Proofcraft

UNSW SYDNEY

MP4161
Advanced Topics in Software Verification

HOL

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Last time...

- natural deduction r[illegible], $\longrightarrow$, $\neg$, iff...
- proof by assumptio[illegible]ule, elim rule
- safe and unsafe rule[illegible]

- indent your proofs! (one space per subgoal)
- prefer implicit backtracking (chaining) or *rule_tac*, instead of *back*
- *prefer* and *defer*
- *oops* and *sorry*

Content

➜ Foundations & Principles

- Intro, Lambda [illegible]atural deduction [1,2]
- Higher Order [illegible] part 1) [2,3[a]]
- Term rewriting [illegible] [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]

- General recursive functions, termination proofs [7[b]]
- Proof automation, Isar (part 2) [8]

- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]

- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Quantifiers

Scope

- Scope of paramete[illegible] subgoal
- Scope of $\forall, \exists, \ldots$: [illegible]; or $\Longrightarrow$

Example:

Scope

- Scope of paramete … subgoal
- Scope of $\forall, \exists, \ldots$: … ; or $\Longrightarrow$

Example:

$$\bigwedge x\ y.\ [\![\ \forall y.\ P\ y \longrightarrow Q\ z\ y;\ \ Q\ x\ y\ ]\!] \Longrightarrow \exists x.\ Q\ x\ y$$

means

Scope

- Scope of paramete[illegible] subgoal
- Scope of $\forall, \exists, \ldots$: [illegible]; or $\Longrightarrow$

Example:

$$\bigwedge x\ y.\ [\![\ \forall y.\ P\ y \longrightarrow Q\ z\ y;\ \ Q\ x\ y\ ]\!] \Longrightarrow \exists x.\ Q\ x\ y$$

means

$$\bigwedge x\ y.\ [\![\ (\forall y_1.\ P\ y_1 \longrightarrow Q\ z\ y_1);\ \ Q\ x\ y\ ]\!] \Longrightarrow (\exists x_1.\ Q\ x_1\ y)$$

Natural deduction for quantifiers

$$\frac{}{\forall x.\ P\ x}\ \text{a} \qquad \frac{P\ x}{R}\ \text{allE}$$

$$\frac{}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x}{R}\ \text{exE}$$

Natural deduction for quantifiers

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}$$

$$\frac{P\ x}{R}\ \text{allE}$$

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Natural deduction for quantifiers

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$$\frac{P\ ?x}{\exists x.\ P\ x}\ \text{exI}\qquad \frac{\exists x.\ P\ x \quad \bigwedge x.\ P\ x \Longrightarrow R}{R}\ \text{exE}$$

- **allI** and **exE** introduce new parameters ($\bigwedge x$).
- **allE** and **exI** introduce new unknowns ($?x$).

Instantiating Rules

apply (**rule_tac** x = "*term*" in *rule*)

Like **rule**, but ?x in *rule* is instantiated by *term* before application.

Similar: **erule_tac**

! x **is in** *rule*, **not in goal** **!**

Two Successful Proofs

1. $\forall x.\ \exists y.\ x = y$

Two Successful Proofs

$. \exists y.\ x = y$

y (rule allI)

$x.\ \exists y.\ x = y$

Two Successful Proofs

1. $\forall x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

best practice

apply (rule_tac x = "x" in exI)

1. $\bigwedge x.\ x = x$

Two Successful Proofs

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apply (rule refl)

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exploration

apply (rule exI)

1. $\bigwedge x.\ x = ?y\ x$

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exploration

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apply (rule refl)

$?y \mapsto \lambda u.u$

Two Successful Proofs

1. $\bigwedge x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

best practice

apply (rule_tac x = "x" in exI)

1. $\bigwedge x.\ x = x$

apply (rule refl)

simpler & clearer

exploration

apply (rule exI)

1. $\bigwedge x.\ x = ?y\ x$

apply (rule refl)

$?y \mapsto \lambda u.u$

shorter & trickier

Two Unsuccessful Proofs

1. $\exists y.\ \forall x.\ x = y$

Two Unsuccessful Proofs

程序代写代做 CS编程辅导

1. $\exists y.\ \forall x.\ x = y$

apply (rule_tac x = ?

