COMP4161
Advanced Topics in Software Verification

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Last time...

- natural deduction ... $\wedge$, $\vee$, $\longrightarrow$, $\neg$, iff...
- proof by assumpti... o rule, elim rule
- safe and unsafe r...

- indent your proofs! (one space per subgoal)
- prefer implicit backtracking (chaining) or *rule_tac*, instead of *back*
- *prefer* and *defer*
- *oops* and *sorry*

Content

➜ Foundations & Principles

- Intro, Lambda, natural deduction [1,2]
- Higher Order Logic (part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7[b]]
- Proof automation, Isar (part 2) [8]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Quantifiers

Scope

- Scope of parame[illegible]le subgoal
- Scope of $\forall, \exists, \ldots$ [illegible]ith ; or $\Longrightarrow$

Example:

$$\bigwedge x\ y.\ [\![\ \forall y.\ P\ y \longrightarrow Q\ z\ y;\ \ Q\ x\ y\ ]\!] \Longrightarrow \exists x.\ Q\ x\ y$$

means

$$\bigwedge x\ y.\ [\![\ (\forall y_1.\ P\ y_1 \longrightarrow Q\ z\ y_1);\ \ Q\ x\ y\ ]\!] \Longrightarrow (\exists x_1.\ Q\ x_1\ y)$$

Natural deduction for quantifiers

$$\frac{\bigwedge x.\ P\ x}{\forall x.\ P\ x}\ \text{allI} \qquad \frac{\forall x.\ P\ x \quad P\ ?x \Longrightarrow R}{R}\ \text{allE}$$

$$\frac{P\ ?x}{\exists x.\ P\ x}\ \text{exI} \qquad \frac{\exists x.\ P\ x \quad \bigwedge x.\ P\ x \Longrightarrow R}{R}\ \text{exE}$$

- **allI** and **exE** introduce new parameters ($\bigwedge x$).
- **allE** and **exI** introduce new unknowns ($?x$).

Instantiating Rules

apply (x = "*term*" in *rule*)

Like **rule**, but ?*x* in *rule* is instantiated by *term* before application.

Similar: **erule_tac**

! *x* **is in** *rule*, **not in goal** !

Two Successful Proofs

. $\exists y.\ x = y$

y (rule alll)

. $\bigwedge x.\ \exists y.\ x = y$

best practice

apply (rule_tac x = "x" in exI)

1. $\bigwedge x.\ x = x$

apply (rule refl)

simpler & clearer

exploration

apply (rule exI)

1. $\bigwedge x.\ x = ?y\ x$

apply (rule refl)

$?y \mapsto \lambda u.u$

shorter & trickier

Two Unsuccessful Proofs

1. $\exists y.\ \forall x.\ x = y$

apply (rule_tac x = ?y in exI)

apply (rule exI)

1. $\forall x.\ x = ?y$

apply (rule allI)

1. $\bigwedge x.\ x = ?y$

apply (rule refl)

$?y \mapsto x$ yields $\bigwedge x'.\ x' = x$

Principle:

$?f\ x_1 \ldots x_n$ can only be replaced by term t

if $params(t) \subseteq x_1, \ldots, x_n$

Safe and Unsafe Rules

Safe allI, exI

Unsafe allE, exE

Create parameters first, unknowns later

Demo: Quantifier Proofs

Parameter names

Paramet  **are chosen by Isabelle**

1. $\forall x.\ \exists y.\ x = y$

apply (rule alll)

1. $\bigwedge x.\ \exists y.\ x = y$

apply (rule_tac x = "x" in exI)

Brittle!

Renaming parameters

1. $\forall x.\ \exists y.\ x = y$

apply (rename_tac N)

1. $\forall N.\ \exists y.\ N = y$

apply (rule_tac x="N" in exI)

In general:
(rename_tac $x_1 \ldots x_n$**) renames the rightmost (inner)** n **parameters to** $x_1 \ldots x_n$

Forward Proof: frule and drule

Rule: $[\![A_1; \ldots; A_m]\!] \Longrightarrow A$

Subgoal: $[\![B_1; \ldots; B_n]\!] \Longrightarrow C$

Substitution: $\sigma(B_i) = \sigma(A_1)$

New subgoals: $\sigma([\![B_1; \ldots; B_n]\!] \Longrightarrow A_2)$

$\sigma([\![B_1; \ldots; B_n]\!] \Longrightarrow A_m)$

$\sigma([\![B_1; \ldots; B_n; A]\!] \Longrightarrow C)$

Like **frule** but also deletes B_i: **apply** (drule $< rule >$)

