ $3B - (-B) = 1$

And $A + \frac{1}{4} = 0$ $\therefore A = -\frac{1}{4}.$ $4B = 1.$ $\therefore B = \frac{1}{4}.$

Now,

$\lim_{a\to\infty} \int_2^a \frac{1}{4}\left(\frac{1}{v-1} - \frac{1}{v+3}\right) dv$

$= \lim_{a\to\infty} \frac{1}{4}\left[\log(v-1) - \log(v+3)\right]_2^a$

$= \lim_{a\to\infty} \frac{1}{4}\left[\log(a-1) - \log(2-1) - \left[\log(a+3) - \log(2+3)\right]\right]$

$= \frac{1}{4}\left[\log\infty - \log 1 - \log\infty + \log 5\right]$

$= \frac{\log 5}{4}$ ans.

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(h) $\int_1^\infty \frac{\ln x}{x}\, dx$

$= \lim_{a\to\infty} \int_1^a \frac{\ln x}{x}\, dx$

Put $\ln x = y$.

$\therefore \frac{1}{x}\, dx = dy$,

$= \lim_{a\to\infty} \int_1^a y.dy$.

$= \lim_{a\to\infty} \left[\frac{y^2}{2}\right]_1^a$

$= \lim_{a\to\infty} \frac{1}{2}\left[(\ln x)^2\right]_1^a$

$= \lim_{a\to\infty} \frac{1}{2}\left[(\ln a)^2 - (\ln 1)^2\right]$

$= \frac{1}{2}\left[(\ln\infty)^2 - 0\right]$

$= \frac{1}{2}\infty = \infty$, divergent,

(i) $\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\, dx$.

$= \int_{-\infty}^{a} \frac{x^2}{9+x^6}\, dx + \int_a^\infty \frac{x^2}{9+x^6}\, dx$

$= \lim_{b\to-\infty} \int_b^a \frac{x^2}{9+x^6}\, dx + \lim_{c\to\infty} \int_a^c \frac{x^2}{9+x^6}\, dx$

Put $x^3 = y$. $\therefore 3x^2 dx = dy$.

$\therefore \lim_{b\to-\infty} \int_b^a \frac{1}{3} \frac{dy}{3^2+y^2} + \lim_{c\to\infty} \int_a^c \frac{1}{3} \frac{dy}{3^2+y^2}$

put $y = 3\tan\theta$. $\therefore 3\sec^2\theta\, d\theta = dy$,

$\frac{1}{3}\left[\lim_{b\to-\infty} \int_b^a \frac{3\sec^2\theta\, d\theta}{3^2+3^2\tan^2\theta} + \lim_{c\to\infty} \int_a^c \frac{3\sec^2\theta\, d\theta}{3^2+3^2\tan^2\theta}\right]$

$= \lim_{b\to-\infty} \int_b^a \frac{1}{9}\, d\theta + \lim_{c\to\infty} \int \frac{1}{9}\, d\theta$

$= \frac{1}{9}\left[\lim_{b\to-\infty} [\theta]_a^b + \lim_{c\to\infty} [\theta]_a^c\right]$

$= \frac{1}{9}\left[\lim_{b\to-\infty} [\tan^{-1}x^3]_a^b + \lim_{c\to\infty} [\tan^{-1}x^3]_a^c\right]$

$= \frac{1}{9}\left[\lim_{b\to-\infty} [\tan^{-1}a^3 - \tan^{-1}b^3] + \lim_{c\to\infty} [\tan^{-1}c^3 - \tan^{-1}a^3]\right]$

$= \frac{1}{9}(-\tan^{-1}(-\infty)) + \frac{1}{9}\tan^{-1}\infty$

$= \frac{1}{9}\tan^{-1}\infty + \frac{1}{9}\tan^{-1}\infty$

$= \frac{1}{9}\times\frac{\pi}{2} + \frac{1}{9}\times\frac{\pi}{2}$

$= \frac{2\pi}{18}$

$= \pi/9$, converges,

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(k) $\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$

$= \lim_{h\to 0} \int_{-2+h}^{14} \frac{dx}{\sqrt[4]{x+2}}$

$= \lim_{h\to 0} \int_{-2+h}^{14} (x+2)^{-1/4}$

$= \lim_{h\to 0} \left[\frac{4}{3}(x+2)^{3/4}\right]_{-2+h}^{14}$

$= \lim_{h\to 0} \frac{4}{3}\left[(14+2)^{3/4} - (-2+h+2)^{3/4}\right]_{-2+h}^{14}$

$= \lim_{h\to 0} \frac{4}{3}\left[(16)^{3/4} - h^{3/4}\right]$

$= \lim_{h\to 0} \frac{4}{3}(8 - h^{3/4})$

$= \frac{4}{3} \times 8$

$= 32/3$, converges,

(l) $\int_{-2}^{3} \frac{1}{x^4} dx$

$= \lim_{h\to 0} \int_{-2}^{0+h} \frac{1}{x^4} dx + \lim_{h'\to 0} \int_{0+h'}^{3} \frac{1}{x^4} dx$

