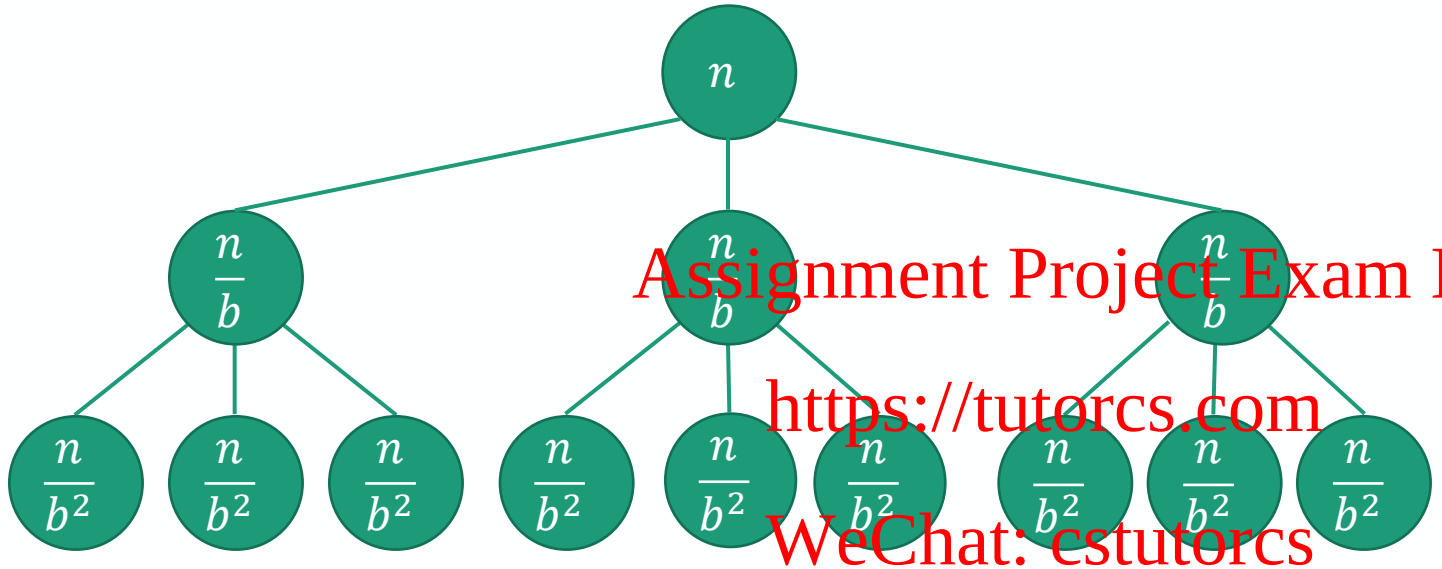
CIT 596: ALGORITHMS & COMPUTATION

Solving Recurrence Relations: Examples

Master Theorem

$$T(n) \leq aT\left(\frac{n}{b}\right) + n^c$$



$$a^i \times \frac{n}{b^i}$$

$$a^{\log_b n} \times \frac{n}{b^{\log_b n}} = a^{\log_b n} \times 1$$

$$n^c \left\{ \begin{array}{l} n^c \\ a\left(\frac{n}{b}\right)^c \\ a^2\left(\frac{n}{b^2}\right)^c \\ \vdots \\ a^i\left(\frac{n}{b^i}\right)^c \\ \vdots \\ a^{\log_b n} \end{array} \right. = n^c \sum_{i=0}^{\log_b n} \left(\frac{a}{b^c}\right)^i$$

Master Theorem

$$n^c \cdot \sum_{i=0}^{\log_b n} \left(\frac{a}{b^c}\right)^i = n^c \cdot \begin{cases} O(1) & \text{if } \frac{a}{b^c} < 1 \\ O(\log n) & \text{if } \frac{a}{b^c} = 1 \\ O\left(\left(\frac{a}{b^c}\right)^{\log_b n}\right) & \text{if } \frac{a}{b^c} > 1 \end{cases}$$

geometric sum!

