COMP4161
Advanced Topics in Software Verification

Gerwin Klein, June Andronick, Miki Tanaka, Johannes Åman Pohjola

T3/2022

Binary Search (java.util.Arrays)


```
public static int binarySearch(int[] a, int key) {
    int low = 0;
    int high = a.length - 1;

    while (low <= high) {
        int mid = (low + high
        int midVal = a[mid];

        if (midVal < key)
             low = mid + 1
         else if (midVal > key)
             high = mid - 1;
         else
             return mid; // key found
     }
     return -(low + 1);  // key not found.
 }
```

Binary Search (`java.util.Arrays`)


```
1:     public static int binarySearch(int[] a, int key) {
2:         int low = 0;
3:         int high = a.length - 1;
4:
5:         while (low <= high) {
6:             int mid = (low + high
7:             int midVal = a[mid];
8:
9:             if (midVal < key)
10:                low = mid + 1
11:             else if (midVal > key)
12:                high = mid - 1;
13:             else
14:                return mid; // key found
15:         }
16:         return -(low + 1);  // key not found.
17:    }
```



```
6:                 int mid = (low + high) / 2;
```


http://googleresearch.blogspot.com/2006/06/extra-extra-read-all-about-it-nearly.html

Organisatorials

V[illegible]on 14:00 – 16:00
[illegible]u 10:00 – 12:00

http://www.cse.unsw.edu.au/~cs4161/

About us: Proofcraft and TS

- **TS** (Trustworthy Systems) is a research group at UNSW
  - track record o[illegible]nd real world impact in verified software
  - biggest achiev[illegible]al verification of **seL4**
- **Proofcraft** is a new [illegible]
  - from former leaders of TS
  - providing services in software verification

- **seL4** is an operating microkernel used around the world in critical systems
  - with a proof of functional correctness and security
    Security↔ Isabelle/HOL model ↔ Haskell model ↔C code ↔Binary
  - 10 000 LOC / more than 1 million lines of proof
  - Open source, `http://sel4.systems`

We are always embarking on exciting new projects. Talk to us!

- *summer student scholarship projects*
- *honours and PhD theses*
- *research assistant and verification engineer positions*

What you will learn

- how to use a theore
- background, how it
- how to prove and s
- how to reason about programs

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Health Warning

Theorem Proving is addictive

Prerequisites

This is an advanced course. It assumes knowledge in

- Functional program
- First-order formal lo

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The following program should make sense to you:

```
map f []      =  []
map f (x:xs)  =  f x : map f xs
```

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This is an advanced course. It assumes knowledge in

- ➜ Functional programming
- ➜ First-order formal logic

The following program should make sense to you:

$$
\begin{array}{lcl}
\text{map f []} & = & \text{[]} \\
\text{map f (x:xs)} & = & \text{f x : map f xs}
\end{array}
$$

You should be able to read and understand this formula:

$$\exists x.\ (P(x) \longrightarrow \forall x.\ P(x))$$

Content — Using Theorem Provers

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- Foundations & Prin
  - Intro, Lambda atural deduction
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- Proof & Specification Techniques
  - Inductively defined sets, rule induction
  - Datatype induction, primitive recursion
  - General recursive functions, termination proofs
  - Proof automation, Isar (part 2)
  - Hoare logic, proofs about programs, invariants
  - C verification
  - Practice, questions, exam prep

Content — Using Theorem Provers

Rough timeline

➜ Foundations & Prin

- Intro, Lambda atural deduction [1,2]
- Higher Order part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7[b]]
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- C verification [9,10]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

To have a chance at succeeding

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- ➜ try the exercises/homework we give, when we do give some
- ➜ **DO NOT CHEAT**
  - Assignments and exams are take-home. This does NOT mean you can work in groups. Each submission is personal.
  - For more info, see Plagiarism Policy[a]

[a] https://student.unsw.edu.au/plagiarism

Credits

some material (in using [illegible] provers part) shamelessly stolen from

Tobias Nipkow, Larry Paulson, Markus Wenzel

David Basin, Burkhardt Wolff

Don't blame them, errors are ours

What is a formal proof?

A derivation in a formal calculus

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A derivation in a formal calculus

Example: $A \wedge B \longrightarrow B \wedge A$ derivable in the following system

Rules:

$$\frac{X \in S}{S \vdash X}\ \text{(assumption)} \qquad \frac{S \cup \{X\} \vdash Y}{S \vdash X \longrightarrow Y}\ \text{(impI)}$$

$$\frac{S \vdash X \quad S \vdash Y}{S \vdash X \wedge Y}\ \text{(conjI)} \qquad \frac{S \cup \{X, Y\} \vdash Z}{S \cup \{X \wedge Y\} \vdash Z}\ \text{(conjE)}$$

What is a formal proof?

