COMP4161
Advanced Topics in Software Verification

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Content

➜ Foundations & Principles

- Intro, Lambda [illegible] atural deduction [1,2]
- Higher Order [illegible] part 1) [2,3[a]]
- Term rewriting [3,4]

➜ Proof & Specification Techniques

- Inductively defined sets, rule induction [4,5]
- Datatype induction, primitive recursion [5,7]
- General recursive functions, termination proofs [7[b]]
- Proof automation, Isar (part 2) [8]
- Hoare logic, proofs about programs, invariants [8,9]
- C verification [9,10]
- Practice, questions, exam prep [10[c]]

[a] a1 due; [b] a2 due; [c] a3 due

Deep Embeddings

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Disadvantages:

➜ Semantically equivalent programs are not obviously equal.

➜ e.g. "IF True THEN SKIP ELSE SKIP = SKIP" is not a true theorem.

➜ Many concepts already present in the logic must be reinvented.

Shallow Embeddings

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- ➜ A definition for each construct, giving its **semantics**.
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Today: a shallow embedding for (interesting parts of) C semantics

Records in Isabelle

Records are *n*-tuples with named components

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record B = A +
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AutoCorres: verified translation from deeply embedded C to monadic representation

- ➜ Specifically designed for humans to do proofs over.

State Monad: Motivation

Model the **semantics** of a (deterministic) computation as a function

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modify – applies its argument to modify the state; returns ():

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Monads, Laws

Formally: a monad **M** is a type constructor with two operations.

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State Monad: Example

A fragment of C:

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void f(int *p) {
    int x = *p;
    if (x < 10) {
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```
record state =
  hp :: int ptr ⇒ int

f p ≡ ptr ⇒ (state ⇒ (unit,state))"
do {
    x ← gets (λs. hp s p);
    if x < 10 then
        modify (hp_update (λh. (h(p := x + 1))))
    else
        return ()
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fail $\equiv \lambda s.$ (undefined, True)

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Computations can **fail**: $'s \Rightarrow (('a \times \ 's) \ \underline{\times \ \text{bool}})$

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guard – fails when given condition applied to the state is False:

guard P $\equiv$ get $\gg=$ (λs. assert (P s))

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select – nondeterministic selection from a set:

$$\text{select}\ A \equiv \lambda s.\ \ ((A \times \{s\}),\text{False})$$

Demo

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Monadic loop, defined **inductively**.

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whileLoop C B

- **condition** C: takes **loop parameter** and **state** as arguments, returns **bool**
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Example: whileLoop $(\lambda p\ s.\ \text{hp}\ s\ p = 0)$ $(\lambda \text{p}.\ \text{return}\ (\text{ptrAdd}\ p\ 1))$ p

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$$\frac{\neg\, C\ r\ s}{(\text{Some}\ (r,s),\ \text{Some}\ (r,s)) \in \text{while\_results}\ C\ B}\ \text{(terminate)}$$

$$\frac{C\ r\ s\quad \text{snd}\ (B\ r\ s)}{(\text{Some}\ (r,s),\ \text{None}) \in \text{while\_results}\ C\ B}\ \text{(fail)}$$

Defining While Loops Inductively

Two-part definition: results and termination

Results: while_results :: $('s \Rightarrow \text{bool}) \Rightarrow$
$('s \Rightarrow ('a \times 's) \text{ set} \times \text{bool})) \Rightarrow$
$'s) \text{ option}) \times ((('a \times 's) \text{ option})) \text{ set}$

$$\frac{\neg\, C\ r\ s}{(\text{Some } (r,s), \text{Some } (r,s)) \in \text{while\_results } C\ B} \text{ (terminate)}$$

$$\frac{C\ r\ s \quad \text{snd } (B\ r\ s)}{(\text{Some } (r,s), \text{None}) \in \text{while\_results } C\ B} \text{ (fail)}$$

$$\frac{C\ r\ s \quad (r',s') \in \text{fst } (B\ r\ s) \quad (\text{Some } (r',\ s'),\ z) \in \text{while\_results } C\ B}{(\text{Some } (r,s),\ z) \in \text{while\_results } C\ B} \text{ (loop)}$$

