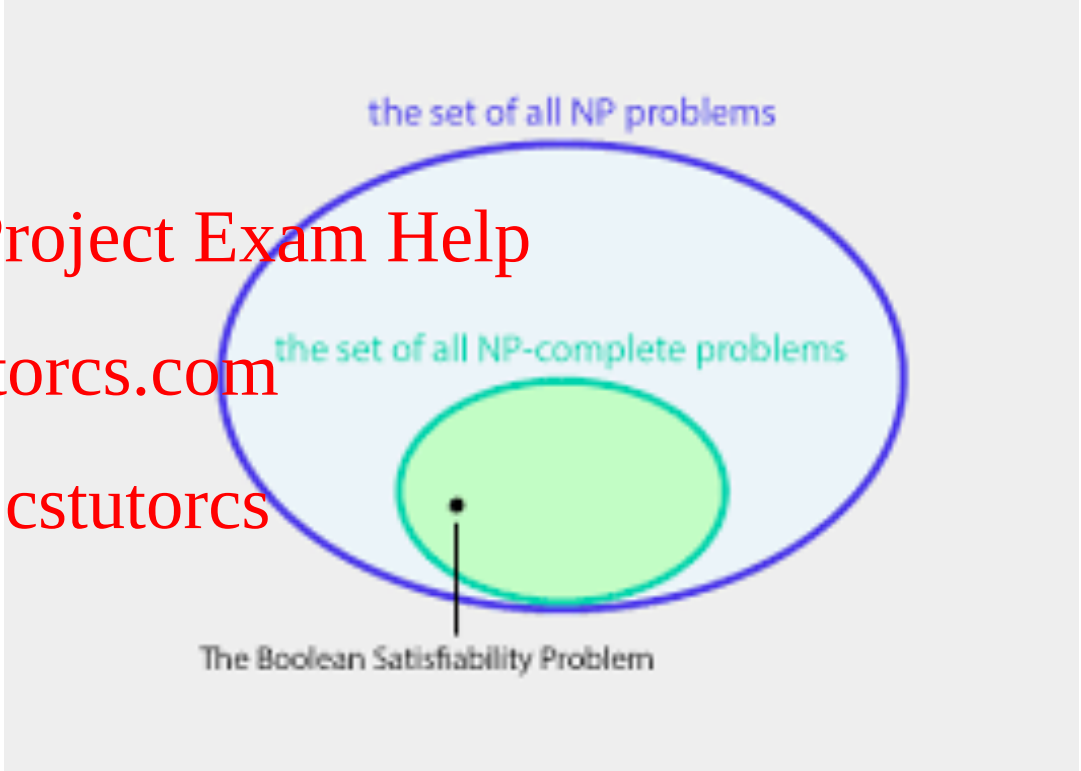
The SAT Problem

- Informal definition
- The SAT problem
- SAT algorithms
- Applications



Why is SAT important?

In theory

Canonical NP-Complete problem

In practice

Applied in many Computer Science problems:

- reachability analysis
- planning
- analysis of gene regulatory network
- fault diagnosis

SAT 2016 competition									
Organizers	Marijn Heule, Matti Järvisalo Tomáš Balyo								
Proceedings	Descriptions of the solvers and benchmarks								
Benchmarks	Available here								
Solvers	Available here								
	Gold	**Silver**	**Bronze**	**Gold**	**Silver**	**Bronze**	**Gold**	**Silver**	**Bronze**
		Agile Track			**Main Track**			**Random Track**	
SAT+UNSAT	Riss	TB_Glucose	CHBR_Glucose	MapleCOMSPS	Riss	Lingeling	Dimetheus	CSCCSat	DCCAlm
		Parallel Track			**No-Limit Track**			**Incremental Library Track**	
SAT+UNSAT	Treengeling	Plingeling	CryptoMiniSat	BreakIDCOMiniSatPS	Lingeling	abcdSAT	CryptoMiniSat	Glucose	Riss
	Best Application Benchmark Solver in the Main Track			**Best Crafted Benchmark Solver in the Main Track**			**Best Glucose Hack in the Main Track**		
SAT+UNSAT	MapleCOMSPS			TC Glucose			Kiel		

What is the Boolean Satisfiability problem?