A derivation in a formal calculus

Example: $A \wedge B \longrightarrow B \wedge A$ is derivable in the following system

Rules:

$$\frac{X \in S}{S \vdash X}\ \text{(assumption)} \qquad \frac{S \cup \{X\} \vdash Y}{S \vdash X \longrightarrow Y}\ \text{(impl)}$$

$$\frac{S \vdash X \quad S \vdash Y}{S \vdash X \wedge Y}\ \text{(conjI)} \qquad \frac{S \cup \{X, Y\} \vdash Z}{S \cup \{X \wedge Y\} \vdash Z}\ \text{(conjE)}$$

Proof:

1.	$\{A, B\} \vdash B$	(by assumption)
2.	$\{A, B\} \vdash A$	(by assumption)
3.	$\{A, B\} \vdash B \wedge A$	(by conjI with 1 and 2)
4.	$\{A \wedge B\} \vdash B \wedge A$	(by conjE with 3)
5.	$\{\} \vdash A \wedge B \longrightarrow B \wedge A$	(by impl with 4)

What is a theorem prover?

Implementation of a formal logic on a computer.

- ➜ fully automated (pr ... logic)
- ➜ automated, but not ... terminating (first order logic)
- ➜ with automation, b ... teractive (higher order logic)

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- ➜ can deliver proofs

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There are other (algorithmic) verification tools:

- ➜ model checking, static analysis, ...
- ➜ usually do not deliver proofs
- ➜ See COMP3153: Algorithmic Verification

Why theorem proving?

- ➜ Analysing systems/[illegible] thoroughly

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Why theorem proving?

- ➜ Analysing systems/ … oroughly
- ➜ Finding design and … n errors early
- ➜ High assurance (ma … machine checked proof)
- ➜ it's not always easy
- ➜ it's fun

Main theorem proving system for this course

Isabelle

➜ used here for applications, learning how to prove

What is Isabelle?

A generic interactive proof assistant

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  more than just yes/no, you can interactively guide the system

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- ➜ **interactive:**
  more than just yes/no, you can interactively guide the system
- ➜ **proof assistant:**
  helps to explore, find, and maintain proofs

If I prove it on the computer, it is correct, right?

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① hardware could be f

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4. compiler could be faulty
5. implementation could be faulty

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⑤ implementation could be faulty

⑥ logic could be inconsistent

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⑤ implementation could be faulty

⑥ logic could be inconsistent

⑦ theorem could mean something else

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probability for

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- faulty implementation reduced by having the right prover architecture

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- wrong theorem reduced by expressive/intuitive logics

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- ➜ wrong theorem reduced by expressive/intuitive logics

No guarantees, but assurance immensly higher than manual proof

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Soundness architectures
careful implementat

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Soundness architectures

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LCF approach, sma nel	HOL4
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If I prove it on the computer, it is correct, right?

Soundness architectures

careful implementat…	PVS ACL2
LCF approach, sma… …nel	HOL4 Isabelle HOL-light
explicit proofs + proof checker	Coq Lean Twelf Isabelle HOL4 Agda

Meta Logic

Meta language:
The language used to t[illegible] another language.

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Examples:
English in a Spanish cl[illegible]h in an English class

Meta Logic

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Examples:
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Meta logic:
The logic used to formalize another logic

Example:
Mathematics used to formalize derivations in formal logic

Meta Logic – Example

Syntax:

Formulae: $F ::= V \quad | \quad F \longrightarrow F \quad | \quad F \wedge F \quad | \quad False$

$- Z]$

Derivable: X a formula, S a set of formulae

Meta Logic – Example

Syntax:

Formulae: $F ::= V \mid F \longrightarrow F \mid F \wedge F \mid False$

$- Z]$

Derivable: X a formula, S a set of formulae

logic / meta logic

$$\frac{X \in S}{S \vdash X} \qquad \frac{S \cup \{X\} \vdash Y}{S \vdash X \longrightarrow Y}$$

$$\frac{S \vdash X \quad S \vdash Y}{S \vdash X \wedge Y} \qquad \frac{S \cup \{X, Y\} \vdash Z}{S \cup \{X \wedge Y\} \vdash Z}$$

Isabelle's Meta Logic

$$\bigwedge \qquad \Longrightarrow \qquad \lambda$$

$\bigwedge$

Syntax: $\bigwedge x.\ F$ other meta level formula)

in ASCII: `!!x. F`

$\bigwedge$

Syntax: $\bigwedge x.\ F$ ... other meta level formula)

in ASCII: `!!x. F`

- ➜ universal quantifier on the meta level
- ➜ used to denote parameters
- ➜ example and more later

$\Longrightarrow$

Syntax: $A \Longrightarrow B$ (A, B other meta level formulae)

in ASCII: `A ==> B`

$\Longrightarrow$

Syntax: $A \Longrightarrow B$ (A, B other meta level formulae)

in ASCII: `A ==> B`

Binds to the right:

$$A \Longrightarrow B \Longrightarrow C \quad = \quad A \Longrightarrow (B \Longrightarrow C)$$

Abbreviation:

$$[\![A; B]\!] \Longrightarrow C \quad = \quad A \Longrightarrow B \Longrightarrow C$$