Examples for Forward Rules

$$\frac{P \wedge Q}{P}$$ conjunct1

$$\frac{P \wedge Q}{Q}$$ conjunct2

$$\frac{P \rightarrow Q \quad P}{Q}$$ mp

$$\frac{\forall x.\ P\ x}{P\ ?x}$$ spec

Forward Proof: OF

r [OF $r_1 \ldots r_n$]

Prove assumption $\ldots$ *theorem* r *with theorem* r_1, *and assumption 2 with theorem* r_2, *and* $\ldots$

Rule r	$[\![A_1; \ldots; A_m]\!] \Longrightarrow A$
Rule r_1	$[\![B_1; \ldots; B_n]\!] \Longrightarrow B$
Substitution	$\sigma(B) \equiv \sigma(A_1)$
r [OF r_1]	$\sigma([\![B_1; \ldots; B_n; A_2; \ldots; A_m]\!] \Longrightarrow A)$

Example:

dvd_add : $[\![\ ?a\ dvd\ ?b; ?a\ dvd\ ?c\ ]\!] \Longrightarrow ?a\ dvd\ ?b + ?c$

dvd_refl : $?a\ dvd\ ?a$

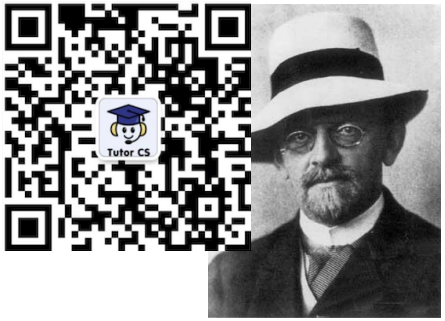
dvd_add[*OF dvd_refl*] : $[\![\ ?a\ dvd\ ?c\ ]\!] \Longrightarrow ?a\ dvd\ ?a + ?c$

Forward proofs: THEN

r_1 [T ] means r_2 [OF r_1]

Demo: Forward Proofs

Hilbert's Epsilon Operator



(David Hilbert, 1862-1943)

$\varepsilon\ x.\ Px$ is a value that satisfies P (if such a value exists)

ε also known as **description operator**.

In Isabelle the ε-operator is written SOME $x.\ P\ x$

$$\frac{P\ ?x}{P\ (\text{SOME}\ x.\ P\ x)}\ \text{somel}$$

More Epsilon

ε Axiom of Choice:

$$\forall x.\ \exists y.\ Q\ x\ y \Longrightarrow \exists f.\ \forall x.\ Q\ x\ (f\ x)$$

Existential and universal quantification can be defined with ε.

Isabelle also knows the definite description operator **THE** (aka ι):

$$\frac{}{(\text{THE } x.\ x = a) = a}\ \text{the\_eq\_trivial}$$

Some Automation

More Proof Methods:

apply (intro <intro-r repeatedly applies intro rules

apply (elim <elim-ru repeatedly applies elim rules

apply clarify applies all safe rules that do not split the goal

apply safe applies all safe rules

apply blast an automatic tableaux prover (works well on predicate logic)

apply fast another automatic search tactic

Epsilon and Automation Demo

We have learned so far...

- Proof rules for pr culus
- Safe and unsafe r
- Forward Proof
- The Epsilon Operator
- Some automation

Isar (Part 1)

A Language for Structured Proofs

Motivation

Is this  $\longrightarrow B) = (B \vee \neg A)$?

Motivation

Is this true: $(A \longrightarrow B) = (B \vee \neg A)$?

YES!

```
apply (rule iffI)
apply (cases A)
 apply (rule disjI1)
 apply (erule impE)
  apply assumption
 apply assumption
apply (rule disjI2)
 apply assumption
apply (rule impI)
apply (erule disjE)
 apply assumption
apply (erule notE)
apply assumption
done
```

or `by blast`

OK it's true. But WHY?

Motivation

WHY is t 

$(A \longrightarrow B) = (B \vee \neg A)$?

Demo

Isar

apply s

What about..

- ➜ unreadable
- ➜ hard to maintain
- ➜ do not scale

- ➜ Elegance?
- ➜ Explaining deeper insights?
- ➜ Large developments?