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Assignment Project Exam Help

Email: tutorcs@163.com

QQ: 749389476

https://tutorcs.com

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apply (rule refl)

$?y \mapsto x$ yields $\bigwedge x'.\ x' = x$

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apply (rule allI)

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apply (rule refl)

$?y \mapsto x$ yields $\bigwedge x'.\ x' = x$

Principle:

$?f\ x_1 \ldots x_n$ can only be replaced by term t

if $params(t) \subseteq x_1, \ldots, x_n$

Safe and Unsafe Rules

Safe and Unsafe Rules

Safe allI, exE

Unsafe allE, exI

Safe and Unsafe Rules

Safe allI, exE

Unsafe allE, exI

Create parameters first, unknowns later

Demo: Quantifier Proofs

Parameter names

Parameter names are chosen by Isabelle

1. $\forall\, x.\ \exists y.\ x = y$

Parameter names

Parameter names are chosen by Isabelle

1. $\forall x.\ \exists y.\ x = y$

apply (rule allI)

1. $\bigwedge x.\ \exists y.\ x = y$

Parameter names

Parameter names are chosen by Isabelle

1. $\forall x. \exists y. x = y$

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1. $\bigwedge x. \exists y. x = y$

apply (rule_tac x = "x" in exI)

Brittle!

Renaming parameters

1. $= y$

ap lll)

1. $x = y$

Renaming parameters

1. $= y$

ap ...allI)

1. $x = y$

apply (rename_tac N)

1. $\bigwedge N.\ \exists y.\ N = y$

Renaming parameters

1. $= y$

ap alll)

1. $x = y$

apply (rename_tac N)

1. $\bigwedge N.\ \exists y.\ N = y$

apply (rule_tac x = "N" in exI)

In general:
(rename_tac $x_1 \ldots x_n$) renames the rightmost (inner) n parameters to $x_1 \ldots x_n$

Forward Proof: frule and drule

apply (frule $<$ *rule* $>$)

Rule: $[\![A_1; \ldots; A_m]\!] \Longrightarrow A$

Subgoal: $1.\ [\![B_1; \ldots; B_n]\!] \Longrightarrow C$

Forward Proof: frule and drule

apply (frule $<$ *rule* $>$)

Rule: $[\![A_1; \dots; A_m]\!] \Longrightarrow A$

Subgoal: 1. $[\![B_1; \dots; B_n]\!] \Longrightarrow C$

Substitution: $\sigma(B_i) \equiv \sigma(A_1)$

Forward Proof: frule and drule

apply (frule < *rule* >)

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Subgoal: $1.\ [\![B_1; \ldots ; B_n]\!] \Longrightarrow C$

Substitution: $\sigma(B_i) \equiv \sigma(A_1)$

New subgoals: $1.\ \sigma([\![B_1; \ldots ; B_n]\!] \Longrightarrow A_2)$

$\vdots$

$\text{m-1}.\ \sigma([\![B_1; \ldots ; B_n]\!] \Longrightarrow A_m)$

$\text{m}.\ \sigma([\![B_1; \ldots ; B_n; A]\!] \Longrightarrow C)$

Forward Proof: frule and drule

apply (frule $<$ *rule* $>$)

Rule: $[\![A_1; \ldots; A_m]\!] \Longrightarrow A$

Subgoal: 1. $[\![B_1; \ldots; B_n]\!] \Longrightarrow C$

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New subgoals: 1. $\sigma([\![B_1; \ldots; B_n]\!] \Longrightarrow A_2)$

$\vdots$

m-1. $\sigma([\![B_1; \ldots; B_n]\!] \Longrightarrow A_m)$

m. $\sigma([\![B_1; \ldots; B_n; A]\!] \Longrightarrow C)$

Like **frule** but also deletes B_i: **apply** (drule $<$ *rule* $>$)

