UNSW SYDNEY

COMP4161
Advanced Topics in Software Verification

Gerwin Klein, June Andronick, Miki Tanaka, Johannes Åman Pohjola

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Content

➜ Foundations & Principles

- Intro, Lambda [illegible] atural deduction [1,2]
- Higher Order [illegible] part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7[b]]
- Proof automation, Isar (part 2) [8]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Last Time

- Sets
- Type Definitions
- Inductive Definition

Inductive Definitions

How They Work

The Nat Example

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- ➜ Objective: **no junk**. Only what must be in X shall be in X.
- ➜ Gives rise to a nice proof principle (rule induction)

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The Set

Definition $\hat{R}$ B-closed iff $\hat{R}\ B \subseteq B$

Definition $\hat{R}^*$ is the least R-closed subset of A

This does always exist:

Fact: $X = \bigcap \{B \subseteq A.\ B \text{ is } R\text{-closed}\}$

Generation from Above

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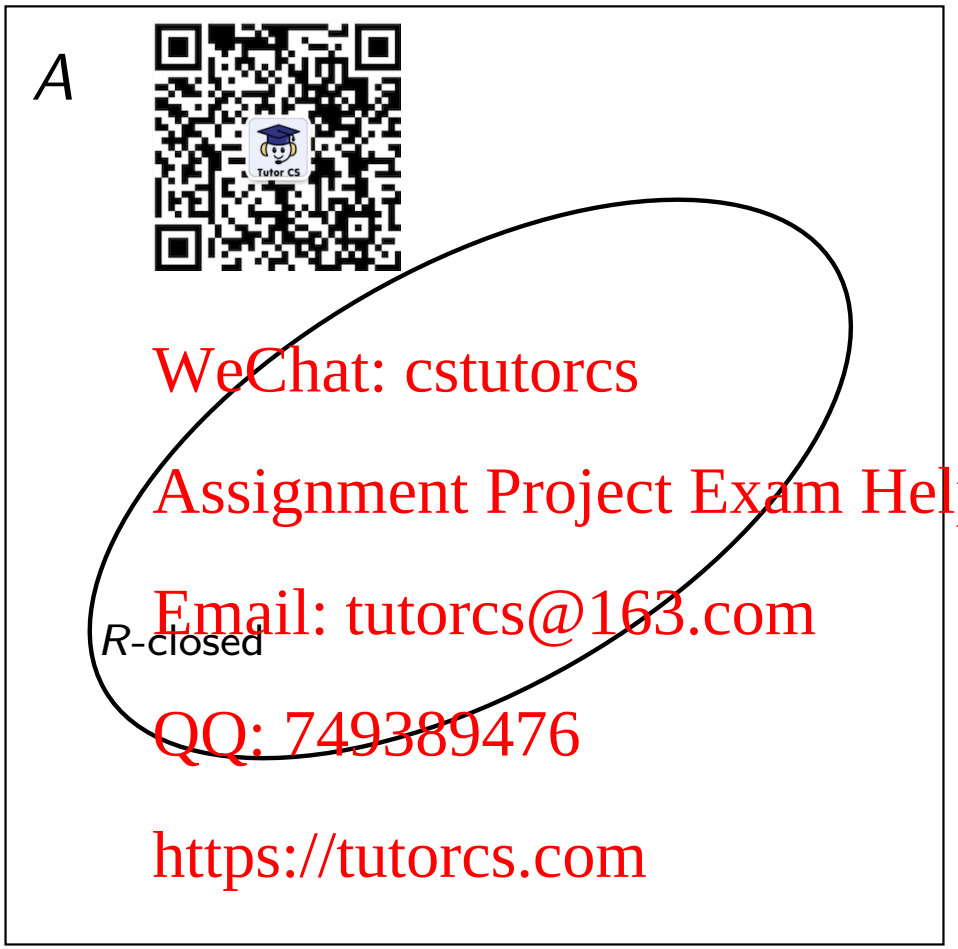