$= \lim_{h\to 0} -\frac{1}{3}\left[\frac{1}{x^3}\right]_{-2}^{h} + \lim_{h'\to 0} \left[-\frac{1}{3}\frac{1}{x^3}\right]_{h'}^{3}$

$= -\frac{1}{3} \lim_{h\to 0} \left[\frac{1}{h^3} + \frac{1}{2^3}\right] - \lim_{h'\to 0} \left[\frac{1}{3}\left(\frac{1}{3^3} - \frac{1}{h'^3}\right)\right]$

$= -\frac{1}{3} \times \infty - \frac{1}{3}\infty = \infty$ divergent,

(m) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$= \lim_{h\to 0} \int_0^{1-h} \frac{1}{\sqrt{1-x^2}}\,dx$

$= \lim_{h\to 0} \left[\sin^{-1}x\right]_0^{1-h}$

$= \lim_{h\to 0} \left[\sin^{-1}(1-h) - \sin^{-1}0\right]$

$= \sin^{-1}(1-0) - \sin^{-1}0$

$= \pi/3 - \pi/6$

$= \frac{\pi}{6}$, converges

(n) $\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx.$

$= \lim_{h\to 0} \int_0^{1-h} \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{h'\to 0} \int_{1+h'}^{9} \frac{1}{\sqrt[3]{x-1}}\,dx$

$= \lim_{h\to 0} \frac{3}{2}\left[(x-1)^{2/3}\right]_0^{1-h} + \lim_{h'\to 0} \frac{3}{2}\left[(x-1)^{2/3}\right]_{1+h'}^{9}$

$= \lim_{h\to 0} \frac{3}{2}\left[(1-h-1)^{2/3} - (0-1)^{2/3}\right] + \frac{3}{2}\lim_{h'\to 0}\left[(9-1)^{2/3} - (1+h'-1)^{2/3}\right]$

$= \frac{3}{2}[0+1] + \frac{3}{2}[4-0]$

$= \frac{3}{2}(5) = 10/3$

$= \frac{3}{2} + \frac{12}{2}$

$= \frac{3}{2} + 6$

$= 15/2$

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(c) $\int_{\pi/2}^{\pi} \operatorname{cosec} x \, dx$

$= \lim_{h \to 0} \int_{\pi/2}^{\pi+h} \operatorname{cosec} x \, dx$

$= \lim_{h \to 0} \int_{\pi/2}^{\pi+h} \sqrt{\cot^2 x - 1} \, dx$

$= \lim_{h \to 0} \frac{3}{2} \left[(\cot^2 x - 1)^{2/3} \right]_{\pi/2}^{\pi+h}$

$= \lim_{h \to 0} \frac{3}{2} \left[(\cot^2(\pi+h) - 1)^{2/3} - (\cot^2 \pi/2 - 1)^{2/3} \right]$

$= \frac{3}{2} \left[(\cot^2 \pi - 1)^{2/3} - (\cot^2 \pi/2 - 1)^{2/3} \right]$

$= \frac{3}{2} [\infty + 1]$

$= \infty$, divergent,

(d) $\int_{-1}^{0} \frac{e^{1/x}}{x^3} \, dx$

=

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Application of Antiderivatives:

How to find the area of region between a curve $y = f(x)$, $a \le x \le b$ and x-axis

(i) Partition $[a, b]$ with zeros of f.

(ii) Integrate f over each interval.

(iii) Add the absolute value of the integrals.

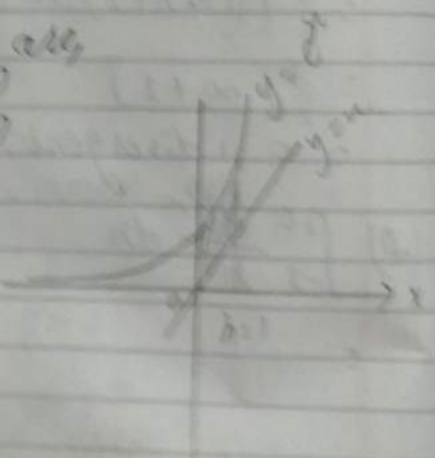
1. Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.

Here,

The given curves are,

$$y = e^x \quad \text{--- (i)}$$

$$y = x \quad \text{--- (ii)}$$



The area bounded by the curves is given by

$$A = \int_0^1 e^x - \int_0^1 x$$

$$= \left[e^x\right]_0^1 - \left[\frac{x^2}{2}\right]_0^1$$

$$= 1.72 - 0.5$$

$$= 1.22$$

$\approx e - 1.5$ square units.