Examples for Forward Rules

$$\frac{P \wedge Q}{P}\ \text{conjunct1} \qquad \frac{P \wedge Q}{Q}\ \text{conjunct2}$$

$$\frac{P \longrightarrow Q \quad P}{Q}\ \text{mp}$$

$$\frac{\forall x.\ P\ x}{P\ ?x}\ \text{spec}$$

Forward Proof: OF

F $r_1 \ldots r_n$]

Prove assumption ... *em r with theorem r_1, and assumption 2 with theore* ... . .

Forward Proof: OF

r [OF $r_1 \dots r_n$]

Prove assumption 1 of theorem r with theorem r_1, and assumption 2 with theorem r_2, and ...

Rule r $[\![A_1; \dots; A_m]\!] \Longrightarrow A$

Rule r_1 $[\![B_1; \dots; B_n]\!] \Longrightarrow B$

Forward Proof: OF

[OF $r_1 \ldots r_n$]

Prove assumption 1 of theorem r with theorem r_1, and assumption 2 with theorem r_2, and . . .

Rule r	$[\![A_1; \ldots; A_m]\!] \Longrightarrow A$
Rule r_1	$[\![B_1; \ldots; B_n]\!] \Longrightarrow B$
Substitution	$\sigma(B) \equiv \sigma(A_1)$

Forward Proof: OF

[illegible]F $r_1 \dots r_n$]

Prove assumption [illegible] *em r with theorem r_1, and assumption 2 with theore* [illegible] *...*

Rule r	$[\![A_1; \dots; A_m]\!] \Longrightarrow A$
Rule r_1	$[\![B_1; \dots; B_n]\!] \Longrightarrow B$
Substitution	$\sigma(B) \equiv \sigma(A_1)$
r [OF r_1]	$\sigma([\![B_1; \dots; B_n; A_2; \dots; A_m]\!] \Longrightarrow A)$

Forward Proof: OF

r [OF $r_1 \ldots r_n$]

Prove assumption 1 of theorem r with theorem r_1, and assumption 2 with theorem r_2, and ...

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Substitution	$\sigma(B) \equiv \sigma(A_1)$
r [OF r_1]	$\sigma([\![B_1; \ldots ; B_n; A_2; \ldots ; A_m]\!] \Longrightarrow A)$

Example:

$dvd\_add :\ [\![?a\ dvd\ ?b; ?a\ dvd\ ?c]\!] \Longrightarrow ?a\ dvd\ ?b + ?c$

$dvd\_refl :\ ?a\ dvd\ ?a$

$dvd\_add[OF\ dvd\_refl] :\ [\![?a\ dvd\ ?c]\!] \Longrightarrow ?a\ dvd\ ?a + ?c$

Forward proofs: THEN

r_1 [means r_2 [OF r_1]

Demo: Forward Proofs

Hilbert's Epsilon Operator

(David Hilbert, 1862–1943)

$\varepsilon\ x.\ Px$ is a value that satisfies P (if such a value exists)

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In Isabelle the ε-operator is written SOME $x.\ P\ x$

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$$\frac{P\ ?x}{P\ (\mathrm{SOME}\ x.\ P\ x)}\ \mathrm{someI}$$

More Epsilon

[illegible] Axiom of Choice:

$$\forall x.\ \exists\ [illegible] \Longrightarrow \exists f.\ \forall x.\ Q\ x\ (f\ x)$$

Existential and universal quantification can be defined with ε.

More Epsilon

Axiom of Choice:

$$\forall x.\ \exists y.\ Q\ x\ y \Longrightarrow \exists f.\ \forall x.\ Q\ x\ (f\ x)$$

Existential and universal quantification can be defined with ε.

