The University of New South Wales

Final Exam

ion 2, 2014

)MP9020

Fou f Computer Science

Time allowed: **2 hours**
Total number of questions: **10**
Maximum number of marks: 50

Not all questions are worth the same.

Answer all questions.

Textbooks, lecture notes, etc. are not permitted, except for up to 2 double-sided A4 sheets containing handwritten notes.

Calculators may not be used.

Answers must be written in ink. Use a pencil or the back of the booklet for rough work. Your rough work will not be marked.

You can answer the questions in any order.

You may take this question paper out of the exam.

Write your answers into the answer booklet provided.

Question 1 (5 marks)

A *leap year* is a year containing one additional day. Nowadays, that day, the 29th of February, is added on every year divisible by 4, unless it is also divisible by 100 but not by 400. For instance, the years 2000 and 2004 were leap years but 1900 was not.

For the purpose of th retend that this rule has always been followed since the year 0 and will alw

For $f, \ell \in \mathbb{N}$ with $f \leq$ a $leap(f, \ell)$ that expresses the exact number of leap years in the interval [the two arguments f and ℓ as well as the arithmetic operations floor, addit d division.

Example: applying the formula to $f = 1896$ and $\ell = 2004$ should yield $leap(1896, 2004) = 27$.

Answer:

$$\begin{aligned} &\lfloor \ell/4 \rfloor - \lfloor (f-1)/4 \rfloor && \text{divisible by 4} \\ &- \lfloor \ell/100 \rfloor - \lfloor (f-1)/100 \rfloor && \text{divisible by 100} \\ &+ \lfloor \ell/400 \rfloor - \lfloor (f-1)/400 \rfloor && \text{divisible by 400} \end{aligned}$$

Question 2 (5 marks)

Prove that $\sum_{i=0}^{n} k^i = \frac{k^{n+1}-1}{k-1}$ for all $n, k \in \mathbb{N}$ with $k > 1$.

Answer: by induction on n. Base case $\sum_{i=0}^{0} k^i = 1 = \frac{k-1}{k-1} = \frac{k^{0+1}-1}{k-1}$. Inductive case

$$\begin{aligned} \sum_{i=0}^{n+1} k^i &= k^{n+1} + \sum_{i=0}^{n} k^i \\ &= k^{n+1} + \frac{k^{n+1}-1}{k-1} && \text{by the ind. hyp.} \\ &= \frac{(k-1)k^{n+1} + k^{n+1} - 1}{k-1} \\ &= \frac{k^{n+2}-1}{k-1} \ . \end{aligned}$$

Question 3 (4 marks)

Prove that $(S \cap T) \times (U \cap V) = (S \times U) \cap (T \times V)$ holds for all sets S, T, U, and V.

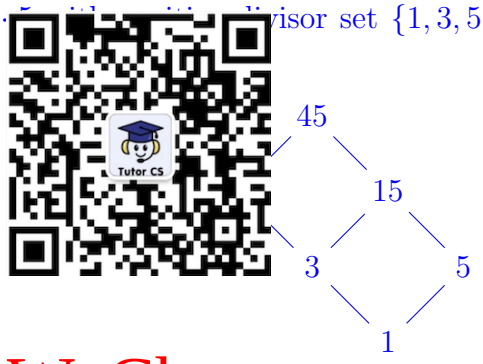
Answer: the claimed equality holds.

$$\begin{aligned} (s,u) \in (S \cap T) \times (U \cap V) &\Leftrightarrow s \in (S \cap T) \wedge u \in (U \cap V) \\ &\Leftrightarrow s \in S \wedge s \in T \wedge u \in U \wedge u \in V \\ &\Leftrightarrow s \in S \wedge u \in U \wedge s \in T \wedge u \in V \\ &\Leftrightarrow (s,u) \in S \times U \wedge (s,u) \in T \times V \\ &\Leftrightarrow (s,u) \in (S \times U) \cap (T \times V) \end{aligned}$$

Question 4 (5 marks)

Consider the partial order $D_{45} = (\{x \in \mathbb{N} : x | 45\}, \sqsubseteq)$ defined by the positive divisors of 45 by $x \sqsubseteq y$ iff $x|y$. Give 3 different topological sorts of D_{45}.

Answer: $45 = 3 \cdot 3 \cdot 5$, with positive divisor set $\{1, 3, 5, 9, 15, 45\}$. We can arrange this in a Hasse diagram



and read off the topological sorts:

$\langle 1, 3, 5, 9, 15, 45 \rangle$
$\langle 1, 5, 3, 9, 15, 45 \rangle$
$\langle 1, 3, 5, 15, 9, 45 \rangle$
$\langle 1, 5, 3, 15, 9, 45 \rangle$
$\langle 1, 5, 3, 9, 15, 45 \rangle$
$\langle 1, 3, 9, 5, 15, 45 \rangle$

Question 5 (6 marks)

Consider the four functions

$$f_1(n) = n^3 \qquad f_2(n) = n^5$$

$$f_3(n) = \begin{cases} n^3 & \text{if } n \text{ is odd} \\ n^5 & \text{otherwise} \end{cases} \qquad f_4(n) = \begin{cases} n^3 & \text{if } n \text{ is prime} \\ n^5 & \text{otherwise} \end{cases}$$

and determine whether

(a) $f_1(n)$ is $\mathcal{O}(f_2(n))$,

(b) $f_2(n)$ is $\mathcal{O}(f_3(n))$,

(c) $f_4(n)$ is $\mathcal{O}(f_3(n))$.