- read: A and B implies C
- used to write down rules, theorems, and proof states

Example: a theorem

mathematics: if $x < 0$ and $y < 0$, then $x + y < 0$

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formal logic: $\vdash x$ [illegible] $0 \longrightarrow x + y < 0$

variation: $x <$ [illegible] $x + y < 0$

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formal logic: $\vdash x$ [illegible] $0 \longrightarrow x + y < 0$

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Isabelle: **lemma** "$x < 0 \wedge y < 0 \longrightarrow x + y < 0$"

variation: **lemma** "$\llbracket x < 0; y < 0 \rrbracket \Longrightarrow x + y < 0$"

Example: a theorem

mathematics: if $x < 0$ and $y < 0$, then $x + y < 0$

formal logic: $\vdash\ x$ [illegible] $0 \longrightarrow x + y < 0$

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Isabelle: **lemma** "$x < 0 \wedge y < 0 \longrightarrow x + y < 0$"

variation: **lemma** "$[\![x < 0; y < 0]\!] \Longrightarrow x + y < 0$"

variation: **lemma**

assumes "$x < 0$" and "$y < 0$" shows "$x + y < 0$"

Example: a rule

logic: $\dfrac{X}{X}$

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Example: a rule

logic: $\frac{X \quad Y}{X \wedge Y}$

variation: $\frac{S \vdash X \quad S \vdash Y}{S \vdash X \wedge Y}$

Isabelle: $[\![X; Y]\!] \Longrightarrow X \wedge Y$

Example: a rule with nested implication

logic:

$$\frac{X \qquad \begin{array}{c}[Y]\\ \vdots\\ Z\end{array}}{}$$

Example: a rule with nested implication

logic:

$$\frac{X \qquad \begin{matrix} Y \\ \vdots \\ Z \end{matrix}}{}$$

variation:

$$\frac{S \cup \{X\} \vdash Z \qquad S \cup \{Y\} \vdash Z}{S \cup \{X \vee Y\} \vdash Z}$$

Example: a rule with nested implication

logic: $\dfrac{X \qquad \begin{array}{c} Y \\ \vdots \\ Z \end{array}}{}$

variation: $\dfrac{S \cup \{X\} \vdash Z \qquad S \cup \{Y\} \vdash Z}{S \cup \{X \vee Y\} \vdash Z}$

Isabelle: $[\![X \vee Y; X \Longrightarrow Z; Y \Longrightarrow Z]\!] \Longrightarrow Z$

λ

Syntax: $\lambda x.\ F$ ther meta level formula)

in ASCII: `%x. F`

λ

Syntax: $\lambda x.\ F$ ([illegible]ther meta level formula)

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- lambda abstraction
- used for functions in object logics
- used to encode bound variables in object logics
- more about this soon

Enough Theory!

Getting started with Isabelle

System Architecture

Isabelle – generic, interactive theorem prover

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Standard ML – logic implemented as ADT

System Architecture

HOL, ZF – obje

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Prover IDE (jEdit) – ...erface

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Isabelle – generic, interactive theorem prover

Standard ML – logic implemented as ADT

User can access all layers!

System Requirements

- **Linux**, **Windows**, o[illegible] **(10.8 +)**
- **Standard ML** (Poly[illegible]entation)
- **Java** (for jEdit)

Premade packages for Linux, Mac, and Windows + info on:

