Advanced Algorithms
COMP4121

Aleks Ignjatovic

School of Computer Science and Engineering
University of New South Wales

Order Statistics

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- Can we do it faster???

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- We will show that it can be done in linear time, by both a deterministic and by a randomised algorithm.
- Why bother with a randomised algorithm if it can be done in linear time with a deterministic algorithm?
- Because in practice the randomised algorithm runs much faster, having much smaller constant c in the bound for the run time $T(n) \leq c \cdot n$.
- It turns out that it is easier to solve (both deterministically and with randomisation) the more general problem of finding the i^{th} smallest element for an arbitrary i than to find just the median.

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7. **else return** RAND-SELECT$(A, q+1, r, i-k)$.

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- Due to reshuffling of elements around the pivot, each iteration of RAND-SELECT costs the length of the array, and you get

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- This is very unlikely to happen; in fact, as we will now see, most of the time the partitions will be reasonably well balanced.

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- Let us find the expected number of partitions between two consecutive balanced partitions.

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- Thus, the expected number of partitions between two balanced partitions is

$$E = 1 \cdot \frac{8}{10} + 2 \cdot \frac{2}{10}\frac{8}{10} + 3 \cdot \left(\frac{2}{10}\right)^2 \cdot \frac{8}{10} + \ldots$$

$$= \frac{8}{10} \cdot \sum_{k=0}^{\infty} (k+1)\left(\frac{2}{10}\right)^k = \frac{8}{10} S$$

where

$$S = 1 + 2 \cdot \frac{2}{10} + 3 \cdot \left(\frac{2}{10}\right)^2 + 4 \cdot \left(\frac{2}{10}\right)^3 + 5 \cdot \left(\frac{2}{10}\right)^4 + \ldots$$

Evaluating $S = \sum_{k=0}^{\infty}(k+1)\left(\frac{2}{10}\right)^k$

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- Trick # 1:

$$
\begin{aligned}
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&= 1 + \frac{2}{10} + \left(\frac{2}{10}\right)^2 + \left(\frac{2}{10}\right)^3 + \left(\frac{2}{10}\right)^4 + \dots \\
&\qquad + \frac{2}{10} + \left(\frac{2}{10}\right)^2 + \left(\frac{2}{10}\right)^3 + \left(\frac{2}{10}\right)^4 + \dots \\
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Evaluating $S = \sum_{k=0}^{\infty}(k+1)\left(\frac{2}{10}\right)^k$

- Summing each row separately we obtain

$$
\begin{aligned}
1 + \frac{2}{10} + \left(\frac{2}{10}\right)^2 + \left(\frac{2}{10}\right)^3 + \left(\frac{2}{10}\right)^4 + \cdots &= \frac{1}{1-\frac{2}{10}} = \frac{10}{8} \\
+ \frac{2}{10} + \left(\frac{2}{10}\right)^2 + \left(\frac{2}{10}\right)^3 + \left(\frac{2}{10}\right)^4 + \cdots &= \frac{2}{10}\,\frac{10}{8} \\
+ \left(\frac{2}{10}\right)^2 + \left(\frac{2}{10}\right)^3 + \left(\frac{2}{10}\right)^4 + \cdots &= \left(\frac{2}{10}\right)^2 \frac{10}{8} \\
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- We can now sum the right hand side column:

$$S = \frac{10}{8}\left(1 + \frac{2}{10} + \left(\frac{2}{10}\right)^2 + \left(\frac{2}{10}\right)^3 + \left(\frac{2}{10}\right)^4 + \ldots\right)$$

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Thus, we obtain

$$E = \frac{8}{10}S = \frac{8}{10}\left(\frac{10}{8}\right)^2 = \frac{10}{8} = \frac{5}{4} < 2$$

A useful digression (Trick #2): $S = \sum_{k=0}^{\infty}(k+1)\left(\frac{2}{10}\right)^k$ evaluated another way

Note that

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}.$$

By differentiating both sides with respect to q we get

$$\sum_{k=1}^{\infty} k\, q^{k-1} = \frac{1}{(1-q)^2}.$$

Substituting $q = 2/10$ we get that $S = (10/8)^2$.

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$$T(n) < 5/4\, n + 5/4\, \frac{9}{10}\, n + 5/4 \left(\frac{9}{10}\right)^2 n + 5/4 \left(\frac{9}{10}\right)^3 n + \ldots$$

$$= \frac{5/4\, n}{1 - \frac{9}{10}} = \frac{50}{4}\, n = 12.5\, n$$

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- Where did we tacitly assume that all elements are distinct?
- How did we estimate the probability of choosing the pivot which results in a balanced partition?