Defining While Loops Inductively

Termination:

$$\begin{aligned}\text{while\_terminates} :: {} & (\text{'}s \Rightarrow \text{bool}) \Rightarrow \\ & (\text{'}s \Rightarrow (\text{'}a \times \text{'}s)\ \text{set} \times \text{bool})) \Rightarrow \\ & \text{'}a \Rightarrow \text{bool}\end{aligned}$$

Defining While Loops Inductively

Termination:

$$\begin{array}{l}\text{while\_terminates} :: ('a \Rightarrow \text{bool}) \Rightarrow \\ \quad ('a \Rightarrow ('a \times 's) \text{ set} \times \text{bool})) \Rightarrow \\ \quad 'a \Rightarrow \text{bool}\end{array}$$

$$\frac{\neg C\ r\ s}{\text{while\_terminates}\ C\ B\ r\ s}(\text{terminate})$$

Defining While Loops Inductively

Termination:

while_terminates :: $('a \Rightarrow 'b \Rightarrow \text{bool}) \Rightarrow$
$('a \Rightarrow 'b \Rightarrow ('a \times 's) \text{ set} \times \text{bool})) \Rightarrow$
$'a \Rightarrow 'b \Rightarrow \text{bool}$

$$\frac{\neg\, C\ r\ s}{\text{while\_terminates}\ C\ B\ r\ s}\ (\text{terminate})$$

$$\frac{C\ r\ s \quad \forall (r',s') \in \text{fst}\ (B\ r\ s).\ \text{while\_terminates}\ C\ B\ r'\ s'}{\text{while\_terminates}\ C\ B\ r\ s}\ (\text{loop})$$

Defining While Loops Inductively

Termination:

while_terminates :: $('a \Rightarrow$ bool$) \Rightarrow$
$('a \Rightarrow ('a \times 's)$ set $\times$ bool$)) \Rightarrow$
$'a \Rightarrow$ bool

$$\frac{\neg\, C\ r\ s}{\text{while\_terminates}\ C\ B\ r\ s}\ (\text{terminate})$$

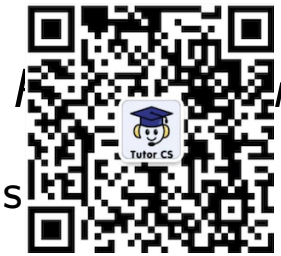
$$\frac{C\ r\ s \quad \forall (r',s') \in \text{fst}\ (B\ r\ s).\ \text{while\_terminates}\ C\ B\ r'\ s'}{\text{while\_terminates}\ C\ B\ r\ s}\ (\text{loop})$$

whileLoop $C\ B \equiv$
$(\lambda r\ s.\ (\{(r',s').\ (\text{Some}\ (r,\ s), \text{Some}\ (r',\ s')) \in \text{while\_results}\ C\ B\},$
$(\text{Some}\ (r,\ s), \text{None}) \in \text{while\_results} \vee$
$\neg \text{while\_terminates}\ C\ B\ r\ s))$

Hoare Logic over Nondeterministic State Monads

Partial correctness:

$$\{\!|P|\!\}\ m\ \{\!|Q|\!\} \equiv \forall s.\ P\ s \longrightarrow (\forall (r,s') \in \mathrm{fst}\ (m\ s).\ Q\ r\ s'$$



➜ Post-condition Q is a function of return-value and result state.