2. Find
and
Here,

2. Find the area of the region enclosed by $x+y^2=0$ and $x+3y^2=2$.

Here, given the curves are,

$x+y^2=0$ — (i) $\therefore y^2=-x$

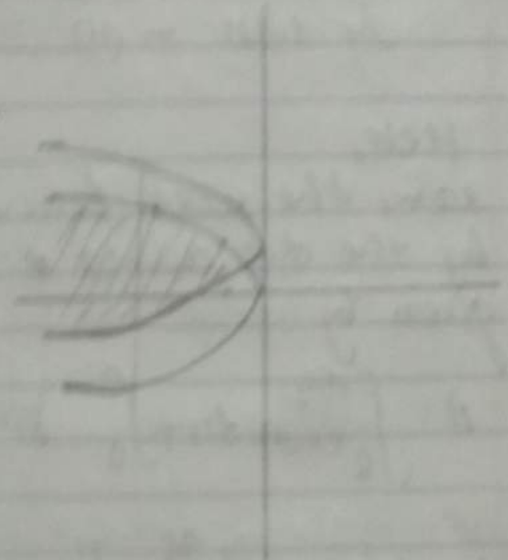
$x+3y^2=2$ — (ii) $\therefore y^2=\frac{1}{3}(2-x)$

(ii) $x+3(-x)=2$

or $x-3x=2$

$-2x=2$

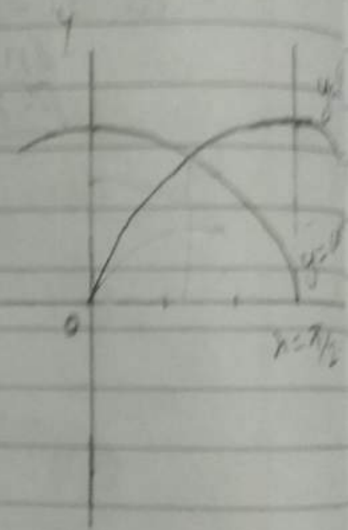
$\boxed{\therefore x=-1}$ $\therefore y=1$



(3) Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

Here,
given curves are:

$$y = \sin x \quad \text{— (i)}$$
$$y = \cos x \quad \text{— (ii)}$$



Here,
Now, the area bounded by the two curves is given by

$$A = \int_0^{\pi/2} \cos x \, dx + \int_0^{\pi/2} \sin x \, dx$$

$$A = [\sin x]_0^{\pi/2} + [-\cos x]_0^{\pi/2}$$
$$= 1 - 0 - 0 + 1$$
$$= 2$$

$2 - (2\sqrt{2} - 2)$

$2 - 2\sqrt{2} + 2$

$4 - 2\sqrt{2}$

$2(2 - \sqrt{2})$

$\int_0^{\pi/4} \sin x + \int_{\pi/4}^{\pi/2} \cos x$

$-\frac{1}{\sqrt{2}} + \cdots - \frac{1}{\sqrt{2}}$ $\quad 2 - \sqrt{2}$

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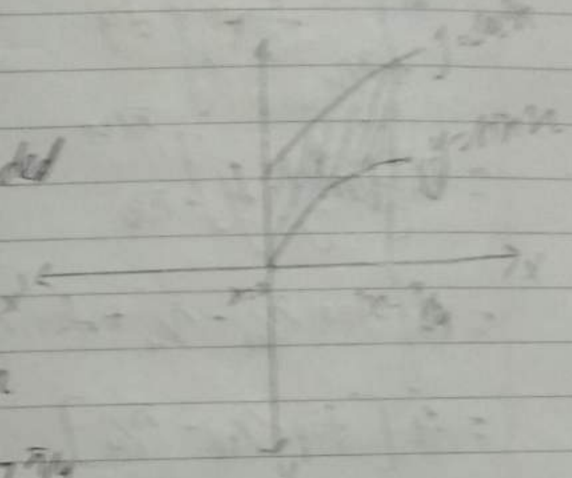
4. Find the area between two curves $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \pi/4$.

Here,

Given two curves are,

$y = \sec^2 x$ — (i)

$y = \sin x$ — (ii)



Now, the area bounded by the curves is given by,

$$A = \int_0^{\pi/4} \sec^2 x - \int_0^{\pi/4} \sin x$$

$$= [\tan x]_0^{\pi/4} - [-\cos x]_0^{\pi/4}$$

$$= \cancel{1} - 0 + \frac{1}{\sqrt{2}} - \cancel{1} = 1/\sqrt{2} \text{ square units.}$$

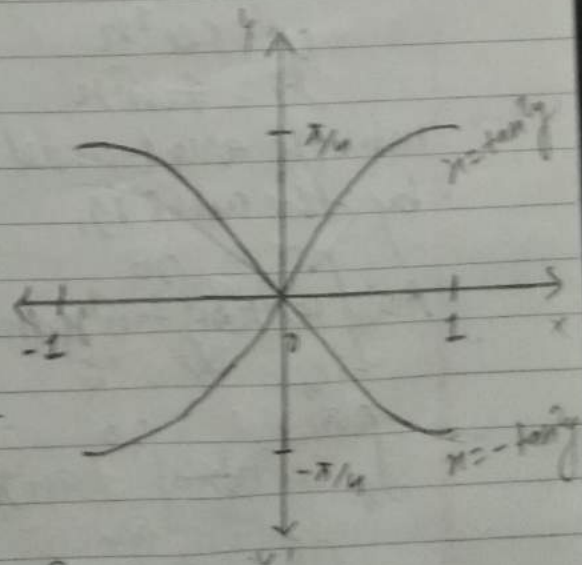
5. Find the area between two curves $x = \tan^2 y$ and $x = -\tan^2 y$; $-\pi/4 \le y \le \pi/4$.

