CIT 596: ALGORITHMS & COMPUTATION

Logarithms and Sums

Logarithms

Logs and exponents are **inverses**, meaning they undo each other, just like addition and subtraction or multiplication and division.

$$(x+a)-a=x \qquad (x-a)+a=x$$

$$(x \cdot a)/a = x \qquad (x/a) \cdot x = x$$

$$a^{\log_a(x)} = x \qquad \log_a(a^x) = x$$

Intuition: $\log_b(x) =$ # of times you divide x by b to get to 1.

$$100 \xrightarrow{1} 50 \xrightarrow{2} 25 \xrightarrow{3} 12.5 \xrightarrow{4} 6.25 \xrightarrow{5} 3.125 \xrightarrow{6} 1.5623 \xrightarrow{7} 0.78125.$$

This is seven divisions, so $\log_2(100) \approx 7$.

Basic Exponent Rules

$$a^m = \underbrace{a\cdots a}_{m\text{ times}} \qquad\qquad a^{-m} = 1/a^m$$

- $a^m a^n = a^{m+n}$ $\quad \underbrace{a\cdots a}_{m\text{ times}} \cdot \underbrace{a\cdots a}_{n\text{ times}} = \underbrace{a\cdots a}_{m+n\text{ times}}$

- $(a^m)^n = a^{mn}$ $\quad \underbrace{\underbrace{a\cdots a}_{m\text{ times}} \cdots \underbrace{a\cdots a}_{m\text{ times}}}_{n\text{ times}} = \underbrace{a\cdots a}_{mn\text{ times}}$

- $\frac{a^m}{a^n} = a^{m-n}$ $\quad \dfrac{\overbrace{a\cdots a}^{m\text{ times}}}{\underbrace{a\cdots a}_{n\text{ times}}} = \dfrac{\overbrace{a\cdots a}^{m-n\text{ times}} \cdot \cancel{\overbrace{a\cdots a}^{n\text{ times}}}}{\cancel{\underbrace{a\cdots a}_{n\text{ times}}}} = \underbrace{a\cdots a}_{m-n\text{ times}} \qquad \dfrac{1}{a^{n-m}} = a^{m-n}$

These rules still hold when m and n aren't positive integers.

Basic Log Rules

To prove them, raise a to each side, then apply an exponent rule and the identity $a^{\log_a(x)} = x$.

- $\log_a(xy) = \log_a(x) + \log_a(y)$

  $a^{\log_a(xy)} = xy = a^{\log_a x} a^{\log_a y} = a^{\log_a x + \log_a y}$

- $\log_a(x^n) = n \cdot \log_a(x)$

  $a^{\log_a(x^n)} = x^n = (a^{\log_a x})^n = a^{n \log_a x}$

- $\log_a(x/y) = \log_a(x) - \log_a(y)$

  $a^{\log_a(x/y)} = \frac{x}{y} = \frac{a^{\log_a x}}{a^{\log_a y}} = a^{\log_a x - \log_a y}$

More Useful Log Rules

- $\log_b(x) = \log_a(x) \cdot \log_b(a)$
  - **Proof:** Take $\log_b$ of both sides of $x = a^{\log_a(x)}$

    $\log_b x = \log_b\left(a^{\log_a x}\right) = \log_a x \cdot \log_b a$
  - **Corollary:** The base doesn't matter in asymptotic notation!

$$\log_a(f(n)) = \Theta(\log_b(f(n))$$

- $m^{\log_b(n)} = n^{\log_b(m)}$
  - **Proof:** $m^{\log_b n} = \left(b^{\log_b m}\right)^{\log_b n} = b^{\log_b m \log_b n} = b^{\log_b n \log_b m}$

    $= \left(b^{\log_b n}\right)^{\log_b m} = n^{\log_b m}$

Sum Formulas

- **Triangular numbers:** $\sum_{i=1}^{n} i = \frac{n \cdot (n+1)}{2} = \Theta(n^2)$

- **Harmonic numbers:** $\sum_{i=1}^{n} \frac{1}{i} \approx \ln(n) = \Theta(\log n)$

- **Geometric sums:** $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r} = \begin{cases} \Theta(1) & r<1 \\ n & r=1 \\ \Theta(r^n) & r>1 \end{cases}$