Weakest Precondition Rules

$\{\!|\ \ |\!\}\ \mathrm{return}\ x\ \{\!|\lambda r\ s.\ P\ r\ s|\!\}$ $\{\!|\ \ |\!\}\ \mathrm{get}\ \{\!|P|\!\}$ $\{\!|\ \ |\!\}\ \mathrm{put}\ x\ \{\!|P|\!\}$

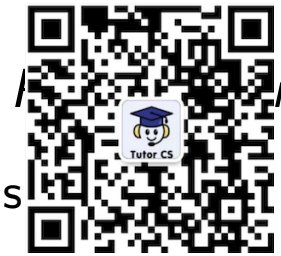
$\{\!|\ \ |\!\}\ \mathrm{gets}\ f\ \{\!|P|\!\}$ $\{\!|\ \ |\!\}\ \mathrm{modify}\ f\ \{\!|P|\!\}$

$\{\!|\ \ |\!\}\ \mathrm{assert}\ P\ \{\!|Q|\!\}$ $\{\!|\ \ |\!\}\ \mathrm{fail}\ \{\!|Q|\!\}$

Hoare Logic over Nondeterministic State Monads

Partial correctness:

$$\{\!|P|\!\}\ m\ \{\!|Q|\!\} \equiv \forall s.\ P\ s \longrightarrow (\forall (r,s') \in \mathrm{fst}\ (m\ s).\ Q\ r\ s'$$



➜ Post-condition Q is a predicate of return-value and result state.

Weakest Precondition Rules

$\{\!|\lambda s.\ P\ x\ s|\!\}\ \mathrm{return}\ x\ \{\!|\lambda r\ s.\ P\ r\ s|\!\}$ $\{\!|\ P\ x\ s|\!\}\ \mathrm{get}\ \{\!|P|\!\}$ $\{\!|\ P\ ()\ x|\!\}\ \mathrm{put}\ x\ \{\!|P|\!\}$

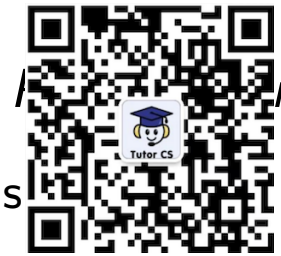
$\{\!|\ P\ (f\ s)\ s|\!\}\ \mathrm{gets}\ f\ \{\!|P|\!\}$ $\{\!|\ P\ ()\ (f\ s)|\!\}\ \mathrm{modify}\ f\ \{\!|P|\!\}$

$\{\!|\ P\ ()\ s|\!\}\ \mathrm{assert}\ P\ \{\!|Q|\!\}$ $\{\!|\ \mathrm{False}\ r\ s|\!\}\ \mathrm{fail}\ \{\!|Q|\!\}$

Hoare Logic over Nondeterministic State Monads

Partial correctness:

$$\{\!|P|\!\}\ m\ \{\!|Q|\!\} \equiv \forall s.\ P\ s \longrightarrow (\forall (r,s') \in \mathrm{fst}\ (m\ s).\ Q\ r\ s'$$



➜ Post-condition Q is a function of return-value and result state.

Weakest Precondition Rules

$$\{\!|\lambda s.\ P\ x\ s|\!\}\ \mathrm{return}\ x\ \{\!|\lambda r\ s.\ P\ r\ s|\!\} \qquad \{\!|\lambda s.\ P\ s\ s|\!\}\ \mathrm{get}\ \{\!|P|\!\} \qquad \{\!|\ \_\ |\!\}\ \mathrm{put}\ x\ \{\!|P|\!\}$$

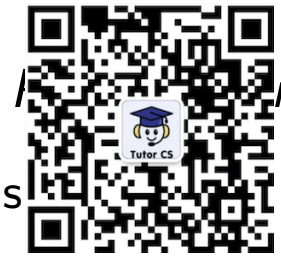
$$\{\!|\ \_\ |\!\}\ \mathrm{gets}\ f\ \{\!|P|\!\} \qquad \{\!|\ \_\ |\!\}\ \mathrm{modify}\ f\ \{\!|P|\!\}$$

$$\{\!|\ \_\ |\!\}\ \mathrm{assert}\ P\ \{\!|Q|\!\} \qquad \{\!|\ \_\ |\!\}\ \mathrm{fail}\ \{\!|Q|\!\}$$

Hoare Logic over Nondeterministic State Monads

Partial correctness:

$$\{\!|P|\!\}\ m\ \{\!|Q|\!\} \equiv \forall s.\ P\ s \longrightarrow (\forall (r,s') \in \mathrm{fst}\ (m\ s).\ Q\ r\ s'$$



➜ Post-condition Q is a function of return-value and result state.