Here

given curves are,

$x = \tan^2 y$

$x = -\tan^2 y$



Now

Area of between curves is,

$$A = \int_{-\pi/4}^{0} \tan^2 y\,dy - \int_{-\pi/4}^{0} -\tan^2 y\,dy$$

$$\int_{0}^{\pi/4} \tan^2 y\,dy - \int_0^{\pi/4} -\tan^2 y\,dy$$

$$= \int_{-\pi/4}^{0} (\tan^2 y + \tan^2 y)\,dy + \int_0^{\pi/4} (\tan^2 y + \tan^2 y)\,dy$$

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$= 2\int_{-\pi/4}^{0} \tan^2 y\,dy + 2\int_{0}^{\pi/4} \tan^2 y\,dy$

$= 2\int_{-\pi/4}^{\pi/4} \tan^2 y\,dy$

$= 2\left[\tan y - y\right]_{-\pi/4}^{\pi/4}$

$= 2\left[\tan \pi/4 - \pi/4 + \tan \pi/4 - \pi/4\right]$

$= 4\left[\tan \pi/4 - \pi/4\right]$

$= 4 - \pi$ square unit

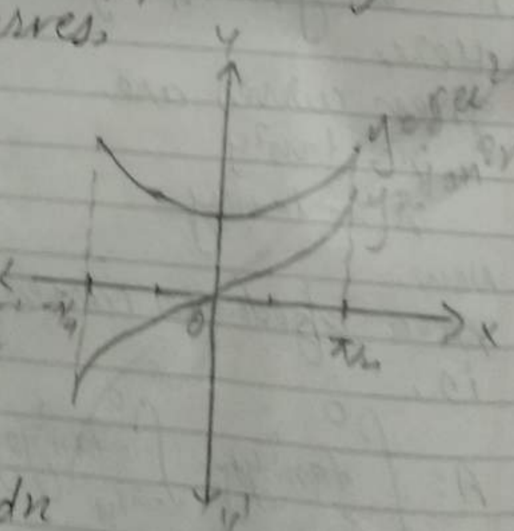
6. Find the area between two curves $y = \sec^2 x$ and $y = \tan^2 x$, $-\pi/4 \le x \le \pi/4$.

Here, given two curves,

$y = \sec^2 x$

$y = \tan^2 x$

Now, the area bounded by the curve is,



$A = \int_{-\pi/4}^{0} \sec^2 x\,dx - \int_{-\pi/4}^{0} \tan^2 x\,dx + \int_{0}^{\pi/4} \sec^2 x\,dx - \int_{-\pi/4}^{0} \tan^2 x\,dx$

$= \int_{-\pi/4}^{0} (\sec^2 x - \tan^2 x)\,dx + \int_{0}^{\pi/4} (\sec^2 x - \tan^2 x)\,dx$

$= \int_{-\pi/4}^{\pi/4} (\sec^2 x - \tan^2 x)\,dx = \left[\tan y - \tan y + y\right]_{-\pi/4}^{\pi/4}$

$= \pi/4 - (-\pi/4) = \pi/2$ square units

Q. 7 Find the area of region between the curve and x-axis.

(i) $f(x) = -x^2 - 2x, [-3, 2]$.

Here, the given curve,

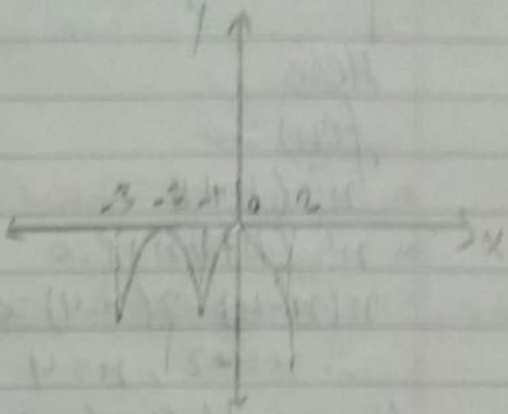
$f(x) = -x^2 - 2x$

Now,

$-x^2 - 2x = 0$

or $x(x+2) = 0$

$\therefore x = 0, x = -2$



Here, the intervals are: $[-3,-2], [-2,0], [0,2]$.

