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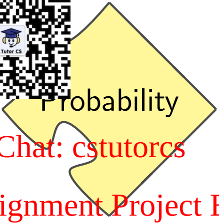
Foundations of Computer Science

Lecture 14: Combinatorics

Topic 4: Probability



		[LLM]	[RW]	[Rosen]
Week 9	Combinatorics	Ch. 14	Ch. 5	Ch. 6, 8
Week 10	Probability	Ch 16, 17	Ch. 9	Ch. 7
Week 10	Statistics	Ch. 18	Ch. 9	Ch. 7

Combinatorics in Computer Science

Combinatorics:

- Computing cost functions in algorithmic analysis
- Identifyiing (in-)efficiencies in data management
- Developing effective techniques for enumerating objects
- Probability calculations

Probability in Computer Science

Probability:

- Artificial Intellig
  - Machine Lea
  - Decision the
  - Image proce g
  - Speech recognition
- Algorithms
  - Algorithm analysis
  - Big Data sampling and analysis
- Security
  - Cryptography
  - Quantum computing
- Networks
  - Network traffic modelling
  - Reliability modelling

Statistics in Computer Science

Statistics:

- Sampling from large data sets
- Identifying anomolies
- Making predictions

Outline

Counting Principles

Basic Counting Rules

Basic Counting Rules: Product

Combinations and Permutations

Alternative Techniques

Difficult Counting Problems

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Counting Techniques

General idea: find methods, algorithms or precise formulae to count the number of elements in various sets or collections derived, in a structured way, from some basic sets.

Examples

Single base set $S = \{s_1, \ldots, s_n\}$, $|S| = n$; find the number of

- all subsets of S
- ordered selections of r different elements of S
- unordered selections of r different elements of S
- selections of r elements from S such that ...
- functions $S \longrightarrow S$ (onto, 1-1)
- partitions of S into k equivalence classes
- graphs/trees with elements of S as labelled vertices/leaves

Example

Example

A restaurant has the following menu:

Starter	Main Course	Dessert
Soup	Fish	Ice-cream
Bread	Beef	Fruit
	Pork	Cheese
	Chicken	

How many:

- 3 course meals (Starter-Main-Dessert) are possible?
- 3 course meals (Any item for each course) are possible?
- 3 course meals (Any item, no duplicates) are possible?
- Meals consisting of 3 items (order is unimportant)?

Example

Example

A restaurant has the following menu:

Starter	Main Course	Dessert
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Bread	Beef	Fruit
	Pork	Cheese
	Chicken	

How many:

- Starter-Main-Dessert?
- Any item for 3 courses?
- Any item, no duplicates, for 3 courses?
- Meals of 3 items?

Basic Counting Rules: Principles

Two simple rules:

- **Union rule** (“or”): If S and T are disjoint $|S \cup T| = |S| + |T|$
- **Product rule** (“and”, “by”): $|S \times T| = |S| \cdot |T|$

These cover many examples, though the rule application is not always obvious.

Common strategies:

- Direct application of the rule
- Relate unknown quantities to known quantities (e.g. $|S| + |T| = |S \cup T| + |S \cap T|$)
- Find a bijection to a set that can be counted

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The Union Rule

Union rule — S and T *disjoint*

$$|S \cup T| = |S| + |T|$$

$S_1, S_2, \ldots, S_n$ pairwise disjoint $(S_i \cap S_j = \emptyset \text{ for } i \neq j)$

$$|S_1 \cup \ldots \cup S_n| = \sum |S_i|$$

Example

How many numbers in $A = [1, 2, \ldots, 999]$ are divisible by 31 or 41?

$\lfloor 999/31 \rfloor = 32$ divisible by 31
$\lfloor 999/41 \rfloor = 24$ divisible by 41
No number in A divisible by both
Hence, $32 + 24 = 56$ divisible by 31 or 41

Consequences of the Union Rule

Fact

For any sets X, Y, Z

$$
\begin{aligned}
|Y \setminus X| &= |Y| - |X \cap Y| \\
|X \cup Y| &= |X| + |Y| - |X \cap Y| \\
|X \cup Y \cup Z| &= |X| + |Y| + |Z| \\
&\quad - |X \cap Y| - |Y \cap Z| - |Z \cap X| \\
&\quad + |X \cap Y \cap Z|
\end{aligned}
$$

Fact

- *If* $|S \cup T| = |S| + |T|$ *then* S *and* T *are disjoint*
- *If* $|\bigcup_{i=1}^{n} S_i| = \sum_{i=1}^{n} |S_i|$ *then* S_i *are pairwise disjoint*
- *If* $|T \setminus S| = |T| - |S|$ *then* $S \subseteq T$

These properties can serve to identify cases when sets are disjoint (resp. one is contained in the other).

Proof.

$|S| + |T| = |S \cup T|$ means $|S \cap T| = |S| + |T| - |S \cup T| = 0$

$|T \setminus S| = |T| - |S|$ means $|S \cap T| = |S|$ means $S \subseteq T$ $\square$

Exercises

RW: 5.3.1 200 peopl[illegible]m or jog, 85 swim and 60 do both. How many jog?