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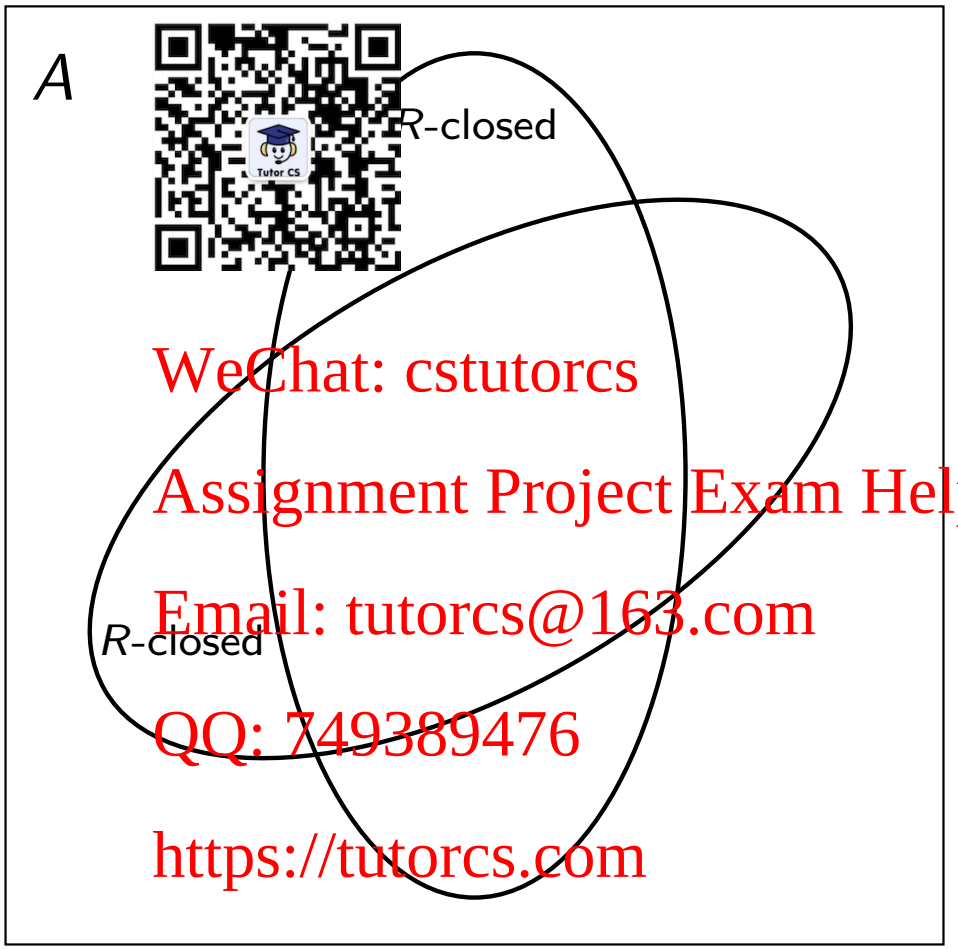

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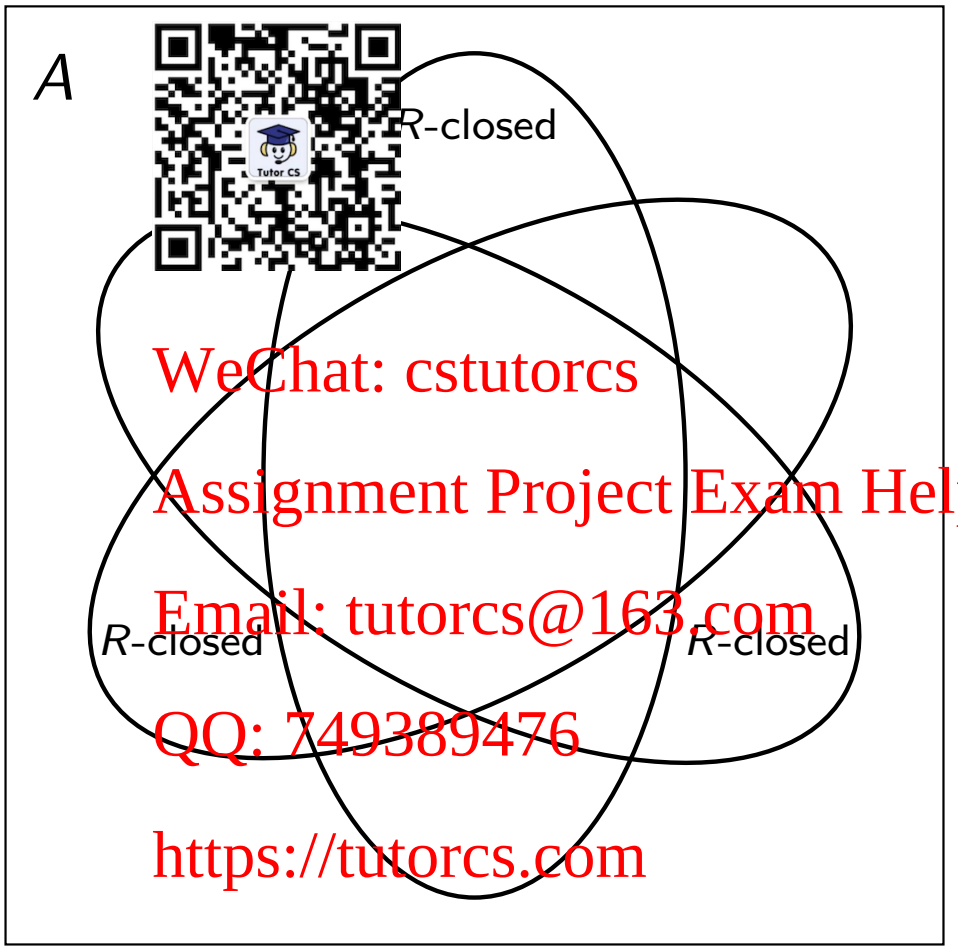