$$n^c \left(\frac{a}{b^c}\right)^{\log_b n} = n^c \frac{a^{\log_b n}}{b^{c \cdot \log_b n}}$$

$$= n^c \frac{a^{\log_b n}}{n^c} = a^{\log_b n}$$

$$= n^{\log_b a}$$

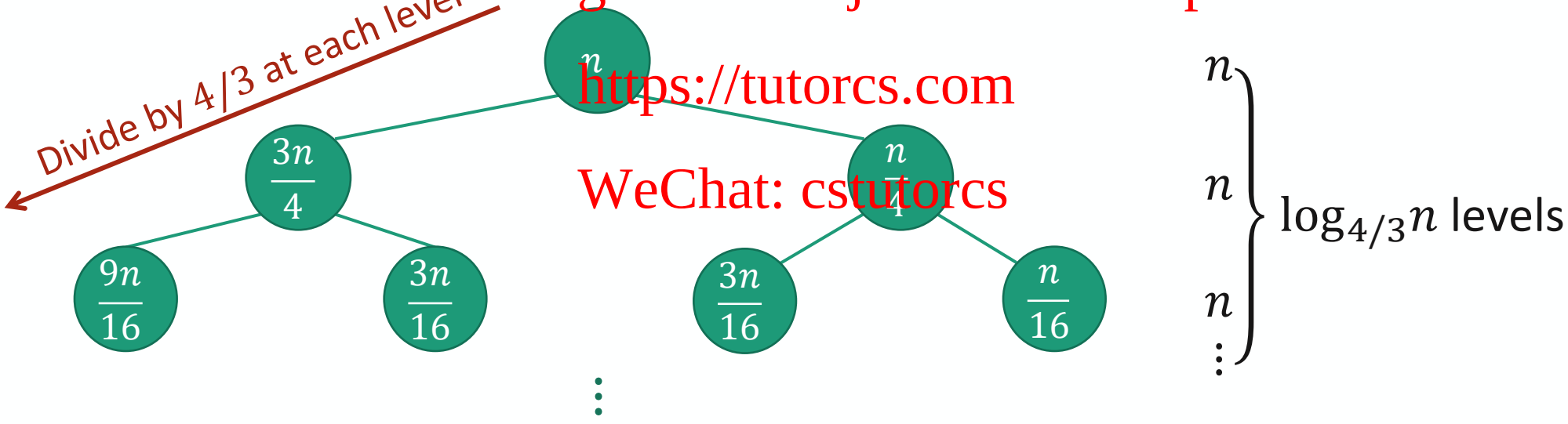
If $T(n) \leq aT\left(\frac{n}{b}\right) + O(n^c)$, then $T(n) = \begin{cases} O(n^c) & \text{if } \log_b a < c \\ O(n^c \log n) & \text{if } \log_b a = c \\ O\left(n^{\log_b a}\right) & \text{if } \log_b a > c \end{cases}$

Examples

- $T(n) \leq 4T\left(\frac{n}{3}\right) + O(n^2)$
  - $a = 4,\ b = 3,\ c = 2,\ \log_b a < c$
  - $T(n) = O(n^c) = O(n^2)$
- $T(n) \leq T\left(\frac{n}{3}\right) + 10$
  - $a = 1,\ b = 3,\ c = 0,\ \log_b a = c$
  - $T(n) = O(n^c \log n) = O(\log n)$
- $T(n) \leq 2T\left(\frac{n}{3}\right) + 6\sqrt{n}$
  - $a = 2,\ b = 3,\ c = \frac{1}{2},\ \log_b a > c$
  - $T(n) = O\left(n^{\log_b a}\right) = O(n^{\log_3 2})$

Examples

- $T(n) \leq T\left(\frac{3n}{4}\right) + T\left(\frac{n}{4}\right) + n$



$$T(n) = O(n \log n)$$