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- Note that if all elements are the same Rand-Select would run in quadratic time no matter which elements are chosen as pivots - they are all equal.
- **Homework:** Modify Rand-Select so that it runs in linear time even when there are many repetitions.

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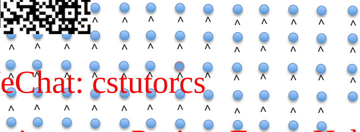
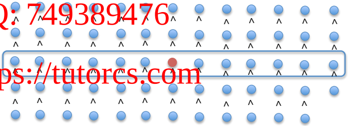
- **Main idea:** Use a recursive call of the very same algorithm to choose a good pivot!

- **Algorithm Select(n, i):**

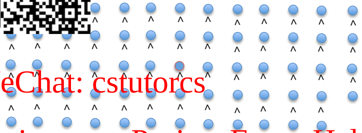
  - Split the numbers in groups of five (the last group might contain less than 5 elements).
  - Order each group by brute force in an increasing order.



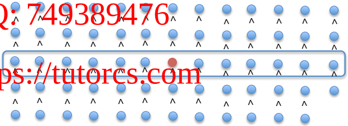
  - Take the collection of all $\lfloor \frac{n}{5} \rfloor$ middle elements of each group (i.e., the medians of each group of five).



- **Algorithm Select**(n, i) :
  - Split the numbers in groups of five (the last group might contain less than 5 elem
  - Order each grou e force in an increasing order.



  - Take the collection of all $\lfloor \frac{n}{5} \rfloor$ middle elements of each group (i.e., the medians of each group of five).



  - Apply recursively SELECT algorithm to find the median p of this collection.

- **Algorithm Select(n, i) continued:**

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  - partition all ele[...]g p as a pivot;

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Deterministic Linear Time Algorithm for Order Statistics

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  - partition all ele g p as a pivot;
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Deterministic Linear Time Algorithm for Order Statistics

- **Algorithm Select(n, i) continued:**
  - partition all elements using p as a pivot;
  - Let k be the number of elements in the subset of all elements smaller than the pivot;
  - **if** $i = k$ **then return** p
  - **else if** $i < k$ **then** recursively SELECT the i^{th} smallest element of the set of elements smaller than the pivot.
  - **else** recursively SELECT the $(i-k)^{th}$ smallest element of the set of elements larger than the pivot.

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- **Note:** This algorithm is the same as RAND-SELECT except for the way how we chose the pivot.

Deterministic Linear Time Algorithm for Order Statistics

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  - **else** recursively SELECT the $(i - k)^{th}$ smallest element of the set of elements larger than the pivot.

- **Note:** This algorithm is the same as RAND-SELECT except for the way how we chose the pivot.
  - instead of choosing pivot randomly we called recursively the very same algorithm to pick the pivot as the median of the middle elements of the groups of five elements.

Deterministic Linear Time Algorithm for Order Statistics

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Deterministic Linear Time Algorithm for Order Statistics

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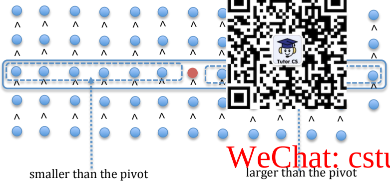

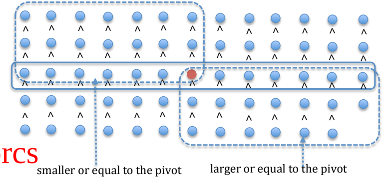

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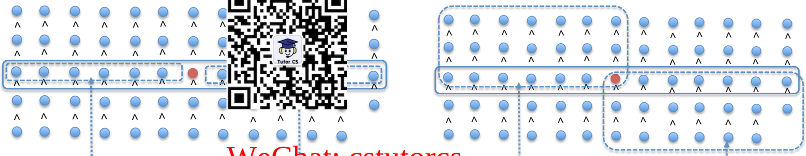

- Note that at least $\lfloor (n/5)/2 \rfloor = \lfloor n/10 \rfloor$ group medians are smaller or equal to the pivot; and at least that many larger than the pivot.