Isabelle also knows the definite description operator **THE** (aka ι):

$$\frac{}{(\text{THE } x.\ x = a) = a}\ \text{the\_eq\_trivial}$$

Some Automation

More Proof Methods:

apply (intro <intro-rul eatedly applies intro rules

apply (elim <elim-rule eatedly applies elim rules

Some Automation

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apply (intro <intro-rul … eatedly applies intro rules

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apply clarify applies all safe rules that do not split the goal

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apply (elim <elim-rule	eatedly applies elim rules
apply clarify	applies all safe rules that do not split the goal
apply safe	applies all safe rules

Some Automation

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apply fast	another automatic search tactic

Epsilon and Automation Demo

We have learned so far...

➜ Proof rules for pred us

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- ➜ Proof rules for pred[illegible]us
- ➜ Safe and unsafe rul[illegible]

We have learned so far...

- Proof rules for pred us
- Safe and unsafe rul
- Forward Proof

We have learned so far...

- ➜ Proof rules for predicate calculus
- ➜ Safe and unsafe rules
- ➜ Forward Proof
- ➜ The Epsilon Operator

We have learned so far...

- Proof rules for pred us
- Safe and unsafe rul
- Forward Proof
- The Epsilon Operator
- Some automation

Isar (Part 1)

A Language for Structured Proofs

Motivation

Is thi $\longrightarrow B) = (B \vee \neg A)$?

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Is thi … $\longrightarrow B) = (B \vee \neg A)$?

YES!

```
apply (rule iffI)
 apply (cases A)
  apply (rule disjI1)
  apply (erule impE)
   apply assumption
  apply assumption
 apply (rule disjI2)
 apply assumption
apply (rule impI)
apply (erule disjE)
 apply assumption
apply (erule notE)
apply assumption
done
```

Motivation

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 apply assumption
apply (erule notE)
apply assumption
done
```

or `by blast`

Motivation

Is this true: $(A \longrightarrow B) = (B \vee \neg A)$?

YES!

```
apply (rule iffI)
 apply (cases A)
  apply (rule disjI1)
  apply (erule impE)
   apply assumption
  apply assumption
 apply (rule disjI2)
 apply assumption
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 apply assumption
apply (erule notE)
apply assumption
done
```

or `by blast`

OK it's true. But WHY?

Motivation

WHY is $(A \longrightarrow B) = (B \vee \neg A)$?

Demo

Isar

apply s

➜ unreadabl

Isar

apply s

➜ unreadabl

➜ hard to m

Isar

apply s

- ➜ unreadabl
- ➜ hard to m
- ➜ do not scale

Isar

apply

➜ unreadabl

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No structure.

Isar

apply scripts

- ➜ unreadable
- ➜ hard to maintain
- ➜ do not scale

What about..

- ➜ Elegance?

No structure.

Isar

apply s

- unreadabl
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What about..

- Elegance?
- Explaining deeper insights?

No structure.

Isar

apply s

➜ unreadable
➜ hard to maintain
➜ do not scale

What about..

➜ Elegance?
➜ Explaining deeper insights?
➜ Large developments?

No structure.

Isar

apply s[illegible]

- ➜ unreadable
- ➜ hard to maintain
- ➜ do not scale

What about..

- ➜ Elegance?
- ➜ Explaining deeper insights?
- ➜ Large developments?

No structure.

Isar!

A typical Isar proof


