COMP4418: Knowledge Representation and Reasoning

Horn Logic

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Horn clauses

Clauses are used two ways:

- as disjunctions: (rain ∨ sleet)
- as implications: (¬child ∨ ¬male

Here focus on 2nd use

Horn clause = at most one +ve literal in c

- positive / definite clause = exactly one +ve literal

  $[\neg p_1, \neg p_2, \ldots, \neg p_n, q]$
- negative clause = no +ve literals

  $[\neg p_1, \neg p_2, \ldots, \neg p_n]$

Note:

$[\neg p_1, \neg p_2, \ldots, \neg p_n, q]$ is a representation for

$(\neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_n \vee q)$ or

$[(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow q]$

So can read as

If p_1 and p_2 and . . . and p_n then q

and write sometimes as

$p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow q$

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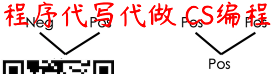
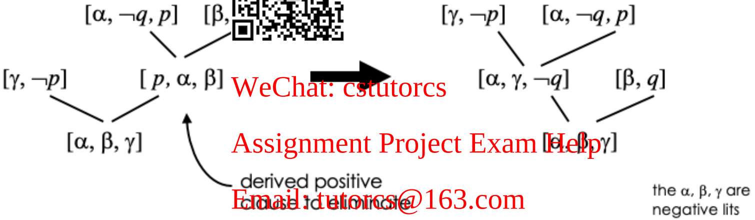
Resolution with Horn clauses

Only two possibilities:



It is possible to rearrange derivations (of [illegible] clauses) so that all new derived clauses are negative clauses



Can also change derivations such that each derived clause is a resolvent of the previous derived one (-ve) and some +ve clause in the original set of clauses

- Since each derived clause is negative, one parent must be positive (and so from original set) and one negative
- Continue working backwards until both parents of derived clause are from the original set of clauses
- Eliminate all other clauses not on direct path

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SLD Resolution

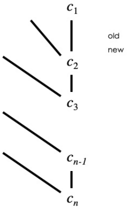
Recurring pattern in derivations

See previously:

- Example 1
- Example 3
- Arithmetic example

But not:

- Example 2
- 3 block example



An *SLD-derivation* of a clause c from a set of clauses S is a sequence of clause $c_1, c_2, \ldots c_n$ such that $c_n = c$, and

1. $c_1 \in S$
2. c_{i+1} is a resolvent of c_i and a clause in S

Write: $S \vdash_{SLD} c$

Note: SLD derivation is just a special form of derivation and where we leave out the elements of S (except c_1)

SLD means S(elected) literals, L(inear) form, D(efinite) clauses

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Completeness of SLD

In general, cannot restrict Resolution steps to always use a clause that is in the original set

Proof:

$S = \{[p, q], [p, \neg q], [\neg p, q], [\neg$

then $S \vdash []$.

Need to resolve some $[l]$ and $[\neg l]$ to get $[]$.

But S does not contain any unit clauses.

So will need to derive both $[l]$ and $[\neg l]$ and then resolve them together.

But can do so for Horn clauses . . .

Theorem: for Horn clauses, $H \vdash []$ iff $H \vdash_{SLD} []$

So: H is unsatisfiable iff $H \vdash_{SLD} []$

This will considerably simplify the search for derivations

Note: in Horn version of SLD-Resolution, each clause $c_1, c_2, \ldots c_n$ will be negative

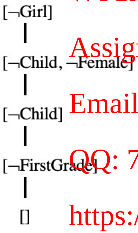
So clauses H must always contain at least one negative clause, c_1.

