COMP4121 Advanced Algorithms

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More randomised algorithms: Gaussian Annulus, Random Projection and Johnson Lindenstrauss Lemmas

Details to be found on pages 12-27 of the Blum, Hopcroft and Kannan textbook

Generating random points in d-dimensional spaces $\mathbb{R}^d$

- Let us consider a Gaussian random variable X with zero mean $(E[X] = \mu = 0)$ and variance $V[X] = v = {}^1/_{2\pi}$; its density is given by

$$\frac{1}{\sqrt{2\pi\, v}} e^{-\frac{x^2}{2v}} = e^{-\pi\, x^2}$$

- Assume that we use such X to generate independently the coordinates $(x_1, \ldots, x_d)$ of a random vector x from $\mathbb{R}^d$.
- Since $E[X] = 0$ we have $E[X^2] = E[(X - E[X])^2] = V[X] = {}^1/_{2\pi}$.
- Thus, also $E\left[\frac{X_1^2+\ldots+X_d^2}{d}\right] = d\,V[X]/d = V[X]$.
- We denote by $|x|$ the norm (here just the length) of a vector x, $|x| = \sqrt{x_1^2 + \ldots + x_d^2}$.
- We now have $E\left[\frac{|x|^2}{d}\right] = V[X] = {}^1/_{2\pi}$.
- Thus the expected value of the square of the length of such a random vector is $E[|x|^2] = {}^d/_{2\pi}$.
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- Also, by the Law of Large Numbers,

$$\mathcal{P}\left(\left|\frac{x_1^2+\ldots+x_d^2}{d} - V[X]\right| > \varepsilon\right) \leq \frac{V[X^2]}{d\,\varepsilon^2}$$

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i.e., thus generated point is at a distance $\Theta(\sqrt{d})$ from the origin.

- If we chose 2 points independently, then

$$E[\langle x, y\rangle] = E[x_1y_1 + \ldots + x_dy_d] = d\,E[XY] = d\,E[X]E[Y] = 0$$

- This means that the expected value of the scalar product of any two vectors with independently randomly chosen coordinates is zero.
- **So, intuitively, vectors with randomly chosen coordinates have approximately the same length $\Theta(\sqrt{d})$ and any two such random vectors are likely to be almost orthogonal!**

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High dimensional spaces are quite counter-intuitive!

- The above should hint that high dimensional spaces are quite counter-intuitive.
- We have to be care[illegible] transfer our 3D intuition to high dimensional spaces.
- We now show two strange facts about high dimensional balls:
  1. Most of the volume of a high dimensional ball is near any of its equators. By this we mean that if we cut a high dimensional ball with any two parallel hyper-planes symmetric with respect to the center and close to the center, then most of the volume of the ball is between these two hyper-planes.
  2. Most of the volume of a high dimensional ball is near its surface. By this we mean that if we consider a ball of radius r and a slightly smaller ball of radius $r(1-\epsilon)$ than most of the volume of the bigger ball is in the annulus outside the smaller ball.
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Generating random points in d-dimensional spaces $\mathbb{R}^d$

- We first note that the volume of a sphere of radius r is proportional to r^d, in the sense that, if we denote by $V(d)$ the volume of the d-dimensional ball of radius 1, then the volume of a d-dimensional ball of radius r is equal to $r^d V(d)$. (This is clear for a d-dimensional cube and it follows for any other solid by the exhaustion method: we approximate the solid by a union of small disjoint hypercubes).
- This implies that the volume of a d-dimensional ball can be represented as the following integral:

$$V(d) = \int_{-1}^{1} \left(\sqrt{1 - x_1^2}\right)^{d-1} V(d-1)dx_1$$

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- Let c be a constant. Consider the portion S of a unit ball consisting of all points such that their x_1 coordinate satisfies $x_1 \geq \frac{c}{\sqrt{d-1}}$.
- Let A be the volume of such a slice S.
- Then

$$A = \int_{\frac{c}{\sqrt{d-1}}}^{1} \left(\sqrt{1-x_1^2}\right)^{d-1} V(d-1)\,dx_1$$

- (Recall that our goal is to show that two randomly and independently generated vectors are almost certainly almost orthogonal; to achieve this we need to perform some mumbo-jumbo whose purpose will be clear a bit later.)