```
      formula_0
      mula_1       by simp
    :
    have formula_n     by blast
    show formula_n+1   by ...
qed
```

A typical Isar proof

formula$_0$
mula$_1$ **by** simp
$\vdots$
have *formula*$_n$ **by** blast
show *formula*$_{n+1}$ **by** ...
qed

proves *formula*$_0 \Longrightarrow$ *formula*$_{n+1}$

A typical Isar proof

$\mathit{formula}_0$

mula_1 **by** simp

⋮

have $\mathit{formula}_n$ **by** blast

show $\mathit{formula}_{n+1}$ **by** ...

qed

proves $\mathit{formula}_0 \Longrightarrow \mathit{formula}_{n+1}$

(analogous to **assumes**/**shows** in lemma statements)

Isar core syntax

proof = **proof** [method] statement* **qed**
  | **by** method

Isar core syntax

proof = **proof** [method] statement* **qed**
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method = (simp . . .) | . . .) | (rule . . .) | . . .

Isar core syntax

proof $=$ **proof** [method] statement* **qed**
 $|$ **by** method

method $=$ (simp . . .) $|$ [illegible]) $|$ (rule . . .) $|$ . . .

statement $=$ **fix** variables $(\bigwedge)$
 $|$ **assume** proposition $(\Longrightarrow)$
 $|$ [**from** name$^+$] (**have** $|$ **show**) proposition proof
 $|$ **next** (separates subgoals)

Isar core syntax

proof = **proof** [method] statement* **qed**
  | **by** method

method = (simp . . .) | [illegible]) | (rule . . .) | . . .

statement = **fix** variables (⋀)
  | **assume** proposition (⟹)
  | [**from** name+] (**have** | **show**) proposition proof
  | **next** (separates subgoals)

proposition = [name:] formula

proof and qed

proof [method] statement* **qed**

lemma "$[\![A; B]\!] \Longrightarrow A$

proof and qed

proof [method] statement* **qed**

lemma "$[\![A; B]\!] \Longrightarrow A$
proof (rule conjI)

proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A$
proof (rule conjI)
    assume A: "A"
    from A **show** "A" **by** assumption

proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A$
proof (rule conjI)
    assume A: "A"
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proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A$
proof (rule conjI)
    assume A: "A"
    from A **show** "A" **by** assumption
next
    assume B: "B"
    from B **show** "B" **by** assumption

proof and qed

proof [method] statement* **qed**

lemma "$\llbracket A; B \rrbracket \Longrightarrow A$
proof (rule conjI)
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➜ **proof** (<method>) applies method to the stated goal

proof and qed

proof [method] statement* **qed**

lemma "$[\![A; B]\!] \Longrightarrow A$

proof (rule conjI)
  assume A: "A"
  from A **show** "A" **by** assumption
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  assume B: "B"
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qed

- ➜ **proof** (<method>) applies method to the stated goal
- ➜ **proof** applies a single rule that fits

proof and qed

proof [method] statement* **qed**

lemma "$[\![A; B]\!] \Longrightarrow A$
proof (rule conjI)
  assume A: "A"
  from A **show** "A" **by** assumption
next
  assume B: "B"
  from B **show** "B" **by** assumption
qed

- **proof** (<method>) applies method to the stated goal
- **proof** applies a single rule that fits
- **proof** - does nothing to the goal

How do I know what to Assume and Show?

he proof state!

lemma "$[\![A; B]\!] \Longrightarrow A$
proof (rule conjI)

How do I know what to Assume and Show?

he proof state!

lemma "$\llbracket A; B \rrbracket \Longrightarrow A$

proof (rule conjI)

➜ **proof** (rule conjI) changes proof state to

1. $\llbracket A; B \rrbracket \Longrightarrow A$
2. $\llbracket A; B \rrbracket \Longrightarrow B$

How do I know what to Assume and Show?

the proof state!

lemma "$\llbracket A; B \rrbracket \Longrightarrow A$

proof (rule conjI)

- **proof** (rule conjI) changes proof state to
  1. $\llbracket A; B \rrbracket \Longrightarrow A$
  2. $\llbracket A; B \rrbracket \Longrightarrow B$
- so we need 2 shows: show "A" and show "B"

How do I know what to Assume and Show?

he proof state!

lemma "$\llbracket A; B \rrbracket \Longrightarrow A$

proof (rule conjI)

- ➜ **proof** (rule conjI) changes proof state to
  1. $\llbracket A; B \rrbracket \Longrightarrow A$
  2. $\llbracket A; B \rrbracket \Longrightarrow B$
- ➜ so we need 2 shows: **show** "A" and **show** "B"
- ➜ We are allowed to **assume** A,
  because A is in the assumptions of the proof state.

The Three Modes of Isar

- **[prove]**:
  goal has been state ds to follow.

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➜ **[prove]**:
goal has been state… …ds to follow.

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proof block has ope… …oal has been proved,
new *from* statemen… …ment or assumptions can follow.

The Three Modes of Isar

- **[prove]**:
  goal has been state[illegible]ds to follow.
- **[state]**:
  proof block has ope[illegible]oal has been proved,
  new *from* statemen[illegible]ment or assumptions can follow.
- **[chain]**:
  from statement has been made, goal statement needs to follow.

The Three Modes of Isar

- **[prove]**:
  goal has been state[illegible]ds to follow.
- **[state]**:
  proof block has ope[illegible]oal has been proved,
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lemma "$[\![A; B]\!] \Longrightarrow A$ [illegible] B"