Informally

Given a propositional formula φ, the Boolean satisfiability (SAT) problem posed on φ means to determine whether there exists a *variable assignment* under which φ *evaluates to true*.

Formally

Let X be a set of propositional variables. Let F be a set of clauses over X.

- F is Satisfiable iff there exists a v: $X \rightarrow \{0,1\}$ s.t. $v \models F$
- F is Unsatisfiable iff $v \not\models F$, for all v: $X \rightarrow \{0,1\}$

Example

$C1 = \{\neg x_1 \vee x_2, \neg x_2 \vee x_3\} = (\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3)$ Sat(C1) = Yes
$x_1 = 0, x_3 = 1,$

$C2 = C1 \cup \{x_1, \neg x_3\} = (x_3 \wedge \neg x_3)$ Sat(C2) = No

Why in CNF?

- There are efficient algorithms that can transform formulae into CNF.

- Most SAT solvers can exploit CNF

  - Easy to detect a conflict
  - Easy to remember partial assignments that don't work (just add 'conflict' clauses)

It's a hard problem..!

A k-SAT problem with input in CNF with at most k literals in each clause…

- Complexity for different values of k:
  - 2-SAT: in P
  - 3-SAT: in NP-complete
  - > 3-SAT: is also NP-complete

A classification of SAT algorithms

- Complete algorithms
  - Proof systems - natural deduction, tableau, etc..
  - Davis-Putman (DP)
    - based on resolution
  - Stalmarck's method
  - Davis-Logemann-Loveland (DLL/DPLL)
    - search-based, basis for most successful solvers
  - Conflict-Driven Clause Learning (CDCL)
- Incomplete algorithms
  - Local search /hill climbing
  - Genetic algorithms

Notation and Terminology

- **Polarity of a variable** in a clause

  The "sign" with which the variable appears in the clause:

  $x_1 \vee \neg x_2$ x_1 positive polarity, x_2 negative polarity

- **Clause shorthand notations**

  Let $C = x_1 \vee x_2 \vee \neg x_3 \vee \neg x_4$, we can write it

  $x_1x_2\neg x_3\neg x_4$ or as $x_1, x_2, \neg x_3, \neg x_4$ or as $\{x_1, x_2, \neg x_3, \neg x_4\}$

- **Resolution**:

  Two clauses C1 and C2 that contain a variable x with opposite polarities, implies a new clause C3 that contains all literals except x and ¬x.

$\{x_1, \neg x_2\}$ $\{x_2, \neg x_3\}$ ⟶ $\{x_1, \neg x_3\}$

Davis-Putman (DP) Algorithm

Give a set S of clauses

For every variable in S

1. for every clause C_i containing the variable and every clause C_j containing the negation of the variable resolve C_i and C_j
2. add the resolvent to the set S of clauses
3. remove from S all original clauses containing the variable or its negation.

Two possible termination cases:

- Derive the empty clause (conclude UNSAT)
- All variables have been selected

Davis-Putman (DP): Example

Let S = {{p, q} {p, ¬q} {¬p, q} {¬p, ¬q} } choose p

S' = {{p, q} {p, ¬q} {¬p, q} {¬p, ¬q} {q} {q, ¬q} {¬q} }

S' = { {q} {q, ¬q} {¬q} } choose q

S'' = { {q} {q, ¬q} {¬q} {q} {¬q} {} }

What about tautologies?

S'' = { {} } EMPTY CLAUSE: UNSAT

Davis-Putman (DP): Example I

Let S = {{p, q} {p, ¬q} {¬p, q} {¬p, ¬q} } choose p

S' = {{p, q} {p, ¬q} {¬p, q} {¬p, ¬q} {q} {q, ¬q} {¬q} }

S' = { {q} {q, ¬q} {¬q} }

S'' = { {q} {q, ¬q} {¬q} }

S'' = { {} }

What about tautologies?

