UNSW SYDNEY

COMP4161
Advanced Topics in Software Verification

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Content

➜ Foundations & Principles

- Intro, Lambda [illegible] atural deduction [1,2]
- Higher Order [illegible] part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7[b]]
- Proof automation, Isar (part 2) [8]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Last Time

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Specification Techniques

Sets

Sets in Isabelle

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- bexI: $[\![P\ x; x \in A]\!] \Longrightarrow \exists x \in A.\ P\ x$
- bexE: $[\![\exists x \in A.\ P\ x; \bigwedge x.\ [\![x \in A; P\ x]\!] \Longrightarrow Q]\!] \Longrightarrow Q$

Demo

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The harder, but safe choice.

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- **typedef**: by definiton as a set

  Example: **typedef** new_type = "{some set}" <proof>

  Introduces a new type as a subset of an existing type.

  The proof shows that the set on the rhs in non-empty.

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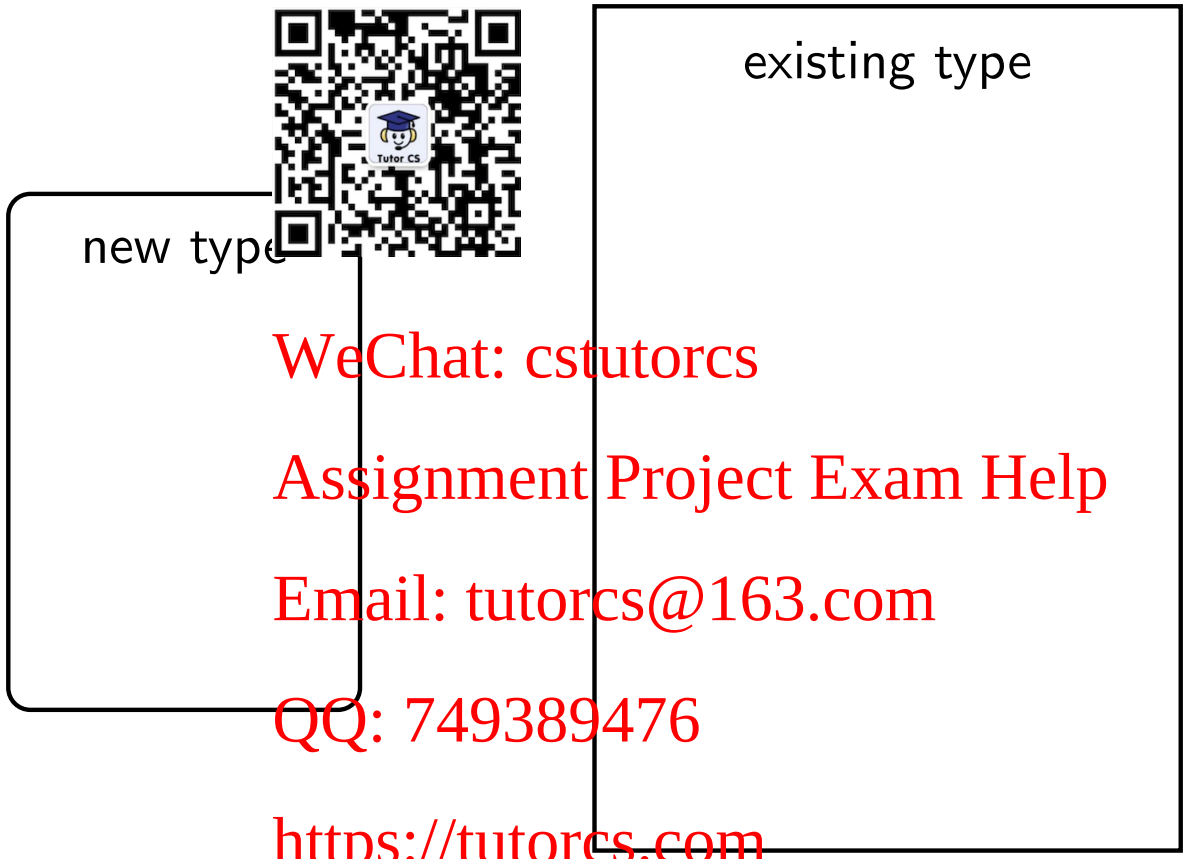