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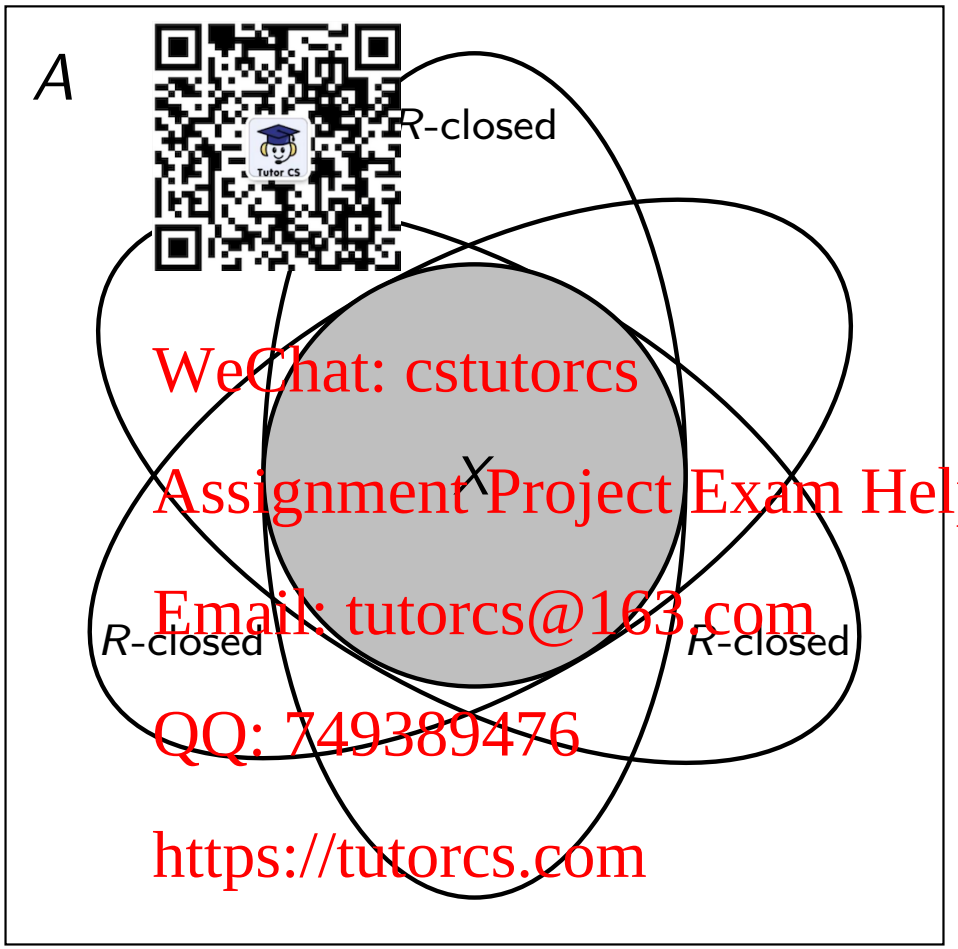

Rule Induction

$$\frac{n \in N}{n + 1 \in N}$$

duction principle

$$[\![P\ 0;\ \bigwedge n.\ P\ n \Longrightarrow P\ (n + 1)]\!] \Longrightarrow \forall x \in N.\ P\ x$$

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In general:

$$\frac{\forall (\{a_1, \ldots a_n\}, a) \in R.\ P\ a_1 \wedge \ldots \wedge P\ a_n \Longrightarrow P\ a}{\forall x \in X.\ P\ x}$$

