COMP9020 MST 12s2 Solutions

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Exercise 1 How many integers not divided by 3 are there between 123 and 66789?

The number of multiples of k in the interval $[n\,..\,m]$ is $\left\lfloor \frac{m}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor$. So we calculate the number of integers between 123 and 66789 as

$$66789 - (123 - 1) = 66667 \tag{1}$$

and the number of integers that are divided by 3 between 123 and 66789 as

$$\left\lfloor \frac{66789}{3} \right\rfloor - \left\lfloor \frac{123-1}{3} \right\rfloor = 22263 - 40 = 22223 \tag{2}$$

Subtracting (2) from (1) gives our result: $66667 - 22223 = \mathbf{44444}$.

Exercise 2 What is $\gcd(9876543213, 9876543210)$?

$$\gcd(9876543213, 9876543210) = \gcd(3, 9876543210) = \gcd(3, 0) = 3$$

since $3|9876543210$.

Exercise 3 Let S, T, U, and V be sets. Prove that $(S \cap T) \times (U \cap V) = (S \times U) \cap (T \times V)$.

$$\begin{aligned}
(s,u) \in (S \cap T) \times (U \cap V) &\Leftrightarrow s \in (S \cap T) \wedge u \in (U \cap V) && \text{Def. of } \times \\
&\Leftrightarrow s \in S \wedge s \in T \wedge u \in U \wedge u \in V && \text{Def. of } \cap \\
&\Leftrightarrow s \in S \wedge u \in U \wedge s \in T \wedge u \in V && \text{commutativity of } \wedge \\
&\Leftrightarrow (s,u) \in (S \times U) \wedge (s,u) \in (T \times V) && \text{Def. of } \times \\
&\Leftrightarrow (s,u) \in (S \times U) \cap (T \times V) && \text{Def. of } \cap \qquad \blacksquare
\end{aligned}$$

Exercise 4 Let $\phi_1 = (p \Rightarrow (q \vee r))$, $\phi_2 = (s \Rightarrow (q \vee p))$, and $\phi = \phi_1 \wedge \phi_2$.

1. Draw Karnaugh maps for the three formulae, ϕ_1, ϕ_2, and ϕ.
2. Read off a minimal DNF for ϕ.
3. Give a minimal CNF for $\neg\phi_1$.

1. (To improve legibility, only the false entries are marked. It turns out to be convenient to give the first two maps over the whole set of propositions rather than just the involved three. That way, the third map is trivially obtained from the first two, and I can just copy&paste the LaTeX code.)

ϕ_1 :

	p	p	$\bar{p}$	$\bar{p}$	
q					$\bar{s}$
q					s
$\bar{q}$	0				s
$\bar{q}$	0				$\bar{s}$
	$\bar{r}$	r	r	$\bar{r}$	

ϕ_2 :

	p	p	$\bar{p}$	$\bar{p}$	
q					$\bar{s}$
q					s
$\bar{q}$			0	0	s
$\bar{q}$					$\bar{s}$
	$\bar{r}$	r	r	$\bar{r}$	

ϕ :

	p	p	$\bar{p}$	$\bar{p}$	
q					$\bar{s}$
q					s
$\bar{q}$	0		0	0	s
$\bar{q}$	0				$\bar{s}$
	$\bar{r}$	r	r	$\bar{r}$	

2. A minimal DNF for ϕ is $q \vee pr \vee \bar{p}\bar{s}$.

3. A minimal CNF for $\neg\phi_1$ is obtained most easily by reading off a minimal DNF for ϕ_1 and then using de Morgan. A minimal DNF for ϕ_1 is $\bar{p} \vee q \vee r$. Hence $p \wedge \bar{q} \wedge \bar{r}$ is a minimal CNF for $\neg\phi_1$.

Exercise 5 Suppose Portia puts a dagger in one of three caskets and places the following inscriptions on the caskets:

Gold casket: The dagger

Silver casket: The dagger t.

Lead casket: At most on a true inscription.

Portia tells her suitor to p es not contain the dagger. Which casket should the suitor choose? Formalise the pro al logic and calculate an answer.

We formalise using six propositions:

g is true iff the dagger is in the gold casket,

s is true iff the dagger is in the silver casket,

ℓ is true iff the dagger is in the lead casket,

G is true iff the inscription on the gold casket is true,

S is true iff the inscription on the silver casket is true, and

L is true iff the inscription on the lead casket is true.