No structure. **Isar!**

A typical Isar proof

p 

formula$_0$

mula$_1$ **by** simp

⋮

have *formula*$_n$ **by** blast

show *formula*$_{n+1}$ **by** ...

qed

proves *formula*$_0 \Longrightarrow$ *formula*$_{n+1}$

(analogous to **assumes**/**shows** in lemma statements)

Isar core syntax

proof = **proof** [method] statement* **qed**
| **by** method

method = (simp ...) | (rule ...) | ...

statement = **fix** variables (⋀)
| **assume** proposition (⟹)
| [**from** name+] (**have** | **show**) proposition proof
| **next** (separates subgoals)

proposition = [name:] formula

proof and qed

proof [method] statement* **qed**

lemma "⟦A; B⟧ ⟹
proof (rule conjI)
  assume A: "A"
  from A **show** "A" **by** assumption
next
  assume B: "B"
  from B **show** "B" **by** assumption
qed

- ➜ **proof** (<method>) applies method to the stated goal
- ➜ **proof** applies a single rule that fits
- ➜ **proof** - does nothing to the goal

How do I know what to Assume and Show?

lemma "$[\![A; B]\!] \Longrightarrow$ **he proof state!**
proof (rule conjI)

- ➜ **proof** (rule conjI) changes proof state to
  1. $[\![A; B]\!] \Longrightarrow A$
  2. $[\![A; B]\!] \Longrightarrow B$
- ➜ so we need 2 shows: **show** "A" and **show** "B"
- ➜ We are allowed to **assume** A, because A is in the assumptions of the proof state.

The Three Modes of Isar

- ➜ **[prove]**:
  goal has been sta[illegible]needs to follow.
- ➜ **[state]**:
  proof block has o[illegible]ubgoal has been proved,
  new *from* statement, goal statement or assumptions can follow.
- ➜ **[chain]**:
  from statement has been made, goal statement needs to follow.

lemma "$[\![A; B]\!] \Longrightarrow A \wedge B$" **[prove]**
proof (rule conjI) **[state]**
  assume A: "A" **[state]**
  from A **[chain]** **show** "A" **[prove]** **by** assumption **[state]**
next **[state]** . . .

Have

Can be used to make intermediate steps.

Example:

lemma "$(x :: \text{nat}) + 1 = 1 + x$"
proof -
  have A: "$x + 1 = \text{Suc}\ x$" **by** simp
  have B: "$1 + x = \text{Suc}\ x$" **by** simp
  show "$x + 1 = 1 + x$" **by** (simp only: A B)
qed

Demo

Backward and Forward

Backward reasoning: . . . **have** "$A \wedge B$" **proof**

- ➜ **proof** picks an **in**[illegible]tomatically
- ➜ conclusion of rule [illegible] with $A \wedge B$

Forward reasoning: . . .

assume AB: "$A \wedge B$"
from AB **have** ". . ." **proof**

- ➜ now **proof** picks an **elim** rule automatically
- ➜ triggered by **from**
- ➜ first assumption of rule must unify with AB

General case: **from** $A_1 \ldots A_n$ **have** R **proof**

- ➜ first n assumptions of rule must unify with $A_1 \ldots A_n$
- ➜ conclusion of rule must unify with R

Fix and Obtain

$v_1 \ldots v_n$

Introduce arbitrary but fixed variables

($\sim$ parameters, $\bigwedge$)

obtain $v_1 \ldots v_n$ **where** <prop> <proof>

Introduces new variables together with property

Fancy Abbreviations

this		previous fact proved or assumed
then		this
thus	=	**then show**
hence	=	**then have**
with $A_1 \ldots A_n$	=	**from** $A_1 \ldots A_n$ this
?thesis	=	the last enclosing goal statement

Demo

Moreover and Ultimately

have X_1: P_1 ...	**have** P_1 ...
have X_2: P_2 ...	**moreover have** P_2 ...
⋮	⋮
have X_n: P_n ...	**moreover have** P_n ...
from $X_1 \ldots X_n$ **show** ...	**ultimately show** ...

wastes lots of brain power
on names $X_1 \ldots X_n$

General Case Distinctions

show *formula*
proof -
  have $P_1 \vee P_2 \vee P_3$ <proof>
  moreover { **assume** P_1 ... **have** ?thesis <proof> }
  moreover { **assume** P_2 ... **have** ?thesis <proof> }
  moreover { **assume** P_3 ... **have** ?thesis <proof> }
  ultimately show ?thesis **by** blast
qed

{ ... } is a proof block similar to **proof** ... **qed**

{ **assume** P_1 ... **have** P <proof> }
stands for $P_1 \Longrightarrow P$

Mixing proof styles

from . . .
have . . .
  apply -  incoming facts assumptions
  apply (. . .)
  ⋮
  apply (. . .)
  done

We have learned so far...

- Isar style proofs
- proof, qed
- assumes, shows
- fix, obtain
- moreover, ultimately
- forward, backward
- mixing proof styles