RW: 5.6.38 (Supp) There are 100 problems, 75 of which are 'easy' and 40 'important'.

What's the smallest number of easy *and* important problems?

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The Product Rule

Product rule

$$|S_1 \times \ldots \times S_k| = |S_1| \cdot |S_2| \cdots |S_k| = \prod_{i=1}^{k} |S_i|$$

If all $S_i = S$ (the same set) and $|S| = m$ then $|S^k| = m^k$

NB

This counts the number of sequences where the first item is from S_1, the second is from S_2, and so on.

Example

Let $\Sigma = \{a, b, c, d, e, f, g\}$.
How many 5-letter words?

$$|\Sigma^5| = |\Sigma|^5 = 7^5 = 16,807$$

How many with no letter repeated?

Product rule: Sequences of selections

Question

How can we count sequences when the underlying set changes?

To count sequences *placement*:

- Define an order hole underlying set
- Select from $[1, n]$, where n is the size of the "remaining" set, and a selection of i represents choosing the i-th element in that set

Example

Let $\Sigma = \{a, b, c, d, e, f, g\}$.

How many 5-letter words with no letter repeated?

$$\prod_{i=0}^{4}(|\Sigma| - i) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2,520$$

Exercises

Exercises

S, T finite. How many functions $S \longrightarrow T$ are there?

RW: 5.1.19 Consider a *complete* graph on n vertices.

(a) No. of paths of length 3

(b) paths of length 3 with all vertices distinct

(c) paths of length 3 with all edges distinct

Exercise

Exercise

RW: 5.3.2 $S = [100 \dots 999]$, thus $|S| = 900$.

(a) How many numb t least one digit that is a 3 or 7?

(b) How many numbers have a 3 *and* a 7?

Combinatorial Symmetry

A **symmetry** of a mathematical object is a bijective mapping from the object to itself which preserves "structure".

A (combinatorial) symmetry defines an equivalence relation where the equivalence classes all have the same size.

We are often interested in counting a set "up to symmetry". That is, counting the number of equivalence classes.

This can also be stated as a constraint that identifies a specific item in each equivalence class (**symmetric constraint**).

Definition

A *k-to-1 function* is a function that maps exactly k inputs to an output.

NB

A k-to-1 function defines the equivalence relation of a combinatorial symmetry and vice-versa.

Product rule: Symmetries and duplications

Question

- *How can we co*[illegible]*ces when we have symmetric constraints?*
- *How can we co*[illegible]*ces when we have duplicates?*

Example

Let $\Sigma = \{a, b, c, d, e\}$.

- How many 5-letter words with no letter repeated and a before b before c?

- How many 5-letter words can be made from a, a, a, d, e?

NB

The answer will be the same.

Product rule: Symmetries and duplications

- S_1 = sequences g for symmetry,
- S_2 = symmetrie
- S = sequences without symmetry

$$S = S_1 \times S_2,$$

so

$$|S_1| = |S|/|S_2|$$

Alternatively, $\frac{1}{|S_2|}$ of the $|S|$ sequences meet the symmetric constraint.

Product rule: Symmetries and duplications

Example

Let $\Sigma = \{a, b, c, d, e$
How many 5-letter w no letter repeated and a before b before c?

Let $\Sigma' = \{a, b, c\}$.

$$S = \prod_{i=0}^{4}(|\Sigma| - i) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

$$S_2 = \prod_{i=0}^{2}(|\Sigma'| - i) = 3 \cdot 2 \cdot 1 = 6$$

$$\text{So } S_1 = 120/6 = 20$$

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Combinatorial Objects: How Many?

permutations

Ordering of all objects from a set S; equivalently: Selecting all objects while *recognising* the order of selection.
The number of permutations of a set of n elements is

$$n! = n \cdot (n-1) \cdots 1, \quad 0! = 1! = 1$$

r-permutations (sequences without repetition)

Selecting any r objects from a set S of size n without repetition while *recognising* the order of selection.
Their number is

$$(n)_r = {}^nP_r = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$$

Permutations with duplicates

Example

How many anagrams of ASSESS?

Label S's: $AS_1S_2ES_3S_4$: $6!$

In each anagram we can label the S's in $4!$ ways.

Suppose there are m anagrams. So $m \cdot 4! = 6!$, i.e. $m = \frac{6!}{4!}$

Example

Number of anagrams of MISSISSIPPI? $\frac{11!}{3!4!2!}$

r-**selections** (or: r-**combinations**)

Collecting any r distinct objects without repetition;

equivalently: selecting r objects from a set S of size n and *not* recognising the order of selection.

Their number is

$$\binom{n}{r} = \frac{(n)_r}{r!} = \frac{n!}{(n-r)!r!} = \frac{n\cdot(n-1)\cdots(n-r+1)}{1\cdot 2\cdots r}$$

NB

These numbers are usually called binomial coefficients due to

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + b^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$$

Also defined for any $\alpha \in \mathbb{R}$ *as* $\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}$

Simple Counting Problems

Example

RW: 5.1.2 Give an example of a counting problem whose answer is

(a) $(26)_{10}$

(b) $\binom{26}{10}$

Draw 10 cards from a half deck (eg. black cards only)
(a) the cards are recorded in the order of appearance
(b) only the complete draw is recorded

Examples

- Number of edges in a complete graph K_n
- Number of diagonals in a convex polygon
- Number of poker hands
- Decisions in games, lotteries etc.