Example 1 (again)

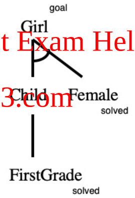
KB:

FirstGrade
FirstGrade → Child
Child ∧ Male → Boy
Kindergarten → Child
Child ∧ Female → Girl
Female

Show KB ∪ {¬Girl} unsatisfiable



or



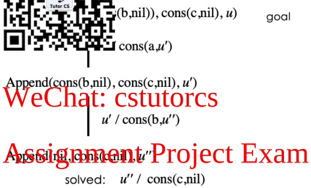
A goal tree whose nodes are atoms, whose root is the atom to prove, and whose leaves are in the KB

Prolog

Horn clauses form the basis of Prolog

Append(nil,y,y)

Append(x,y,z) $\rightarrow$ Append(cons [illegible] ns(w,z))



[illegible](b,nil)), cons(c,nil), u) goal

[illegible] cons(a,u')

Append(cons(b,nil), cons(c,nil), u')

u' / cons(b,u'')

Append(nil, cons(c,nil), u'')

solved: u'' / cons(c,nil)

So goal succeeds with u = cons(a,cons(b,cons(c,nil)))

that is: Append([a b],[c],[a b c])

With SLD derivation, can always extract answer from proof

$H \vdash \exists x \alpha(x)$ iff for some term t, $H \vdash \alpha(t)$

Different answers can be found by finding other derivations

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Back-chaining procedure

Satisfiability of a set of Horn clauses with exactly one negative clause

```
Solve $[q_1, q_2, \ldots, q_n] =$                /* to establish conjunction of $q_i$ */
    If $n = 0$ then return **YES**;         /* empty clause detected */
    For each $d \in KB$ do
        If $d = [q_1, \neg p_1, \neg p_2, \ldots, \neg p_m]$    /* match first q */
            and                                    /* replace $q$ by -ve lits */
          Solve $[p_1, p_2, \ldots, p_m, q_2, \ldots, q_n]$      /* recursively */
        then return **YES**
    end for;                       /* can't find a clause to eliminate q */
    Return **NO**
```

Depth-first, left-right, back-chaining

- depth-first because attempt p_i before trying q_i
- left-right because try q_i in order, 1, 2, 3, ...
- back-chaining because search from goal q to facts in KB p

This is the execution strategy of Prolog

First-order case requires unification etc.

Problems with back-chaining

Can go into infinite loop

- tautologous clause: $[p, \neg p]$
- corresponds to Prolog program with p :- p

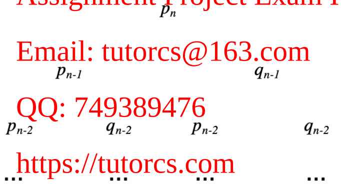
Previous back-chaining algorithm is

Example:

- consider $2n$ atoms: $p_1, \ldots, p_n,$ q
- and $4n - 4$ clauses:

  $(p_i \Rightarrow p_{i+1}), (q_i \Rightarrow q_{i+1}),$

  $(p_i \Rightarrow q_{i+1}), (q_i \Rightarrow q_{i+1}).$
- with goal p_n has execution tree like this:



- search eventually fails after 2^n steps!

Is this inherent in Horn clauses?

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Forward-chaining

Simple procedure to determine if Horn KB $\models q$.

main idea: mark atoms as solved

1. If q is marked as solved, then return YES.
2. Is there a $\{p_1, \neg p_2, \ldots, \neg p_n\} \in$ KB such that $p_2, \ldots, p_n$ are marked as solved, but the positive literal p_1 is not marked as solved?

   no: return **NO**

   yes: mark p_1 as solved, and go to 1.

FirstGrade example:

Marks: FirstGrade, Child, Female, Girl

then done!

Observe:

- only letters in KB can be marked, so at most a linear number of iterations
- not goal-directed, so not always desirable

A similar procedure with better data structures will run in *linear* time overall

First-order undecidability

Even with just Horn clauses, in the first-order case we still have the possibility of generating an infinite branch of resolvents

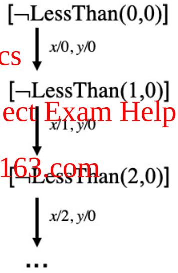
KB: LessThan(succ(x),y) → LessTh

Q: LessThan(zero,zero)

As with full Resolution, there is no way to detect when this will happen

So there is no procedure that will test for satisfiability of first-order Horn clauses

the question is undecidable



As with full clauses, the best that can be expected is to give control of the deduction to the *user*

To some extent this is what is done in Prolog, but we will see more in “Procedural Control”

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