Now,

$$\int_{-3}^{-2} -x^2 - 2x\,dx = \left[-\frac{x^3}{3} - x^2\right]_{-3}^{-2} = \left[\frac{-(-8)}{3} - 4 - \left[\frac{-(-27)}{3} - 9\right]\right]$$

$$= \frac{8}{3} - 4 - \frac{27}{3} + 9 = \frac{8-12-27+27}{3} = -\frac{4}{3}$$

$$\int_{-2}^{0} -x^2 - 2x\,dx = \left[0 - \left[\frac{-(-8)}{3} - 4\right]\right] = \frac{-8}{3} + 4 = \frac{4}{3}$$

$$\int_{0}^{2} -x^2 - 2x\,dx = \left[\frac{-8}{3} - 4 - 0\right] = \frac{-20}{3}$$

Thus the total area is,

$$A = \left|-\frac{4}{3}\right| + \left|\frac{4}{3}\right| + \left|-\frac{20}{3}\right|$$

$$= \frac{4}{3} + \frac{4}{3} + \frac{20}{3}$$

$$= \frac{28}{3} \text{ square unit.}$$

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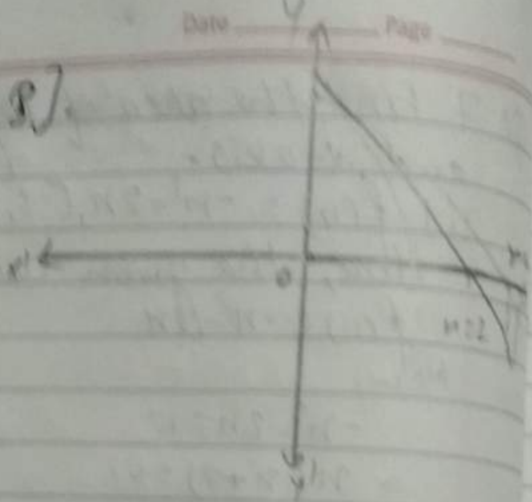
(ii) $f(x) = x^2 - 6x + 8$ $[0,3]$.



Here

$f(x) = 0$

$\Rightarrow x^2 - 6x + 8 = 0$

$\Rightarrow x^2 - (4+2)x + 8 = 0$

$x(x-4) - 2(x-4) = 0$

$\therefore x = 2,\ x = 4$

Hence it divides into $[0,2][2,3]$ subintervals.

Now

$$\int_0^2 x^2 - 6x + 8\,dx = \left[\frac{x^3}{3} - 3x^2 + 8x\right]_0^2 = \left[\frac{8}{3} - 12 + 16\right] = \frac{20}{3}$$

$$\int_2^3 x^2 - 6x + 8\,dx = \left[\frac{27}{3} - 27 + 24 - \frac{8}{3} + 12 - 16\right] = \frac{27-8}{3} - 3$$

$$= \frac{19}{3} - 7 = -\frac{2}{3}$$

Now, the total area is,

$$A = \left|\frac{20}{3}\right| + \left|-\frac{2}{3}\right| = \frac{22}{3} \text{ squ. unit.}$$

(iii) $f(x) = x^3 - 4x$; $-2 \le x \le 2$.

Here

$f(x) = x^3 - 4x$

$\Rightarrow x^3 - 4x = 0$,

$x = 0, \pm 2$.

Now the function in given interval is sub divided into $[-2,0]$, $[0,2]$.

Now,

$$\int_{-2}^0 x^3 - 4x\,dx = \left[\frac{x^4}{4} - 2x^2\right]_{-2}^0 = -\left[\frac{16}{4} - 4x^2\right] = \frac{16}{4}$$

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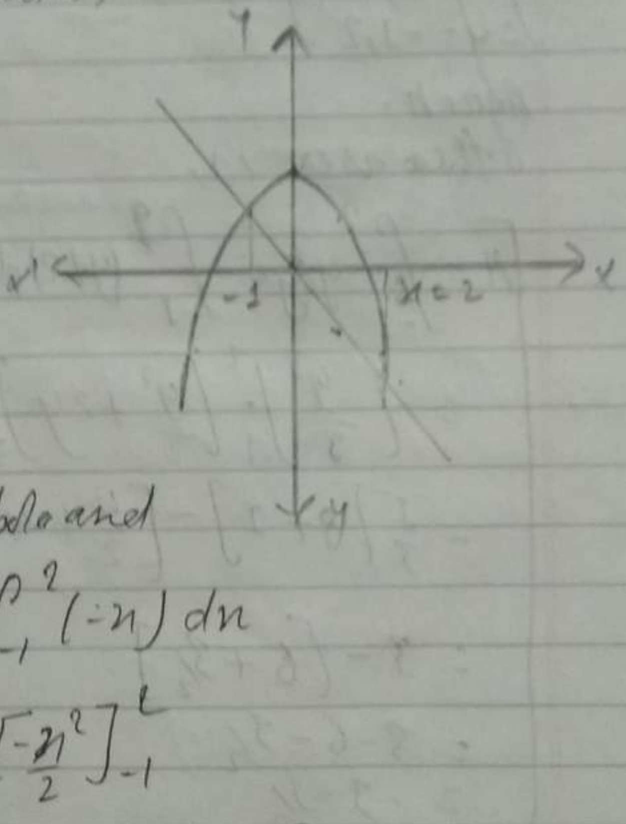
$$\int_0^2 x^3 - 4x\,dx = \left[\frac{16}{4} - 2\times 4\right] = 4.$$

Now,
the total area is,

$$A = |4| + |4| = 8 \text{ squ unit.}$$

8. Find the area of region enclosed by the parabola $y = 2 - x^2$ and line $y = -x$.

Here,
given parabola,
$y = 2 - x^2$ and line,
$y = -x$.