`http://mirror.cse.unsw.edu.au/pub/isabelle/`

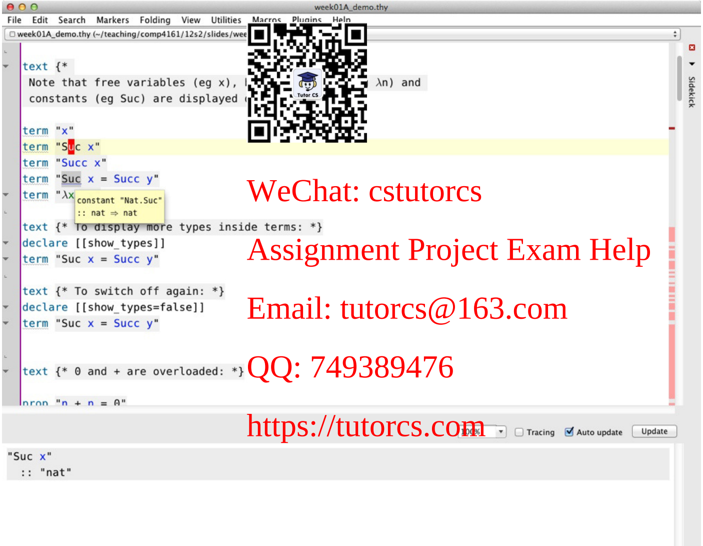
Documentation

Available from `http://isabelle.in.tum.de`

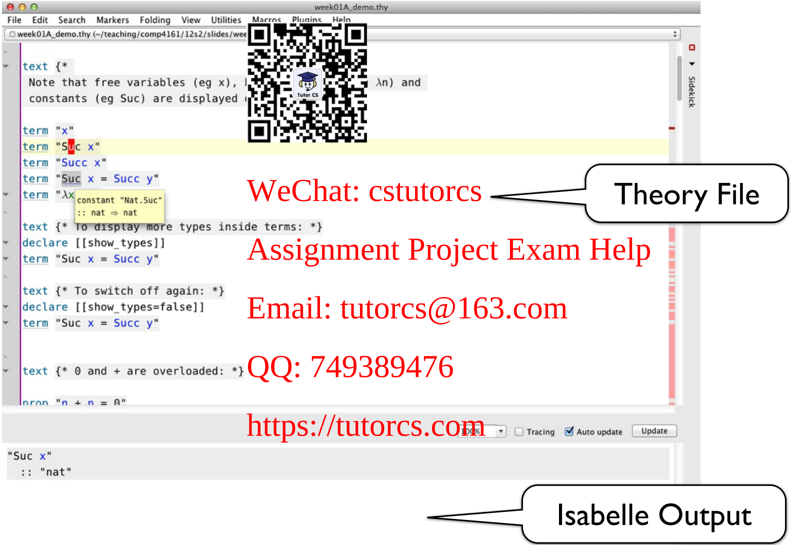
- ➜ Learning Isabelle
  - Concrete Sema
  - Tutorial on Isa (LNCS 2283)
  - Tutorial on Isar
  - Tutorial on Locales
- ➜ Reference Manuals
  - Isabelle/Isar Reference Manual
  - Isabelle Reference Manual
  - Isabelle System Manual
- ➜ Reference Manuals for Object-Logics

Demo

jEdit/PIDE


week01A_demo.thy
File Edit Search Markers Folding View Utilities Macros Plugins Help
week01A_demo.thy (~/teaching/comp4161/12s2/slides/wee
text {*
Note that free variables (eg x),
λn) and
constants (eg Suc) are displayed
term "x"
term "Suc x"
term "Succ x"
term "Suc x = Succ y"
term "λx
constant "Nat.Suc"
:: nat ⇒ nat
text {* To display more types inside terms: *}
declare [[show_types]]
term "Suc x = Succ y"
text {* To switch off again: *}
declare [[show_types=false]]
term "Suc x = Succ y"
text {* 0 and + are overloaded: *}
prop "n + n = 0"
Tracing
Auto update
Update
"Suc x"
:: "nat"
Sidekick

jEdit/PIDE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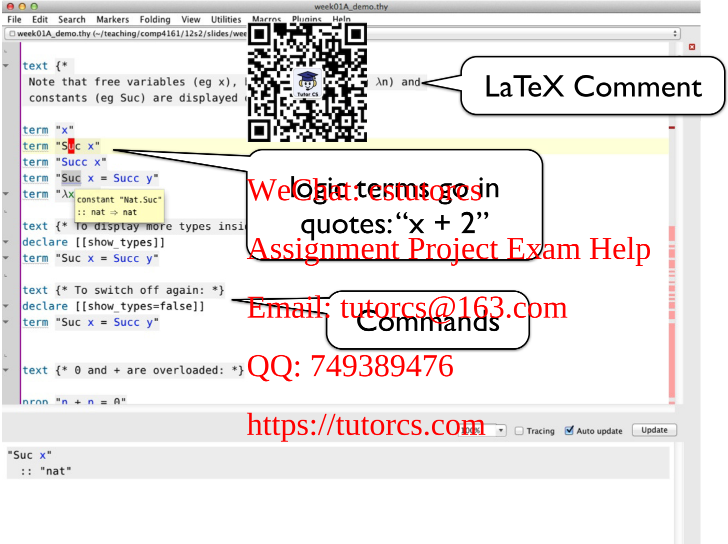
Theory File
Isabelle Output

jEdit/PIDE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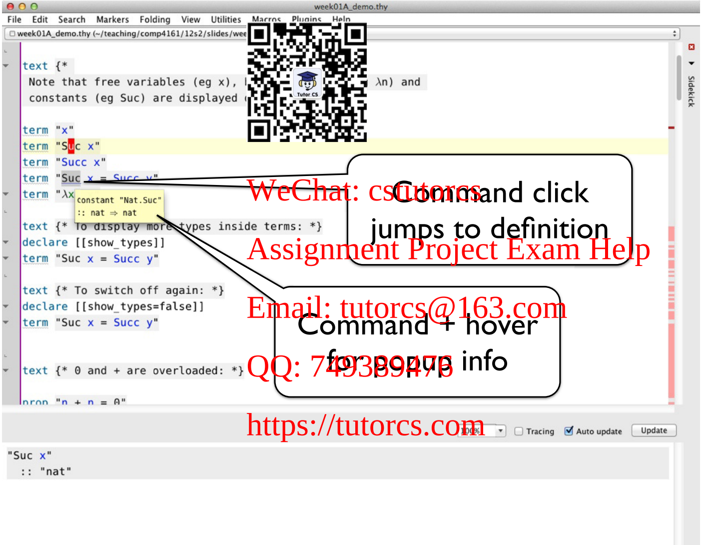
LaTeX Comment
logic terms go in
quotes: "x + 2"
Commands

