# COM[illegible]0

Foundations of Computer Science



Lecture 16: Statistics

# Outline



Random Variables and Expectation

Linearity of Expectation

Expected Time to Success

Standard Deviation and Variance

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Random Variables and Expectation

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# Random Variables



## Definition

An (integer) **random** is a function from $\Omega$ to $\mathbb{Z}$.
In other words, it ass number value with every outcome.

Random variables are often denoted by $X, Y, Z, \ldots$
We extend arithmetic to random variables in the natural way.



## Definition



Given random variables $X$, $Y$ and integer $k$:



$$\begin{aligned}
X + Y : &\quad \omega \mapsto X(\omega) + Y(\omega) \\
XY : &\quad \omega \mapsto X(\omega).Y(\omega) \\
X - k : &\quad \omega \mapsto X(\omega) - k \\
kX : &\quad \omega \mapsto k.X(\omega)
\end{aligned}$$

# Examples

**Example**



Random variable $X$: value of rolling one die
$\Omega = \{1, 2, 3, 4, 5, 6\}$
$X(i) = i$

**Example**



Random variable $X_s$: sum of rolling two dice



$\Omega = \{(1, 1), (1, 2), \ldots, (6, 6)\}$



$X_s((1, 1)) = 2 \qquad X_s((1, 2)) = 3 = X_s((2, 1)) \quad \ldots$



**Question**

*Is* $X_s = X + X$*? No.*



$X_s = X + Y$ where $X$ and $Y$ are independent and identically distributed (i.i.d)

# Expectation



## Definition

The **expected value** (also called "expectation" or "average") of a random variable $X$ is

$$E(X) = \sum_{k \in \mathbb{Z}} P(X = k) \cdot k$$



## NB

*Expectation is a truly universal concept; it is the basis of all decision making, of estimating gains and losses, in all actions under risk. Historically, a rudimentary concept of expected value arose long before the notion of probability.*

# Examples



**Example**

The expected value when rolling one die is:

$$E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \ldots + \frac{1}{6} \cdot 6 = 3.5$$



**Example**



The expected sum when rolling two dice is



$$E(X_s) = \frac{1}{36} \cdot 2 + \frac{2}{36} \cdot 3 + \ldots + \frac{6}{36} \cdot 7 + \ldots + \frac{1}{36} \cdot 12 = 7$$

# Examples





**Example**

$\boxed{\text{RW: 9.3.3}}$ Buy one lottery ticket for \$1. The only prize is \$1M. Each ticket has probability $6 \cdot 10^{-7}$ of winning.

$\Omega = \{win, lose\}$ $\quad X_L(win) = \$999,999$ $\quad X_L(lose) = -\$1$

$E(X_L) = 6 \cdot 10^{-7} \cdot \$999,999 + (1 - 6 \cdot 10^{-7}) \cdot -\$1 = -\$0.4$

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# Linearity of expectation



**Theorem (linearity [illegible]ted value)**

*For any random vari[illegible] and integer k:*

$$E(X + Y) = E(X) + E(Y) \qquad E(k \cdot X) = k \cdot E(X)$$



**Example**

The expected sum when rolling two dice can be computed as



$$E(X_s) = E(X) + E(Y) = 3.5 + 3.5 = 7$$

# Example



## Example

$E(S_n)$, where $S_n \stackrel{\text{def}}{=} |\text{HEADS in } n \text{ tosses}|$

- 'hard way'

  $E(S_n) = \sum_{k=0}^{n} P(S_n = k) \cdot k = \sum_{k=0}^{n} \frac{1}{2^n}\binom{n}{k} \cdot k$

  since there are $\binom{n}{k}$ sequences of $n$ tosses with $k$ HEADS, and each sequence has the probability $\frac{1}{2^n}$

  $= \frac{1}{2^n} \sum_{k=1}^{n} \frac{n}{k}\binom{n-1}{k-1} k = \frac{n}{2^n} \sum_{k=0}^{n-1} \binom{n-1}{k} = \frac{n}{2^n} \cdot 2^{n-1} = \frac{n}{2}$

  using the 'binomial identity' $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

- 'easy way'

  $E(S_n) = E(S_1^1 + \cdots + S_1^n) = \sum_{i=1\ldots n} E(S_1^i) = nE(S_1) = n \cdot \frac{1}{2}$

Note: $S_n \stackrel{\text{def}}{=} |\text{HEADS in } n \text{ tosses}|$ while each $S_1^i \stackrel{\text{def}}{=} |\text{HEADS in 1 toss}|$

# Observations



**Fact**

*If $X_1, X_2, \ldots, X_n$ are independent, identically distributed random variables, then $E(X_1 + X_2 + \ldots + X_n) = E(nX_1) = nE(X_1)$.*



**NB**

$X_1 + X_2 + \ldots + X_n$ *and* $nX_1$ *are very different random variables.*

**Exercise**

You face a quiz cons [illegible] ix true/false questions, and your plan is to guess the a [illegible] each question (randomly, with probability 0.5 of bei [illegible] There are no negative marks, and answering four or more questions correctly suffices to pass. What is the probability of passing and what is the expected score?

# Exercises

**Exercise**



RW: 9.3.7

An urn has $m + n =$ [illegible]es, $m \geq 0$ red and $n \geq 0$ blue.
7 marbles selected at [illegible] without replacement.
What is the expected [illegible] of red marbles drawn?