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$$A = \int_{\frac{c}{\sqrt{d-1}}}^{1} \left(\sqrt{1-x_1^2}\right)^{d-1} V(d-1)\,dx_1$$

- (Recall that our goal is to show that two randomly and independently generated vectors are almost certainly almost orthogonal; to achieve this we need to perform some mumbo-jumbo whose purpose will be clear a bit later).

- We need the inequality $1 - x \leq e^{-x}$ for all $0 \leq x \leq 1$.
- For a proof, see the refresher notes on probability and statistics, available at the course web site.
- Using such an inequali[illegible]at we have

$$A = \int_{\frac{c}{\sqrt{d-1}}}^{1} \left(\sqrt{1-x_1^2}\right)^{d-1} V(d-1)dx_1$$

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$$A < \int_{\frac{c}{\sqrt{d-1}}}^{1} \left(e^{-x_1^2}\right)^{\frac{d-1}{2}} V(d-1)dx_1$$

Over the slice S we have $x_1 \geq \frac{c}{\sqrt{d-1}}$ which implies $\frac{x_1\sqrt{d-1}}{c} \geq 1$ and this implies

$$A < \int_{\frac{c}{\sqrt{d-1}}}^{1} \frac{x_1\sqrt{d-1}}{c} e^{-x_1^2\frac{d-1}{2}} V(d-1)dx_1$$

- The purpose of adding this term is to allow substitution of variable x_1 with variable $u = x_1^2$; then $du = 2x_1 dx_1$.

Generating random points in d-dimensional spaces $\mathbb{R}^d$

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Generating random points in d-dimensional spaces $\mathbb{R}^d$

- Thus, with $u = x_1^2$ we have $du = 2x_1 dx_1$ and so, from

$$A < \int_{\frac{c}{\sqrt{d-1}}}^{1} \frac{x_1\sqrt{d-1}}{c} e^{-x_1^2\frac{d-1}{2}} V(d-1)\,dx_1$$

we obtain

$$A < \int_{\frac{c^2}{d-1}}^{1} \frac{\sqrt{d-1}}{c} e^{-u\frac{d-1}{2}} V(d-1)\frac{1}{2}\,du$$

$$< \frac{\sqrt{d-1}}{2c} V(d-1) \int_{\frac{c^2}{d-1}}^{\infty} e^{-u\frac{d-1}{2}}\,du$$

$$= \frac{\sqrt{d-1}}{2c} V(d-1) \frac{1}{\frac{d-1}{2}} e^{-u\frac{d-1}{2}} \Big|_{u=\frac{c^2}{d-1}}^{u=\infty}$$

$$= \frac{\sqrt{d-1}}{2c} V(d-1) \frac{1}{\frac{d-1}{2}} e^{-\frac{d-1}{2}\frac{c^2}{d-1}}$$

$$= \frac{1}{c\sqrt{d-1}} V(d-1) e^{-\frac{c^2}{2}}$$

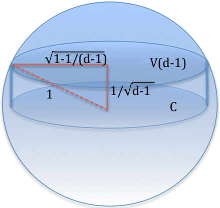
- So we have obtained that the volume A of the slice S of the unit ball which lies above the hyperplane $x_1 = \frac{c}{\sqrt{d-1}}$ satisfies