jEdit/PIDE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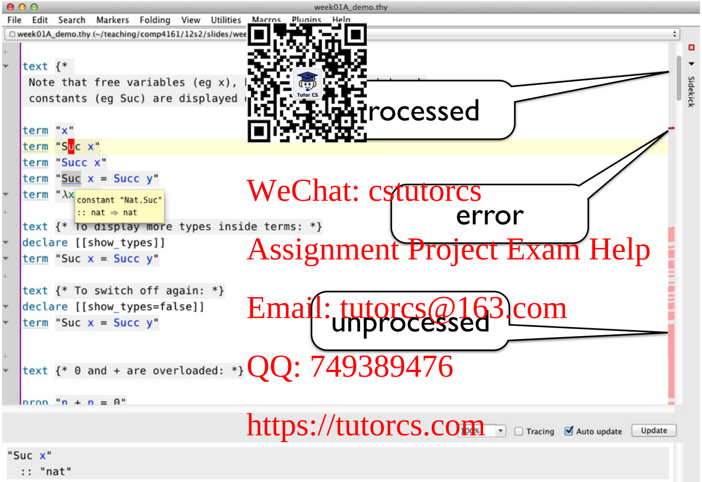
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processed
error
unprocessed
Tracing
Auto update
Update
"Suc x"
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Sidekick

Exercises

- Download and insta... om
  `http://mirror.cs ... 1.au/pub/isabelle/`
- Step through the de... m the lecture web page
- Write your own theory file, look at some theorems in the library, try 'find_theorems'
- How many theorems can help you if you need to prove something containing the term "Suc(Suc x)"?
- What is the name of the theorem for associativity of addition of natural numbers in the library?

λ-Calculus

Content

➜ Foundations & Principles

- Intro, Lambda [illegible] atural deduction [1,2]
- Higher Order [illegible] part 1) [2,3[a]]
- Term rewriting [illegible] [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7[b]]
- Proof automation, Isar (part 2) [8]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

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[a] a1 due; [b] a2 due; [c] a3 due

λ-calculus

Alonzo Church

- lived 1903–1995
- supervised people li[illegible]ing, Stephen Kleene
- famous for Church-[illegible]s, lambda calculus, first undecidability results
- invented λ calculus in 1930's

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λ-calculus

- originally meant as foundation of mathematics
- important applications in theoretical computer science
- foundation of computability and functional programming

untyped λ-calculus

- turing complete mo… utation
- a simple way of wri… unctions

untyped λ-calculus

- turing complete model of computation
- a simple way of writing functions

Basic intuition:

instead of	$f(x) = x + 5$
write	$f = \lambda x.\ x + 5$

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Basic intuition:

instead of $f(x) = x + 5$

write $f = \lambda x.\ x + 5$

$\lambda x.\ x + 5$

- a term
- a nameless function
- that adds 5 to its parameter

Function Application

For applying arguments to functions

ad of $f(a)$

$f\ a$

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Example: $\quad (\lambda x.\ x + 5)\ (a + b)$ evaluates to

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Evaluating: (computation!) in $(\lambda x.\ t)\ a$ replace x by a in t

Example: $(\lambda x.\ x + 5)\ (a + b)$ evaluates to $(a + b) + 5$

That's it!

Now Formal

Syntax

Terms: $t \quad | \ c \ | \ (t\ t) \ | \ (\lambda x.\ t)$

$v, x \quad \in C, \quad V, C$ sets of names

Syntax

Terms: $t ::= v \mid c \mid (t\ t) \mid (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

- v, x variables
- c constants
- $(t\ t)$ application
- $(\lambda x.\ t)$ abstraction

Conventions

- leave out parentheses where possible
- list variables instead of multiple λ

Example: instead of $(\lambda y. (\lambda x. (x\ y)))$ write $\lambda y\ x.\ x\ y$

Conventions

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Rules:

- ➜ list variables: $\lambda x.\ (\lambda y.\ t) = \lambda x\ y.\ t$
- ➜ application binds to the left: $x\ y\ z = (x\ y)\ z \neq x\ (y\ z)$
- ➜ abstraction binds to the right: $\lambda x.\ x\ y = \lambda x.\ (x\ y) \neq (\lambda x.\ x)\ y$
- ➜ leave out outermost parentheses

Getting used to the Syntax

Example:

$\lambda x\ y\ z.\ x\ z\ (y\ z) =$

Getting used to the Syntax

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Getting used to the Syntax

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$\lambda x\ y\ z.\ x\ z\ (y\ z) =$

$\lambda x\ y\ z.\ (x\ z)\ (y\ z) =$

$\lambda x\ y\ z.\ ((x\ z)\ (y\ z)) =$

$\lambda x.\ \lambda y.\ \lambda z.\ ((x\ z)\ (y\ z)) =$

Getting used to the Syntax

Example:

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$\lambda x\ y\ z.\ ((x\ z)\ (y\ z)) =$