Weakest Precondition Rules

$$\{\!|\lambda s.\ P\ x\ s|\!\}\ \mathrm{return}\ x\ \{\!|\lambda r\ s.\ P\ r\ s|\!\} \qquad \{\!|\lambda s.\ P\ s\ s|\!\}\ \mathrm{get}\ \{\!|P|\!\} \qquad \{\!|\lambda s.\ P\ ()\ x|\!\}\ \mathrm{put}\ x\ \{\!|P|\!\}$$

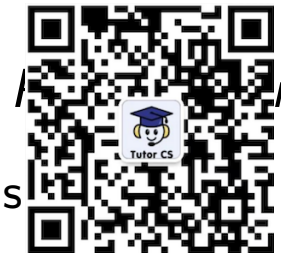
$$\{\!|\ \ |\!\}\ \mathrm{gets}\ f\ \{\!|P|\!\} \qquad \{\!|\ \ |\!\}\ \mathrm{modify}\ f\ \{\!|P|\!\}$$

$$\{\!|\ \ |\!\}\ \mathrm{assert}\ P\ \{\!|Q|\!\} \qquad \{\!|\ \ |\!\}\ \mathrm{fail}\ \{\!|Q|\!\}$$

Hoare Logic over Nondeterministic State Monads

Partial correctness:

$$\{\!|P|\!\}\ m\ \{\!|Q|\!\} \equiv \forall s.\ P\ s \longrightarrow (\forall (r,s') \in \mathrm{fst}\ (m\ s).\ Q\ r\ s')$$



➜ Post-condition Q is a predicate of return-value and result state.

Weakest Precondition Rules

$\{\!|\lambda s.\ P\ x\ s|\!\}$ return x $\{\!|\lambda r\ s.\ P\ r\ s|\!\}$ $\quad$ $\{\!|\lambda s.\ P\ s\ s|\!\}$ get $\{\!|P|\!\}$ $\quad$ $\{\!|\lambda s.\ P\ ()\ x|\!\}$ put x $\{\!|P|\!\}$

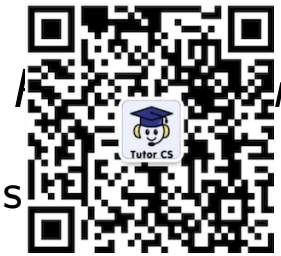
$\{\!|\lambda s.\ P\ (f\ s)\ s|\!\}$ gets f $\{\!|P|\!\}$ $\quad$ $\{\!|$ $|\!\}$ modify f $\{\!|P|\!\}$

$\{\!|$ $|\!\}$ assert P $\{\!|Q|\!\}$ $\quad$ $\{\!|$ $|\!\}$ fail $\{\!|Q|\!\}$

Hoare Logic over Nondeterministic State Monads

Partial correctness:

$$\{\!|P|\!\}\ m\ \{\!|Q|\!\} \equiv \forall s.\ P\ s \longrightarrow (\forall (r,s') \in \mathrm{fst}\ (m\ s).\ Q\ r\ s'$$



➜ Post-condition Q is a function of return-value and result state.

