CO580 Algorithms

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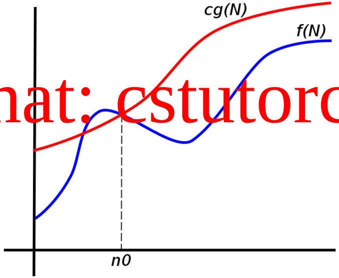
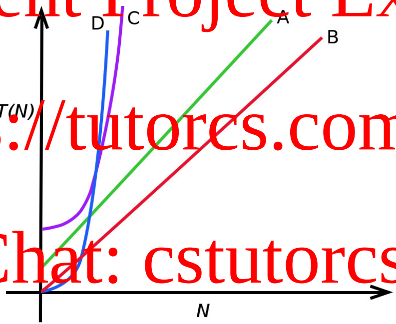


Recall



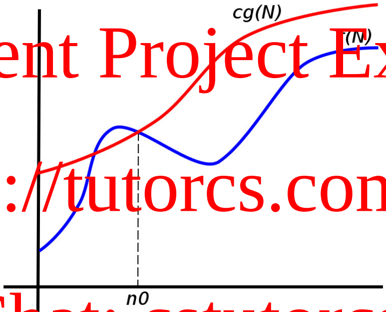
Asymptotic Notation

Algorithm performance is often expressed using asymptotic notation which captures the key ideas we discussed.

- Functions with similar growth are grouped into sets.
- The sets denote a bound on the functions.
- A function f is in
  - $O(g)$ if g is an asymptotic upper bound for f;
  - $\Omega(g)$ if g is an asymptotic lower bound for f;
  - $\Theta(g)$ if g is an asymptotically tight bound for f.

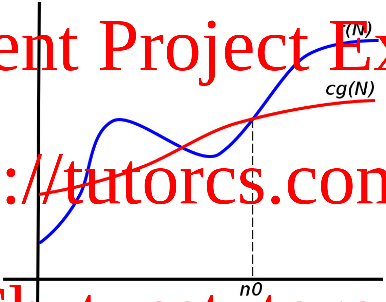
  where g is a characteristic function.
- The definitions of O, Ω and Θ are broad — coefficients are not significant.
- So, (A) and (B) above are both in $O(N)$, but (C) and (D) are not because they grow too fast.

Big O: Upper Bound



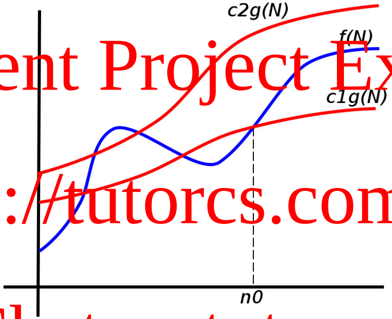
$$O(g(N)) = \left\{ f(N) \mid \begin{array}{l} \text{there are positive constants } c \text{ and } n_0 \\ \text{such that } 0 \leq f(N) \leq c\,g(N) \text{ for all } N \geq n_0 \end{array} \right\}$$

Big Omega: Lower Bound



$$\Omega(g(N)) = \left\{ f(N) \;\middle|\; \begin{array}{l} \text{there are positive constants } c \text{ and } n_0 \\ \text{such that } 0 \le c\, g(N) \le f(N) \text{ for all } N \ge n_0 \end{array} \right\}$$

Big Theta: Tight Bound



$$\Theta(g(N)) = \left\{ f(N) \;\middle|\; \begin{array}{l} \text{there are positive constants } c_1,\ c_2 \text{ and } n_0 \\ \text{such that} \\ 0 \leq c_1\, g(N) \leq f(N) \leq c_2\, g(N) \text{ for all } N \geq n_0 \end{array} \right\}$$

Asymptotic Notation

Even though $O(N)$ etc. are sets, bounds are usually stated like this:

- $N + 5 = O(N)$
- $T(N) = O(N^2)$
- (rather than $T(N) \in O(N^2)$)

Also, even though asymptotic notation applies to functions, it is (abusively) applied to algorithms too.

- We say "SimpleSearch is $O(N)$"

We use the same notation to talk about other resources:

- We say "the space complexity of MergeSort is $\Theta(N)$"

Space Complexity

The SimpleSearch procedure requires:

- $\Theta(1)$ space for the best case
- $\Theta(1)$ space for the worst case
- $\Theta(1)$ space for any input

"1" is the normal reference function for any constant

- The space used by the input is ignored
- If not this would mask differences due to algorithm
- SimpleSearch only needs space for a few local variables (e.g. a loop counter). This does not depend on N.

Better Search

- So, we have a $O(N)$ search algorithm. Can you do any better?

$k = 10$

	0	1	2	3	4	5	6
a	5	6	7	21	23	29	50

- You have already seen Binary Search.
- It uses the fact that elements are ordered.
- Checking an element in the middle means you can discount half the remaining data.

Binary Search: Design

Question

Binary Search creates regions in a. What properties should the algorithm maintain for it to be correct?