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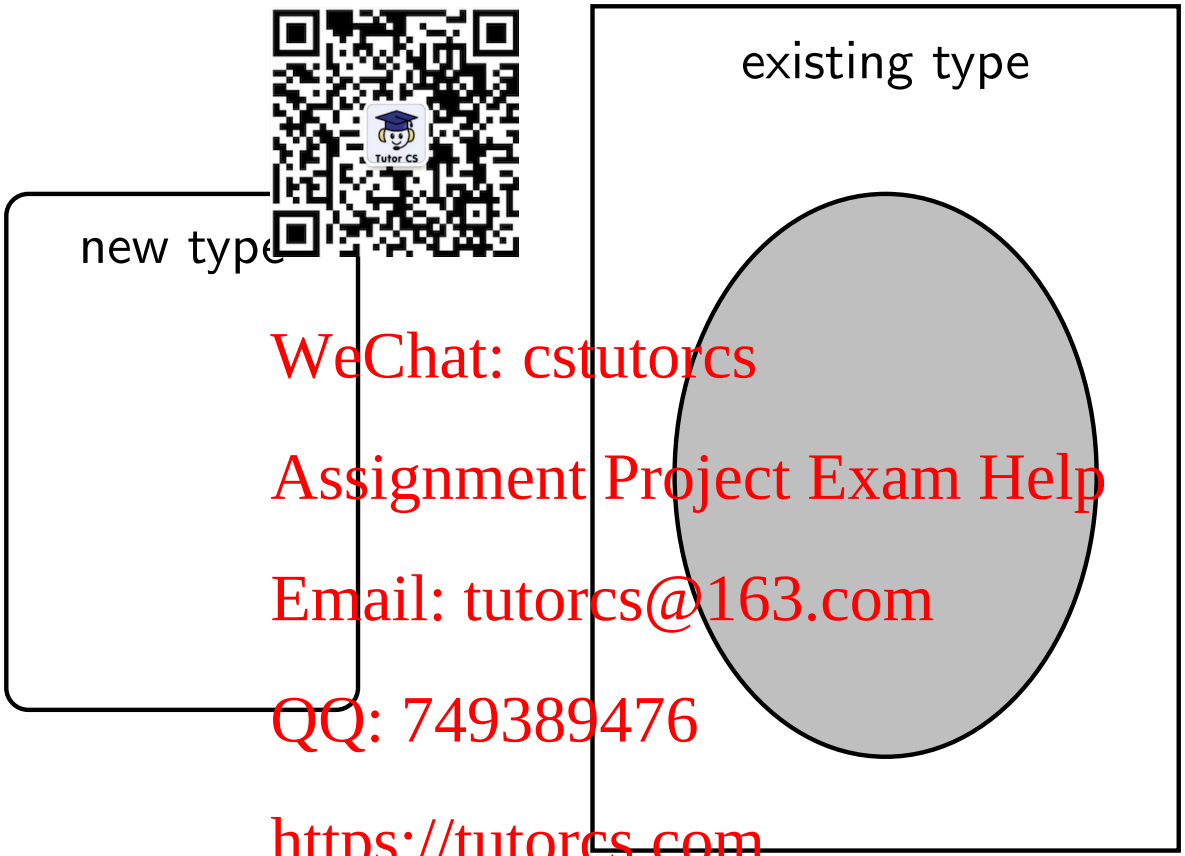