$\lambda x.\ \lambda y.\ \lambda z.\ ((x\ z)\ (y\ z)) =$

$(\lambda x.\ (\lambda y.\ (\lambda z.\ ((x\ z)\ (y\ z)))))$

Computation

Intuition: replace parameter by argument
this is call[illegible]tion

Remember: $(\lambda x.\ t$ [illegible] uated (noted $\longrightarrow_\beta$) to
t where [illegible] placed by a

Example

$(\lambda x\ y.\ Suc\ x\ =\ y)\ 3 \longrightarrow_\beta$

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$(\lambda x\ y.\ f\ (y\ x))\ \ 5\ \ (\lambda x.\ x) \longrightarrow_\beta$

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$(\lambda y.\ f\ (y\ 5))\ \ (\lambda x.\ x) \longrightarrow_\beta$
$f\ ((\lambda x.\ x)\ 5) \longrightarrow_\beta$

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$f\ ((\lambda x.\ x)\ 5) \longrightarrow_\beta$
$f\ 5$

Defining Computation

β reduction:

$$
\begin{array}{rclcrcl}
 & & & & t & \longrightarrow_\beta & s[x \leftarrow t] \\
s & \longrightarrow_\beta & s' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s'\ t) \\
t & \longrightarrow_\beta & t' & \Longrightarrow & (s\ t) & \longrightarrow_\beta & (s\ t') \\
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\end{array}
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Still to do: define $s[x \leftarrow t]$

Defining Substitution

Easy conc problem: variable capture.

$(\lambda x.\ x\ z)[z \leftarrow x]$

Defining Substitution

Easy conc[illegible] problem: variable capture.

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We do **not** want: $(\lambda x.\ x\ x)$ as result.

What do we want?

Defining Substitution

Easy conc[illegible] problem: variable capture.

$(\lambda x.\ x\ z)[z \leftarrow x]$

We do **not** want: $(\lambda x.\ x\ x)$ as result.

What do we want?

In $(\lambda y.\ y\ z)\ [z \leftarrow x] = (\lambda y.\ y\ x)$ there would be no problem.

So, solution is: rename bound variables.

Free Variables

Bound variables: in $(\lambda x.\ t)$, x is a bound variable.

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Free variables FV of

$$
\begin{array}{lcl}
FV\ (x) & = & \{x\} \\
FV\ (c) & = & \{\} \\
FV\ (s\ t) & = & FV(s) \cup FV(t) \\
FV\ (\lambda x.\ t) & = & FV(t) \setminus \{x\}
\end{array}
$$

Example: $FV(\ \ \lambda x.\ (\lambda y.\ (\lambda x.\ x)\ y)\ y\ x\ \ )$

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Example: $FV(\ \lambda x.\ (\lambda y.\ (\lambda x.\ x)\ y)\ y\ x\ ) = \{y\}$

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Example: $FV(\ \lambda x.\ (\lambda y.\ (\lambda x.\ x)\ y)\ y\ x\ ) = \{y\}$

Term t is called **closed** if $FV(t) = \{\}$

Free Variables

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\end{aligned}
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Example: $FV(\ \lambda x.\ (\lambda y.\ (\lambda x.\ x)\ y)\ y\ x\ ) = \{y\}$

Term t is called **closed** if $FV(t) = \{\}$

The substitution example, $(\lambda x.\ x\ z)[z \leftarrow x]$, is problematic because the bound variable x is a free variable of the replacement term "x".

Substitution

$$
\begin{aligned}
x\ [x \leftarrow t] &= t \\
y\ [x \leftarrow t] &= y & \text{if } x \neq y \\
c\ [x \leftarrow t] &= c \\
(s_1\ \ s_2)\ [x \leftarrow t] &=
\end{aligned}
$$

Substitution

$$
\begin{array}{lll}
x\ [x \leftarrow t] & = t & \\
y\ [x \leftarrow t] & = y & \text{if } x \neq y \\
c\ [x \leftarrow t] & = c & \\
 & & \\
(s_1\ s_2)\ [x \leftarrow t] & = (s_1[x \leftarrow t]\ s_2[x \leftarrow t]) & \\
 & & \\
(\lambda x.\ s)\ [x \leftarrow t] & = &
\end{array}
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 & & \\
(s_1\ \ s_2)\ [x \leftarrow t] & = (s_1 [x \leftarrow t]\ \ s_2 [x \leftarrow t]) & \\
 & & \\
(\lambda x.\ s)\ [x \leftarrow t] & = (\lambda x.\ s) & \\
(\lambda y.\ s)\ [x \leftarrow t] & = &
\end{array}
$$

Substitution

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\\
(s_1\ s_2)\ [x \leftarrow t] & = (s_1[x \leftarrow t]\ s_2[x \leftarrow t]) & \\
\\
(\lambda x.\ s)\ [x \leftarrow t] & = (\lambda x.\ s) & \\
(\lambda y.\ s)\ [x \leftarrow t] & = (\lambda y.\ s[x \leftarrow t]) & \text{if } x \neq y \text{ and } y \notin FV(t) \\
(\lambda y.\ s)\ [x \leftarrow t] & = &
\end{array}
$$