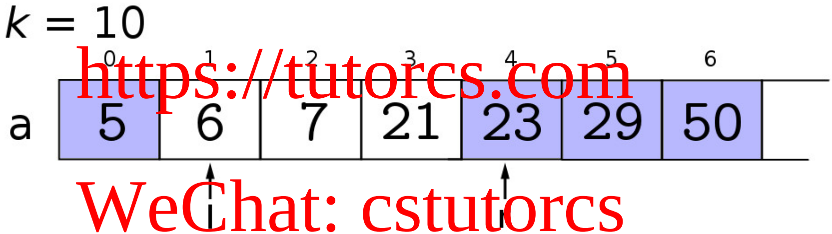
$k = 10$

	0	1	2	3	4	5	6
a	5	6	7	21	23	29	50

Loop Invariants: A Design Tool

A loop invariant is a property that is true before every iteration of a loop.

- Used to ensure/prove correctness; also helps in design



In Binary Search we assert that:

- Elements left of index l are known to be less than k;
- Elements at index r or above are known to be greater than k;
- so $a[l, \ldots, r-1]$ is unsearched.

Loop Invariants

A loop invariant must satisfy each of these:

initialisation The invariant must be true before the loop begins

maintenance If the invariant is true before a loop iteration, then it is still true before the next

termination When the loop ends the invariant implies a useful property of the algorithm

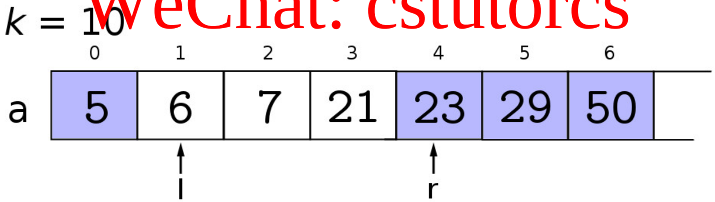
A tricky problem can be solved by coming up with an idea for an invariant

- The three conditions help see how (and if) it would work in detail

Loop Invariants

For Binary Search:

- **initialisation**: The whole of a should be unsearched, which gives initial values for l and r
- **maintenance**: The invariants must hold before each iteration, which gives the form of updates of l and r
- **termination**: If the loop ends nothing should be unsearched, which gives the loop condition



Loop Invariants

- Elements left of l are less than k
- Elements r and above are greater than k
- $a[l, \dots, r-1]$ is unsearched

Binary Search($a[1, \dots, N], k$)

```
l = 1, r = N + 1                      // all unsearched
while l < r                           // more to search
  m = l + (r-l) / 2
  if (k == a[m]) return True
  else if (k < a[m])  r = m           // search up to m-1
  else                l = m + 1       // search down to m+1
return False
```

Performance

What is the worst case time complexity of Binary Search?

Binary Search($a[1, \ldots, N]$, k)

```
                                       Cost  Executions
l = 1, r = N + 1                        c1      1
while l < r                             c2      ??
  m = l + (r-l) // 2                    c3      ??
  if (k == a[m])                        c4      ??
    return True                         c5      0
  else if (k < a[m])                    c6      ??
    r = m                               c7      ??
  else
    l = m + 1                           c8      ??
return False                            c9      1
```


Intuition: loop executes $\log_2 N$ times.

Performance

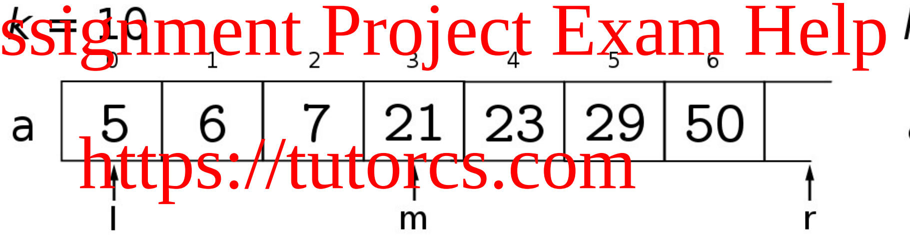
Alternative: analyse the recursive form of the program.

BinSearch(a, l, r, k)


```
                                            Cost
  if (l >= r)                                c1
    return False                             c2
  m = l + (r-l) / 2                          c3
  if (k == a[m])                             c4
    return True                              c5
  else if (k < a[m])                         c6
    return BinSearch(a, l, m, k)             T(N')
  else
    return BinSearch(a, m+1, r, k)           T(N'')
```


- where N' and N'' are numbers left to search
- Exercise: what are N' and N'' in the worst case? Be exact.

Worst Case Recursion



- `m` is always placed at `l` $+ \lfloor N/2 \rfloor$
- if N is odd: $N' = N'' = \lfloor N/2 \rfloor$
- if N is even: $N' = \lfloor N/2 \rfloor$, $N'' = \lfloor N/2 \rfloor - 1$
- So the worst case is when $k < a[0]$
  - If $N > 0$, will have $\lfloor N/2 \rfloor$ unsearched elements

Performance

We now have enough information to write a worst case formula for $T(N)$

BinSearch(a, l, r, k)


```
                                              Cost
if (l >= r)                                   c1
  return False                                c2
m = l + (r-l) / 2                             c3
if (k == a[m])                                c4
  return True                                 c5
else if (k < a[m])                            c6
  return BinSearch(a, l, m, k)                T(floor(N/2))
else
  return BinSearch(a, m+1, r, k)              ?
```