Now,

$$-x = 2 - x^2$$

or $x^2 - x - 2 = 0$

or $x^2 - (2-1)x - 2 = 0$

or $x(x-2) + 1(x-2) = 0$

$\therefore x = -1, 2.$

Now,
Area enclosed by parabola and line is,

$$A = \int_{-1}^{2} (2 - x^2)\,dx - \int_{-1}^{2} (-x)\,dx$$

$$= \left[2x - \frac{x^3}{3}\right]_{-1}^{2} - \left[-\frac{x^2}{2}\right]_{-1}^{2}$$

$$= \left[4 - \frac{8}{3} - \left(-2 + \frac{1}{3}\right)\right] + \left[\frac{4}{2} - \frac{1}{2}\right]$$

$$= \left(6 - \frac{8}{3} - \frac{1}{3}\right) + \left[\frac{3}{2}\right]$$

$$\cancel{= \frac{-16 + 2 + 9}{6} = -\frac{5}{6}}$$

$$= 6 - 3 + \frac{3}{2} = 3 + \frac{3}{2} = \frac{9}{2} \text{ squ. unit}$$

9. Find the area of the region enclosed by parabola $x = y^2$ and line $x = y+2$ in first quadrant.

Here
given curve and line,

$x = y^2$

$x = y+2$

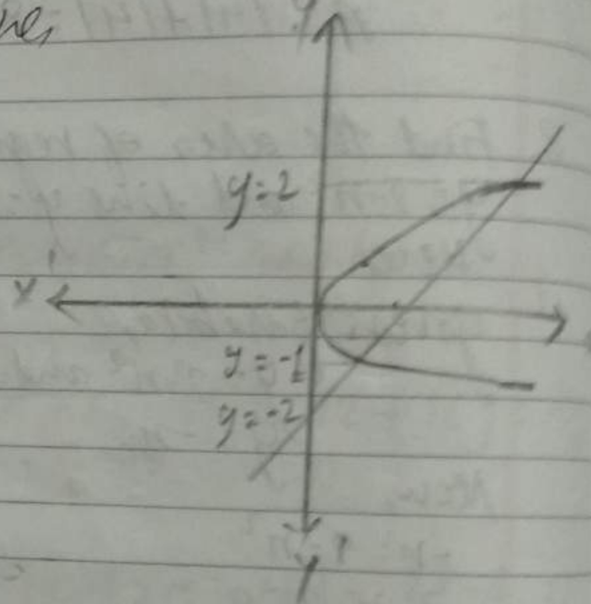
Now,

$y^2 - y - 2 = 0$

$y^2 - (2-1)y - 2 = 0$

$y(y-2) + 1(y-2) = 0$

$\therefore y = -1, 2$



Again,
the area is,

$$A = \int_{-1}^{2} y^2 dy - \int_{-1}^{2} (y+2) dy$$

$$= \left[\frac{y^3}{3}\right]_{-1}^{2} - \left[\frac{y^2}{2} + 2y\right]_{-1}^{2}$$

$$= \frac{1}{3}[8+1] - \left[\frac{4}{2} + 4 - \left[\frac{1}{2} - 2\right]\right]$$