Substitution

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\begin{array}{lll}
x\ [x \leftarrow t] & = t & \\
y\ [x \leftarrow t] & = y & \text{if } x \neq y \\
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(\lambda y.\ s)\ [x \leftarrow t] & = (\lambda y.\ s[x \leftarrow t]) & \text{if } x \neq y \text{ and } y \notin FV(t) \\
(\lambda y.\ s)\ [x \leftarrow t] & = (\lambda z.\ s[y \leftarrow z][x \leftarrow t]) & \text{if } x \neq y \\
 & & \text{and } z \notin FV(t) \cup FV(s)
\end{array}
$$

Substitution Example

$$(x\ \ (\lambda x.\ x) \ \cdots\ x))[x \leftarrow y]$$

Substitution Example

$$
\begin{aligned}
& (x\ \ (\lambda x.\ x \ldots x))[x \leftarrow y] \\
= \ & (x[x \leftarrow y]) \ldots)[x \leftarrow y])\ \ ((\lambda y.\ z\ x)[x \leftarrow y])
\end{aligned}
$$

Substitution Example

$$
\begin{array}{rl}
 & (x\ \ (\lambda x.\ x)\ \ (\lambda y.\ z\ x))[x \leftarrow y] \\
= & (x[x \leftarrow y])\ \ ((\lambda x.\ x)[x \leftarrow y])\ \ ((\lambda y.\ z\ x)[x \leftarrow y]) \\
= & y\ (\lambda x.\ x)\ (\lambda y.\ z\ y)
\end{array}
$$

α Conversion

Bound names are irrelevant:
$\lambda x.\ x$ and $\lambda y.\ y$ denot function.

α **conversion:**
$s =_\alpha t$ means $s = t$ up ng of bound variables.

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 & & & & (\lambda x.\ t) & \longrightarrow_\alpha & (\lambda y.\ t[x \leftarrow y]) \ \text{ if } y \notin FV(t) \\
s & \longrightarrow_\alpha & s' & \Longrightarrow & (s\ t) & \longrightarrow_\alpha & (s'\ t) \\
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\end{array}
$$

$$s =_\alpha t \quad \text{iff} \quad s \longrightarrow_\alpha^* t$$

($\longrightarrow_\alpha^*$ = transitive, reflexive closure of $\longrightarrow_\alpha$ = multiple steps)

α Conversion

Equality in Is[illegible]quality modulo α conversion:

if $s =_\alpha t$ [illegible]d t are syntactically equal.

Examples:

$x\ (\lambda x\ y.\ x\ y)$

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\end{array}
$$

Back to β

We have defined β reduction: $\longrightarrow_\beta$

Some notation and cor

➜ β **conversion:** $s = \ldots s \longrightarrow_\beta^* n \wedge t \longrightarrow_\beta^* n$

Back to β

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Some notation and cor

- β **conversion:** $s =$. $s \longrightarrow_\beta^* n \wedge t \longrightarrow_\beta^* n$
- t is **reducible** if the uch that $t \longrightarrow_\beta s$

Back to β

We have defined β reduction: $\longrightarrow_\beta$

Some notation and cor[illegible]

- β **conversion:** $s =$ [illegible]. $s \longrightarrow_\beta^* n \wedge t \longrightarrow_\beta^* n$
- t is **reducible** if the[illegible]uch that $t \longrightarrow_\beta s$
- $(\lambda x.\ s)\ t$ is called a **redex** (reducible expression)

Back to β

We have defined β reduction: $\longrightarrow_\beta$

Some notation and concepts:

- **β conversion:** $s =_\beta t$ iff $\exists n.\ s \longrightarrow_\beta^* n \wedge t \longrightarrow_\beta^* n$
- t is **reducible** if there is an s such that $t \longrightarrow_\beta s$
- $(\lambda x.\ s)\ t$ is called a **redex** (reducible expression)
- t is reducible iff it contains a redex

Back to β

We have defined β red[illegible] $\longrightarrow_\beta$

Some notation and co[illegible]

- ➜ β **conversion:** $s =$ [illegible]. $s \longrightarrow_\beta^* n \wedge t \longrightarrow_\beta^* n$
- ➜ t is **reducible** if the[illegible]uch that $t \longrightarrow_\beta s$
- ➜ $(\lambda x.\ s)\ t$ is called a **redex** (reducible expression)
- ➜ t is reducible iff it contains a redex
- ➜ if it is not reducible, t is in **normal form**

Does every λ term have a normal form?

Example:

$$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \longrightarrow_\beta$$

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Does every λ term have a normal form?

No!