For each of the three parts, either give values n_0 and c that prove the big-oh relationship, or assume that there are such values n_0 and c, and then derive a contradiction.

Answer:

(1, 2): f_1 is $\mathcal{O}(f_2)$. Constants $n_0 = 0$ and $c = 1$ work as witnesses.

(2, 3): f_2 is not $\mathcal{O}(f_3)$. Suppose otherwise for constants n_0 and c. Consider the smallest odd number m at least $\max(n_0 + 1, \sqrt{c})$. Then $f_2(m) = m^5 \geq m^3 c = c f_3(m)$.

(4, 3): f_4 is not $\mathcal{O}(f_3)$. Suppose otherwise for constants n_0 and c. Consider the smallest composite number m at least $\max(n_0 + 1, \sqrt{c})$. Then $f_4(m) = m^5 \geq m^3 c = c f_3(m)$.

Question 6 (6 marks)

For each of the following recurrences, what is $T(n)$'s order of growth?

$$T(n) = 2T(n-1) + 4n^2 + 3n + 2 \tag{1}$$

$$T(n) = 2T(\frac{n}{2}) + 4n^3 + 3n^2 + 2n + 1 \tag{2}$$

$$T(n) = 2T(\frac{n}{2}) + 4n^2 + 3n + 2 \tag{3}$$

Answer: by the mast

(a) $\mathcal{O}(2^n)$

(b) $\mathcal{O}(n^3 \log n)$

(c) $\mathcal{O}(n^2)$

Question 7 (5 marks)

Suppose license plates are constructed using three letters picked (without replacement) from the word `SYDNEY` followed by two decimal digits picked (again without replacement) from the word `012345566778899`.

(a) How many license plates can be constructed if the letters and digits have to be different (e.g. `DYS54` is allowed but `YSY54` is not because `Y` occurs twice),

(b) How many license plates can be constructed if the letters and digits need not be different (e.g. `YSY54` is allowed and so is `END55` but not `SDD64`)?

Answer:

(a) remove `Y` from the first and `56789` from the second word to arrive at `SYDNE` and `0123456789`. Choosing three different letters from the first $= 5 \cdot 4 \cdot 3 = 60$ possibilities. Choosing two from the second word $= 10 \cdot 9 = 90$ possibilites. Together that's $60 * 90 =$ **5400** different license plates.

(b) We start with 60 choices as before for the three letters from the first word where all letters are different. To those we add the ones containing two `Y`s. There are 4 other letters and 3 positions for the other letter $= 12$ different words with two `Y`s. The two numbers can be chosen from 10 digits so that's $10^2 = 100$ possibilites. But we've accounted for certain combinations that we cannot have—the combinations of the form dd for $d \in \{0, 1, 2, 3, 4\}$. There are 5 of those. Altogether we have $(60 + 12) \cdot (100 - 5) =$ **6840**.

Question 8 (5 marks)

Let G_k be a graph on k vertices, consisting of a cycle of $k - 1$ vertices plus a central node connected to all vertices on the cycle. Given $k > 3$, what is the chromatic number of G_k?

Answer: $\chi(G_k) \in \{3, 4\}$: the cycle may be odd or even, needing 2 or 3 colours, the central node takes one more.

Question 9 (5 marks)

You have applied to join a chess club and been told that to qualify you must play three games against X, winning two games in a row. In other words, the sequences "WWL" (for "win,win,loss"), "WWW", and "LWW" will get you qualified whereas "WLW" and anything with fewer than two w[illegible] qualified.

*"Who gets the white p[illegible]nd are told you and X alternate and you can choose whether to start the fi[illegible]e or with black.

Knowing that the prob[illegible] is better with the white pieces (first-move advantage) should you choose wh[illegible] first game?

Use $0 < P_b < P_w < 1$ for the probability of winning when playing with the black, respectively, white pieces to model the problem and prove your answer correct.

Answer: I should choose to start with black. Let $p, q \in (0, 1)$ be the probability of me winning with black and white, respectively, so $p < q$. The probability of winning two games in a row when starting the first game with black is the sum of the probabilities of winning the first two, the last two, or all three games: $pq(1-p) + (1-p)qp + qpp = (2(1-p) + p)pq = (2-p)pq$ whereas the same calculation for beginning with white results in $qp(1-q) + (1-q)pq + qpq = (2(1-q) + q)qp = (2-q)qp$. Since $p < q$, $(2-p)pq > (2-q)pq$.

Question 10 (4 marks)

Suppose you are taking an exam consisting of 50 multiple choice questions. Each question has four possible answers exactly one of which is correct. A correct answer scores 2, an incorrect answer scores -1 and blank scores 0. You did not study at all, and decide to randomly guess all the answers and leave no blanks. What should you expect to score in the exam? Derive the correct answer to this question mathematically.

Answer: $2 \cdot \frac{1}{4} \cdot 50 - 1 \cdot \frac{3}{4} \cdot 50 = 25 - 37.5 = \mathbf{-12.5}$