The Three Modes of Isar

- ➜ **[prove]**:
  goal has been stated, proof needs to follow.
- ➜ **[state]**:
  proof block has opened or subgoal has been proved,
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lemma "$\llbracket A; B \rrbracket \Longrightarrow A \wedge B$" **[prove]**

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  from A **[chain]**

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lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$" **[prove]**
proof (rule conjI) **[state]**
  assume A: "A" **[state]**
  from A **[chain]** **show** "A" **[prove]** **by** assumption **[state]**
next **[state]** . . .

Have

Can be [illegible] ake intermediate steps.

Example:

Have

Can be used to make intermediate steps.

Example:

lemma "$(x :: nat) + 1 = 1 + x$"

Have

Can be used to make intermediate steps.

Example:

lemma "$(x :: \text{nat}) + 1 = 1 + x$"
proof -
  have A: "$x + 1 = \text{Suc } x$" **by** simp
  have B: "$1 + x = \text{Suc } x$" **by** simp
  show "$x + 1 = 1 + x$" **by** (simp only: A B)
qed

Demo

Backward and Forward

Backward reasoning: … **have** "$A \wedge B$" **proof**

Backward and Forward

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Backward and Forward

Backward reasoning: . . . **have** "$A \wedge B$" **proof**

- ➜ **proof** picks an **intr** … natically
- ➜ conclusion of rule m … ith $A \wedge B$

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Forward reasoning: .

assume AB: "$A \wedge B$"
from AB **have** ". . ." **proof**

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assume AB: "$A \wedge B$"
from AB **have** ". . ." **proof**

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Backward reasoning: . . . **have** "$A \wedge B$" **proof**

- ➜ **proof** picks an **intr**[illegible]matically
- ➜ conclusion of rule m[illegible]ith $A \wedge B$

Forward reasoning: . [illegible]

assume AB: "$A \wedge B$"

from AB **have** ". . ." **proof**

- ➜ now **proof** picks an **elim** rule automatically
- ➜ triggered by **from**

Backward and Forward

Backward reasoning: . . . **have** "$A \wedge B$" **proof**

- ➜ **proof** picks an **intro** rule automatically
- ➜ conclusion of rule must unify with $A \wedge B$

Forward reasoning: . . .

assume AB: "$A \wedge B$"
from AB **have** ". . ." **proof**

- ➜ now **proof** picks an **elim** rule automatically
- ➜ triggered by **from**
- ➜ first assumption of rule must unify with AB

Backward and Forward

Backward reasoning: . . . **have** "$A \wedge B$" **proof**

- **proof** picks an **intr**[illegible]atically
- conclusion of rule m[illegible]ith $A \wedge B$

Forward reasoning: . [illegible]

assume AB: "$A \wedge B$"
from AB **have** ". . ." **proof**

- now **proof** picks an **elim** rule automatically
- triggered by **from**
- first assumption of rule must unify with AB

General case: from $A_1 \ldots A_n$ **have** R **proof**

- first n assumptions of rule must unify with $A_1 \ldots A_n$
- conclusion of rule must unify with R

Fix and Obtain

$v_1 \dots v_n$

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Introdu[illegible]bitrary but fixed variables

($\sim$ parameters, $\bigwedge$)

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Introduces new variables together with property

Fancy Abbreviations

this previous fact proved or assumed

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this previous fact proved or assumed

then m this

Fancy Abbreviations

this		previous fact proved or assumed
then		**m** this
thus	=	**then show**

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this	=	the previous fact proved or assumed
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thus	=	**then show**
hence	=	**then have**
with $A_1 \ldots A_n$	=	**from** $A_1 \ldots A_n$ this

Fancy Abbreviations

this	=	the previous fact proved or assumed
then	=	**from** this
thus	=	**then show**
hence	=	**then have**
with $A_1 \ldots A_n$	=	**from** $A_1 \ldots A_n$ this
?thesis	=	the last enclosing goal statement

Demo

Moreover and Ultimately

have X_1: P_1 ...
have X_2: P_2 ...
$\vdots$
have X_n: P_n ...
from $X_1 \ldots X_n$ **show** ...

Moreover and Ultimately

have X_1: P_1 ...
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wastes lots of brain power
on names $X_1 \ldots X_n$

Moreover and Ultimately

have X_1: P_1
have X_2: P_2
$\vdots$
have X_n: P_n ...
from $X_1 \ldots X_n$ **show** ...

have P_1 ...
moreover **have** P_2 ...
$\vdots$
moreover **have** P_n ...
ultimately **show** ...

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General Case Distinctions

show *formula*
proof -

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  have $P_1 \vee P_2$ <proof>

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  moreover … P_1 . . . **have** ?thesis <proof> }

General Case Distinctions

show *formula*
proof -
  have $P_1 \lor P_2$ <proof>
  moreover { **assume** P_1 ... **have** ?thesis <proof> }
  moreover { **assume** P_2 ... **have** ?thesis <proof> }

General Case Distinctions

show *formula*
proof -
  have $P_1 \vee P_2$ [illegible]oof>
  moreover [illegible] P_1 ... **have** ?thesis <proof> }
  moreover { **assume** P_2 ... **have** ?thesis <proof> }
  moreover { **assume** P_3 ... **have** ?thesis <proof> }

General Case Distinctions