We need to model that there's a dagger in precisely one of the caskets.

$$(g \Leftrightarrow \bar{s}\bar{\ell}) \wedge (s \Leftrightarrow \bar{g}\bar{\ell}) \wedge (\ell \Leftrightarrow \bar{g}\bar{s}) \tag{3}$$

The inscriptions are modeled as follows.

$$G \Leftrightarrow g \tag{4}$$
$$S \Leftrightarrow \neg s \tag{5}$$
$$L \Leftrightarrow ((G \Rightarrow \bar{S}\bar{L}) \wedge (S \Rightarrow \bar{G}\bar{L}) \wedge (L \Rightarrow \bar{G}\bar{S})) \tag{6}$$

A quick check reveals that both g and s are consistent with (3)–(6). Our only hope is hence ℓ. We deduce the following consequences from ℓ being true:

$$\bar{G} \qquad \text{by (3) and (4)} \tag{7}$$
$$S \qquad \text{by (3) and (5)} \tag{8}$$
$$L \Leftrightarrow (\bar{L} \wedge \bar{L}) \qquad \text{by (6)–(8)} \tag{9}$$

But $L \Leftrightarrow \bar{L}$ is equivalent to false, contradicting our assumption about ℓ being true. We conclude that the suitor must pick the lead casket to be safe.

Exercise 6 In $\mathbb{B}^5$, what is the value of $(0,0,1,1,1) \wedge (0,1,0,1,0)$?

Recall that "$\wedge$" in $\mathbb{B}^n$ corresponds to the bit-wise "and", so the result is

$$(0 \wedge 0, 0 \wedge 1, 1 \wedge 0, 1 \wedge 1, 1 \wedge 0) = (0,0,0,1,0) \ .$$

Exercise 7 How many Boolean algebra isomorphisms of $\mathcal{P}(\{a,b,c\})$ onto $\mathbb{B}^3$ are there? Explain your answer briefly.

Such an isomorphism is completely determined by how we map the atoms, which in this case are the singleton sets $\{a\}$, $\{b\}$, and $\{c\}$. They must be mapped onto the atoms of $\mathbb{B}^3$, which are 001, 010, and 100. There are $3! = 3 \cdot 2 \cdot 1 = 6$ onto functions between sets of size 3, so our answer is **6**.

Exercise 8 Let S be a finite set and let $n \in \mathbb{N}$. How many

1. functions,
2. onto functions,
3. binary relations, and
4. n-ary relations

are there on S? Explain your answers.

1. $|S|^{|S|}$ — for every element a free choice between all elements
2. $|S|!$ — onto functions on finite sets are 1–1 (permutations)
3. $2^{(|S|^2)}$ — size of the powerset of the set of pairs
4. $2^{(|S|^n)}$ — size of the powerset of the set of n-tuples

There was a severe typo in the exam whence I had to discard the last question.

Exercise 9 Let S, T, and U be sets. Let $R_1 \subseteq S \times T$ and $R_2 \subseteq T \times U$. Prove that $(R_1; R_2)^{\leftarrow} = (R_2{}^{\leftarrow}); (R_1{}^{\leftarrow})$.

(I use the notation $\exists x\,(P(x))$ to express "there exists an x such that $P(x)$ is true.")

$$
\begin{aligned}
(u,s) \in (R_1;R_2)^{\leftarrow} &\Leftrightarrow (s,u) \in R_1;R_2 && \text{Def. of } .^{\leftarrow}\\
&\Leftrightarrow \exists t\,((s,t) \in R_1 \wedge (t,u) \in R_2) && \text{Def. of ;}\\
&\Leftrightarrow \exists t\,((t,s) \in R_1^{\leftarrow} \wedge (u,t) \in R_2^{\leftarrow}) && \text{Def. of } .^{\leftarrow}\\
&\Leftrightarrow \exists t\,((u,t) \in R_2^{\leftarrow} \wedge (t,s) \in R_1^{\leftarrow}) && \text{commutativity of } \wedge\\
&\Leftrightarrow (u,s) \in (R_2^{\leftarrow});(R_1^{\leftarrow}) && \text{Def. of ;} \quad \blacksquare
\end{aligned}
$$