COMP4161
Advanced Topics in Software Verification

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- Higher Order [illegible] part 1) [2,3[a]]
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- Proof automation, Isar (part 2) [8]
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- C verification [9,10]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Last Time on HOL

- Defining HOL

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- Higher Order Abstr

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Term Rewriting

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$$l_2 = r_2$$
$$\vdots$$
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Applications in:

- **Mathematics** (algebra, group theory, etc)
- **Functional Programming** (model of execution)
- **Theorem Proving** (dealing with equations, simplifying statements)

Term Rewriting: The Idea

use $l_1 \longrightarrow r_1$, $l_2 \longrightarrow r_2$, ..., $l_n \longrightarrow r_n$ as reduction rules

decide $l = r$ by deciding $l \overset{*}{\longleftrightarrow} r$

Arrow Cheat Sheet

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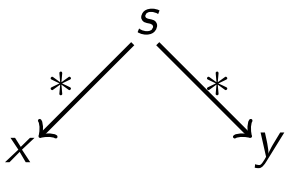
Fact: $\longrightarrow$ is Church-Rosser iff it is confluent.

Confluence



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n set of reduction rules confluent?

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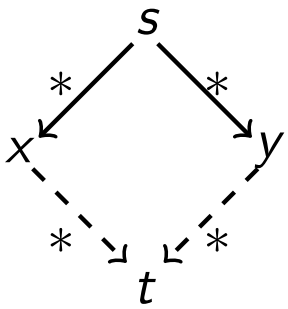

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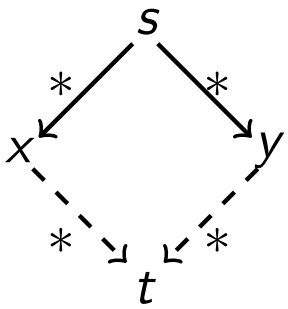

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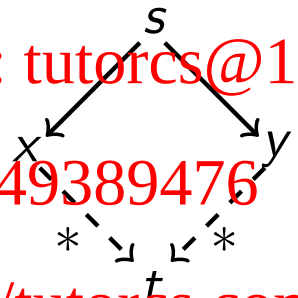

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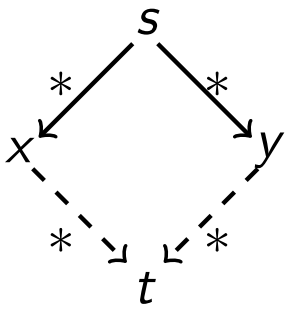

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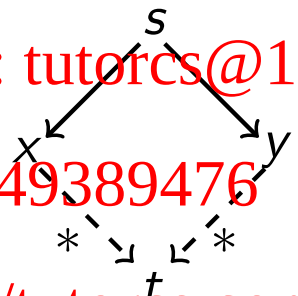

Fact: local confluence and termination $\Longrightarrow$ confluence

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② $<_r$ is well founded, because $<$ is well founded on $\mathbb{N}$

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True for most orders that don't treat certain parts of terms as special cases.

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We show that the rewrite system defined by these rules is terminating.

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\text{osize}'\ (P \longrightarrow Q) & x = & 2^x + (\text{osize}'\ P\ (x+1)) + (\text{osize}'\ Q\ (x+1)) \\
\text{osize}\ \ P & = & \text{osize}'\ P\ 0
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\end{array}
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The other rules decrease the depth of the things osize counts, so decrease osize.

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apply simp

➜ uses simplification rules

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Term rewriting in Isabelle is called **Simplifier**

Simply simp

- ➜ uses simplification rules
- ➜ (almost) blindly from left to right

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Apply simp

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- ➜ until no rule is applicable.

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apply simp

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termination: not guaranteed (may loop)

Term Rewriting in Isabelle

Term rewriting in Isabelle is called **Simplifier**

Apply simp

- ➜ uses simplification rules
- ➜ (almost) blindly from left to right
- ➜ until no rule is applicable.

termination: not guaranteed
(may loop)

confluence: not guaranteed
(result may depend on which rule is used first)

Control

➜ Equations turned in[illegible]ation rules with **[simp]** attribute

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Control

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  apply (simp add: <[illegible]and **apply** (simp del: <rules>)
- ➜ Using only the specified set of equations:

  apply (simp only: <rules>)

Demo

We have seen today...

- Equations and Tern

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We have seen today...

- ➜ Equations and Tern
- ➜ Confluence and Ter

reduction systems

We have seen today...

- ➜ Equations and Tern
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reduction systems

Exercises

➜ Show, via a pen-and-paper proof, that the osize function is monotonic with respect to the terms from that example.