Exercises

RW: 5.1.6 From a group of 12 men and 16 women, how many committees can be chosen consisting of

(a) 7 members?

(b) 3 men and 4 women?

(c) 7 women or 7 men?

RW: 5.1.7 As above, but any 4 people (male or female) out of 9 and two, Alice and Bob, unwilling to serve on the same committee.

Counting Poker Hands

Exercises

RW: 5.1.15 A poker [illegible]ists of 5 cards drawn without replacement from a s[illegible]eck of 52 cards

$$\{A, 2\text{-}10, J, Q, K\} \times \{\text{club, spade, heart, diamond}\}$$

(a) Number of "4 of a kind" hands (e.g. 4 Jacks)

(b) Number of non-straight flushes, i.e. all cards of same suit but *not* consecutive (e.g. 8,9,10,J,K)

Selecting items summary

Selecting k items from a set of n items:

With replacement	Order matters	Examples	Formula
Yes	Yes	Words of length k (sequences of length k)	n^k
No	Yes	k-permutations	$(n)_k$
No	No	Subsets of size k	$\binom{n}{k}$
Yes	No	Multisets of size k	$\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$

"Balls in boxes"

Have n "distinguishable" boxes.
Have k balls which are either

1. Indistinguishable
2. Distinguishable

How many ways to place balls in boxes with

- A. At most one
- B. Any number of

balls per box?

NB

Suppose K is a set with $|K| = k$ and N is a set with $|N| = n$:

- *2A counts the number of injective functions from K to N*
- *2B counts the number of functions from K to N*

"Balls in boxes"

Case	[illegible]	Balls per box	Number
1A	Indist.	At most 1	$\binom{n}{k}$
1B	Indist.	Any number	$\binom{n+k-1}{k}$
2A	Dist.	At most 1	$(n)_k$
2B	Dist.	Any number	n^k

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Alternative techniques

What if the current techniques are unwieldy?
Other techniques for obtaining an exact count:

- Find a different approach for counting
- Make use of symmetries
- Make use of recursion
- Write a program (running time?)

Example

How many sequences of 15 coin flips have an even number of heads?

- Using "balls in boxes": $\binom{15}{0} + \binom{15}{2} + \ldots + \binom{15}{14}$
- Use symmetry: $\frac{1}{2} \times 2^{15}$
- Use recursion: $\mathrm{Even}(n) = \mathrm{Odd}(n-1) + \mathrm{Even}(n-1)$; $\mathrm{Odd}(n) = \mathrm{Even}(n-1) + \mathrm{Odd}(n-1)$

Example

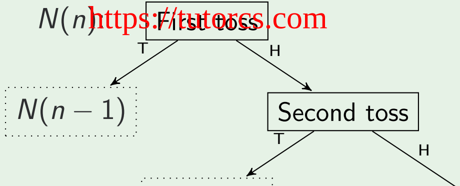
Example

How many sequences of n coin flips do not contain HH?

$$
\begin{aligned}
C(0)& \\
C(1)& \\
C(n)& = C(n-1) + C(n-2) + 2^{n-2}
\end{aligned}
$$

$$
\begin{aligned}
N(0) &= 1 \\
N(1) &= 2 \\
N(2) &= 3 \\
N(n) &= N(n-1) + N(n-2)
\end{aligned}
$$

We can summarise all possible outcomes in a **recursive tree**



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Example (Ramsay

An example of a *Ramsey number* is $R(3,3) = 6$, meaning that

> *"K_6 is the smallest complete graph such that if all edges are painted using two colours, then there must be at least one monochromatic triangle"*

This serves as the basis of a game called S-I-M (invented by Simmons), where two adversaries connect six dots, respectively using blue and red lines. The objective is to *avoid* closing a triangle of one's own colour. The second player has a winning strategy, but the full analysis requires a computer program.

Using Programs to Count

Two dice, a red die and a black die, are rolled.
(Note: one *die*, two *dice*)

Write a program to list all pairs $\{(R, B) : R > B\}$

Similarly, for three dice, list all triples $R > B > G$

Generally, for n dice, all of which are m-sided ($n \leq m$), list all *decreasing* n-tuples

NB

In order to just find the number of such n-tuples, it is not necessary to list them all. One can write a recurrence relation for these numbers and compute (or try to solve) it.

Approximate Counting

NB

A Count may be a p[illegible]*e or an* **estimate***.*

The latter should be asymptotically correct or at least give a good asymptotic bound, whether upper or lower. If S *is the base set,* $|S| = n$ *its size, and we denote by* $c(S)$ *some collection of objects from* S *we are interested in, then we seek constants* a, b *such that*

$$a \leq \lim_{n \to \infty} \frac{est(|c(S)|)}{|c(S)|} \leq b$$

In other words $est(|c(S)|) \in \Theta(|c(S)|)$.