$$A < V(d-1)\frac{e^{-\frac{c^2}{2}}}{c\sqrt{d-1}}$$

- We now want to show that this volume is small compared to half the volume of the entire ball. However, we do not want to use $V(d)$, we want an expression involving $V(d-1)$ which can cancel $V(d-1)$ in the above bound.
- Note that the volume of the whole hemisphere H is larger than the volume of a cylinder below the intersection of the ball and the plane $x_1 = \frac{1}{\sqrt{d-1}}$, which has the volume C which is the product of the d-1 dimensional "surface area" of its base which is equal to $V(d-1)\left(\sqrt{1-\frac{1}{d-1}}\right)^{d-1}$ and its height which is $1/\sqrt{d-1}$.
- Thus, the volume C of such a cylinder

$$C = V(d-1)\left(\sqrt{1-\frac{1}{d-1}}\right)^{d-1}\frac{1}{\sqrt{d-1}}$$

$$= V(d-1)\left(1-\frac{1}{d-1}\right)^{\frac{d-1}{2}}\frac{1}{\sqrt{d-1}}$$



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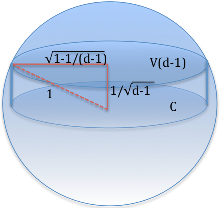

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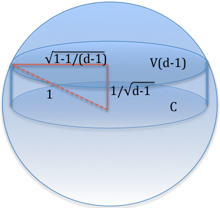

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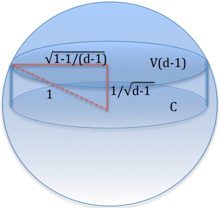

- We now need yet another inequality: for all $\alpha \geq 1$ and all $0 < x < 1$,

$$(1-x)^{\alpha} \geq 1 - \alpha x$$

(This inequality can be proved by considering $\beta(x) = (1-x)^{\alpha} - 1 + \alpha x$; again, $\beta(0) = 0$ and $\beta'(x) = -\alpha(1-x)^{\alpha-1} + \alpha = \alpha(1-(1-x)^{\alpha-1}) \geq 0$ which implies that $\beta(x)$ is non-decreasing and thus non-negative.)

- Applying the above inequality with $\alpha = (d-1)/2$ and $x = 1/(d-1)$ we obtain

$$\begin{aligned} C &= V(d-1)\left(1-\frac{1}{d-1}\right)^{\frac{d-1}{2}}\frac{1}{\sqrt{d-1}} \\ &\geq V(d-1)\left(1-\frac{1}{d-1}\,\frac{d-1}{2}\right)\frac{1}{\sqrt{d-1}} \\ &\geq V(d-1)\frac{1}{2\sqrt{d-1}} \end{aligned}$$

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- So we got that the volume A of the slice of the unit ball which is above the plane $x_1 = \frac{c}{\sqrt{d-1}}$ and the volume H of the whole hemisphere satisfy

$$A < V(d-1)\frac{e^{-\frac{c^2}{2}}}{c\sqrt{d-1}}$$

$$H \geq V(d-1)\frac{1}{2\sqrt{d-1}}$$

- Thus, the ratio of the volume of the slice of the sphere and the whole hemisphere satisfies

$$\frac{A}{H} < \frac{V(d-1)\frac{e^{-\frac{c^2}{2}}}{c\sqrt{d-1}}}{V(d-1)\frac{1}{2\sqrt{d-1}}} = \frac{2}{c}e^{-\frac{c^2}{2}}$$

- **Corollary:** If we pick uniformly at random a point from the unit ball, the probability that its coordinate x_1 satisfies $|x_1| > \frac{c}{\sqrt{d-1}}$ is less than $\frac{2}{c}e^{-\frac{c^2}{2}}$.