$$= 3 - [6 + 3/2]$$

$$= 3 - 6 - 3/2$$

$$= -3 - 3/2$$

The area in the first quadrant is,

$$\int_0^2 (y+2) dy - \int_0^2 y^2 dy,$$

$$= \left[\frac{y^2}{2} + 2y\right]_0^2 - \left[\frac{y^3}{3}\right]_0^2$$

$$= 2 + 4 - 8/3 = 6 - 8/3 = 10/3 \text{ squ. unit},$$

10. Find the area of the region enclosed by parabola $y^2-4x=4$ and line $4x-y=16$.

Here,

given parabola is,

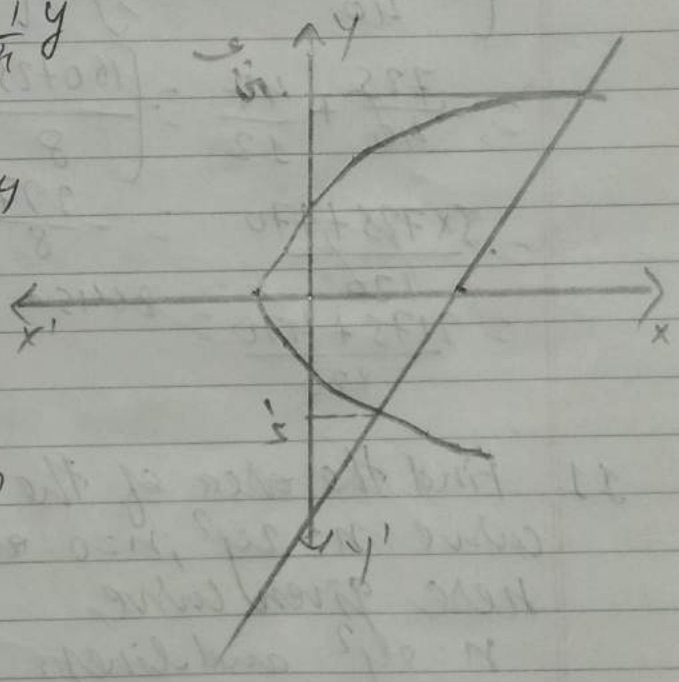
$$y^2-4x=4$$

$$\therefore x=\frac{1}{4}y^2-1 \text{ and}$$

line is,

$$4x-y=16$$

$$x=4+\frac{1}{4}y$$



Now,

$(4x-16)^2-4x=4$.

$16x^2-128x+256-4x=4$

$16x^2-132x+256-4=0$

$16x^2-132x+252=0$

$4x^2-33x+63=0$

$4x^2-(12+21)x+63=0$

$4x(x-3)-21(x-3)=0$

$(x-3)(4x-21)=0$

$\therefore x=3,\ 21/4$.

$y^2-\frac{4}{4}(16+y)=4$

$y^2-16-y-4=0$

$y^2-y-20=0$

$y^2-(5-4)y-20=0$

$y(y-5)+4(y-5)=0$

$\therefore y=-4,5$.

Now,

$$-\int_{-4}^{5} 4+\frac{1}{4}y\,dy+\int_{-4}^{5}\frac{1}{4}y^2-1\,dy$$

$$= \left[4y + \frac{1}{8}y^2\right]_{-4}^{5} + \left[\frac{y^3}{12} - y\right]_{-4}^{5}$$

$$= \left[4\times5 + \frac{1}{8}25 - \left(4\times-4 + \frac{1}{8}16\right)\right] + \left[\frac{125}{12} - 5 - \left(\frac{-64}{12} + 4\right)\right]$$

$$= \left[20 + \frac{25}{8} + 16 - \frac{16}{8}\right] + \left[\frac{125}{12} - 5 + \frac{64}{12} - 4\right]$$

$$= \left[\frac{\cancel{40 + 125 + 640 - 80}}{40}\right] + \left[\frac{125 - 60 + 64 - 48}{12}\right]$$

$$= -\frac{725}{40} + \frac{47}{12} = \left[\frac{160 + 25 + 128 - 16}{8}\right] + \left[\frac{81}{12}\right]$$

$$= \frac{3\times725 + 470}{120} = -\frac{297}{8} + \frac{81}{12} = \frac{-1782 + 324}{48}$$

$$= \frac{2175 + 470}{120} = 26450 \qquad = -\frac{243}{8} \text{ squ. units,}$$

11. Find the area of the region bounded by curve $x = 2y^2$, $x = 0$ and $y = 3$.

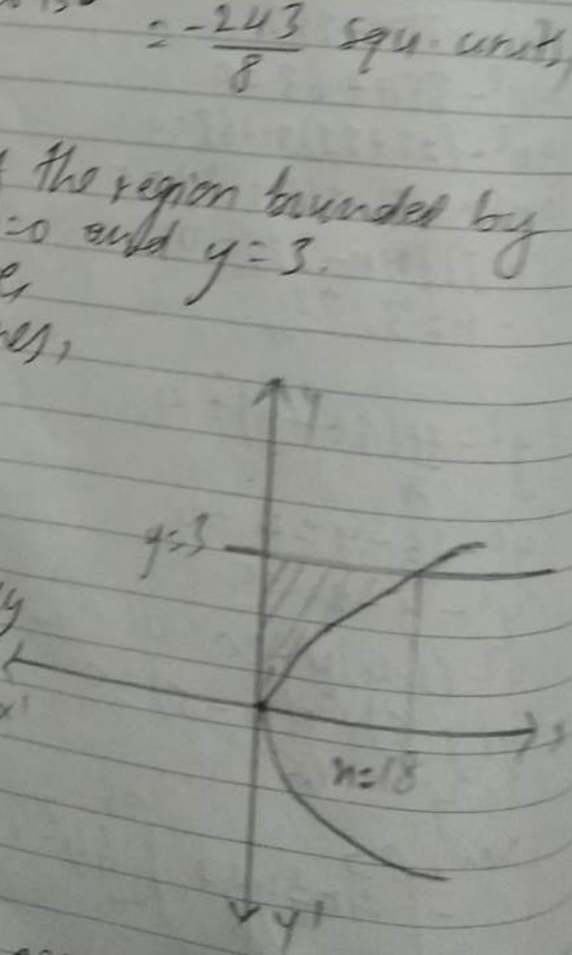
here given curve,

$x = 2y^2$ and lines,

$x = 0$, $y = 3$.

