CIT 596: ALGORITHMS & COMPUTATION

Karatsuba Multiplication

Multiplying Recursively

The grade-school multiplication algorithm is iterative and runs in $O(n^2)$ time on n-digit numbers. Can we do better by using recursion?

$$
\begin{aligned}
&1234 \cdot 5678 \\
&= (1200 + 34)(5600 + 78) \\
&= 1200 \cdot 5600 + 1200 \cdot 78 + 34 \cdot 5600 + 34 \cdot 78 \\
&= (12 \cdot 56) \cdot 10000 + (12 \cdot 78 + 34 \cdot 56) \cdot 100 + 34 \cdot 78
\end{aligned}
$$

Multiplying Recursively

```
MULTREC(x, y)
    if n == 1
        return xy
    a = first n/2 digits of x
    b = last n/2 digits of x
    c = first n/2 digits of y
    d = first n/2 digits of y
    p_f = MULTREC(a, c)
    p_o = MULTREC(a, d)
    p_i = MULTREC(b, c)
    p_l = MULTREC(b, d)
    return 10^n p_f + 10^{n/2}(p_o + p_i) + p_l
```


- If n is odd, we can pad x and y with a zero:

$$12345 \mapsto 012345.$$

- $O(n)$ time to split the inputs.
- Four recursive calls on inputs of size $n/2$.
- $O(n)$ time to shift and add.
- $T(n) \leq 4T\left(\frac{n}{2}\right) + O(n)$

Recursion Tree Method

- $T(n) \leq 4T\left(\frac{n}{2}\right) + O(n)$
- Let c be a constant such that $T(n) \leq 4T\left(\frac{n}{2}\right) + cn$ for all sufficiently large n.

level

0

1

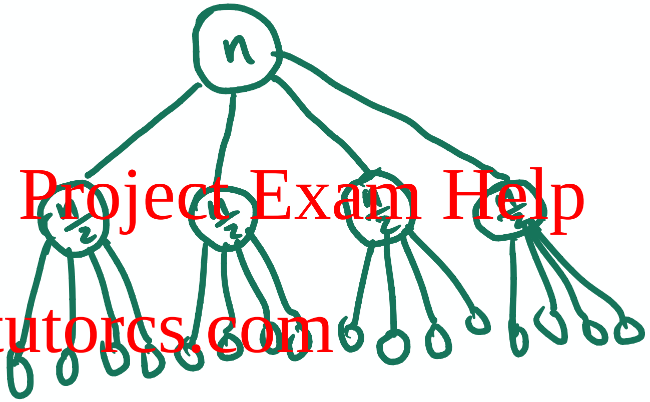
2

$\vdots$

i

$\vdots$

$\log_2 n$



cn

$4 \cdot \frac{cn}{2} = 2cn$

$16 \cdot \frac{cn}{4} = 4cn$

$\vdots$

$4^i \cdot \frac{cn}{2^i} = 2^i cn$

$\vdots$

$\sum_{i=0}^{\log_2 n} 2^i cn$

Simplifying the Sum

$$T(n) \le \sum_{i=0}^{\log_2 n} 2^i cn = cn \sum_{i=0}^{\log_2 n} 2^i$$

$$= cn \cdot O(2^{\log_2 n})$$

$$= cn \cdot O(n)$$

$$= O(n^2)$$

Gauss's Trick

Since we compute $10^n ac + 10^{n/2}(ad + bc) + bd$, we don't need the values of ad and bc; we only need their sum.

$$(a + b)(c + d) = ac + ad + bc + bd$$

$$ad + bc = (a + b)(c + d) - ac - bd$$

We only need to perform three multiplications!

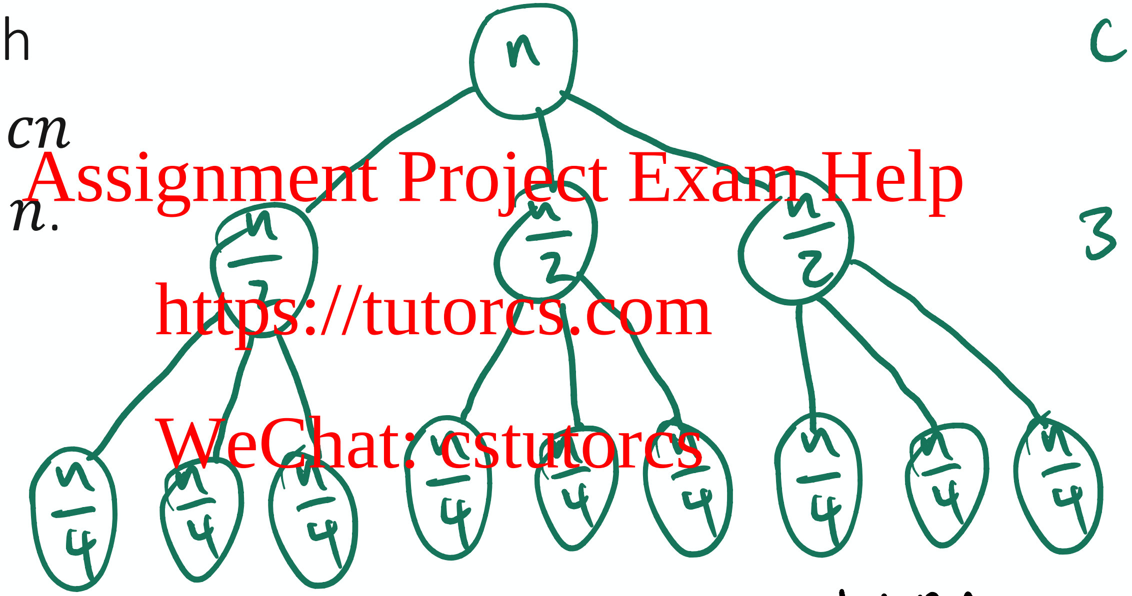
Karatsuba Multiplication

$\textsc{Karatsuba}(x, y)$
- **if** $n == 1$
  - **return** xy
- a, b, c, d as before
- $p_f = \textsc{Karatsuba}(a, c)$
- $p_l = \textsc{Karatsuba}(b, d)$
- $q = \textsc{Karatsuba}(a + b, c + d)$
- **return** $10^n p_f + 10^{n/2}(q - p_f - p_l) + p_l$

- $O(n)$ time to split the inputs.
- ***Three*** recursive calls on inputs of size $n/2$.
- $O(n)$ time to shift and add.
- $T(n) \leq 3T\left(\frac{n}{2}\right) + O(n)$

Running Time Analysis of KARATSUBA

Let c be a constant such that $T(n) \leq 3T\left(\frac{n}{2}\right) + cn$ for all sufficiently large n.



cn

$3\,\frac{cn}{2}$

$9\,\frac{cn}{4}$

$\vdots$

$cn\left(\frac{3}{2}\right)^i$

$$cn \cdot \sum_{i=0}^{\log_2 n} \left(\frac{3}{2}\right)^i = cn \cdot O\left(\left(\frac{3}{2}\right)^{\log_2 n}\right) = cn \cdot O\left(\frac{3^{\log_2 n}}{2^{\log_2 n}}\right)$$

$$= cn \cdot O\left(\frac{3^{\log_2 n}}{n}\right) = O\left(3^{\log_2 n}\right) = O\left(n^{\log_2 3}\right) = O\left(n^{1.585}\right)$$