```
show formula
proof -
  have P1 ∨ P2 ... oof>
  moreover   ... P1   ...  have ?thesis  <proof> }
  moreover   { assume P2  ...  have ?thesis  <proof> }
  moreover   { assume P3  ...  have ?thesis  <proof> }
  ultimately show ?thesis by blast
qed
```

General Case Distinctions

show *formula*
proof -
  have $P_1 \vee P_2$ … oof>
  moreover … P_1 … **have** ?thesis <proof> }
  moreover { **assume** P_2 … **have** ?thesis <proof> }
  moreover { **assume** P_3 … **have** ?thesis <proof> }
  ultimately show ?thesis **by** blast
qed

{ … } is a proof block similar to **proof** … **qed**

General Case Distinctions

show *formula*
proof -
  have $P_1 \vee P_2 \vee P_3$ <proof>
  moreover { **assume** P_1 ... **have** ?thesis <proof> }
  moreover { **assume** P_2 ... **have** ?thesis <proof> }
  moreover { **assume** P_3 ... **have** ?thesis <proof> }
  ultimately show ?thesis **by** blast
qed

{ ... } is a proof block similar to **proof** ... **qed**

{ **assume** P_1 ... **have** P <proof> }

stands for $P_1 \Longrightarrow P$

Mixing proof styles

from ...
have ...
  apply - incoming facts assumptions
  apply (...)
  ⋮
  apply (...)
  done

We have learned so far...

- ➜ Isar style proofs

We have learned so far...

- Isar style proofs
- proof, qed

We have learned so far...

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- proof, qed
- assumes, shows

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- Isar style proofs
- proof, qed
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We have learned so far...

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- proof, qed
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- proof, qed
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