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Rules with side conditions

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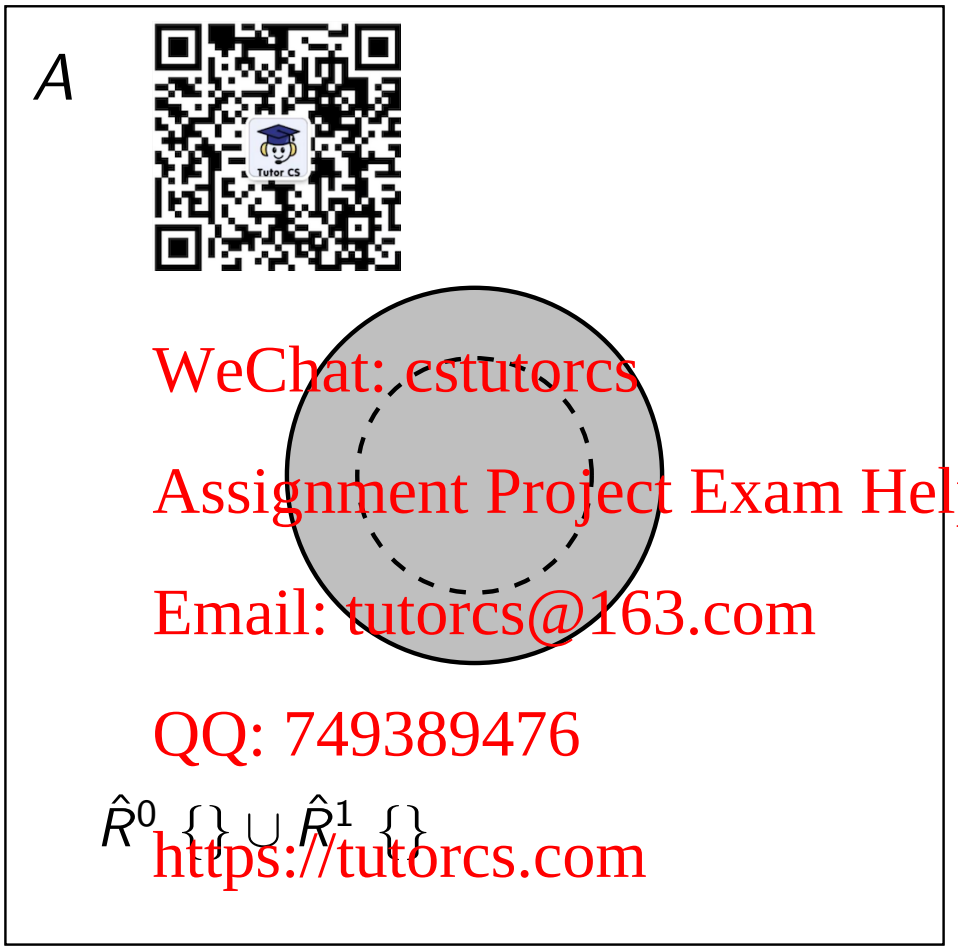

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Does this always work?

Knaster-Tarski Fixpoint Theorem:

Let $(A, \leq)$ be a comple[redacted] and $f :: A \Rightarrow A$ a monotone function. Then the fixpoints of f[redacted] a complete lattice.

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Lattice:
Finite subsets have a greatest lower bound (meet) and least upper bound (join).

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Implications:

- ➜ least and greatest fixpoints exist (complete lattice always non-empty).
- ➜ can be reached by (possibly infinite) iteration. (Why?)

Exercise

Formalize this lecture in Isabelle:

- Define **closed** f A : (α set) $\Rightarrow$ α set $\Rightarrow$ bool
- Show closed f A $\wedge$ $\Longrightarrow$ closed f $(A \cap B)$ if f is monotone (**mono** is predefined)
- Define **lfpt** f as the intersection of all f-closed sets
- Show that lfpt f is a fixpoint of f if f is monotone
- Show that lfpt f is the least fixpoint of f
- Declare a constant R :: (α set $\times$ α) set
- Define $\hat{R}$ :: α set $\Rightarrow$ α set in terms of R
- Show soundness of rule induction using R and lfpt $\hat{R}$

We have learned today ...

- ➜ Formal background [illegible] definitions
- ➜ Definition by inters[illegible]
- ➜ Computation by ite[illegible]
- ➜ Formalisation in Isabelle