- **This also means that most of the volume of the unit ball is between the hyperplanes $x_1 = -\frac{c}{\sqrt{d-1}}$ and $x_1 = \frac{c}{\sqrt{d-1}}$, i.e., near the equator.**
- Also, the annulus consisting of all x such that $r \leq |x| \leq 1$ for $r < 1$ has volume $V(d) - r^d V(d) = V(d)(1 - r^d)$ so **if d is large almost all of the volume of the unit ball is near its surface.**

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- If we randomly and independently choose 2 points x, x' from the unit ball, we can rotate the ball so that x is in the direction of x_1 coordinate.
- Thus, the above two fact guarantee that with high probability $|x| \approx 1$ and $|x'| \approx 1$ as well as that the projection of x' onto x has, with high probability $\geq 1 - \frac{2}{c}e$ [illegible] ue smaller than $\frac{c}{\sqrt{d-1}}$, i.e., that x and x' are almost orthogon[illegible]
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- It shows that if we draw randomly a relatively small number n of vectors compared to the dimension d, we are almost guaranteed that they will be of approximately the same length and that form an almost orthogonal basis.
- They will span a space onto which we will project vectors from $\mathbb{R}^d$ into a space of dimension n.
- The purpose is to reduce the dimensionality of the space.
- **Theorem:** Assume we draw independently and uniformly n points $x_1, \ldots, x_n$ from the unit ball. Then with probability $1 - \frac{3}{2}\frac{1}{n}$ for all $1 \leq i, j \leq n$, $i \neq j$,
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- Thus, taking $\varepsilon = \frac{2 \ln n}{d}$, we get that for each i

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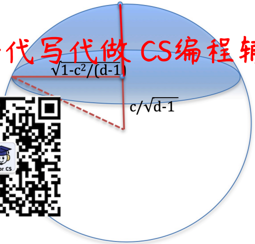
- To prove the second claim we rotate the sphere so that x_j is the polar axis.



- We now note that, since $|x_j| \le 1$

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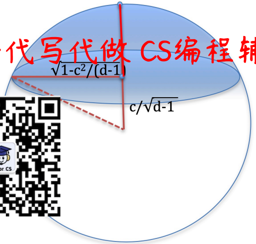

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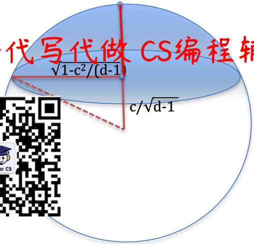

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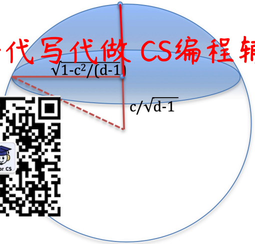

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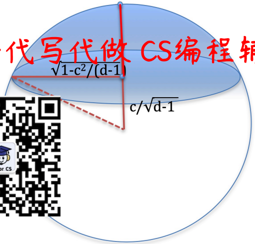

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Gaussian Annulus Theorem

- We need the following lemma which we do not prove; its proof is based on tail inequalities which can be found in the BHK book. Recal that for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ [illegible] $= \sqrt{x_1^2 + \ldots + x_d^2}$.
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If we draw from such a distribution a random vector x, then the probability that x belongs to the annulus (hollow ball of outer radius $\sqrt{d} + \beta$ with the shell of thickness 2β)

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is at least $1 - 3e^{-\frac{\beta^2}{96}}$.

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Gaussian Annulus Theorem

- We need the following lemma which we do not prove; its proof is based on tail inequalities which can be found in the BHK book. Recal that for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ [illegible] $\sqrt{x_1^2 + \ldots + x_d^2}$.
- **Lemma:** Let X be a [illegible]onal spherical Gaussian with zero mean and unit variance in each c[illegible]e., the with the probability density

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- For any vector v define $f(v)$, called *the random projection*, by

$$f(v) = (\langle v, u_1\rangle, \dots, \langle v, u_k\rangle)$$

- **Theorem: (Random Projection Lemma)**

$$\mathcal{P}\left(\Big| |f(v)| - \sqrt{k}|v| \Big| \geq \varepsilon\sqrt{k}|v|\right) \leq 3e^{-\gamma\, k\, \varepsilon^2}$$

