Proofcraft

UNSW SYDNEY

MP4161

Advanced Topics in Software Verification

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T3/2022

Last time...

- ➜ λ calculus syntax
- ➜ free variables, substitution
- ➜ β reduction
- ➜ α and η conversion
- ➜ β reduction is confluent
- ➜ λ calculus is expressive (Turing complete)
- ➜ λ calculus is inconsistent (as a logic)

Content

- Foundations & Principles
  - Intro, Lambda [illegible]atural deduction [1,2]
  - Higher Order [illegible]part 1) [2,3[a]]
  - Term rewriting[illegible] [3,4]

- Proof & Specification Techniques
  - Inductively defined sets, rule induction [4,5]
  - Datatype induction, primitive recursion [5,7]
  - General recursive functions, termination proofs [7[b]]
  - Proof automation, Isar (part 2) [8]
  - Hoare logic, proofs about programs, invariants [8,9]
  - C verification [9,10]
  - Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

λ calculus is inconsistent

Can find term R such $=_\beta$ `not`(R R)

There are more terms t make sense:

`1 2`, `true false`, etc.

λ calculus is inconsistent

Can find term R such that $R\ R =_\beta$ `not`$(R\ R)$

There are more terms that do not make sense:

`1 2`, `true false`, etc.

Solution: rule out ill-formed terms by using types.

(Church 1940)

Introducing types

Idea: assign a type to each "sensible" λ term.

Examples:

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  Write: $(\lambda x.\ x) :: \alpha \Rightarrow \alpha$

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Idea: assign a type to each "sensible" λ term.

Examples:

- for *term t has type* α *write* $t :: \alpha$
- if x has type α then $\lambda x.\ x$ is a function from α to α
  Write: $(\lambda x.\ x) :: \alpha \Rightarrow \alpha$
- for $s\ t$ to be sensible:
  s must be a function
  t must be right type for parameter

  If $s :: \alpha \Rightarrow \beta$ and $t :: \alpha$ then $(s\ t) :: \beta$

That's about it

Now formally again

Syntax for $\lambda^{\rightarrow}$

Terms: $t ::= v \mid c \mid (t\ t) \mid (\lambda x.\ t)$

$v \in V,\ c \in C,\quad V, C$ sets of names

Types: $\tau ::= b \mid \nu \mid \tau \Rightarrow \tau$

$b \in \{\mathtt{bool}, \mathtt{int}, \ldots\}$ base types

$\nu \in \{\alpha, \beta, \ldots\}$ type variables

$$\alpha \Rightarrow \beta \Rightarrow \gamma \quad = \quad \alpha \Rightarrow (\beta \Rightarrow \gamma)$$

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Context Γ:

Γ: function from variable and constant names to types.

Term t has type τ in context Γ: $\quad \Gamma \vdash t :: \tau$

Examples

$\Gamma \vdash (\lambda x.\ x) ::$

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A term t is **well typed** or **type correct**
if there are Γ and τ such that $\Gamma \vdash t :: \tau$

Type Checking Rules

Variables:
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Example Type Derivation:

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Remember:

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Example: $[] \vdash \lambda$[illegible]l $\Rightarrow$ `bool`
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A term can have more than one type.

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$\tau \lesssim \sigma$ if there is a substitution S such that $\tau = S(\sigma)$

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$\texttt{int} \Rightarrow \texttt{bool} \quad \lesssim \quad \alpha \Rightarrow \beta$

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Type checking and type inference on $\lambda^{\rightarrow}$ are decidable.

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Definition of β reduction stays the same.

Fact: Well typed terms stay well typed during β reduction

Formally: $\Gamma \vdash s :: \tau \wedge s \longrightarrow_\beta t \Longrightarrow \Gamma \vdash t :: \tau$

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This property is called **subject reduction**

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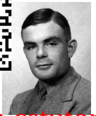
β redu[illegible] $\lambda^{\to}$ always terminates.



(Alan Turing, 1942)

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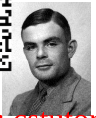

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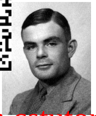

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- ➜ **$=_\beta$ is decidable**
  To decide if $s =_\beta t$, reduce s and t to normal form (always exists, because $\longrightarrow_\beta$ terminates), and compare result.
- ➜ **$=_{\alpha\beta\eta}$ is decidable**
  This is why Isabelle can automatically reduce each term to $\beta\eta$ normal form.

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Checkpoint:

- ➜ untyped lambda calculus is turing complete (all computable functions can be expressed)
- ➜ but it is inconsistent
- ➜ $\lambda^{\rightarrow}$ "fixes" the inconsistency problem by adding types
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But wait... typed functional languages are turing complete!

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So...

