REVIEW OF PROBABILITY

Another example: $X =$ “the sum of two die rolls”

X can take on values in $[2..12]$

- e.g. $X = 5$ can result from one of $\{(1,4),(2,3),(3,2),(4,1)\}$

Probability of each outcome is $1/36$ so $\Pr[X = 5] = 4 \cdot \frac{1}{36} = \frac{1}{9}$

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability $(\times\ 36)$	1	2	3	4	5	6	5	4	3	2	1

Expectation of a random variable: weighted average of the values it can take

$$\mathbf{E}[X] = \sum_{x=2}^{12} (x \cdot \Pr[X = x]) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \cdots + 12 \cdot \frac{1}{36} = 7$$

EXPECTATION

Could do a messy sum to find that $\mathrm{E}[X] = 7$ but there is a better way.

Let us define two random variables: X_1 and X_2, the results of the first and second die rolls, respectively. It's easy to see that

$$\mathrm{E}[X_1] = \mathrm{E}[X_2] = 3.5$$

(note that $\mathrm{E}[X]$ doesn't have to be in the range of X for any X).

Since

$$X = X_1 + X_2$$

by definition of X, X_1, and X_2, we get:

Linearity of Expectation: $\mathrm{E}[X] = \mathrm{E}[X_1] + \mathrm{E}[X_2]$

EXPERIMENT: TOSS 10 COINS

Random Variables: $X_1, X_2, \dots, X_{10}$

We define $X_i = 1$ if i^{th} toss is H and 0 if T. Then

$$\Pr[X_i = 0] = 0.5 = \Pr[X_i = 1] \Rightarrow \mathrm{E}[X_i] = 0 \cdot 0.5 + 1 \cdot 0.5 = 0.5$$

$$X = \text{number of } H\text{s} = \sum_{\mathrm{i}=1}^{10} X_{\mathrm{i}}$$

$$\mathrm{E}[X] = \sum_{i=1}^{10} \mathrm{E}[X_i] = 10 \cdot 0.5 = 5$$

Indicator random variable: random variable that is 1 if some event occurs and 0 otherwise

- e.g. the X_{i} here are indicator random variables

If X_{i} is an indicator random variable, $\mathrm{E}[X_i] = \Pr[X_i = 1]$

EXPECTATION EXAMPLE: HAT CHECKING

- n people go to a restaurant, take off their hats and throw them in a pile.
- Afterwards, they each take a random hat from the pile.
- What is the expected number of people who get their hat back?

Let $X_i =$ the indicator random variable for the event "the i^{th} person gets their hat back."

$$\mathrm{E}[X_i] = 1/n$$

Define X to be the number of people getting their hat back.

$$X = \sum_{i=1}^{n} X_i \Rightarrow \mathrm{E}[X] = \sum_{i=1}^{n} \mathrm{E}[X_i] = n \cdot \frac{1}{n} = 1$$

On average, 1 person gets their hat back!