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# Example

## Example



Find the average waiting time for the first HEAD, with no upper bound on the 'durati[illegible]llows for all possible sequences of tosses, regardless of [illegible] times TAILS occur initially).

$$A = E(X_w) = \sum_{k=1}^{\infty} k \cdot P(X_w = k) = \sum_{k=1}^{\infty} k \frac{1}{2^k}$$

$$= \frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \dots$$



This can be evaluated by breaking the sum into a sequence of geometric progressions

$$\begin{aligned}
&\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots \\
&= \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\right) + \left(\frac{1}{2^2} + \frac{1}{2^3} + \dots\right) + \left(\frac{1}{2^3} + \dots\right) + \dots \\
&= 1 + \frac{1}{2} + \frac{1}{2^2} + \dots \\
&= 2
\end{aligned}$$

# Expected time to success

There is also a recursive 'trick' for solving the sum



$$A = \sum_{k=1}^{\infty} \frac{k}{2^k} = \sum_{k=1}^{\infty} \frac{k-1}{2^k} + \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{k-1}{2^{k-1}} + 1 = \frac{1}{2} A + 1$$

Now $A = \frac{A}{2} + 1$ ar

**NB**



*A much simpler but equally valid argument is that you expect 'half' a* HEAD *in 1 toss, so you ought to get a 'whole'* HEAD *in 2 tosses.*



**Theorem**



*If the probability of success is $p$ then:*

- *The expected number of (indep.) trials before 1 success is* $\frac{1}{p}$
- *The expected number of (indep.) trials before $k$ successes is* $\frac{k}{p}$

# Exercise



## Exercise

$\boxed{\text{RW: 9.4.12}}$ A die is rolled until the first 4 appears. What is the expected waiting time?

## Example



To find an object $\mathcal{X}$ in an unsorted list $L$ of elements, one needs to search linearly through $L$. Let the probability of $\mathcal{X} \in L$ be $p$, hence there is $1-p$ likelihood of $\mathcal{X}$ being absent altogether. Find the expected number of comparison operations.

If the element is in the list, then the number of comparisons averages to $\frac{1}{n}(1+\ldots+n)$; if absent we need $n$ comparisons. The first case has probability $p$, the second $1-p$. Combining these we find

$$E_n = p\frac{1+\ldots+n}{n} + (1-p)n = p\frac{n+1}{2} + (1-p)n = (1-\frac{p}{2})n + \frac{p}{2}$$

As one would expect, increasing $p$ leads to a lower $E_n$.

# Success vs Expected value



**Question**

*Does high probability of success lead to a high expected value?*

Generally, no.



**Example**

Buying more tickets in the lottery increases your chances of winning, but the expected value of winnings *decreases*.

# Example



## Example

Roulette (outcomes $0, 1, \ldots, 36$). Win: $35 \times$ bet



**Strategy 1**: Bet $\$1$ on a single number



- Probability of winning: $\frac{1}{37}$
- Expected winnings: $\frac{1}{37}.(\$35) + \frac{36}{37}(-\$1) \approx -2.7c$

# Example

## Example



Roulette (outcomes 6). Win: $35 \times$ bet

**Strategy 2**: Place $ 24 numbers, selected from among 0 to 36.

- Probability of winning: $\frac{24}{37} \approx 65\%$
- Expected winnings:
  - If one of the numbers comes up, win $35 from the bet on that number and lose $23 from the bets on the remaining numbers, thus collecting $12.
    This happens with probability $p = \frac{24}{37}$.
  - With probability $q = \frac{13}{37}$ none of the numbers appear, leading to loss of $24.



So expected winnings are:

$p \cdot \$12 - q \cdot \$24 = \$12\frac{24}{37} - \$24\frac{13}{37} = -\$\frac{24}{37} \approx -65c = 24 \times -2.7c$

# Gambler's ruin



Many so-called 'winn[illegible]ns' that purport to offer a winning strategy do somethin[illegible] they provide a scheme for frequent relatively moderate w[illegible]t the cost of an occasional very big loss.

It turns out (it is a formal theorem) that there can be *no system* that converts an 'unfair' game into a 'fair' one. In the language of decision theory, 'unfair' denotes a game whose individual bets have negative expectation.

It can be easily checked that any individual bets on roulette, on lottery tickets or on just about any commercially offered game have negative expected value.

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# Standard Deviation and Variance



**Definition**

For random variable $X$ with expected value (or: **mean**) $\mu = E(X)$, the **standard deviation** of $X$ is

$$\sigma = \sqrt{E((X-\mu)^2)}$$

and the **variance** of $X$ is

$$\sigma^2$$



Standard deviation and variance measure how spread out the values of a random variable are. The smaller $\sigma^2$ the more confident we can be that $X(\omega)$ is close to $E(X)$, for a randomly selected $\omega$.

**NB**

*The variance can be calculated as* $E((X-\mu)^2) = E(X^2) - \mu^2$

# Example



## Example

Random variable $X_d$ [illegible] of a rolled die

$$\mu = E(X_d) = 3.5$$



$$E(X_d^2) = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 9 + \frac{1}{6}\cdot 16 + \frac{1}{6}\cdot 25 + \frac{1}{6}\cdot 36 = \frac{91}{6}$$

Hence, $\sigma^2 = E(X_d^2) - \mu^2 = \frac{35}{12} \quad \rightarrow \quad \sigma \approx 1.71$

## Exercises

RW: 9.5.10 (Supp) T[illegible]endent experiments are performed.
$P$(1st experiment su[illegible] 0.7
$P$(2nd experiment succeeds) $= 0.2$
Random variable $X$ counts the number of successful experiments.





a. Expected value of $X$?



b. Probability of exactly one success?



c. Probability of at most one success?

e. Variance of $X$?