Eliminate tautologies first

S'' = { {q} {¬q} } choose q

S'' = { {} } EMPTY CLAUSE: UNSAT

Davis-Putman (DP): Example II

Let S = { {p,¬q} {q}}

choose q

S' = {{p, ¬q} {q} {p}}

S' = {{p}}

No further variable can be chosen

Satisfiable and any assignment MUST make p true

BUT

- p appears only positively in the given set S of clauses.
- we could remove the clause that includes p from the beginning and "remember" that p has to be true.

S = { {p,¬q} {q}} [p]

S = {{q}} [p, q]

S = { } [p, q]

Basic Rules

Pure Literal rule

- Given a CNF formula F, expressed as a set S of clauses, a literal is pure in F if it only occurs only as a positive or as a negative literal in S.

$S = \{\{\neg x_1, x_2\}\{x_3, \neg x_2\}\{x_4, \neg x_5\}\{x_5, \neg x_4\}\}$

$\neg x_1$ and x_3 are pure literals

- Clauses containing pure literals can be removed from the formula (i.e. assign pure literals to the values that satisfy the clauses)

$S = \{\{\neg x_1, x_2\}\{x_3, \neg x_2\}\{x_4, \neg x_5\}\{x_5, \neg x_4\}\}$

$S' = \{\{x_4, \neg x_5\}\{x_5, \neg x_4\}\}$

Remember $[\neg x_1, x_3]$

Assignment must make x_1 = false and x_3 = true

Basic Rules

Unit clause

Clause with a single literal. It is satisfied by assuming the literal to be true.

Unit propagation rule

Once a literal has been assigned, its assignment needs to be propagated to other clauses:

- Every clause (other than the unit clause itself), that contains the unit clause, is removed
- In every clause that contains the negation of the unit clause, this negated literal is deleted

$S = \{ \{x_1, x_2\} \ \{\neg x_1\ x_3\} \ \{\neg x_3\ x_4\} \ \ \{x_1\}\}$

removed $\neg x_1$ deleted unchanged unchanged

$S' = \{ \{x_3\} \ \{\neg x_3\ x_4\} \ \ \{x_1\}\}$ X_1 become then pure literal

Using a partial assignment: a clause can be unit clause when all literal except one are assigned based on a given partial assignment.

Improved Davis-Putman (DP) Algorithm

DP(M, S): M is a partial model of clauses processed so far
S is set of clauses still to process

DP([], S) (At the beginning)

Iteratively apply the following steps:

- Select a variable x
- Apply resolution rule between every pair of clauses of the form $(x \vee \alpha)$ and $(\neg x \vee \beta)$ in S

- Remove all clauses containing either x or ¬x
- Apply the pure literal rule and unit propagation

Terminate when either the empty clause or empty formula (equivalently, a formula containing only pure literals) is derived

Davis-Putman (DP) Algorithm II: example

$M = \{\}$

$S = \{\{x_1, \neg x_2, \neg x_3\}\{\neg x_1, \neg x_2, \neg x_3\}\{x_2, x_3\}\{x_3, x_4\}\{x_3, \neg x_4\}\}$

$S_1 = \{\{\neg x_2, \neg x_3\}\{x_2, x_3\}\{x_3, x_4\}\{x_3, \neg x_4\}\}$

$S_2 = \{\{\neg x_3, x_3\}\{x_3, x_4\}\{x_3, \neg x_4\}\}$ tautology

$S_3 = \{\{x_3, x_4\}\{x_3, \neg x_4\}\}$ x_3 pure literal

$S_4 = \{\}$ $M = \{x_3\}$

Satisfiable provided that x_3 is true.

Improved Davis-Putman (DP) Algorithm

Key ideas:

- Resolution eliminates 1 variable at each step
- Use of pure literal rule and unit propagation

But still inefficient…..

DLL (or DPLL) Algorithm

Basic idea:

- Instead of eliminating variables, split on a given variable at each step
- Also apply the pure literal rule and unit propagation

Basic Rules and Notations for DLL

Let F be the initial set of clauses and let M = {} be initial assignment.
Let Sc be the current set of clauses:

UNSAT(Sc)	If Sc contains the {}, then F is unsatisfiable
SAT(Sc)	If Sc = {} then F is satisfiable
MULT(L, Sc)	If a literal occurs more than once in a clause in Sc, then all but one can be deleted
SUB(C, Sc)	A clause C in Sc can be deleted, if it is a superset of another clause in Sc
TAUT(C, Sc)	Delete from Sc any clause that contains a literal L and its negation ¬L
PURE(L, Sc)	If L is a pure literal in Sc, delete all clauses that contain L, and add L to M (i.e. M = M ∪ {L}).