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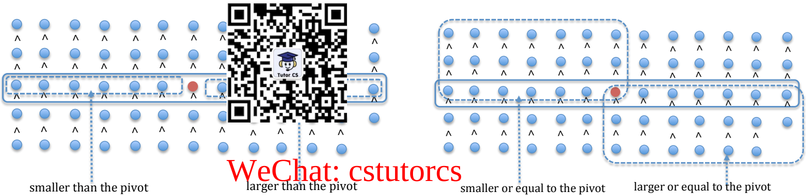

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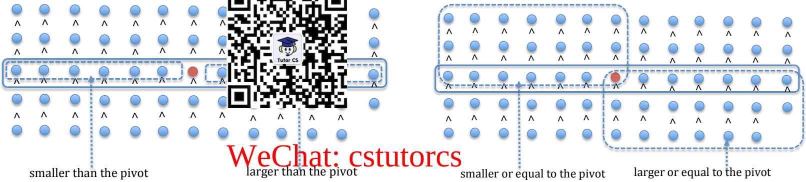

- Note that at least $\lfloor (n/5)/2 \rfloor = \lfloor n/10 \rfloor$ group medians are smaller or equal to the pivot; and at least that many larger than the pivot.
- But this implies that at least $\lfloor 3n/10 \rfloor$ of the total number of elements are smaller than the pivot, and that many elements larger than the pivot.
- (the same caveat: we are assuming all elements are distinct; otherwise we have to slightly tweak the algorithm to split all elements equal to the pivot evenly between the two groups.)

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Deterministic Linear Time Algorithm for Order Statistics

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Deterministic Linear Time Algorithm for Order Statistics

- What is the run time of our algorithm?

$$T(n) \leq T(n/5) + T(7n/10) + Cn.$$

- Let us show that $T(n) \leq 10Cn$ for all n. Assume that this is true for all $k < n$ and let us prove it is true for n as well.

Deterministic Linear Time Algorithm for Order Statistics

- What is the run time of our algorithm?

$$T(n) \le T(n/5) + T(7n/10) + Cn.$$

- Let us show that $T(n) \le 11Cn$ for all n. Assume that this is true for all $k < n$ and let us prove it is true for n as well.
  - Note: this is a proof using the following form of induction: $\varphi(0)$ & $(\forall n)((\forall k < n)\varphi(k) \to \varphi(n)) \to (\forall n)\varphi(n)$.

- What is the run time of our algorithm?

$$T(n) = T(n/5) + T(7n/10) + Cn.$$

- Let us show that $T(n) < 11Cn$ for all n. Assume that this is true for all $k < n$ and let us prove it is true for n as well.
  - Note: this is a proof using the following form of induction: $\varphi(0)\ \&\ (\forall n)((\forall k < n)\varphi(k) \to \varphi(n)) \to (\forall n)\varphi(n)$.
- Thus, assume $T(n/5) < 11C \cdot n/5$ and $T(7n/10) < 11C \cdot 7n/10$;

Deterministic Linear Time Algorithm for Order Statistics

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- Thus, assume $T(n/5) < 11C \cdot n/5$ and $T(7n/10) < 11C \cdot 7n/10$; then

$$T(n) \leq T(n/5) + T(7n/10) + Cn < 11C \cdot \frac{n}{5} + 11C \cdot \frac{7n}{10} + Cn$$

$$= 109\frac{Cn}{10} < 11C \cdot n$$

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- Let us show that $T(n) < 11Cn$ for all n. Assume that this is true for all $k < n$ and let us prove it is true for n as well.
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- Thus, assume $T(n/5) < 11C \cdot n/5$ and $T(7n/10) < 11C \cdot 7n/10$; then

$$T(n) \leq T(n/5) + T(7n/10) + Cn < 11C \cdot \frac{n}{5} + 11C \cdot \frac{7n}{10} + Cn$$
$$= 109\frac{Cn}{10} < 11C \cdot n$$

which proves out statement that $T(n) < 11C \cdot n$.

Deterministic Linear Time Algorithm for Order Statistics

- Note that this algo[illegible] a genuine recursion (rather than just an iteration) so its execution involves lots of traffic on the stack, which makes this algorithm slow in practice; the randomised version of it, RAND-SELECT, significantly outperforms it.

Deterministic Linear Time Algorithm for Order Statistics

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- Similarly RAND-QUICKSORT in practice outperforms MERGESORT, which, unlike RAND-QUICKSORT, is guaranteed to run in time $O(n \log n)$.