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- Let us define $f^*(v) = f(v)/\sqrt{k}$;
- if we divide by $\sqrt{k}$ both sides of the inner inequality in

$$\mathcal{P}\left(\Big|\,|f(v)| - \sqrt{k}|v|\,\Big| \geq \varepsilon\sqrt{k}|v|\right) \leq 3e^{-\gamma\, k\, \varepsilon^2}$$

  we obtain the following consequence:

- **Corollary:**

$$\mathcal{P}(|\,|f^*(v)| - |v|\,| \geq \varepsilon|v|) \leq 3e^{-\gamma\, k\, \varepsilon^2}$$

- Thus, the modified projection mapping $f^*(v)$ “almost preserves” length of vectors.
- This corollary is needed to prove the following important theorem with lots of practical applications.

- **Theorem: (Johnson-Lindenstrauss Lemma)** For any ε, $0 < \varepsilon < 1$, and any integer n, assume that k satisfies $k \geq \frac{3}{\gamma\,\varepsilon^2} \ln n$. Then for any set of n points given by vectors in $\mathbb{R}^d$, with the probability of at least $1 - \frac{3}{2n}$, the random pr $\mathbb{R}^d \to \mathbb{R}^k$ has the property that for ALL pairs of points v_i, v_j

$$||f^*(v_i - v_j)| - |v_i - v_j|| \leq \varepsilon |v_i - v_j|$$

Thus, $f^*(v)$ "almost" preserves distances between points given by vectors v_i, despite reduction of dimensionality from $d >> k$ to only k.

- **Proof:** By applying the Random Projection Lemma, we get that for any pair v_i, v_j,

$$\mathcal{P}\left(||f^*(v_i - v_j)| - |v_i - v_j|| > \varepsilon |v_i - v_j|\right) \leq 3e^{-\gamma\,k\,\varepsilon^2} \leq 3e^{-\gamma \frac{3}{\gamma\,\varepsilon^2} \ln n\,\varepsilon^2}$$
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- Since there are $\binom{n}{2} < \frac{n^2}{2}$ pairs, the probability that at least one of the above inequalities fails is less than $\frac{n^2}{2} \cdot \frac{3}{n^3} = 1 - \frac{3}{2n}$. (Union bound again).
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Johnson-Lindenstrauss Lemma

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- Given a database of a large number of papers and a new paper p, find a paper p_i from the database which is the "most similar" to p, in the sense that the same keywords appear with approximately same frequencies.

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- Solution: choose a large enough k, i.e., $k \geq \frac{288}{\varepsilon^2}\ln(|DB| + |Q|)$ where $|DB|$ is the size of the database and $|Q|$ the expected number of queries to be made.
- Recall that $\gamma = 1/96$ … 288. However, with more careful analysis, the requirement $k \geq$ … can be reduced to $k \geq \frac{24}{3\varepsilon^2 - 2\varepsilon^3}\ln n$.
- A good dictionary contains more than 100,000 words.
- So if you want to compare English documents according to the relative frequency of words, each document has to be represented as a vector in $100,000$ dimensional space!
- Assume that your database has 100,000 documents stored; you want to find the most similar documents in your database for a batch of $10,000$ new documents. How many random vectors should you choose to project onto?
- If we set $\varepsilon = 0.1$ we obtain

$$k \geq \frac{24}{3\varepsilon^2 - 2\varepsilon^3}\ln(|DB| + |Q|) = \frac{24}{3 \times .01 - 2 \times .001} \times \ln(110,000) = 9950,$$

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- As each query y arrives also replace $y \in \mathbb{R}^d$ with its projection $f^*(y) = f(y)/\sqrt{k}$.
- Do the search for the nearest neighbour of $f(y)/\sqrt{k}$ in the projected k dimensional space instead of the search for the nearest neighbour of y in the space of dimension $d >> k$.

Applications of the Johnson-Lindenstrauss Lemma

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