Example:

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(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \longrightarrow_\beta \\
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$$

(but: $(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ (\lambda x.\ x\ x)) \longrightarrow_\beta \lambda y.\ y$)

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$$
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\end{array}
$$

(but: $(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ (\lambda x.\ x\ x)) \longrightarrow_\beta \lambda y.\ y$)

λ calculus is not terminating

β reduction is confluent

Confluence: $s \longrightarrow^*_\beta x \wedge s \longrightarrow^*_\beta y \Longrightarrow \exists t.\ x \longrightarrow^*_\beta t \wedge y \longrightarrow^*_\beta t$

$$\begin{tikzcd} & s \arrow[dl] \arrow[dr, "*"] & \\ x & & y \end{tikzcd}$$

β reduction is confluent

Confluence: $s \longrightarrow^*_\beta x \land s \longrightarrow^*_\beta y \Longrightarrow \exists t.\ x \longrightarrow^*_\beta t \land y \longrightarrow^*_\beta t$

$$\begin{tikzcd}
 & s \arrow[dl] \arrow[dr, "*"] & \\
x \arrow[dr, dashed, "*"'] & & y \arrow[dl, dashed, "*"] \\
 & t &
\end{tikzcd}$$

Order of reduction does not matter for result

Normal forms in λ calculus are unique

β reduction is confluent

Example:

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a$

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a$

β reduction is confluent

Example:

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a$ ... $y.\ y)\ (a\ a)$

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a$ ... y

β reduction is confluent

Example:

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a$ … $y.\ y)\ (a\ a) \longrightarrow_\beta \lambda y.\ y$

$(\lambda x\ y.\ y)\ ((\lambda x.\ x\ x)\ a$ … y

η Conversion

Another case of trivially equal functions: $t = (\lambda x.\ t\ x)$

η Conversion

Another case of trivially equal functions: $t = (\lambda x.\ t\ x)$

Definition:

$$
\begin{array}{llllrll}
 & & & & (\lambda x.\ t\ x) & \longrightarrow_\eta & t \qquad \text{if } x \notin FV(t) \\
s & \longrightarrow_\eta & s' & \Longrightarrow & (s\ t) & \longrightarrow_\eta & (s'\ t) \\
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\end{array}
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$$s =_\eta t \quad \text{iff} \quad \exists n.\ s \longrightarrow_\eta^* n \wedge t \longrightarrow_\eta^* n$$

Example: $(\lambda x.\ f\ x)\ (\lambda y.\ g\ y) \longrightarrow_\eta$

η Conversion

Another case of trivially equal functions: $t = (\lambda x.\ t\ x)$

Definition:

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- ➜ η reduction is confluent and terminating.
- ➜ $\longrightarrow_{\beta\eta}$ is confluent.
  $\longrightarrow_{\beta\eta}$ means $\longrightarrow_\beta$ and $\longrightarrow_\eta$ steps are both allowed.
- ➜ **Equality in Isabelle is also modulo η conversion.**

In fact ...

Equality in Isa… odulo α, β, and η conversion.

We will see later why that is possible.

Isabelle Demo

So, what can you do with λ calculus?

λ calculus is very expressive, you can encode:

- logic, set theory
- turing machines, fu[illegible]grams, etc.

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\texttt{or} & \equiv \lambda x\ y.\ \texttt{if}\ x\ \texttt{true}\ y
\end{array}$$

More Examples

Encoding natural numbers (Church Numerals)

$$
\begin{aligned}
0 &\equiv \lambda f\ x.\ x \\
1 &\equiv \lambda f\ x.\ f\ x \\
2 &\equiv \lambda f\ x.\ f\ (f\ x) \\
3 &\equiv \lambda f\ x.\ f\ (f\ (f\ x)) \\
\ldots
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Numeral n takes arguments f and x, applies f n-times to x.

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\texttt{add} &\equiv \lambda m\ n.\ \lambda f\ x.\ m\ f\ (n\ f\ x)
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Fix Points

$$(\lambda x\ f.\ f\ (x\ x\ f))\ \ (\lambda x\ f.\ f\ (x\ x\ f))\ \ t \longrightarrow_\beta$$

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$(\lambda xf.\ f\ (x\ x\ f))\ (\lambda xf.\ f\ (x\ x\ f))$ is Turing's fix point operator

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  allows arbitrary quantification over predicates
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- ➜ **Whitehead & Russell** (Principia Mathematica, 1910-1913):
  Fix the problem
- ➜ **Church** (1930): λ calculus as logic, true, false, $\wedge$, ... as λ terms

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with	$\{x \mid P\ x\} \equiv \lambda x.\ P\ x$	$x \in M \equiv M\ x$
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you can write	$R \equiv \lambda x.\ \texttt{not}\ (x\ x)$	
and get	$(R\ R) =_\beta \texttt{not}\ (R\ R)$	
because	$(R\ R) = (\lambda x.\ \texttt{not}\ (x\ x))\ R \longrightarrow_\beta \texttt{not}\ (R\ R)$	

We have learned so far...

- ➜ λ calculus syntax
- ➜ free variables, subst
- ➜ β reduction
- ➜ α and η conversion
- ➜ β reduction is confluent
- ➜ λ calculus is very expressive (turing complete)
- ➜ λ calculus results in an inconsistent logic