Weakest Precondition Rules

$$\{\!|\lambda s.\ P\ x\ s|\!\}\ \mathrm{return}\ x\ \{\!|\lambda r\ s.\ P\ r\ s|\!\} \qquad \{\!|\lambda s.\ P\ s\ s|\!\}\ \mathrm{get}\ \{\!|P|\!\} \qquad \{\!|\lambda s.\ P\ ()\ x|\!\}\ \mathrm{put}\ x\ \{\!|P|\!\}$$

$$\{\!|\lambda s.\ P\ (f\ s)\ s|\!\}\ \mathrm{gets}\ f\ \{\!|P|\!\} \qquad \{\!|\lambda s.\ P\ ()\ (f\ s)|\!\}\ \mathrm{modify}\ f\ \{\!|P|\!\}$$

$$\{\!|\quad\quad|\!\}\ \mathrm{assert}\ P\ \{\!|Q|\!\} \qquad \{\!|\quad\quad|\!\}\ \mathrm{fail}\ \{\!|Q|\!\}$$

Hoare Logic over Nondeterministic State Monads

Partial correctness:

$\{\!|P|\!\}\ m\ \{\!|Q|\!\} \equiv \forall s.\ P$ [illegible] $r,s') \in \text{fst}\ (m\ s).\ \ Q\ r\ s'$

➜ Post-condition Q is [illegible] of return-value and result state.

Weakest Precondition Rules

$\{\!|\lambda s.\ \ P\ x\ s|\!\}$ return x $\{\!|\lambda r\ s.\ \ P\ r\ s|\!\}$ $\quad \{\!|\lambda s.\ P\ s\ s|\!\}$ get $\{\!|P|\!\}$ $\quad \{\!|\lambda s.\ P\ ()\ x|\!\}$ put x $\{\!|P|\!\}$

$\{\!|\lambda s.\ P\ (f\ s)\ s|\!\}$ gets f $\{\!|P|\!\}$ $\quad \{\!|\lambda s.\ P\ ()\ (f\ s)|\!\}$ modify f $\{\!|P|\!\}$

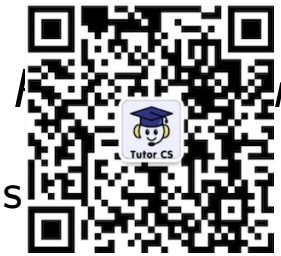
$\{\!|\lambda s.\ \ P \longrightarrow Q\ ()\ s|\!\}$ assert P $\{\!|Q|\!\}$ $\quad \{\!| \quad |\!\}$ fail $\{\!|Q|\!\}$

Hoare Logic over Nondeterministic State Monads

Partial correctness:

$$\{\!|P|\!\}\ m\ \{\!|Q|\!\} \equiv \forall s.\ P\ s \longrightarrow (r,s') \in \text{fst}\ (m\ s).\ Q\ r\ s'$$

➜ Post-condition Q is a function of return-value and result state.



Weakest Precondition Rules

$\{\!|\lambda s.\ P\ x\ s|\!\}$ return x $\{\!|\lambda r\ s.\ P\ r\ s|\!\}$ $\quad$ $\{\!|\lambda s.\ P\ s\ s|\!\}$ get $\{\!|P|\!\}$ $\quad$ $\{\!|\lambda s.\ P\ ()\ x|\!\}$ put x $\{\!|P|\!\}$

$\{\!|\lambda s.\ P\ (f\ s)\ s|\!\}$ gets f $\{\!|P|\!\}$ $\quad$ $\{\!|\lambda s.\ P\ ()\ (f\ s)|\!\}$ modify f $\{\!|P|\!\}$

$\{\!|\lambda s.\ P \longrightarrow Q\ ()\ s|\!\}$ assert P $\{\!|Q|\!\}$ $\quad$ $\{\!|\lambda\_.\ \text{True}|\!\}$ fail $\{\!|Q|\!\}$

More Hoare Logic Rules

$$\frac{\{\!|}{}\ \text{  }\ \frac{}{|\!\}\ \textbf{if}\ P\ \textbf{then}\ f\ \textbf{else}\ g\ \{\!|S|\!\}}$$

