COMP4161
Advanced Topics in Software Verification

$$\{P\} \ldots \{Q\}$$

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Content

➜ Foundations & Principles

- Intro, Lambda [illegible] atural deduction [1,2]
- Higher Order [illegible] part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7[b]]
- Proof automation, Isar (part 2) [8]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

A Crash Course in Semantics

(For more,
see Concrete Semantics)

IMP - a small Imperative Language

Commands:
datatype com

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ign vname aexp (_ := _)
ni com com (_; _)
d bexp com com (IF _ THEN _ ELSE _)
While bexp com (WHILE _ DO _ OD)

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type_synonym aexp = state ⇒ nat
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Example Program

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E $A \neq 0$ DO
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Expressions are functions from state to bool or nat:

$B := (\lambda\sigma.\ 1)$;
WHILE $(\lambda\sigma.\ \sigma\ A \neq 0)$ DO
$B := (\lambda\sigma.\ \sigma\ B * \sigma\ A)$;
$A := (\lambda\sigma.\ \sigma\ A - 1)$
OD

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How to define execution of a program?

- ➜ A wide field of its own
- ➜ Some choices:
  - Operational (inductive relations, big step, small step)
  - Denotational (programs as functions on states, state transformers)
  - Axiomatic (pre-/post conditions, Hoare logic)

Structural Operational Semantics

$$\frac{}{\langle P, \sigma \rangle \to \sigma}$$

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$$\frac{b\,\sigma = \text{True} \quad \langle c_1, \sigma \rangle \to \sigma'}{\langle \text{IF } b \text{ THEN } c_1 \text{ ELSE } c_2, \sigma \rangle \to \sigma'}$$

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Demo: The Definitions in Isabelle

Proofs about Programs

Now we know:

- ➜ What programs are
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Example:

Show that example program from slide 6 implements the factorial.

lemma $\langle \text{factorial}, \sigma \rangle \to \sigma' \Longrightarrow \sigma' B = \text{fac}\ (\sigma A)$

(where $\text{fac}\ 0 = 1$, $\text{fac}\ (\text{Suc}\ n) = (\text{Suc}\ n) * \text{fac}\ n$)

Demo: Example Proof

Too tedious

Inc eded for each loop

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Is there something easier?

Floyd/Hoare

Idea: describe meaning of program by pre/post conditions

Examples:

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$\{x = n\} \quad \text{IF } y < 0 \text{ THEN } x := x + y \text{ ELSE } x := x - y \quad \{x = n - |y|\}$

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Proofs: have rules that directly work on such triples

Meaning of a Hoare-Triple

$\{P\}$ c $\{Q\}$

What are the assertions Q?

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- ➜ Other choice: syntax and semantics for assertions (deep embedding)

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$$\models \{P\}\ c\ \{Q\} \quad \equiv \quad \forall \sigma\ \sigma'.\ P\ \sigma \wedge \langle c, \sigma \rangle \to \sigma' \longrightarrow Q\ \sigma'$$

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This lecture: partial correctness only (easier)

Hoare Rules

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Are the Rules Correct?

Soundness: $\vdash \{P\}\ c\ \{Q\}$ $\Rightarrow$ $\{P\}\ c\ \{Q\}$

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Proof: by rule induction on $\vdash \{P\}\ c\ \{Q\}$

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Demo: Hoare Logic in Isabelle