here,

Area of region is,



$$A = \int_0^3 y - 3\,dy - \int_0^3 2y^2 dy$$

$$= \left[\frac{y^2}{2} - 3y\right]_0^3 - \left[\frac{2y^3}{3}\right]_0^3$$

$$= \frac{9^2}{2} - 3\times9 - \frac{2\times 3^3}{3}$$

 $= \cancel{162 - 54 - 3888} = \frac{27 - 162 - 54}{6}$

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12. Find the area bounded by x-axis and curve $y = 4 - x^2$.

Here,

curve is,

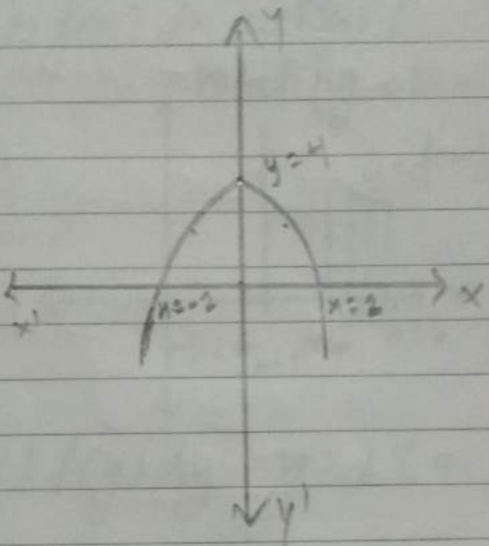
$$y = 4 - x^2$$

when $f(x) = 0$ then,

$x = \pm 2$.

Now,

Area of bounded region is,



$$A = \int_{-2}^{2} 4 - x^2 \, dx$$

$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= 8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right)$$

$$A = 16 - \frac{16}{3} = \frac{32}{3} \text{ squ. unit}$$

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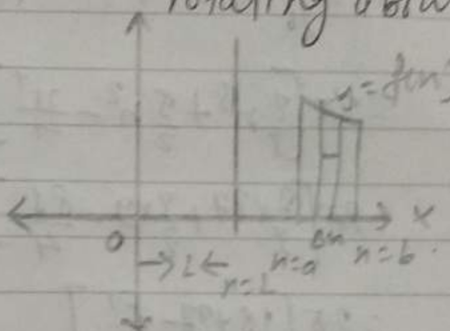
Volume (V) $= \int_a^b 2\pi (x-L) f(x)\, dx$, $x = L \le a$

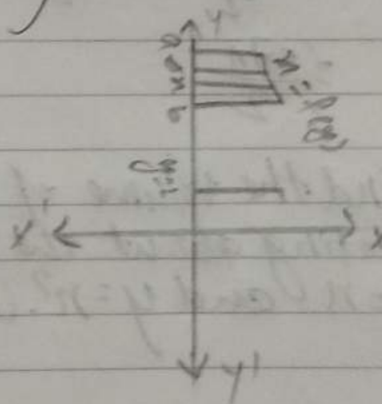
Volume (V) $= \int_a^b 2\pi x f(x)\, dx$ when $L = 0$

rotating about y-axis.



Volume (V) $= \int_a^b 2\pi (y-L) f(y)\, dy$ $y = L \le a$.

volume (V) $= \int_a^b 2\pi y f(y)\, dy$ when $L = 0$



Exercise 6.2

1) The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate a solid. Find the volume of solid.

Here,

$f(x) = 3x - x^2 = 0 \therefore x = 0, 3$

Now,

Volume of solid is $= 2\pi \int_0^3 .$

$V = \int_0^3 2\pi (x-(-1))(3x - x^2)\, dx$

$$V = 2\pi \int_0^3 (x+1)(3x - x^2)\,dx$$

$$= 2\pi \int_0^3 (3x^2 - x^3 + 3x - x^2)\,dx$$

$$= 2\pi \int_0^3 (2x^2 + 3x - x^3)\,dx$$

$$= 2\pi \left[\frac{2}{3}x^3 + \frac{3}{2}x^2 - \frac{x^4}{4}\right]_0^3$$

$$= 2\pi \left[\frac{2}{3}\times 27 + \frac{3}{2}\times 9 - \frac{81}{4}\right]$$

$$= 2\pi \left[18 + \frac{27}{2} - \frac{81}{4}\right]$$

$$= 2\pi \left[\frac{72 + 54 - 81}{\cancel{4}_2}\right]$$

$$= \frac{\cancel{2}\,45\pi}{\cancel{2}} = \frac{45\pi}{2} \text{ Ans.}$$

$$\cancel{= \frac{27\pi}{2}}$$

2. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.