- ➜ typed functional lan[illegible] turing complete
- ➜ but $\lambda^{\rightarrow}$ is not...
- ➜ How does this work

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So...

➜ typed functional languages are turing complete

➜ but $\lambda^{\rightarrow}$ is not...

➜ How does this work?

➜ By adding one single constant, the Y operator (fix point operator), to $\lambda^{\rightarrow}$

➜ This introduces the non-termination that the types removed, but in a safe way that prevents inconsistency

$$Y :: (\tau \Rightarrow \tau) \Rightarrow \tau$$

$$Y\ t \longrightarrow_\beta t\ (Y\ t)$$

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$$Y :: (\tau \Rightarrow \tau) \Rightarrow \tau$$

$$Y\ t \longrightarrow_\beta t\ (Y\ t)$$

Fact: If we add Y to $\lambda^{\rightarrow}$ as the only constant, then each computable function can be encoded as closed, type correct $\lambda^{\rightarrow}$ term.

- ➜ Y is used for recursion
- ➜ lose decidability (what does $Y\ (\lambda x.\ x)$ reduce to?)
- ➜ (Isabelle/HOL doesn't have Y; recursion is more restricted)

Types and Terms in Isabelle

Types: $\tau \quad ::= \quad \mathtt{b} \mid {'\nu} \mid {'\nu} :: C \mid \tau \Rightarrow \tau \mid (\tau, \ldots, \tau)\ K$

$\mathtt{b} \in \{\mathtt{bool}, \mathtt{int}, \ldots\}$ base types

$\nu \in \{\alpha, \beta, \ldots\}$ type variables

$K \in \{\mathtt{set}, \mathtt{list}, \ldots\}$ type constructors

$C \in \{\mathtt{order}, \mathtt{linord}, \ldots\}$ type classes

Terms: $t \quad ::= \quad v \mid c \mid ?v \mid (t\ t) \mid (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

Types and Terms in Isabelle

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$\mathtt{b} \in \{\mathtt{bool},$ [illegible] base types

$\nu \in \{\alpha, \beta,$ [illegible] variables

$K \in \{\mathtt{set},$ [illegible] type constructors

$C \in \{\mathtt{order}, \mathtt{linord}, \dots\}$ type classes

Terms: $t \ ::= \ v \ \mid \ c \ \mid \ ?v \ \mid \ (t\ t) \ \mid \ (\lambda x.\ t)$

$v, x \in V, \quad c \in C, \quad V, C$ sets of names

➜ **type constructors**: construct a new type out of a parameter type.
Example: `int list`

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- **type constructors**: construct a new type out of a parameter type. Example: `int list`
- **type classes**: restrict type variables to a class defined by axioms. Example: $\alpha :: \mathit{order}$
- **schematic variables**: variables that can be instantiated.

Type Classes

➜ similar to Haskell's , but with semantic properties

class order =

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  class linorder = order +
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- can be instantiated

  instance nat :: "{order, linorder}" **by** ...

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- ➜ convention: lemma must be true for all x
- ➜ **during the proof**, x must **not** be instantiated

Solution:
Isabelle has **free** (x), **bound** (x), and **schematic** (?X) variables.

Only schematic variables can be instantiated.

Free converted into schematic after proof is finished.

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Examples:

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?X \wedge ?Y & =_{\alpha\beta\eta} & x \wedge x \\
?P\ x & =_{\alpha\beta\eta} & x \wedge x \\
P\ (?f\ x) & =_{\alpha\beta\eta} & ?Y\ x
\end{array}
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Find substitution σ on variables for terms s, t such that $\sigma(s) = \sigma(t)$

In Isabelle:
Find substitution σ on schematic variables such that $\sigma(s) =_{\alpha\beta\eta} \sigma(t)$

Examples:

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?X \wedge ?Y & =_{\alpha\beta\eta} \quad x \wedge x & [?X \leftarrow x, ?Y \leftarrow x] \\
?P\ x & =_{\alpha\beta\eta} \quad x \wedge x & [?P \leftarrow \lambda x.\ x \wedge x] \\
P\ (?f\ x) & =_{\alpha\beta\eta} \quad ?Y\ x & [?f \leftarrow \lambda x.\ x, ?Y \leftarrow P]
\end{array}
$$

Higher Order: schematic variables can be functions.

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- Unification modulo $\alpha\beta$ (Higher Order Unification) is semi-decidable
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- Higher Order Unification has possibly infinitely many solutions

But:

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- Important fragments (like Higher Order Patterns) are decidable

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But:

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➜ Important fragments (like Higher Order Patterns) are decidable

Higher Order Pattern:

➜ is a term in β normal form where

➜ each occurrence of a schematic variable is of the form $?f\ t_1\ \dots\ t_n$

➜ and the $t_1\ \dots\ t_n$ are η-convertible into n distinct bound variables

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