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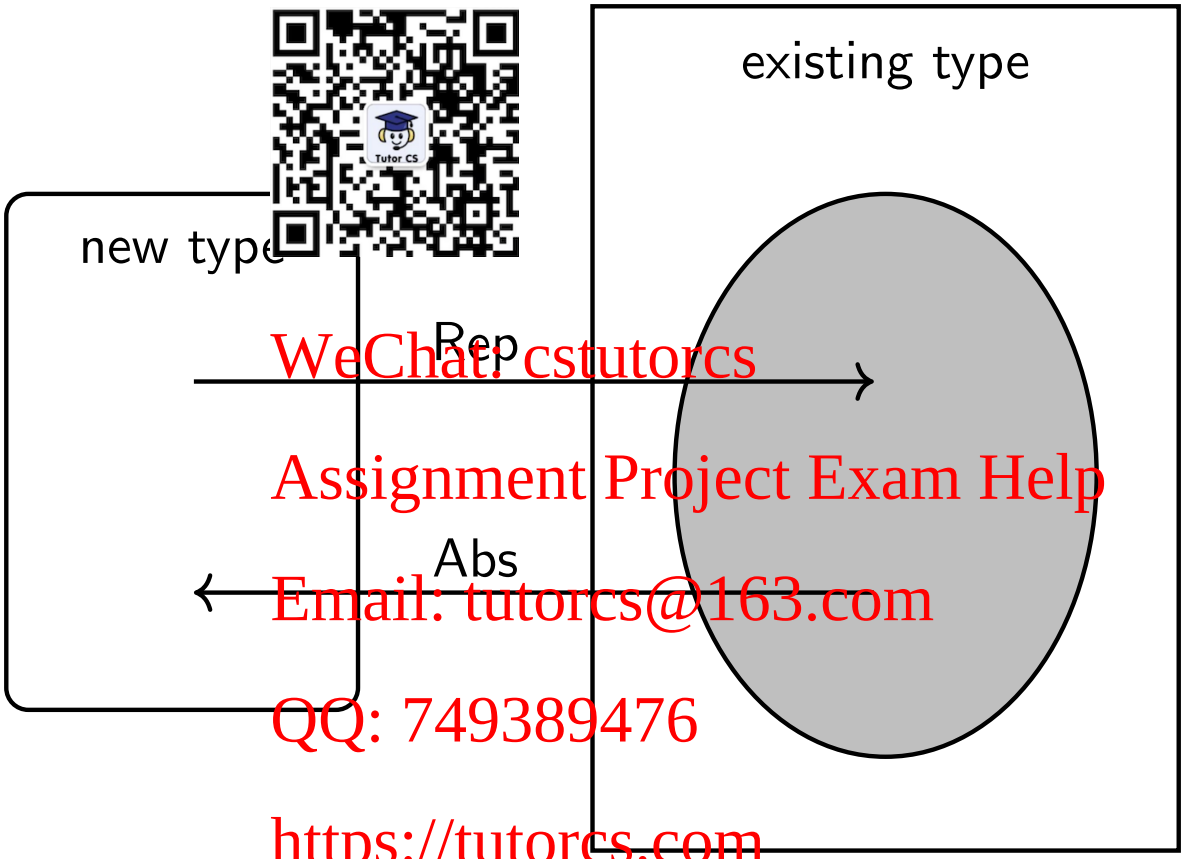

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- functions Abs_Prod, Rep_Prod
- both injective
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④ We now can:

- define constants Pair, fst, snd in terms of Abs_Prod and Rep_Prod
- derive all characteristic theorems
- forget about Rep/Abs, use characteristic theorems instead

Demo

Introducing new Types

Inductive Definitions

Example

$$\frac{}{\langle \text{skip}, \sigma \rangle \longrightarrow \sigma} \qquad \frac{[\![e]\!]\sigma = v}{\langle x := e, \sigma \rangle \longrightarrow \sigma[x \mapsto v]}$$

$$\frac{\langle c_1, \sigma \rangle \longrightarrow \sigma' \quad \langle c_2, \sigma' \rangle \longrightarrow \sigma''}{\langle c_1 ; c_2, \sigma \rangle \longrightarrow \sigma''}$$

$$\frac{[\![b]\!]\sigma = \text{False}}{\langle \text{while } b \text{ do } c, \sigma \rangle \longrightarrow \sigma}$$

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Big: which set?

Simpler Example

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- ➜ Alternative (greatest set) occasionally also useful: coinduction

Rule Induction

$$\frac{n \in N}{n + 1 \in N}$$

duction principle

$$[\![P\ 0;\ \bigwedge n.\ P\ n \Longrightarrow P\ (n + 1)]\!] \Longrightarrow \forall x \in N.\ P\ x$$

Demo

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We have learned today ...

- Sets

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- ➜ Sets
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