Basic Rules and Notations for DLL

Let F be the initial set of clauses and let M = {} be initial assignment.
Let Sc be the current set of clauses:

UNIT(L, Sc)	If Sc contain unit clause {L}, then delete all clauses in Sc that include L, remove the element ¬L from the remaining clauses, and add L to M (i.e. M = M ∪ {L}).
SPLIT(L,Sc)	If Sc contains two clauses of the form {….{C_k, L}…{C_m, ¬L}…} branch the computation. • One branch has M = M ∪ {L} and Sc is generated as in UNIT rule assuming L to be the fact. • Other branch has M = M ∪ {¬L} and Sc is generated as in UNIT rule assuming ¬L to be the fact.

DLL Procedure (pseudo code)

DLL(M, S): boolean;
%M is a (partial) model so far and S are clauses still to process

1. If SAT(S) return true; %M is a (partial) model
2. If UNSAT(S) return false; % S has no models
3. If SUB(C, S) return DLL(M, S \ C);
4. If PURE(L, S) return DLL([L | M], S'), S' = S'\ { C_i | L in C_i}; %Make L true
5. If UNIT(L, S) return DLL([L | M], S'), S' generated from S by removing clauses containing L, and removing ¬L from rest.
6. If UNIT(¬L, S) return DLL([¬L | M], S'), S' generated from S by removing clauses containing ¬L, and removing L from rest.
7. Otherwise, SPLIT(L, S) return DLL([L | M], S') ∨ DLL([¬L | M], S"), S' formed as in Step 5 , and S" formed as in Step 6.

DLL Example

$S = \{\{x_1, x_2\}\ \{x_4, \neg x_2, \neg x_3\}\ \{\neg x_1, x_3\}\ \{\neg x_4\}\ \}$ $\quad M = \{\}$

UNIT($\neg x_4$, S) $\quad M = \{\neg x_4\}$

$S_1 = \{\{x_1, x_2\}\ \{\neg x_2, \neg x_3\}\ \{\neg x_1, x_3\}\}$

SPLIT(x_1, S_1) $\quad M = \{x_1, \neg x_4\}$

SPLIT($\neg x_1$, S_1) $\quad M = \{\neg x_1, \neg x_4\}$

$S_1 = \{\{\neg x_2, \neg x_3\}\ \{x_3\}\}$

UNIT(x_3, S_1) $\quad M = \{x_3, x_1, \neg x_4\}$

$S_2 = \{\{\neg x_2\}\}$

PURE($\neg x_2$, S_2) $\quad M = \{\neg x_2, x_3, x_1, \neg x_4\}$

$S_3 = \{\ \}$

Satisfiable

$S_1 = \{\{x_2\}\ \{\neg x_2, \neg x_3\}\}$

UNIT(x_2, S_1) $\quad M = \{x_2, \neg x_1, \neg x_4\}$

$S_2 = \{\{\neg x_3\}\}$

PURE($\neg x_3$, S_2) $\quad M = \{\neg x_3, x_2, \neg x_1, \neg x_4\}$

$S_3 = \{\ \}$

Satisfiable

Properties of DLL

- MULT, SUB, TAUT, UNIT rules are equivalence preserving

$$S' \equiv S$$

- PURE and SPLIT are unsatisfiability preserving.

  - PURE: S is unsatisfiable $\equiv$ S' is unsatisfiable

  - SPLIT: S is unsatisfiable $\equiv$ S' is unsatisfiable and S" is unsatisfiable.

- Theorem
  - DLL(M, S) halts with False if S has no models.
  - DLL(M, S) halts with True and returns at least one (partial) model of S if S is satisfiable.

Summary

- Introduced the notion of satisfiability (SAT)
- Described simple DP algorithm, based on resolutions
- Introduced simple simplification rules
  - TAUT, SUB, PURE literals
- Unit propagation and SPLIT rules
- DLL/DPLL algorithm