More Hoare Logic Rules

$$\frac{P \implies \{\!|Q|\!\}\ f \qquad \neg P \implies \{\!|R|\!\}\ g\ \{\!|S|\!\}}{\{\!|\lambda s.(P \longrightarrow Q\ s) \wedge (\neg P \longrightarrow R\ s)|\!\}\ \textbf{if}\ P\ \textbf{then}\ f\ \textbf{else}\ g\ \{\!|S|\!\}}$$

More Hoare Logic Rules

$$\frac{P \implies \{\!|Q|\!\}\ f \quad \neg P \implies \{\!|R|\!\}\ g\ \{\!|S|\!\}}{\{\!|\lambda s.(P \longrightarrow Q\ s) \wedge (\neg P \longrightarrow R\ s)|\!\}\ \textbf{if}\ P\ \textbf{then}\ f\ \textbf{else}\ g\ \{\!|S|\!\}}$$

$$\frac{\bigwedge x.\ \{\!|B\ x|\!\}\ g\ x\ \{\!|C|\!\} \quad \{\!|A|\!\}\ f\ \{\!|B|\!\}}{\{\!|A|\!\}\ \textbf{do}\ \{x \leftarrow f; g\ x\}\ \{\!|C|\!\}}$$

More Hoare Logic Rules

$$\frac{P \implies \{\!|R|\!\}\ f\ \{\!|S|\!\} \qquad \neg P \implies \{\!|R|\!\}\ g\ \{\!|S|\!\}}{\{\!|\lambda s.(P \longrightarrow Q\ s) \wedge (\neg P \longrightarrow R\ s)|\!\}\ \textbf{if}\ P\ \textbf{then}\ f\ \textbf{else}\ g\ \{\!|S|\!\}}$$

$$\frac{\bigwedge x.\ \{\!|B\ x|\!\}\ g\ x\ \{\!|C|\!\} \qquad \{\!|A|\!\}\ f\ \{\!|B|\!\}}{\{\!|A|\!\}\ \textbf{do}\ \{x \leftarrow f; g\ x\}\ \{\!|C|\!\}}$$

$$\frac{\{\!|R|\!\}\ m\ \{\!|Q|\!\} \qquad \bigwedge s.\ P\ s \implies R\ s}{\{\!|P|\!\}\ m\ \{\!|Q|\!\}}$$

More Hoare Logic Rules

$$\frac{P \implies \{\!|R|\!\}\ f\ \{\!|S|\!\} \qquad \neg P \implies \{\!|R|\!\}\ g\ \{\!|S|\!\}}{\{\!|\lambda s.(P \longrightarrow Q\ s) \wedge (\neg P \longrightarrow R\ s)|\!\}\ \textbf{if}\ P\ \textbf{then}\ f\ \textbf{else}\ g\ \{\!|S|\!\}}$$

$$\frac{\bigwedge x.\ \{\!|B\ x|\!\}\ g\ x\ \{\!|C|\!\} \qquad \{\!|A|\!\}\ f\ \{\!|B|\!\}}{\{\!|A|\!\}\ \textbf{do}\ \{x \leftarrow f; g\ x\}\ \{\!|C|\!\}}$$

$$\frac{\{\!|R|\!\}\ m\ \{\!|Q|\!\} \qquad \bigwedge s.\ P\ s \implies R\ s}{\{\!|P|\!\}\ m\ \{\!|Q|\!\}}$$

$$\frac{\bigwedge r.\ \{\!|\lambda s.\ I\ r\ s \wedge C\ r\ s|\!\}\ B\ \{\!|I|\!\} \qquad \bigwedge r\ s.\ [\![I\ r\ s;\ \neg C\ r\ s]\!] \implies Q\ r\ s}{\{\!|I\ r|\!\}\ \mathsf{whileLoop}\ C\ B\ r\ \{\!|Q|\!\}}$$

Demo

We have seen today

- Deep and shallow e
- Isabelle records
- Nondeterministic S with Failure
- Monadic Weakest F Rules